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Two New Bailey Lattices and Their Applications

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Abstract: In our present investigation, we develop two new Bailey lattices. We describe a number of q-multisums new forms with multiple variables for the basic hypergeometric series which arise as consequences of these two new Bailey lattices. As applications, two new transformations for basic hypergeometric by using the unit Bailey pair are derived. Besides it, we use this Bailey lattice to get some kind of mock theta functions. Our results are shown to be connected with several earlier works related to the field of our present investigation.

Keywords: bailey transform; q-series; bailey pair; basic hypergeometric series

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1. Introduction, Motivation and Preliminaries

Throughout the paper, we use the standard q-notations (see [1]). For $|q| < 1$, we define the q-shifted factorials as:

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

It is easy to see that:

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

For convenience, we also adopt the following compact notation for the multiple q-shifted factorial:

$$(a_1,a_2,\ldots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\ldots(a_m;q)_n,$$

where n is an integer or ∞.

The q-binomial coefficients are defined by

$$\begin{bmatrix} N \\ j \end{bmatrix}_q = \frac{(q;q)_N}{(q;q)_j(q;q)_{N-j}}.$$

The basic hypergeometric series ${}_r\phi_s$ is defined as:

$${}_r\phi_s(a_1,a_2,\ldots,a_r b_1,b_2,\ldots,b_s;q,z) = \sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(q,b_1,b_2,\ldots,b_s;q)_n}\left((-1)^n q^{n(n-1)/2}\right)^{1+s-r} z^n.$$

The study of basic (or q-) hypergeometric functions and q-polynomials is of great interest in many areas of mathematics and physics, including the Theory of Partitions, and is also helpful in a wide selection of areas including, Particle Physics, Lie Theory, Theory of Heat Conduction and Statistics, Quantum Mechanics, Mechanical Engineering, Combinatorial Analysis, Cosmology, Non-Linear Electric Circuit Theory and Finite Vector Spaces, (see, for example, [2] (pp. 350–351); see also [3–6]).

We would like to remark that Srivastava's published review article [7] is potentially useful for researcher, in which it is shown that the so-called (p, q)-calculus is an insignificant, unimportant and inconsequential variation of the classical q-calculus, the extra parameter p being unnecessary or superfluous (see, for details, [7] (p. 340)).

Approximately 35 q-series identities and deemed mock theta functions were studied from different viewpoints and perspectives by the many authors in the 21st century. For example, some types of q-series were studied by Watson [8], some were found in Ramanujan's lost notebook and studied by Hickerson, Choi and Andrews [9–13]. Some other well-known mathematicians like Gordon, Berndt, McIntosh and Chan have studied the q-series from deferent perspectives (see for example [14–17]).

The following definition of Bailey pair is due to Bailey (see [18]).

Definition 1 (see [18])**.** *A pair of sequences $\{\alpha_L\}$ and $\{\beta_L\}$ related by the equation*

$$\beta_L = \sum_{r=0}^{L} \frac{\alpha_r}{(q;q)_{L-r}(aq;q)_{L+r}} \tag{1}$$

is called a Bailey pair relative to a.

In the 1940s and 1950s, Bailey and Slater systematically use the fact that, subject to convergence conditions, if β_L is given by (1), then we have the identity

$$\sum_{n\geq 0}(\rho_1;q)_n(\rho_2;q)_n(aq/\rho_1\rho_2)^n\beta_n = \frac{(aq/\rho_1;q)_\infty(aq/\rho_2;q)_\infty}{(aq;q)_\infty(aq/\rho_1\rho_2;q)_\infty}\sum_{n\geq 0}\frac{(\rho_1;q)_n(\rho_2;q)_n(aq/\rho_1\rho_2)^n}{(aq/\rho_1;q)_n(aq/\rho_2;q)_n}\alpha_n. \tag{2}$$

what is known as Bailey Lemma.

The Bailey transformation [18] is given by Bailey, which is a special case of a certain general type series transformation. The iteration of the Bailey pairs gives us the Bailey chain [19]:

$$(\alpha,\ \beta) \to (\alpha',\ \beta') \to (\alpha'',\ \beta'') \to \dots$$

and so, by the applications of this Bailey chain, one can immediately get a number of sequences of Bailey pairs.

