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Asymptotic Properties of Solutions to Discrete Volterra Monotone Type Equations

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Abstract: We investigate the higher order nonlinear discrete Volterra equations. We study solutions with prescribed asymptotic behavior. For example, we establish sufficient conditions for the existence of asymptotically polynomial, asymptotically periodic or asymptotically symmetric solutions. On the other hand, we are dealing with the problem of approximation of solutions. Among others, we present conditions under which any bounded solution is asymptotically periodic. Using our techniques, based on the iterated remainder operator, we can control the degree of approximation. In this paper we choose a positive non-increasing sequence u and use $\mathrm{o}(u_n)$ as a measure of approximation.

Keywords: discrete Volterra equation; prescribed asymptotic behavior; asymptotically polynomial solution; asymptotically periodic solution; degree of approximation

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Citation: Migda, J.; Migda, M.; Schmeidel, E. Asymptotic Properties of Solutions to Discrete Volterra Monotone Type Equations. *Symmetry* **2021**, *13*, 918. https://doi.org/10.3390/sym13060918

Academic Editor: Alina Alb Lupas

Received: 24 April 2021
Accepted: 14 May 2021
Published: 21 May 2021

1. Introduction

We denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Assume τ is an integer, $m \in \mathbb{N}$,

$$f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}, \quad K : \mathbb{N} \times \mathbb{N} \to \mathbb{R}, \quad b : \mathbb{N} \to \mathbb{R}$$

and consider difference equations of the form

$$\Delta^m x_n = b_n + \sum_{k=1}^{n} K(n,k) f(k, x_{k-\tau}). \tag{1}$$

By a solution of (1) we mean a sequence $x : \mathbb{N} \to \mathbb{R}$ satisfying (1) for all large n. We say that Equation (1) is of monotone type if one of the following conditions is satisfied:

(a) f is non-decreasing with respect to the second variable and $(-1)^m K(n,k) \geq 0$ for all (n,k);
(b) f is non-increasing with respect to the second variable and $(-1)^m K(n,k) \leq 0$ for all (n,k).

By studying the hereditary influences in population growth models Vito Volterra obtained an equation of the form

$$x^{(n)}(t) = h(t) + \int_0^t K(t,s) x(s) ds$$

which was termed the Volterra integro-differential equation. The non-linear Volterra integro-differential equation of the form

$$x^{(n)}(t) = h(t) + \int_0^t K(t,s) F(s, x(s)) ds \tag{2}$$

appears also in many problems. Volterra equations are frequently used to describe many real world phenomena concerning biology, chemistry, physics, mechanics, economy, medicine, population dynamics, and others. For more information on the theory and applications of linear and non-linear Volterra integro-differential equations we refer readers to the books by Burton [1] and Wazwaz [2] and, for example, the papers [3–6].

In the last four decades many authors have studied the qualitative properties of solutions of discrete Volterra equations. In particular, asymptotic properties of solutions of first order Volterra difference equations were considered, e.g., in [7–20] or [21]. For example, in [19] the necessary and sufficient condition for boundedness of all solutions of the linear Volterra equation

$$\Delta x(n) = \sum_{i=0}^{n} A(n,i)x(i)$$

are obtained. In [8], the authors established conditions under which every solution of the system of linear Volterra equations

$$x(n+1) = h(n) + \sum_{i=0}^{n} H(n,i)x(i)$$

is convergent. Some population models described by Volterra difference equations can be found in the recent monograph by Raffoul [22]. However, there are relatively few papers devoted to the higher order discrete Volterra equations, see [23–25].

In this paper we investigate asymptotic behavior of solutions to Equation (1) which is a discrete analog of Equation (2). We mainly deal with problems of two types. The first is the problem of the existence of solutions with prescribed asymptotic behavior. The second problem is the approximation of a given solution of Equation (1). Studies on solutions with prescribed asymptotic behavior are usually based on the application of the Schauder or Darboux type theorems. In this case, conditions of the continuity type are superimposed on the function f. We use the Knaster-Tarski theorem. Using the Knaster-Tarski theorem, we replace the conditions of the continuity type with the conditions of the monotonicity type. This allows us to apply our results to, e.g., floor function, ceiling function, or other locally constant functions. To our knowledge, the asymptotic properties of solutions to the Volterra equations of the monotonic type have not been studied. We believe that the case of monotonic type equations, e.g., with a locally constant function f, is important in the application of numerical methods.

We use techniques from [26] based on the use of the iterated remainder operator. This allows us to control the degree of approximation of solutions. In this paper, we choose a positive non-increasing sequence u and use $\mathrm{o}(u_n)$ as a measure of approximation. Two particularly important approximation cases can be obtained when u is a power sequence or a geometric sequence. More precisely, if $u_n = n^s$ for some fixed $s \in (-\infty, 0]$, then we have the so-called harmonic approximation. If $u_n = \lambda^n$, where $\lambda \in (0,1)$ is fixed, then we have the geometric approximation. It is worth noting that even in the case of $u_n = 1$, i.e., in the case when $\mathrm{o}(1)$ is a degree of approximation, our results are new.

The organization of the paper is as follows. In Section 2, we introduce some notations and terminology. Moreover, we present two basic lemmas. In Section 3, we present and prove two theorems. They are the main results of the paper. In Section 4, we present a number of different consequences of Theorems 1 and 2. Section 5 provides examples, remarks and additional results. Some conclusions are given in Section 6.

