



Article Newton's Law of Cooling with Generalized Conformable Derivatives

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Abstract: In this communication, using a generalized conformable differential operator, a simulation of the well-known Newton's law of cooling is made. In particular, we use the conformable $t^{1-\alpha}$, $e^{(1-\alpha)t}$ and non-conformable $t^{-\alpha}$ kernels. The analytical solution for each kernel is given in terms of the conformable order derivative $0 < \alpha \le 1$. Then, the method for inverse problem solving, using Bayesian estimation with real temperature data to calculate the parameters of interest, is applied. It is shown that these conformable approaches have an advantage with respect to ordinary derivatives.

Keywords: fractional calculus; conformable derivative; Newton law of cooling

PACS: 47.54.Bd; 47.55.pb; 45.10.Hj

1. Introduction

The fractional calculus idea was first suggested by Leibniz and L'Hopital in a letter three centuries ago; it is an area of classical mathematics which deals with derivatives and integrals of arbitrary orders [1,2]. Along its history, many fractional derivative definitions have been introduced [3,4]. All of them are defined via fractional integrals, thus they inherit nonlocal properties from integrals. Heredity and nonlocality are properties of these definitions, important in many application fields. In particular, in recent years, the notion of conformable derivative was introduced in terms of an incremental quotient, which opened a new direction in this area: the conformable calculus. The conformable derivative definition was first given by Khalil et al. in [5,6], with $0 < \alpha < 1$, this operator shows a similarity to the integer order derivative and overcomes many of the shortcomings of the classical fractional derivatives [1,2]. In particular, if f is differentiable the result is obtained by multiplying the first derivative by a certain factor with fractional power. Recently, a new non conformable derivative definition has been introduced in [7]. Although these definitions are valid and work in the case $0 < \alpha < 1$, a general definition was needed for conformable derivatives of any order, integer or not, generalizing the well known conformable derivatives to higher orders [8]. On the other hand, along with this theoretical development, the applications of fractional and generalized calculus have been increased to various areas of science and technology [9–18], and have shown advances in relation to the known integer order. In [19], the conformable derivative was applied to the Newton's law of cooling, having as solution the Kohlraush stretched exponential function. In [20], it was performed an experimental setup to verify the effectiveness of the conformable derivative.



Citation: Vivas-Cortez, M.; Fleitas, A.; Guzmán, P.M.; Nápoles, J.E.; Rosales, J.J. Newton's Law of Cooling with Generalized Conformable Derivatives. *Symmetry* **2021**, *13*, 1093. https://doi.org/10.3390/sym13061093

Academic Editors: Olaniyi S Iyiola and Erkan Nane

Received: 17 March 2021 Accepted: 11 May 2021 Published: 21 June 2021

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Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). In this communication, we start with the ordinary Newton's law of cooling and construct its corresponding fractional equation using the generalized conformable derivative [7]. After that, based on a set of experimental data, we studied Newton's law of cooling using different kernels, conformable and non-conformable. Then, the method of the inverse problem solving using Bayesian estimation is used to calculate the parameters of interest. It is shown that these conformable approaches have an advantage with respect to ordinary derivative.

2. Preliminary

In [7] a generalized conformable derivative was defined in the following way (see also [16]). Given a function $f : [0, +\infty) \to \mathbb{R}$. Then, the N-derivative of f of order α is defined by

$$N_F^{\alpha}f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon F(t, \alpha)) - f(t)}{\varepsilon},$$
(1)

for all t > 0, $\alpha \in (0, 1)$ being $F(t, \alpha)$ some function. If f is α —differentiable in some $(0, \alpha)$, and $\lim_{t\to 0^+} N_F^{(\alpha)} f(t)$ exists, then define $N_F^{(\alpha)} f(0) = \lim_{t\to 0^+} N_F^{(\alpha)} f(t)$. The definition given in this way is a natural generalization of the ordinary derivative. This definition generalizes the conformable derivative given in [5] and its properties have been studied.

The $N_F^{\alpha}f(t)$ derivative defined above (1) is a local type derivative, because it is about a limit of a certain incremental quotient. So, its geometric interpretation should be similar to the interpretation of the ordinary derivative: the slope of the tangent line to the graph of the function at the point (t, f(t)), since [7]

$$N_F^{\alpha}f(t) = F(t,\alpha)f'(t), \qquad (2)$$

where f' is an ordinary derivative. The factor $F(t, \alpha)$ provides us the information about the shape of the curve that represents the coordinate f(t). Even more, it is clear that the influence of the kernel $F(t, \alpha)$, implies some variation of the "speed of convergence" (greater or lesser) with respect to the limit of the incremental quotient for the ordinary derivative.