In the 1980s, it was seen by Andrews that the work of Bailey actually led to the process which provided new pairs satisfying (1) from known ones [19,20]. He then gave a certain new pair of sequences $(\alpha_L,\ \beta_L)$, which satisfy (1) a Bailey pair relative to a and showed that if $(\alpha_L,\ \beta_L)$ is such a sequence, then so is $(\alpha'_L,\ \beta'_L)$ with

$$\alpha'_L = \frac{(\rho_1;q)_L(\rho_2;q)_L(aq/\rho_1\rho_2)^L}{(aq/\rho_1;q)_L(aq/\rho_2;q)_L}\alpha_L \tag{3}$$

and

$$\beta'_L = \sum_{j=0}^{L}\frac{(\rho_1;q)_j(\rho_2;q)_j(aq/\rho_1\rho_2;q)_{L-j}(aq/\rho_1\rho_2)^j}{(q;q)_{L-j}(aq/\rho_1;q)_L(aq/\rho_2;q)_L}\beta_j. \tag{4}$$

One of the features of the work of Andrews [19,20] is that it transforms a Bailey pair relative to a into a new Bailey pair relative to a. Generally, the transformation of a Bailey pair relative to a into a Bailey pair relative to b is easily possible. For example, in [21] (Theorem 3.1) for a Bailey pair $(\alpha_L(b), \beta_L(b))$, Warnaar gave their first result relative to b.

In particular, he fixed a non-negative number N and $b = aq^N$ and showed that the pair $(\alpha'_L(a), \beta'_L(a))$ with

$$\alpha'_L(a) = (1-aq^{2L})(aq;q)_N \frac{(\rho_1;q)_L(\rho_2;q)_L(\frac{aq}{\rho_1\rho_2})^L}{(\frac{aq}{\rho_1};q)_L(\frac{aq}{\rho_2};q)_L}$$
$$\times \sum_{j=0}^{N}(-1)^j a^j q^{2Lj-j(j+1)/2} \begin{bmatrix} N \\ j \end{bmatrix} \frac{(aq;q)_{2L-j-1}}{(aq;q)_{2L-j+N}} \alpha_{L-j}(b) \tag{5}$$

and

$$\beta'_L(a) = \sum_{r=0}^{L} \frac{(\rho_1;q)_r(\rho_2;q)_r\left(\frac{aq}{\rho_1\rho_2};q\right)_{L-r}}{(q;q)_{L-r}(aq/\rho_1;q)_L(aq/\rho_2;q)_L}\left(\frac{aq}{\rho_1\rho_2}\right)^r \beta_r(b). \tag{6}$$

is a Bailey pair. Warnaar [21] (Theorem 3.2) also proved a very similar result to [21] (Theorem 3.1) and has called it the second Warnaar–Bailey lemma (see for details [21] (Theorem 3.2)).

It was also pointed out by Agarwal, Andrews and Bressoud [22] (see also [23]) that successive Bailey pairs are not always linearly arranged, but that even within the limitations of fixed ρ and σ, they showed that if $(\alpha_L,\ \beta_L)$ is a Bailey pair relative to a, that is, these sequences satisfy Equation (1) for all $L \geq 0$. Furthermore, if $\alpha' = \{\alpha'_L\}$ by

$$\alpha'_0 = \alpha_0,$$
$$\alpha'_L = (1-a)\left(\frac{a}{\rho\sigma}\right)^L \frac{(\sigma;q)_L(\rho;q)_L}{(a/\rho;q)_L(a/\sigma;q)_L}\left[\frac{\alpha_L}{1-aq^{2L}} - \frac{aq^{2L-2}\alpha_{L-1}}{1-aq^{2L-2}}\right], \tag{7}$$

for all $L \geq 1$ and $\beta' = \{\beta'_L\}$ by

$$\beta'_L = \sum_{k=0}^{L} \frac{(\sigma;q)_k(\rho;q)_k(\frac{a}{\rho\sigma};q)_{L-k}}{(q;q)_{L-k}(a/\rho;q)_L(a/\sigma;q)_L}\left(\frac{a}{\rho\sigma}\right)^k \beta_k, \tag{8}$$

then $(\alpha'_L,\ \beta'_L)$ is a Bailey pair relative to aq^{-1}, that is, these sequences satisfy Equation (1) with a replaced by aq^{-1} for all $L \geq 0$.

Moreover, Schilling and Warnaar [24] studied these Bailey lattice in a systematic way. In fact they applied the well-known Bailey lemma and obtained more general q-series identities, which generalize previously known results (see [24]). Very recently, Jia and Zeng (see [25]) have proven a general expansion formula in Askey–Wilson polynomials using Bailey transform. For some more recent investigations involving the basic q-transformation, q-hypergeometric series, q-multisums and mock theta functions, we may refer the interested reader to [18,21–24,26–28] .