2. Preliminaries

We denote by $\mathbb{Z}$ the set of all integers and $\mathbb{R}^{\mathbb{N}}$ is the space of all sequences $x : \mathbb{N} \to \mathbb{R}$. We will use the convention $x_n = x_1$ whenever $x \in \mathbb{R}^{\mathbb{N}}$ and $n < 1$. Let $m \in \mathbb{N}$. We will use the following notations

$$\mathrm{A}(m) = \left\{ x \in \mathbb{R}^{\mathbb{N}} : \sum_{i_1=1}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} |x_{i_m}| < \infty \right\}.$$

For any $x \in \mathrm{A}(m)$ we define the sequence $r^m(x)$ by

$$r^m(x)(n) = \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} x_{i_m}. \tag{3}$$

Then

$$r^m(x)(n) = \mathrm{o}(1) \tag{4}$$

and

$$r^m(x)(n) = \sum_{j=n}^{\infty} \binom{m-1+j-n}{m-1} x_j = \sum_{k=0}^{\infty} \binom{m+k-1}{m-1} x_{n+k} \tag{5}$$

for any $x \in \mathrm{A}(m)$ and any $n \in \mathbb{N}$. Moreover

$$\Delta^m (r^m(x))(n) = (-1)^m x_n \tag{6}$$

for any $x \in \mathrm{A}(m)$ and any $n \in \mathbb{N}$. It is easy to see that if $x, z \in \mathrm{A}(m)$ and $x \le z$, then $r^m(x) \le r^m(z)$. For more information about the operator r^m see [26]. We will use the following consequence of the Knaster-Tarski fixed point theorem.

Lemma 1. ([27], Lemma 4.9). *Let $y, \rho \in \mathbb{R}^{\mathbb{N}}$ and let S denote the set*

$$\{x \in \mathbb{R}^{\mathbb{N}} : |x - y| \le |\rho|\}$$

with natural order defined by: $x \le z$ if $x_n \le z_n$ for any $n \in \mathbb{N}$. Then every non-decreasing map $T : S \to S$ has a fixed point.

We will also need the following lemma.

Lemma 2. ([28], Lemma 2.3). *Assume u is a positive and non-decreasing sequence,*

$$a \in \mathbb{R}^{\mathbb{N}}, \quad m \in \mathbb{N}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{n^{m-1}|a_n|}{u_n} < \infty.$$

Then there exists a sequence $z \in \mathbb{R}^{\mathbb{N}}$ such that $z_n = \mathrm{o}(u_n)$ and $\Delta^m z_n = a_n$.

For $k \in \mathbb{N}$ we use the factorial notation

$$n^{\underline{k}} = n(n-1)\cdots(n-k+1).$$

Moreover, we will use the ceiling function $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ defined by

$$\lceil t \rceil = \min\{n \in \mathbb{Z} : n \ge t\}.$$

3. Main Results

We present two theorems in this section. In Theorem 1 we deal with the problem of the existence of solutions with prescribed asymptotic behavior. More precisely, for a given solution y of the equation $\Delta^m y_n = b_n$ and a given positive and non-increasing sequence u

we present the sufficient conditions for the existence of a solution x to Equation (1) such that $x_n = y_n + \mathrm{o}(u_n)$. The proof of this theorem is based on the Knaster-Tarski fixed point theorem. To use the Knaster-Tarski theorem, it is necessary to assume that Equation (1) is of the monotone type.

Theorem 2 is devoted to the problem of approximating the solutions of (1). For a given solution x of Equation (1) and a given positive and non-increasing sequence u, we establish the sufficient conditions for the existence of a solution y to equation $\Delta^m y_n = b_n$ such that $x_n = y_n + \mathrm{o}(u_n)$. In Theorem 2, we do not need to assume that Equation (1) is of monotone type.

Theorem 1. *Assume $u, w : \mathbb{N} \to (0, \infty)$, $g : [0, \infty) \to [0, \infty)$, g is locally bounded,*

$$|f(n,t)| \le g(|tw_n|) \quad \text{for} \quad (n,t) \in \mathbb{N} \times \mathbb{R}, \tag{7}$$

$$\sum_{n=1}^{\infty} \frac{n^{m-1}}{u_n} \sum_{k=1}^{n} |K(n,k)| < \infty, \tag{8}$$

w is bounded, $\Delta u_n \le 0$, and (1) is of monotone type. Then, for any solution y of the equation $\Delta^m y_n = b_n$ such that, $y_{n-\tau} = \mathrm{O}(w_n^{-1})$ there exists a solution x of (1) with the property $x_n = y_n + \mathrm{o}(u_n)$.

Proof. Assume $y \in \mathbb{R}^{\mathbb{N}}$, $\Delta^m y_n = b_n$, and $y_{n-\tau} = \mathrm{O}(w_n^{-1})$. Let

$$T = \{x \in \mathbb{R}^{\mathbb{N}} : |x - y| \le 1\}.$$

By (7), there exists a constant K such that if $x \in T$ and $n \in \mathbb{N}$, then

$$|w_n x_{n-\tau}| = |w_n x_{n-\tau} - w_n y_{n-\tau} + w_n y_{n-\tau}|$$

$$\le |w_n||x_{n-\tau} - y_{n-\tau}| + |w_n y_{n-\tau}| \le K.$$

Since g is locally bounded, there exists a positive constant M such that $g([0,K]) \subset [0, M]$. Therefore, using (7) we have

$$g(|w_n x_{n-\tau}|) \le M \qquad \text{and} \qquad |f(n, x_{n-\tau})| \le g(|x_{n-\tau} w_n|) \le M \tag{9}$$

for $x \in T$ and $n \in \mathbb{N}$. Let $\alpha : \mathbb{N} \to \mathbb{R}$ be defined by

$$\alpha_n = \sum_{k=1}^{n} |K(n,k)|.$$

The sequence u is positive and non-increasing. Hence, using (8), we have $\alpha \in \mathrm{A}(m)$. So there exists an index p such that $Mr^m(\alpha)(n) \le 1$ for $n \ge p$. Let $\mu, \rho \in \mathbb{R}^{\mathbb{N}}$,

$$\mu_n = 0 \quad \text{for} \quad n < p, \qquad \mu_n = 1 \quad \text{for} \quad n \ge p, \qquad \rho_n = \mu_n M r^m(\alpha)(n).$$