On the other hand, Newton's law of cooling states that the rate of heat loss of a body is proportional to the temperature difference between the body and its environment. The transfer of heat is important in the processes because it is a type of energy that is in motion due to a temperature difference, and it is present in processes such as condensation, vaporization, crystallization, climate changes, chemical reactions and so forth, where heat transfer has its own mechanisms and each of them has particular characteristics. In particular, what Newton's law affirms is that the cooling of a body is directly proportional to the difference between the instantaneous temperature of the body T(t), t > 0 and the environment T_e :

$$\frac{dT}{dt} = -k(T(t) - T_e), \qquad T(0) = T_0,$$
(3)

where T_0 is the initial temperature at t = 0. This expression is not very precise and is considered only a valid approximation for small differences between T and T_e . Newton's law of cooling is generally limited to simple cases where the mode of energy transfer is convection, from a solid surface to a surrounding fluid in motion. In any case the above expression is useful to show how the cooling of a body follows approximately a law of exponential decay,

$$T(t) = T_e + (T_0 - T_e)e^{-kt}.$$
(4)

Figure 1 shows the behavior of the function (4).



Figure 1. Shows the curve associated with the temperature T(t) for the ordinary model (4).

To determine the value of k, we suppose that after some given time t_1 , the temperature changes from T to T_1 , with these conditions we can find the value of k from (4), getting,

$$k = \frac{1}{t_1} ln \left(\frac{T_0 - T_e}{T_1 - T_e} \right).$$
(5)

3. Results

In this work, we model the Newton's law of cooling using a conformable and nonconformable kernels in a generalized conformable differential operator. From the numerical study carried out, some methodological observations are made.

First of all, we must ensure that by transforming Equation (3) to its corresponding fractional equation, the physical parameters in it retain their physical units. Following [21], it is necessary to take into account "the distortion" suffered by the derivative when going from the ordinary case to the non-integer. We can use the physical parameter $[k] = s^{-1}$ to pass from the ordinary differential operator to the fractional one, as follows

$$\frac{d}{dt} = k^{1-\alpha} \frac{d^{\alpha}}{dt^{\alpha}}, \qquad 0 < \alpha \le 1.$$
(6)

In this way, the differential operator retains its dimensionality $[d/dt] = s^{-1}$, and α is the order of derivative. Substituting (6) in (3), we obtain the fractional Newton's law of cooling, as

$$\frac{d^{\alpha}T}{dt^{\alpha}} = -k^{\alpha}(T(t) - T_e), \qquad 0 < \alpha \le 1.$$
(7)

A similar equation has been solved by applying Caputo and Riemann-Liouville type fractional derivatives for water, mustard oil and mercury [17]. In [19] the conformable derivative was applied to the same equation, and later applied to experimental data for the water [20]. In this case, a closed expression for the parameter k was obtained, which cannot be obtained by using fractional derivatives [17].

In order to find a solution as accurate as possible that predicts the experimental results, in this work we analyse different kernels, conformable and nonconformable. Using the property (2) together with (6), we get

$$\frac{df}{dt} = k^{1-\alpha} \frac{d^{\alpha} f}{dt^{\alpha}} = k^{1-\alpha} N^{\alpha} f(t) = k^{1-\alpha} F(t,\alpha) \frac{df(t)}{dt}.$$
(8)

Replacing this operator in the expression (3), we obtain the general conformable Newton's low of cooling

$$\frac{dT}{dt} = -\frac{k^{\alpha}}{F(t,\alpha)} \Big(T(t) - T_e \Big).$$
(9)

This equation is an ordinary differential equation with parameter $0 < \alpha \le 1$, when $T(0) = T_0$, it has the particular solution

$$T(t,\alpha) = T_e + (T_0 - T_e) \exp\left(-\int \frac{k^{\alpha}}{F(t,\alpha)} dt\right), \qquad 0 < \alpha \le 1,$$
(10)

where the function $F(t, \alpha)$ plays the role of a generalized conformable kernel. We consider different types of particular kernels.