Motivated by the above-mentioned works, here in this paper, we consider the more general transformation, by using and substituting the symmetry character for specializing some variables, give a more direct proof of the result. Furthermore, by making use of the new Bailey lattices, we obtain certain new forms of the q-multisums with multiple variables. As an example, we get two new transformations for basic hypergeometric q-series. Furthermore, we obtain a new relation between q-multisums and mock theta functions. Our results, which we have presented in this article, are shown to be connected with a number of earlier works on this subject.

2. A Set of Lemmas

To prove our main results, we need the following lemmas.

Lemma 1. *Fix N a nonnegative integer and* $\left|\frac{a^3q^{3+2L+N}}{\rho_1^2\sigma_1^2}\right| < 1$, *we have*

$$\sum_{r=0}^{L-j}\frac{(1-aq^{2r+2j})}{(1-aq^{2j})}\frac{(aq^{2j};q)_r}{(aq^{2j+N+1};q)_r}\frac{(q^{-N};q)_r}{(q;q)_r}\frac{(\rho_1^2q^{2j},\rho_2^2q^{2j};q^2)_r}{(a^2q^{2+2j}/\rho_1^2,a^2q^{2+2j}/\rho_2^2;q^2)_r}$$
$$\times\frac{(q^{-2L+2j};q^2)_r}{(a^2q^{2+2L+2j};q^2)_r}\left(-\frac{a^3q^{3+2L+N}}{\rho_1^2\sigma_2^2}\right)^r = \frac{(a^2q^{4j+2};q^2)_{L-j}(a^2q^2/\rho_1^2\rho_2^2;q^2)_{L-j}}{(a^2q^{2j+2}/\rho_1^2;q^2)_{L-j}(a^2q^{2j+2}/\rho_2^2;q^2)_{L-j}}$$
$$\times\sum_{r=0}^{L-j}\frac{(\rho_1^2q^{2j},\rho_2^2q^{2j},-aq^{2j+1+N},-aq^{2j+2+N},q^{-2L+2j};q^2)_r}{(q^2,a^2q^{2+4j+2N},-aq^{2j+1},-aq^{2j+2},\frac{\rho_1^2\rho_2^2}{a^2}q^{-2L+2j};q^2)_r}q^{2r}.$$

Proof. We start with the terminating transformation [29] (Equation (1.3)):

$$ {}_{10}\phi_9\left[\begin{matrix} a,a^{\frac12}q,-a^{\frac12}q,b,x,-x,y,-y,q^{-n},-q^{-n} \\ a^{\frac12},-a^{\frac12},aq/b,aq/x,-aq/x,aq/y,-aq/y,-aq^{1+n},aq^{1+n}\end{matrix};q,-\frac{a^3q^{3+2n}}{bx^2y^2}\right]$$

$$=\frac{(a^2q^2;q^2)_n(a^2q^2/x^2y^2;q^2)_n}{(a^2q^2/x^2;q^2)_n(a^2q^2/y^2;q^2)_n}\times {}_5\phi_4\left[\begin{matrix}x^2,y^2,-aq/b,,-aq^2/b,q^{-2n}\\ -aq,-aq^2,a^2q^2/b^2,\frac{x^2y^2}{a^2}q^{-2n}\end{matrix};q^2,q^2\right].$$

In the above transformation, by taking

$$a\to aq^{2j},\ n\to(L-j),\ b\to q^{-N},x\to\rho_1q^j,y\to\rho_2q^j$$

and after some elementary and straightforward simplification, we can complete the proof Lemma 1. □

Lemma 2 (see [19,20])**.** *Let $(\alpha_L,\ \beta_L)$ form a Bailey pair relative to a, that is, these sequences satisfy Equation (1) for all $L\ge 0$. Then, so does $(\alpha_L',\ \beta_L')$ with*

$$\alpha_L'=\frac{(\rho_1;q)_L(\rho_2;q)_L(aq/\rho_1\rho_2)^L}{(aq/\rho_1;q)_L(aq/\rho_2;q)_L}\alpha_L$$

and

$$\beta_L'=\sum_{j=0}^{L}\frac{(\rho_1;q)_j(\rho_2;q)_j(aq/\rho_1\rho_2;q)_{L-j}(aq/\rho_1\rho_2)^j}{(q;q)_{L-j}(aq/\rho_1;q)_L(aq/\rho_2;q)_L}\beta_j.$$