Define an operator $G : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by

$$G(x)(n) = \sum_{k=1}^{n} K(n,k) f(k, x_{k-\tau}).$$

If $x \in T$, then

$$|G(x)(n)| \le \sum_{k=1}^{n} |K(n,k)||f(k, x_{k-\tau})| \le M \sum_{k=1}^{n} |K(n,k)| = M\alpha_n.$$

Hence $G(x) \in \mathrm{A}(m)$ for any $x \in T$. Let

$$S = \{x \in \mathbb{R}^{\mathbb{N}} : \ |x - y| \leq \rho\}.$$

Then $S \subset T$. Define an operator $A : S \to \mathbb{R}^{\mathbb{N}}$ by

$$A(x) = y + (-1)^m \mu r^m (G(x)).$$

If $x \in S$, then

$$|A(x) - y| = |\mu r^m (G(x))| \leq \mu r^m (|G(x)|) \leq \rho.$$

Hence $A(S) \subset S$. Now we assume that the condition (a) of the definition of monotonicity of (1) is fulfilled. The proof in the case (b) is analogous. Let $x, z \in S$, $x \leq z$. If $n \in \mathbb{N}$, then

$$(-1)^m G(x)(n) = \sum_{k=1}^{n} (-1)^m K(n,k) f(k, x_{k-\tau})$$

$$\leq \sum_{k=1}^{n} (-1)^m K(n,k) f(k, z_{k-\tau}) = (-1)^m G(z)(n).$$

Hence $(-1)^m G(x) \leq (-1)^m G(z)$. Since the operator r^m is non-decreasing, we get

$$A(x) = y + (-1)^m \mu r^m (G(x)) \leq y + (-1)^m \mu r^m (G(z)) = A(z).$$

By Lemma 1, there exists a sequence $x \in S$ such that $A(x) = x$. Then, for $n \geq p$, we have

$$x_n = y_n + (-1)^m r^m (G(x))(n). \tag{10}$$

Hence

$$\Delta^m x_n = \Delta^m y_n + G(x)(n) = b_n + \sum_{k=1}^{n} K(n,k) f(k, x_{k-\tau})$$

for $n \geq p$. Therefore x is a solution of (1). Now we will show that

$$r^m (G(x))(n) = \mathrm{o}(u_n).$$

Define sequences $\beta, \gamma^+, \gamma^-$ by

$$\beta_n = \frac{|G(x)(n)|}{u_n}, \quad \gamma_n^+ = \max(0, G(x)(n)), \quad \gamma_n^- = -\min(0, G(x)(n)).$$

Then $0 \leq \gamma^+ \leq |G(x)|$. Hence $\gamma^+ \in \mathrm{A}(m)$ and using (5) we get

$$r^m(\gamma^+)(n) = \sum_{k=0}^{\infty} \binom{m+k-1}{m-1} \gamma^+_{n+k} \leq \sum_{k=0}^{\infty} \binom{m+k-1}{m-1} |G(x)(n+k)|$$

$$= \sum_{k=0}^{\infty} \binom{m+k-1}{m-1} u_{n+k} \beta_{n+k} \leq \sum_{k=0}^{\infty} \binom{m+k-1}{m-1} u_n \beta_{n+k} = u_n r^m(\beta)(n).$$

Therefore,

$$0 \leq \frac{r^m(\gamma^+)(n)}{u_n} \leq r^m(\beta)(n).$$

By (3), $r^m(\beta)(n) = \mathrm{o}(1)$. Hence $r^m(\gamma^+)(n) = \mathrm{o}(u_n)$. Analogously, $r^m(\gamma^-)(n) = \mathrm{o}(u_n)$. Thus

$$r^m(G(x))(n) = r^m(\gamma^+ - \gamma^-)(n) = r^m(\gamma^+)(n) - r^m(\gamma^-)(n) = \mathrm{o}(u_n).$$

Now, using (10), we obtain $x_n = y_n + \mathrm{o}(u_n)$. The proof is complete. □

Theorem 2. *Assume $u, w : \mathbb{N} \to (0, \infty)$, $g : [0, \infty) \to [0, \infty)$, g is locally bounded,*

$$|f(n,t)| \leq g(|tw_n|) \quad \textit{for} \quad (n,t) \in \mathbb{N} \times \mathbb{R}, \tag{11}$$

$$\sum_{n=1}^{\infty} \frac{n^{m-1}}{u_n} \sum_{k=1}^{n} |K(n,k)| < \infty, \tag{12}$$

w is bounded, and $\Delta u_n \leq 0$. Then for any solution x of (1) such that $x_{n-\tau} = \mathrm{O}(w_n^{-1})$ there exists a solution y of the equation $\Delta^m y_n = b_n$ such that, $x_n = y_n + \mathrm{o}(u_n)$.