Khalil Kernel: In this case, the generalized kernel $F(t, \alpha)$ takes the particular form $F(t, \alpha) = t^{1-\alpha}$. Then, from (10), we have the particular solution

$$\Gamma(t,\alpha) = T_e + (T_0 - T_e)e^{-\frac{k^\alpha}{\alpha}t^\alpha},$$
(11)

fulfilled the initial condition $T(0) = T_0$. Assuming that after a time τ , the temperature is T_1 , then the convection coefficient is [19],

$$k(\alpha) = \left[\frac{\alpha}{\tau^{\alpha}} \ln\left(\frac{T_0 - T_e}{T_1 - T_e}\right)\right]^{1/\alpha}.$$
(12)

It is easy to see that, when $\alpha \to 1$ we get (4) and (5). Figure 2 shows the behaviour of $k(\alpha)$ depending on the some values of α (12), associated with the conformable model. Figure 3 shows the behaviour of temperature with regard to time associated with the model with derivative conformable with kernel $t^{1-\alpha}$ (11), for different values of α .



Figure 2. Shows the behavior of $k(\alpha)$ (12) depending on the some values of α .



Figure 3. Shows the curves associated with the temperature $T(t, \alpha)$ (11), associated with some values of α ; {1 (black), 0.9 (yellow), 0.75 (blue), 0.5 (green), 0.25 (red).

Conformable kernel: This particular kernel has the form $F(t, \alpha) = e^{(1-\alpha)t}$, then from (10), we have

$$T(t,\alpha) = T_e + (T_0 - T_e) \exp\left(-\frac{k^{\alpha}}{\alpha - 1}e^{(\alpha - 1)t}\right).$$
(13)

Then, after a certain time τ the body has the temperature T_1 , the convection coefficient is obtained in a closed form

$$k(\alpha) = \left[\frac{1-\alpha}{e^{(\alpha-1)\tau}}\ln\left(\frac{T_1-T_e}{T_0-T_e}\right)\right]^{1/\alpha}.$$
(14)

Figure 4 shows the behaviour of $k(\alpha)$ (14) for some values of α , and Figure 5 shows some curves which represent the Equation (13) varying the variable *t* for some values of α using the conformable kernel $e^{(1-\alpha)t}$.



Figure 4. Shows the behavior of $k(\alpha)$ depending on the values of α , (14).



Figure 5. Shows the curves associated with the direct type problem (13), for the values of α {1 (black), 0.9 (yellow), 0.75 (blue), 0.5 (green), 0.25 (red)}.

Non conformable kernel: Finally, we consider the kernel $F(t, \alpha) = t^{-\alpha}$. In this case, from (10), we get the solution

$$T(t,\alpha) = T_e + (T_0 - T_e)e^{-\frac{k^{\alpha}}{\alpha+1}t^{\alpha+1}},$$
(15)

with the corresponding convective coefficient

$$k(\alpha) = \left[\frac{\alpha+1}{\tau^{\alpha+1}}\ln\left(\frac{T_0 - T_e}{T_1 - T_e}\right)\right]^{1/\alpha}.$$
(16)

The dependence of $k(\alpha)$ and the temperature $T(t, \alpha)$ with respect to α are given in Figures 6 and 7, for the models with non-conformable derivative (15) and (16) with kernel $t^{-\alpha}$. In the case of Figure 7, the curve of the ordinary model (4) with discontinuous lines in black is also shown.



Figure 6. Shows the behaviour of $k(\alpha)$ (16), depending on the values of α .



Figure 7. Shows the curves associated with the direct type problem (15), for the values of α {1 (black), 0.9 (yellow), 0.75 (blue), 0.5 (green), 0.25 (red)}. The ordinary model shown with discontinuous lines in black color.

Remark 1. We must point out that, in the case of conformable kernels the ordinary case has not been drawn, because for $\alpha = 1$, these derivatives are reduced to the classical derivative, therefore, they coincide with the model. In the non-conformable case, when $\alpha = 1$ this does not happen, hence the need to compare with the ordinary model.