Lemma 3 ([1,29])**.** *We have the following transformation:*

$$ {}_6\phi_5\left[\begin{matrix}a,a^{1/2}q,-a^{1/2}q,b,q^{-r},-q^{-r}\\ a^{1/2},-a^{1/2},aq/b,-aq^{1+r},aq^{1+r}\end{matrix};q,-aq^{1+2r}/b\right]=\frac{(a^2q^2;q^2)_r(-aq/b;q)_{2r}}{(-aq;q)_{2r}(a^2q^2/b^2;q^2)_r}.$$

Lemma 4 ([1] (II.12))**.** *We have the following transformation:*

$$ {}_3\phi_2\left[\begin{matrix}a,b,q^{-r}\\ c,abq^{1-r}/c\end{matrix};q,q\right]=\frac{(c/a,c/b;q)_r}{(c,c/ab;q)_r}.$$

Our further investigation is organized as follows. In Section 3, we give some important results, related to Bailey pairs. In Section 4, we derive the new Rogers–Ramanujan identities using new Bailey lattices. This is made explicit in two new Bailey transformations, Theorems 3 and 5. Furthermore, using these Bailey transformations, we get two new basic hypergeometric q-series. In Section 5, we get the analogous results to those that were proved by Lovejoy [30,31].

3. Main Results and Their Demonstration

Theorem 1. *Fix N a nonnegative integer and set $b = aq^N$. Let $(\alpha_L(b^2,q^2), \beta_L(b^2,q^2))$ be a Bailey pair with $\left|\frac{a^3q^{3+2L+N}}{\rho_1^2\rho_2^2}\right| < 1$. Then, so is $(\alpha'_L(a^2,q^2), \beta'_L(a^2,q^2))$ with:*

$$\alpha'_L(a^2,q^2) = (1-aq^{2L})(aq;q)_N \frac{(\rho_1^2;q^2)_L(\rho_2^2;q^2)_L(\frac{a^2q^2}{\rho_1{}^2\rho_2{}^2})^L}{(\frac{a^2q^2}{\rho_1^2};q^2)_L(\frac{a^2q^2}{\rho_2^2};q^2)_L}$$
$$\times \sum_{j=0}^{N}(-1)^j a^j q^{2Lj-j(j+1)/2}\begin{bmatrix} N \\ j \end{bmatrix}_q \frac{(aq;q)_{2L-j-1}}{(aq;q)_{2L-j+N}}\alpha_{L-j}(b^2,q^2)$$

and

$$\beta'_L(a^2,q^2) = \sum_{r=0}^{L} \frac{(\rho_1^2;q^2)_r(\rho_2^2;q^2)_r\left(\frac{a^2q^2}{\rho_1^2\rho_2^2};q^2\right)_{L-r}(-bq;q)_{2r}}{(q^2;q^2)_{L-r}(a^2q^2/\rho_1^2;q^2)_L(a^2q^2/\rho_2^2;q^2)_L(-aq;q)_{2r}}\left(\frac{a^2q^2}{\rho_1^2\rho_2^2}\right)^r\beta_r(b^2,q^2).$$

Proof. In order show the pair $(\alpha'_L(a^2,q^2)$,$\beta'_L(a^2,q^2)$) satisfy (1), we first need to recall the following formula:

$$I = \sum_{r=0}^{L}\frac{\alpha'_r(a^2,q^2)}{(q^2;q^2)_{L-r}(a^2q^2;q^2)_{L+r}} \tag{9}$$

Now, substituting the expression in (9) for $\alpha'_L(a^2,q^2)$, we have

$$I = \sum_{r=0}^{L}\frac{(1-aq^{2r})}{(q^2;q^2)_{L-r}(a^2q^2;q^2)_{L+r}}\frac{(\rho_1^2,\rho_2^2;q^2)_r}{(a^2q^2/\rho_1^2;q^2)_r(a^2q^2/\rho_2^2;q^2)_r}\left(\frac{a^2q^2}{\rho_1^2\rho_2^2}\right)^r(aq;q)_N$$
$$\times\sum_{j=0}^{N}(-1)^j a^j q^{2rj-j(j+1)/2}\begin{bmatrix} N \\ j \end{bmatrix}_q\left(\frac{(aq;q)_{2r-j-1}}{(aq;q)_{2r-j+N}}\right)\alpha_r(b^2,q^2).$$