Proof. Assume x is a solution of (1) such that $x_{n-\tau} = \mathrm{O}(w_n^{-1})$. There exists a positive constant K such that $|w_n x_{n-\tau}| \leq K$ for any n. Since g is locally bounded, there exists a positive constant M such that $g([0,K]) \subset [0,M]$. By (11) we have

$$|f(k, x_{k-\tau})| \leq g(|w_k x_{k-\tau}|) \leq M \tag{13}$$

for any $k \in \mathbb{N}$. Define a sequence β by

$$\beta_n = \Delta^m x_n - b_n.$$

Since x is a solution of (1), we have

$$|\beta_n| = \left| \sum_{k=1}^{n} K(n,k) f(k, x_{k-\tau}) \right| \leq M \sum_{k=1}^{n} |K(n,k)|$$

for large n. Hence there exists a constant $P \geq M$ such that

$$|\beta_n| \leq P \sum_{k=1}^{n} |K(n,k)|$$

for any $n \in \mathbb{N}$. Using (12) we get

$$\sum_{n=1}^{\infty} \frac{n^{m-1}|\beta_n|}{u_n} \leq P \sum_{n=1}^{\infty} \frac{n^{m-1}}{u_n} \sum_{k=1}^{n} |K(n,k)| < \infty.$$

By Lemma 2, there exists a sequence z such that

$$z_n = \mathrm{o}(u_n) \quad \text{and} \quad \Delta^m z_n = \beta_n.$$

Let $y = x - z$. Then

$$\Delta^m y_n = \Delta^m x_n - \Delta^m z_n = \Delta^m x_n - \beta_n = \Delta^m x_n - \Delta^m x_n + b_n = b_n$$

for any $n \in \mathbb{N}$. Moreover $x_n = y_n + z_n = y_n + \mathrm{o}(u_n)$. □

We say that a sequence $w \in \mathbb{R}^{\mathbb{N}}$ is standard if

$$w_{n+1} = \mathrm{O}(w_n) \quad \text{and} \quad w_{n-1} = \mathrm{O}(w_n).$$

For example, if $s \in \mathbb{R}$, then the sequence $w_n = n^s$ is standard. If $\lambda > 0$, then the sequence $w_n = \lambda^n$ is standard. It is easy to see that a sum of two standard sequences is standard. In particular any polynomial sequence is standard. The sequence $w_n = n^n$ is not standard.

Remark 1. *Assume $w \in \mathbb{R}^{\mathbb{N}}$ is a positive standard sequence. Then the sequence w_n^{-1} is also standard. In this case, condition $y_{n-\tau} = \mathrm{O}(w_n^{-1})$ in Theorem 1 can be replaced by condition $y_n = \mathrm{O}(w_n^{-1})$. Similarly, condition $x_{n-\tau} = \mathrm{O}(w_n^{-1})$ in Theorem 2 can be replaced by condition $x_n = \mathrm{O}(w_n^{-1})$.*

4. Consequences

4.1. Solutions with Prescribed Asymptotic Behavior

In this subsection we present some consequences of Theorem 1.

Corollary 1. *Assume the assumptions of Theorem 1 are satisfied and moreover*

$$\sum_{n=1}^{\infty} \frac{n^{m-1}|b_n|}{u_n} < \infty. \tag{14}$$

Then, for any polynomial φ such that $\deg(\varphi) < m$ and $\varphi(n-\tau) = \mathrm{O}(w_n^{-1})$ there exists a solution x of (1) such that $x_n = \varphi(n) + \mathrm{o}(u_n)$.

Proof. By Lemma 2, there exists a sequence z such that

$$z_n = \mathrm{o}(u_n) \quad \text{and} \quad \Delta^m z_n = b_n.$$

It is easy to check that conditions $z_n = \mathrm{o}(u_n)$, $w_n = \mathrm{O}(1)$, $\Delta u_n \le 0$ imply $z_{n-\tau} = \mathrm{O}(w_n^{-1})$. Let $y = \varphi + z$. Then y is a solution of the equation $\Delta^m y_n = b_n$ and, by Theorem 1, there exists a solution x of (1) such that $x_n = y_n + \mathrm{o}(u_n)$. Hence we get

$$x_n = \varphi(n) + z_n + \mathrm{o}(u_n) = \varphi(n) + \mathrm{o}(u_n) + \mathrm{o}(u_n) = \varphi(n) + \mathrm{o}(u_n).$$

□

Corollary 2. *Assume the assumptions of Theorem 1 are satisfied. Then for any bounded solution y of the equation $\Delta^m y_n = b_n$ there exists a solution x of (1) such that $x_n = y_n + \mathrm{o}(u_n)$.*

Proof. It is easy to see that boundedness of y implies the condition $y_{n-\tau} = \mathrm{O}(w_n^{-1})$. Hence the assertion is a consequence of Theorem 1. □

Condition (7) in Theorem 1 is complicated. Below by reducing the generality, we simplify this condition.

Corollary 3. *Assume $\alpha, \beta \in (0,\infty)$, $u : \mathbb{N} \to (0,\infty)$, $\Delta u_n \le 0$,*

$$p = \frac{\beta}{\alpha}, \quad |f(n,t)| \le \frac{|t|^\alpha}{n^\beta} \quad \textit{for all} \quad (n,t), \quad \sum_{n=1}^{\infty} \frac{n^{m-1}}{u_n} \sum_{k=1}^{n} |K(n,k)| < \infty,$$

and (1) is of monotone type. Then, for any solution y of the equation $\Delta^m y_n = b_n$ such that $y_n = \mathrm{O}(n^p)$ there exists a solution x of (1) such that $x_n = y_n + \mathrm{o}(u_n)$.