4. Materials and Methods

In the same direction, an observation equation is associated to the model

$$y_i = g(T(t_i)) + \varepsilon_i, \qquad i = 1, \dots, n, \tag{17}$$

where y_i correspond to the i - th observed value under uncertainty from a solution of (9) associated with temperature values at the discrete time $t_i \in [0; t]; i = 1, 2, ..., n; g$ is the observation function and ϵ_i are measurement errors, which are considered as independent and identically distributed (i.i.d.) random variables from a normal distribution, with mean zero and constant variance σ^2 , denoted by $\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$.

For the model defined in (9) and (17), the parameter of interest is $\phi = (\alpha, k, Te, T_0, \sigma^2)$. The prior distributions proposed are: $k \sim \mathcal{G}(\gamma_k, \mu_k)$, $Te \sim \mathcal{G}(\gamma_e, \mu_e)$, $T_0 \sim \mathcal{G}(\gamma_0, \mu_0)$ and $\tau \sim \mathcal{G}(\gamma_\tau, \mu_\tau)$; where $\mathcal{G}(\gamma, \mu)$ denotes the Gamma distribution with shape parameter γ and rate parameter μ . \mathcal{U} is the continuous uniform distribution on the interval (0,1); $\alpha \sim \mathcal{U}(0,1), \tau = 1/\sigma^2$.

The parameters introduced in the prior distributions are called hyperparameters. The prior distributions have been defined in order to reflect the information already known about the possible values of the parameters of interest. Taking for granted prior independence of the parameters, the joint prior distribution can be written as: $p(\phi|hyperparameters) = p(\alpha)p(k|\gamma_k,\mu_k), p(Te|\gamma_{Te},\mu_{Te}), p(T_0|\gamma_{T0},\mu_{T_0}), p(\tau|\gamma_{\tau},\mu_{\tau})$, where $p(a|\gamma_a,\mu_a), p(\alpha)$ and $p(\tau|\gamma_{\tau},\mu_{\tau})$ have been previously defined.

Let $O = (O_1, O_2, ..., O_n)$ denote observed data at times, which are independent and identically distributed $(t_1, t_2, ..., t_n)$ from the model defined by (9) and (17), the likelihood function is given by

$$L(O|\phi) = \prod_{i=1}^{n} f_O(O_i) = \frac{1}{(\sigma\sqrt{2\pi})^n} \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (O_i - g(T(t_i)))^2\right\},$$
(18)

where $T(t_i)$, i = 1, ..., n is a solution of (9). Then, applying Bayes theorem, the posterior distribution of the parameters of interest is given by

$$p(\phi|O) = \frac{L(O|\phi)p(\phi)}{\int_{\ominus} L(O|\phi)p(\phi)d\phi'},$$
(19)

where \ominus denotes the parameter space of ϕ . It is known that,

$$p(\phi|O) \propto L(O|\phi)p(\phi).$$
 (20)

Adopting a loss quadratic function, the Bayesian point estimation is the posterior mean of $\hat{\phi}_B$, which is given by $\hat{\phi}_B = E(\phi|O)$. Markov Chain Monte Carlo simulations (MCMC) are employed to work with (19). One of the most popular MCMC techniques is the Metropolis-Hastings [22]. WinBUGS and JAGS are some computer programs that implement MCMC algorithms.

With the objective of evaluation the behaviour of the solution of the differential equation with initial condition (9), according to different fractional approaches (classical, Khalil, conformable, non-conformable) in problems of direct and inverse type, was used the R software [23] and the packages associated with the Bayesian estimate [24].

Now, we show an application of the inverse problem resolution using a Bayesian estimation with real temperature data [20], for different kernels; Khalil $F(t, \alpha) = t^{1-\alpha}$ (*blue*), Conformable Derivative with $F(t, \alpha) = e^{(1-\alpha)t}$ (*red*), non-conformable $F(t, \alpha) = t^{-\alpha}$ (*yellow*) and Ordinary $F(t, \alpha) = 1$ (*black*). Figure 8 shows the adjustments corresponding to the observations (black points) associated with real data.



Figure 8. Temperature vs. *t*.

The errors in the adjustments according to the different approaches are the following: Khalil et al. (0.2034), $F(t, \alpha) = e^{(\alpha - 1)t}$ (3.0500), non-conformable (8.8700) and Ordinary (3.1902). Figure 9 shows the trace and estimated posterior distributions of the parameters of interest using Khalil fractional derivatives.



Figure 9. Trace and estimated posterior densities of parameters.