Transforming $j \to (r-j)$ and interchanging the order of summation in the above identity, gives

$$I = \sum_{j=0}^{L}\sum_{r=0}^{L-j}\frac{(1-aq^{2r+2j})(aq;q)_N}{(q^2;q^2)_{L-r-j}(a^2q^2;q^2)_{L+r+j}}\frac{(\rho_1^2,\rho_2^2;q^2)_{r+j}}{(a^2q^2/\rho_1^2;q^2)_{r+j}(a^2q^2/\rho_2^2;q^2)_{r+j}}\left(\frac{a^2q^2}{\rho_1^2\rho_2^2}\right)^{r+j}$$
$$\times(-1)^r a^r q^{2(r+j)r-r(r+1)/2}\begin{bmatrix} N \\ r \end{bmatrix}_q\frac{(aq;q)_{r+2j-1}}{(aq;q)_{r+2j+N}}\alpha_j(b^2,q^2).$$
$$= \sum_{j=0}^{L}\frac{(\rho_1^2,\rho_2^2;q^2)_j(aq;q)_N(aq;q)_{2j-1}\alpha_j(b^2,q^2)}{(q^2;q^2)_{L-j}(a^2q^2;q^2)_{L+j}(a^2q^2/\rho_1^2;q^2)_j(a^2q^2/\rho_2^2;q^2)_j(aq;q)_{2j+N}}\left(\frac{a^2q^2}{\rho_1^2\rho_2^2}\right)^j$$
$$\times\sum_{r=0}^{L-j}\frac{(1-aq^{2r+2j})(\rho_1^2q^{2j},\rho_2^2q^{2j};q^2)_r}{(q^{2+2L+2j};q^2)_{-r}(a^2q^{2+2L-2j};q^2)_r(a^2q^{2+2j}/\rho_1^2,a^2q^{2+2j}/\rho_2^2;q^2)_r}$$
$$\times\frac{(-1)^r a^r q^{2(r+j)r-r(r+1)/2}(q;q)_N(aq^{2j};q)_r}{(q;q)_r(q;q)_{N-r}(aq^{2j+N+1};q)_r}\left(\frac{a^2q^2}{\rho_1^2\rho_2^2}\right)^r.$$

After some simplification, the inner summation can be written as:

$$\sum_{r=0}^{L-j}(1-aq^{2r+2j})\frac{(aq^{2j};q)_r}{(aq^{2j+N+1};q)_r}\frac{(q^{-N};q)_r}{(q;q)_r}\frac{(\rho_1^2q^{2j},\rho_2^2q^{2j};q^2)_r}{(a^2q^{2+2j}/\rho_1^2,a^2q^{2+2j}/\rho_2^2;q^2)_r}$$
$$\times\frac{(q^{-2L+2j};q^2)_r}{(a^2q^{2+2L+2j};q^2)_r}\left(-\frac{a^3q^{3+2L+N}}{\rho_1^2\sigma_1^2}\right)^r.$$

Comparing with Lemma 1, we have

$$I=\sum_{j=0}^{L}\frac{(\rho_1^2,\rho_2^2;q^2)_j(aq;q)_N(aq;q)_{2j-1}(1-aq^{2j})\alpha_j(b^2,q^2)}{(q^2;q^2)_{L-j}(a^2q^2;q^2)_{L+j}(a^2q^2/\rho_1^2;q^2)_j(a^2q^2/\rho_2^2;q^2)_j(aq;q)_{2j+N}}\left(\frac{a^2q^2}{\rho_1\rho_2}\right)^j$$
$$\times\frac{(a^2q^{4j+2};q^2)_{L-j}(a^2q^2/\rho_1^2\rho_2^2;q^2)_{L-j}}{(a^2q^{2j+2}/\rho_1^2;q^2)_{L-j}(a^2q^{2j+2}/\rho_2^2;q^2)_{L-j}}\sum_{r=0}^{L-j}\frac{(\rho_1^2q^{2j},\rho_2^2q^{2j};q^2)_r}{(q^2,a^2q^{2+4j+2N};q^2)_r}$$
$$\times\frac{(-aq^{2j+1+N},-aq^{2j+2+N},q^{-2L+2j};q^2)_r}{(-aq^{2j+1},-aq^{2j+2},\frac{\rho_1^2\rho_2^2}{a^2}q^{-2L+2j};q^2)_r}q^{2r}$$
$$=\sum_{j=0}^{L}\sum_{r=0}^{L-j}\frac{(\rho_1^2;q^2)_{r+j}(\rho_2^2;q^2)_{r+j}\left(\frac{a^2q^2}{\rho_1^2\rho_2^2}\right)_{L-r-j}(-aq^{N+1};q)_{2r+2j}}{(q^2;q^2)_{L-r-j}(a^2q^2/\rho_1^2;q^2)_L(a^2q^2/\rho_2^2;q^2)_L(-aq;q)_{2r+2j}(a^2q^{2N+2};q^2)_{2j+r}}$$
$$\times\frac{\alpha_j(b^2,q^2)}{(q^2;q^2)_r}\left(\frac{a^2q^2}{\rho_1^2\rho_2^2}\right)^{r+j}.$$