Proof. Let y be a solution of the equation $\Delta^m y_n = b_n$ such that $y_n = \mathrm{O}(n^p)$. Define a sequence w and a function $g : [0,\infty) \to [0,\infty)$ by

$$w_n = \frac{1}{n^p}, \quad g(t) = t^\alpha.$$

Since $(n-\tau)^p = \mathrm{O}(n^p)$, we have $y_{n-\tau} = \mathrm{O}(w_n^{-1})$. Hence all assumptions of Theorem 1 are satisfied. Therefore, there exists a solution x of (1) such that $x_n = y_n + \mathrm{o}(u_n)$. □

Corollary 4. *Assume the assumptions of Corollary 3 are satisfied and moreover*

$$\sum_{n=1}^{\infty} \frac{n^{m-1}|b_n|}{u_n} < \infty.$$

Then, for any polynomial φ such that $\deg(\varphi) < m$ and $\varphi(n) = \mathrm{O}(n^p)$ there exists a solution x of (1) such that $x_n = \varphi(n) + \mathrm{o}(u_n)$.

Proof. Using Corollary 1 instead of Theorem 1 in the proof of Corollary 3 we obtain the result. □

Assuming boundedness of the function f we obtain an especially simple version of Theorem 1.

Corollary 5. *Assume f is bounded, $u : \mathbb{N} \to (0, \infty)$, $\Delta u_n \leq 0$,*

$$\sum_{n=1}^{\infty} \frac{n^{m-1}}{u_n} \sum_{k=1}^{n} |K(n,k)| < \infty,$$

and (1) is of monotone type. Then, for any solution y of the equation $\Delta^m y_n = b_n$ there exists a solution x of (1) such that $x_n = y_n + \mathrm{o}(u_n)$.

Proof. Let y be a solution of the equation $\Delta^m y_n = b_n$. Choose a positive constant M such that $|f(n,t)| \leq M$ for any (n,t). Define a function $g : [0,\infty) \to [0,\infty)$ by $g(t) = M$ for any $t \in [0,\infty)$ and let w be an arbitrary bounded positive sequence such that $y_{n-\tau} = \mathrm{O}(w_n^{-1})$. Then, all assumptions of Theorem 1 are satisfied. Hence there exists a solution x of (1) such that $x_n = y_n + \mathrm{o}(u_n)$. □

Corollary 6. *Assume the assumptions of Corollary 5 are satisfied and moreover*

$$\sum_{n=1}^{\infty} \frac{n^{m-1}|b_n|}{u_n} < \infty.$$

Then, for any polynomial φ such that $\deg(\varphi) < m$ there exists a solution x of (1) such that $x_n = \varphi(n) + \mathrm{o}(u_n)$.

Proof. Using Corollary 1 instead of Theorem 1 in the proof of Corollary 5 we obtain the result. □

4.2. Approximation of Solutions

This section is devoted to the consequences of Theorem 2.

Corollary 7. *Assume the assumptions of Theorem 2 are satisfied and moreover*

$$\sum_{n=1}^{\infty} \frac{n^{m-1}|b_n|}{u_n} < \infty.$$

Then, for any solution x of (1) such that $x_{n-\tau} = \mathrm{O}(w_n^{-1})$ there exists a polynomial φ, such that $\deg(\varphi) < m$ and $x_n = \varphi(n) + \mathrm{o}(u_n)$.

Proof. By Theorem 2 there exists a solution y of the equation $\Delta^m y_n = b_n$ such that, $x_n = y_n + \mathrm{o}(u_n)$. By Lemma 2, there exists a sequence z such that

$$z_n = \mathrm{o}(u_n) \quad \text{and} \quad \Delta^m z_n = b_n.$$

Let $\varphi = y - z$. Then $\Delta^m \varphi = \Delta^m y - \Delta^m z = b - b = 0$. Hence φ is a polynomial such that $\deg(\varphi) < m$. Moreover

$$x_n = y_n + \mathrm{o}(u_n) = \varphi(n) + z_n + \mathrm{o}(u_n) = \varphi(n) + \mathrm{o}(u_n) + \mathrm{o}(u_n) = \varphi(n) + \mathrm{o}(u_n).$$

□

Corollary 8. *Assume the assumptions of Theorem 2 are satisfied. Then, for any bounded solution x of (1) there exists a solution y of the equation $\Delta^m y_n = b_n$ such that $x_n = y_n + \mathrm{o}(u_n)$.*

Proof. It is easy to see that boundedness of x implies the condition $x_{n-\tau} = \mathrm{O}(w_n^{-1})$. Hence the assertion is a consequence of Theorem 2. □

Corollary 9. *Assume $\alpha, \beta \in (0, \infty)$, $u : \mathbb{N} \to (0, \infty)$, $\Delta u_n \le 0$, $p = \beta/\alpha$,*

$$|f(n,t)| \le \frac{|t|^\alpha}{n^\beta} \quad \textit{for all} \quad (n,t), \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{n^{m-1}}{u_n} \sum_{k=1}^{n} |K(n,k)| < \infty.$$

Then, for any solution x of (1) such that $x_n = \mathrm{O}(n^p)$ there exists a solution y of the equation $\Delta^m y_n = b_n$ such that $x_n = y_n + \mathrm{o}(u_n)$.