Derivative	α	Te	T_0	k	τ
Ordinary	_	2.3619	8.4119	2.5933	0.1191
$F(t,\alpha) = e^{(1-\alpha)t}$	0.3703	1.9094	2.0215	13.0428	0.0961
Khalil et al.	0.7923	2.1510	8.8381	1.9983	0.02607
non-Conformable	0.6666	2.3872	7.1298	10.8981	0.3973

The estimates of the parameters α , *k*, *Te*, *T*₀, and τ related to the real data [20], according to the different approach are:

Figure 10 shows the adjustments corresponding to the observations (black points) associated with real data.



Figure 10. Temperature vs. *t*.

The errors in the adjustments, according to the different approaches are the following: Khalil et al. (2.6412), $F(t, \alpha) = e^{(\alpha-1)t}$ (5.5393), non-conformable (17.4638) and Ordinary (6.5927).

The estimates of the parameters α , k, Te, T_0 , and τ related to the real data, according to the different approach are:

Derivative	α	Te	<i>T</i> ₀	k	τ
Ordinary	_	3.2904	9.5120	0.0891	0.2127
$F(t,\alpha) = e^{(1-\alpha)t}$	1.2440	3.0916	0.0711	0.9477	0.1557
Khalil et al.	2.6813	10.1773	0.0516	0.7107	0.1127
non-Conformable	1.2440	3.0916	0.0711	0.9477	0.1557

Figure 11 shows the adjustments corresponding to the observations (black points) associated with real data.



Figure 11. Temperature vs. t.

The errors in the adjustments according to the different approaches are the following: Khalil et al.(1.7671), $F(t, \alpha) = e^{(\alpha-1)t}$ (5.5023), non-conformable (21.2764) and Ordinary (1.9976).

The estimates of the parameters α , k, Te, T_0 , and τ related to the real data, according to the different approach are:

Derivative	α	Te	T_0	k	τ
Ordinary	_	3.4686	9.9086	0.0406	0.0703
$F(t,\alpha) = e^{(1-\alpha)t}$	2.9407	3.1779	0.0407	0.9876	0.0791
Khalil et al.	3.2229	10.0060	0.0357	0.9354	0.0640
non-Conformable	2.940	3.1779	0.04079	0.9876	0.0791

5. Conclusions

In this work, a simulation of the well-known Newton's Law of Cooling using a generalized differential operator with different kernels $F(t, \alpha)$ is made. From the results obtained, the conformable derivative $F(t, \alpha) = t^{1-\alpha}$ [5] shows the best fit, verifying the results obtained in [20]. However, the following questions arise:

- (1) In any model, the conformable derivatives provide the best fit? We believe that each model must be analysed separately, for two fundamental questions: first, because it is clear that if $\alpha \to 1$ we obtain the ordinary derivative, then there will always be a "critical" value of α from which this model will fit as well as ordinary model. For asymptotic properties (when $t \to \infty$) the non-conformable model used $e^{t^{-\alpha}}$ behaves like the ordinary model, hence values of *t* large enough, this model better represents the behaviour of the model (in this regard you can consult [13]).
- (2) The distortion factor used, see (6), is the same in all models, which leads us to the conclusion that said correction factor must depend on the kernel used and not be unique, that is, we must use a factor of the type $F(k^{-1}, \alpha)$. This would improve the fit of the remaining models. The application of this type of generalized equations of Newton's law of cooling can be applied in the study of the thermal dynamics of systems with complex spatial profiles (see [25]). Estimating the time of death is a fundamental problem in forensic medicine, and to calculate it, Newton's cooling

model and its corresponding generalization, the Marshall and Hoare model, have been applied (cf. [17]). The fractional and generalized generalization of the Marshall and Hoare model will be analysed in detail in a future work.

Author Contributions: All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Department of Electrical Engineering and by the Division of Engineering, Campus Irapuato-Salamanca, both from the University of Guanajuato (México).

Institutional Review Board Statement: Not Applicable.

Informed Consent Statement: Not Applicable.

Data Availability Statement: Not Applicable.

Acknowledgments: M. Vivas-Cortez thanks to Dirección de Investigación from Pontificia Universidad Católica del Ecuador for the technical support given to the research project entitled: Algunas desigualdades de funciones convexas generalizadas (Some inequalities of generalized convex functions).

Conflicts of Interest: The authors declare no conflict of interest.

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