Shifting $r\to(r-j)$, interchanging sums and recalling $(\alpha_L(b^2,q^2),\beta_L(b^2,q^2))$ is a Bailey pair, we have

$$I=\sum_{r=0}^{L}\sum_{j=0}^{L-r}\frac{(\rho_1^2;q^2)_r(\rho_2^2;q^2)_r\left(\frac{a^2q^2}{\rho_1^2\rho_2^2};q^2\right)_{L-r}(-aq^{N+1};q^2)_{2r}}{(q^2;q^2)_{L-r}(a^2q^2/\rho_1^2;q^2)_L(a^2q^2/\rho_2^2;q^2)_L(-aq;q^2)_{2r}}\left(\frac{a^2q^2}{\rho_1^2\rho_2^2}\right)^r$$
$$\times\frac{\alpha_j(b^2,q^2)}{(a^2q^{2N+2};q^2)_{j+r}(q^2;q^2)_{r-j}}$$
$$=\sum_{r=0}^{L}\frac{(\rho_1^2;q^2)_r(\rho_2^2;q^2)_r\left(\frac{a^2q^2}{\rho_1^2\rho_2^2};q^2\right)_{L-r}(-bq;q^2)_{2r}\left(\frac{a^2q^2}{\rho_1^2\rho_2^2}\right)^r}{(q^2;q^2)_{L-r}(a^2q^2/\rho_1^2;q^2)_L(a^2q^2/\rho_2^2;q^2)_L(-aq;q^2)_{2r}}\sum_{j=0}^{L-r}\frac{\alpha_j(b^2,q^2)}{(b^2q^2;q^2)_{r+j}(q^2;q^2)_{r-j}}$$
$$=\sum_{r=0}^{L}\frac{(\rho_1^2;q^2)_r(\rho_2^2;q^2)_r\left(\frac{a^2q^2}{\rho_1^2\rho_2^2};q^2\right)_{L-r}(-bq;q^2)_{2r}}{(q^2;q^2)_{L-r}(a^2q^2/\rho_1^2;q^2)_L(a^2q^2/\rho_2^2;q^2)_L(-aq;q)_{2r}}\left(\frac{a^2q^2}{\rho_1^2\rho_2^2}\right)^r\beta_L(b^2,q^2)$$
$$=\beta'_L(a^2,q^2).$$

So, the pair $(\alpha'_L(a^2,q^2),\beta'_L(a^2,q^2))$ is also a Bailey pair. The proof of the Theorem 1 is now completed. □

If we take $N=1$ in Theorem 1, we have the following corollary.

Corollary 1. *Set $b = aq$. Let $(\alpha_L(b^2, q^2), \beta_L(b^2, q^2))$ be a Bailey pair. Then, so is $(\alpha'_L(a^2, q^2), \beta'_L(a^2, q^2))$ with*

$$\alpha'_L(a^2, q^2) = (1 - aq^{2L})(1 - aq) \frac{(\rho_1^2; q^2)_L (\rho_2^2; q^2)_L (\frac{a^2q^2}{\rho_1{}^2\rho_2{}^2})^L}{(\frac{a^2q^2}{\rho_1^2}; q^2)_L (\frac{a^2q^2}{\rho_2^2}; q^2)_L}$$
$$\times \left[\frac{(aq; q)_{2L-1}}{(aq; q)_{2L+1}} \alpha_L(b^2, q^2) - aq^{2L-1} \frac{(aq; q)_{2L-2}}{(aq; q)_{2L}} \alpha_{L-1}(b^2, q^2) \right] \tag{10}$$

and

$$\beta'_L(a^2, q^2) = \sum_{r=0}^{L} \frac{(\rho_1^2; q^2)_r (\rho_2^2; q^2)_r \left(\frac{a^2q^2}{\rho_1^2\rho_2^2}; q^2 \right)_{L-r} (1 + aq^{2r+1})}{(q^2; q^2)_{L-r} (a^2q^2/\rho_1^2; q^2)_L (a^2q^2/\rho_2^2; q^2)_L (1 + aq)}$$
$$\times \left(\frac{a^2q^2}{\rho_1^2\rho_2^2} \right)^r \beta_r(b^2, q^2). \tag{11}$$

To prove the next Theorem (Theorem 2 below), we make use of Lemmas 3 and 4, and followed the same steps as we have done in the proof of Theorem 1, we can easily get the proof of Theorem 2, so we choose to omit the details involved.