Proof. Let x be a solution of (1) such that $x_n = \mathrm{O}(n^p)$. Define a sequence w and a function $g : [0, \infty) \to [0, \infty)$ by $w_n = n^{-p}$, $g(t) = t^\alpha$. Since $(n-\tau)^p = \mathrm{O}(n^p)$, we have $y_{n-\tau} = \mathrm{O}(w_n^{-1})$. Hence all assumptions of Theorem 2 are satisfied. Therefore, there exists a solution y of the equation $\Delta^m y_n = b_n$ such that $x_n = y_n + \mathrm{o}(u_n)$. □

Corollary 10. *Assume f is bounded, $u : \mathbb{N} \to (0, \infty)$, $\Delta u_n \le 0$, and*

$$\sum_{n=1}^{\infty} \frac{n^{m-1}}{u_n} \sum_{k=1}^{n} |K(n,k)| < \infty.$$

Then for any solution x of (1) there exists a solution y of the equation $\Delta^m y_n = b_n$ such that $x_n = y_n + \mathrm{o}(u_n)$.

Proof. Let x be a solution of (1) and let $g = M$ be a positive constant function such that $|f(n,t)| \le M$ for any $(n,t) \in \mathbb{N} \times \mathbb{R}$. There exists a bounded positive sequence w such that $x_{n-\tau} = \mathrm{O}(w_n^{-1})$. Then, all assumptions of Theorem 2 are satisfied. Hence there exists a solution y of the equation $\Delta^m y_n = b_n$ such that $x_n = y_n + \mathrm{o}(u_n)$. □

Corollary 11. *Assume f is bounded, $u : \mathbb{N} \to (0, \infty)$, $\Delta u_n \le 0$,*

$$\sum_{n=1}^{\infty} \frac{n^{m-1}}{u_n} \sum_{k=1}^{n} |K(n,k)| < \infty, \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{n^{m-1}|b_n|}{u_n} < \infty.$$

Then, for any solution x of (1) there exists a polynomial φ such that $\deg(\varphi) < m$ and $x_n = \varphi(n) + \mathrm{o}(u_n)$.

Proof. Using Corollary 7 instead of Theorem 2 in the proof of Corollary 10 we obtain the result. □

5. Examples, Remarks, and Additional Results

We start with an example illustrating Theorem 1.

Example 1. *Let $m = 3$, $\tau = 0$,*

$$b_n = \frac{6}{(n+5)^{\underline{4}}}, \quad K(n,k) = -\frac{k}{n(n+1)2^{n+4}}, \quad f(n,t) = \lceil t \rceil^2, \quad g(t) = (t+1)^2.$$

Then, Equation (1) takes the form

$$\Delta^3 x_n = \frac{6}{(n+5)^{\underline{4}}} - \sum_{k=1}^{n} \frac{k}{n(n+1)2^{n+4}} \lceil x_k \rceil^2. \tag{15}$$

So, (1) is of monotone type. It is easy to check that $y_n = \frac{n+1}{n+2}$ *is a solution of equation*

$$\Delta^3 y_n = \frac{6}{(n+5)^{\underline{4}}}. \tag{16}$$

Set $w_n = 1$, $u_n = \left(\frac{2}{3}\right)^n$. *Then* $y_n = \mathrm{O}(w_n^{-1})$. *It is easy to check that*

$$\sum_{k=1}^{\infty} \frac{n^2 3^n}{2^n} \sum_{k=1}^{n} \frac{k}{n(n+1)2^{n+4}} < \infty.$$

Thus, by Theorem 1, there exists a solution x of (15) such that $x_n = y_n + \mathrm{o}(u_n)$. *For example the sequence*

$$x_n = \frac{n+1}{n+2} + \frac{1}{2^{n+2}}$$

is such a solution.

Condition

$$\sum_{n=1}^{\infty} \frac{n^{m-1}}{u_n} \sum_{k=1}^{n} |K(n,k)| < \infty, \tag{17}$$

may be difficult to verify. The following lemma may facilitate the verification of this condition.

Lemma 3. *Assume* $m \in \mathbb{N}$, $u : \mathbb{N} \to (0,\infty)$, $K : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$, $K^* \in \mathbb{R}^{\mathbb{N}}$,

$$K^*(n) = n \max(|K(n,1)|, |K(n,2)|, \ldots, |K(n,n)|), \tag{18}$$

and at least one of the following conditions is satisfied

$$\liminf_{n\to\infty} \frac{\ln u_n - \ln K^*(n)}{\ln n} > m, \quad \liminf_{n\to\infty} n \ln\left(\frac{K^*(n)u_{n+1}}{K^*(n+1)u_n}\right) > m,$$

$$\liminf_{n\to\infty} n\left(\frac{K^*(n)u_{n+1}}{K^*(n+1)u_n} - 1\right) > m.$$

Then, the condition (17) is satisfied.

Proof. Using ([27], Lemma 4.4, Lemma 4.5) and ([29], Lemma 6.4) we get

$$\sum_{n=1}^{\infty} \frac{n^{m-1}K^*(n)}{u_n} < \infty.$$

Since $\sum_{k=1}^{n} |K(n,k)| \le K^*(n)$ for any n, we obtain (17). □

Example 2. *Let* $\alpha, \beta \in \mathbb{R}$, $m \in \mathbb{N}$. *Define a kernel K and a sequence u by*

$$K(n,k) = \frac{k^\alpha}{3^{\sqrt{n}}}, \quad u_n = n^\beta.$$

If K^* *is defined by (18), then* $K^*(n) = n^{\alpha+1}/3^{\sqrt{n}}$ *and*

$$n \ln\left(\frac{K^*(n)u_{n+1}}{K^*(n+1)u_n}\right) = n \ln\left(\left(\frac{n}{n+1}\right)^{\alpha+1}\left(\frac{n+1}{n}\right)^{\beta}\frac{3^{\sqrt{n+1}}}{3^{\sqrt{n}}}\right)$$

$$= n(\alpha+1)\ln\frac{n}{n+1} + n\beta \ln\frac{n+1}{n} + n(\sqrt{n+1} - \sqrt{n})\ln 3 \to \infty.$$

Hence, by Lemma 3 we get (17).