Theorem 2. *Fix N a nonnegative integer and set $b = aq^N$. Let $(\alpha_L(b^2, q^2), \beta_L(b^2, q^2))$ be a Bailey pair. Then, so is $(\alpha'_L(a^2, q^2), \beta'_L(a^2, q^2))$ with:*

$$\alpha'_L(a^2, q^2) = (1 - aq^{2L})(aq; q)_N \sum_{j=0}^{N} \frac{(\rho_1^2; q^2)_{L-j} (\rho_2^2; q^2)_{L-j} (\frac{b^2q^2}{\rho_1{}^2\rho_2{}^2})^{L-j}}{(\frac{b^2q^2}{\rho_1^2}; q^2)_{L-j} (\frac{b^2q^2}{\rho_2^2}; q^2)_{L-j}}$$
$$\times (-1)^j a^j q^{2Lj - j(j+1)/2} \begin{bmatrix} N \\ j \end{bmatrix}_q \frac{(aq; q)_{2L-j-1}}{(aq; q)_{2L-j+N}} \alpha_{L-j}(b^2, q^2)$$

and

$$\beta'_L(a^2, q^2) = \sum_{r=0}^{L} \frac{(\rho_1^2; q^2)_r (\rho_2^2; q^2)_r \left(\frac{b^2q^2}{\rho_1^2\rho_2^2}; q^2 \right)_{L-r} (-bq; q)_{2L}}{(q^2; q^2)_{L-r} (b^2q^2/\rho_1^2; q^2)_L (b^2q^2/\rho_2^2; q^2)_L (-aq; q)_{2L}} \left(\frac{b^2q^2}{\rho_1^2\rho_2^2} \right)^r \beta_r(b^2, q^2).$$

Remark 1. If we set $N = 0$, in Theorem 2, we will then arrived at Lemma 2.

Corollary 2. *Set $b = aq$ and let $(\alpha_L(b^2, q^2), \beta_L(b^2, q^2))$ be a Bailey pair. Then, so is $(\alpha'_L(a^2, q^2), \beta'_L(a^2, q^2))$ with*

$$\alpha'_L(a^2, q^2) = (1 - aq^{2L})(1 - aq) \frac{(\rho_1^2; q^2)_L (\rho_2^2; q^2)_L (\frac{b^2q^2}{\rho_1{}^2\rho_2{}^2})^L}{(\frac{b^2q^2}{\rho_1^2}; q^2)_L (\frac{b^2q^2}{\rho_2^2}; q^2)_L}$$
$$\times \frac{(aq; q)_{2L-1}}{(aq; q)_{2L+1}} \alpha_L(b^2, q^2) - (1 - aq^{2L})(1 - aq) aq^{2L-1}$$
$$\frac{(\rho_1^2; q^2)_{L-1} (\rho_2^2; q^2)_{L-1} (\frac{b^2q^2}{\rho_1{}^2\rho_2{}^2})^{L-1}}{(\frac{b^2q^2}{\rho_1^2}; q^2)_{L-1} (\frac{b^2q^2}{\rho_2^2}; q^2)_{L-1}} \frac{(aq; q)_{2L-2}}{(aq; q)_{2L}} \alpha_{L-1}(b^2, q^2) \tag{12}$$

and

$$\beta'_L(a^2,q^2) = \sum_{r=0}^{L} \frac{(\rho_1^2;q^2)_r(\rho_2^2;q^2)_r\left(\frac{b^2q^2}{\rho_1^2\rho_2^2};q^2\right)_{L-r}(1+aq^{2L+1})}{(q^2;q^2)_{L-r}(b^2q^2/\rho_1^2;q^2)_L(b^2q^2/\rho_2^2;q^2)_L(1+aq)}\left(\frac{b^2q^2}{\rho_1^2\rho_2^2}\right)^r\beta_r(b^2,q^2). \quad (13)$$

4. Two New Bailey Transformations

The Bailey chain is a well-known and frequently used technique in the theory of partitions as described in the introduction section. It arose from Bailey's [18] realization that the Rogers–Ramanujan identities could be derived from the simple Bailey chains.