The following lemma can be the basis for the theory of 'geometric approximation' of the solutions of Equation (1).

Lemma 4. *Assume $m \in \mathbb{N}$, $K : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$, $K^* \in \mathbb{R}^{\mathbb{N}}$ is defined by (18), and*

$$\limsup_{n\to\infty} \sqrt[n]{K^*(n)} < \lambda < 1. \tag{19}$$

Then

$$\sum_{n=1}^{\infty} \frac{n^{m-1}}{\lambda^n} \sum_{k=1}^{n} |K(n,k)| < \infty. \tag{20}$$

Proof. Define a sequence $\omega \in \mathbb{R}^{\mathbb{N}}$ and a number μ by

$$\omega_n = \frac{n^{m-1}K^*(n)}{\lambda^n}, \quad \mu = \limsup_{n\to\infty} \sqrt[n]{K^*(n)}.$$

Then

$$\limsup_{n\to\infty} \sqrt[n]{\omega_n} = \frac{\mu}{\lambda} < 1 \Rightarrow \sum_{n=1}^{\infty} \frac{n^{m-1}K^*(n)}{\lambda^n} < \infty$$

and we get (20). □

It is clear that, in Lemma 4, condition (19) can be replaced by condition:

$$\limsup_{n\to\infty} \frac{K^*(n+1)}{K^*(n)} < \lambda < 1. \tag{21}$$

Example 3. *Let*

$$K(n,k) = \frac{(-1)^m 2^k}{n!}, \quad f(n,t) = \frac{t}{n^m}, \quad \lambda \in (0,1) \quad \textit{and} \quad b \in \mathbb{R}^{\mathbb{N}}.$$

Then

$$K^*(n) = \frac{n2^n}{n!}, \quad \lim_{n\to\infty} \frac{K^*(n+1)}{K^*(n)} = \lim_{n\to\infty} \frac{2}{n} = 0 < \lambda.$$

Hence

$$\sum_{n=1}^{\infty} \frac{n^{m-1}}{\lambda^n} \sum_{k=1}^{n} |K(n,k)| < \infty.$$

Therefore, by Corollary 3, for any solution y of the equation $\Delta^m y_n = b_n$ such that $y_n = \mathrm{O}(n^m)$, there exists a solution x of the equation

$$\Delta^m x_n = b_n + \sum_{k=1}^{n} \frac{(-1)^m 2^k x_{n-\tau}}{(n!)n^m}$$

such that $x_n = y_n + \mathrm{o}(\lambda^n)$.

Now we turn to the problem of asymptotically periodic solutions to Equation (1). Let $q \in \mathbb{N}$. We say that a sequence $\beta \in \mathbb{R}^{\mathbb{N}}$ is q-balanced if it is q-periodic and

$$\beta_1 + \beta_2 + \cdots + \beta_q = 0.$$

Example 4. *If $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, then the sequence*

$$(\alpha_1, \alpha_2, \alpha_3, -\alpha_1, -\alpha_2, -\alpha_3, \alpha_1, \alpha_2, \alpha_3, -\alpha_1, -\alpha_2, -\alpha_3, \dots)$$

is 6-balanced. More generally, we say that a sequence $\gamma \in \mathbb{R}^{\mathbb{N}}$ is q-symmetric if

$$x_{n+q} = -x_n$$

for any $n \in \mathbb{N}$. It is easy to see that any q-symmetric sequence γ is 2q-balanced.

Lemma 5. ([27], Lemma 7.7). *Assume $m, q \in \mathbb{N}$ and $\beta \in \mathbb{R}^{\mathbb{N}}$ is q-balanced. Then there exists a q-periodic sequence $\gamma \in \mathbb{R}^{\mathbb{N}}$ such that $\Delta^m \gamma = \beta$.*

Corollary 12. *Assume the assumptions of Theorem 1 are satisfied, $q \in \mathbb{N}$, and the sequence b is q-balanced. Then there exists a q-periodic solution y of the equation $\Delta^m y_n = b_n$ such that for any $c \in \mathbb{R}$ there exists an asymptotically q-periodic solution x of (1) such that $x_n = c + y_n + \mathrm{o}(u_n)$.*

Proof. By Lemma 5 there exists a q-periodic solution y of the equation $\Delta^m y_n = b_n$. Let $c \in \mathbb{R}$. Then the sequence $c + y$ is bounded and $\Delta^m(c + y) = b$. By Corollary 2 there exists a solution x of (1) such that $x_n = c + y_n + \mathrm{o}(u_n)$. □

Remark 2. *If the assumptions of Theorem 1 are satisfied, $q \in \mathbb{N}$, a sequence $\gamma \in \mathbb{R}^{\mathbb{N}}$ is q-symmetric and $\Delta^m \gamma = b$, then, by Corollary 2, there exists an asymptotically symmetric solution x of (1), such that $x_n = \gamma_n + \mathrm{o}(u_n)$.*

Below we establish conditions under which any bounded solution of (1) is asymptotically periodic.

Corollary 13. *Assume the assumptions of Theorem 2 are satisfied, $q \in \mathbb{N}$, and the sequence b is q-balanced. Then, for any bounded solution x of (1) there exists a q-periodic sequence y such that $x_n = y_n + \mathrm{o}(u_n)$.*

Proof. Let x be a bounded solution of (1). By Corollary 8 there exists a solution y of the equation $\Delta^m y_n = b_n$ such that $x_n = y_n + \mathrm{o}(u_n)$. By Lemma 5 there exists a q-periodic sequence $\gamma \in \mathbb{R}^{\mathbb{N}}$ such that $\Delta^m \gamma = b$. Let $\lambda = y - \gamma$. Then $\Delta^m \lambda = \Delta^m y - \Delta^m \gamma = b - b = 0$. Hence λ is a polynomial sequence. Moreover,

$$\lambda_n = y_n - \gamma_n = x_n - \gamma_n - \mathrm{o}(u_n).$$

Hence λ is bounded. Therefore, the sequence λ is constant and $y = \lambda + \gamma$ is q-periodic. □

We say that a sequence $x \in \mathbb{R}^{\mathbb{N}}$ is (f, τ)-bounded if the sequence $f(k, x_{k-\tau})$ is bounded. For (f, τ)-bounded solutions of Equation (1) we have the following simple version of Theorem 2.

Theorem 3. *Assume*

$$u : \mathbb{N} \to (0, \infty), \quad \Delta u_n \le 0, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{n^{m-1}}{u_n} \sum_{k=1}^{n} |K(n,k)| < \infty. \tag{22}$$

Then, for any (f, τ)-bounded solution x of (1) there exists a solution y of the equation $\Delta^m y_n = b_n$ such that, $x_n = y_n + \mathrm{o}(u_n)$.

Proof. Let x be an (f, τ)-bounded solution of (1) and let M be a positive constant such that

$$|f(k, x_{k-\tau})| \le M$$

for any $k \in \mathbb{N}$. Now, repeating the second part of the proof of Theorem 2 we get the result. □

Corollary 14. *Assume* (22) *and*

$$\sum_{n=1}^{\infty} \frac{n^{m-1}|b_n|}{u_n} < \infty. \tag{23}$$

Then, for any (f,τ)*-bounded solution* x *of* (1) *there exists a polynomial* φ *such that* $\deg(\varphi) < m$ *and* $x_n = \varphi(n) + \mathrm{o}(u_n)$.

Proof. Assume x is an (f,τ)-bounded solution of (1). By Theorem 3 there exists a sequence y such that $\Delta^m y = b$ and $x_n = y_n + \mathrm{o}(u_n)$. By Lemma 2 there exists a sequence z such that $z_n = \mathrm{o}(u_n)$ and $\Delta^m z = b$. Let $\varphi = y - z$. Then φ is a polynomial sequence, $\deg(\varphi) < m$, and $x_n = \varphi(n) + z_n + \mathrm{o}(u_n) = \varphi(n) + \mathrm{o}(u_n)$. □

Below we present conditions under which any solution of (1) is asymptotically polynomial.

Corollary 15. *Assume* (22), (23)*, and* f *is bounded. Then for any solution* x *of* (1) *there exists a polynomial* φ *with the property* $\deg(\varphi) < m$ *and* $x_n = \varphi(n) + \mathrm{o}(u_n)$.

Proof. If f is bounded, then any sequence $x \in \mathbb{R}^{\mathbb{N}}$ is (f,τ)-bounded. Hence the assertion follows from Corollary 14. □

Finally, we present a version of Theorem 1 relating to the case of an ordinary difference equation. In this case, our result is also new.

Theorem 4. *Assume* $u, w : \mathbb{N} \to (0,\infty)$, $g : [0,\infty) \to [0,\infty)$, g *is locally bounded,*

$$|f(n,t)| \le g(|tw_n|) \quad \textit{for} \quad (n,t) \in \mathbb{N} \times \mathbb{R}, \tag{24}$$

$$a \in \mathbb{R}^{\mathbb{N}}, \quad \sum_{n=1}^{\infty} \frac{n^{m-1}|a_n|}{u_n} < \infty, \tag{25}$$

w *is bounded,* $\Delta u_n \le 0$*, and one of the following conditions is satisfied:*

(a) f *is non-decreasing with respect to the second variable and* $(-1)^m a_n \ge 0$ *for all* n*,*

(b) f *is non-increasing with respect to the second variable and* $(-1)^m a_n \le 0$ *for all* n*.*

Then, for any solution y *of the equation* $\Delta^m y_n = b_n$ *such that,* $y_{n-\tau} = \mathrm{O}(w_n^{-1})$ *there exists a solution* x *of the equation*

$$\Delta^m x_n = a_n f(n, x_{n-\tau}) + b_n \tag{26}$$

such that $x_n = y_n + \mathrm{o}(u_n)$.

Proof. Let us define a map $K : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ by

$$K(n,k) = \begin{cases} a_n & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}.$$

Then, the assumptions of Theorem 1 are satisfied and Equation (1) takes the form (26). Hence, using Theorem 1, we obtain the result. □

6. Conclusions

One of the main tools used in this paper is the Knaster-Tarski fixed point Theorem. We believe that this theorem can be used to study the asymptotic properties of solutions to discrete equations of various types, e.g., neutral type equations, Sturm-Liouville Equations or other Equations with quasi-differences. It also seems that the results presented in this paper can be generalized using the asymptotic pair technique from [29]. Of course, from Theorem 4 one can draw conclusions analogous to the results from Section 4. We leave it to the reader.

Author Contributions: Conceptualization, J.M.; Methodology, J.M., M.M., and E.S.; Formal Analysis, J.M., M.M., and E.S.; Resources, M.M. and E.S.; Writing—Original Draft Preparation, J.M., M.M., and E.S. All authors have read and agreed to the published version of the manuscript.

Funding: Scientific research by the second author was carried out at the Poznan University of Technology as part of the Rector's grant 2021 (research project No. 0213/SIGR/2154).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare that they have no competing interests.

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