Similarly, we can have the new Rogers–Ramanujan identities using this new Bailey lattices. This is made explicit in the following two new Bailey transformation Theorem 3 and Theorem 5. Using these Bailey transformations, we get two new basic hypergeometric q-series.

Theorem 3. *Let α, β be sequences satisfying Equation (1) with $b = aq^N$ where N is non negative, then we have*

$$\begin{aligned}
\beta_m^{(0)}(a^2q^{-2N},q^2) &= \sum_{m\geq m_1\geq\cdots\geq m_i\geq\cdots\geq m_t\geq 0} \frac{(\rho_1;q^2)_{m_1}\ldots(\rho_t;q^2)_{m_t}}{\left[\left(\frac{a^2q^{2-2N}}{\rho_1};q^2\right)_m\right]\ldots\left[\left(\frac{a^2q^{2-2N}}{\rho_i};q^2\right)_{m_{i-1}}\right]} \\
&\times \frac{(\sigma_1;q^2)_{m_1}\ldots(\sigma_t;q^2)_{m_t}}{\left[\left(\frac{a^2q^2}{\rho_{i+1}};q^2\right)_{m_i}\right]\ldots\left[\left(\frac{a^2q^2}{\rho_t};q^2\right)_{m_{t-1}}\right]}\frac{\left(\frac{a^2q^{2-2N}}{\sigma_1\rho_1};q^2\right)_{m-m_1}\ldots\left(\frac{a^2q^{2-2N}}{\sigma_i\rho_i};q^2\right)_{m_{i-1}-m_i}}{\left[\left(\frac{a^2q^{2-2N}}{\sigma_1};q^2\right)_m\right]\ldots\left[\left(\frac{a^2q^{2-2N}}{\sigma_i};q^2\right)_{m_{i-1}}\right]} \\
&\times \frac{\left(\frac{a^2q^2}{\sigma_{i+1}\rho_{i+1}};q^2\right)_{m_i-m_{i-1}}\ldots\left(\frac{a^2q^2}{\sigma_t\rho_t};q^2\right)_{m_{t-1}-m_t}}{\left[\left(\frac{a^2q^2}{\sigma_{i+1}};q^2\right)_{m_i}\right]\ldots\left[\left(\frac{a^2q^2}{\sigma_t};q^2\right)_{m_{t-1}}\right]}\frac{(a^2q^{2-2N})^{(m_1+\cdots+m_i)}(a^2q^2)^{(m_{i+1}+\cdots+m_t)}}{[(q^2;q^2)_{m-m_1}]\ldots[(q^2;q^2)_{m_{t-1}-m_t}]} \\
&\times \frac{(-aq;q)_{2m_i}B_m^{(t)}(a^2,q^2)}{(-aq^{1-N};q)_{2m_i}(\rho_1\sigma_1)^{m_1}\ldots(\rho_i\sigma_i)^{m_i}(\rho_{i+1}\sigma_{i+1})^{m_{i+1}}\ldots(\rho_t\sigma_t)^{m_t}}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{m}\sum_{j=0}^{N}\frac{1}{(q^2;q^2)_{m-k}(a^2q^{2-2N};q^2)_{m+k}}\frac{(\rho_1;q^2)_k\ldots(\rho_i;q^2)_k(\rho_{i+1};q^2)_{k-j}\ldots(\rho_t;q^2)_{k-j}}{\left[\left(\frac{a^2q^{2-2N}}{\rho_1};q^2\right)_k\right]\ldots\left[\left(\frac{a^2q^{2-2N}}{\rho_i};q^2\right)_k\right]} \\
&\times \frac{(\sigma_1;q^2)_k\ldots(\sigma_i;q^2)_k(\sigma_{i+1};q^2)_{k-j}\ldots(\sigma_t;q^2)_{k-j}}{\left[\left(\frac{a^2q^2}{\rho_{i+1}};q^2\right)_{k-j}\right]\ldots\left[\left(\frac{a^2q^2}{\rho_t};q^2\right)_{k-j}\right]}\frac{(a^2q^{2-2N})^{(ki)}(a^2q^2)^{(k-j)(t-i)}}{\left(\frac{a^2q^{2-2N}}{\sigma_1};q^2\right)_k\ldots\left(\frac{a^2q^{2-2N}}{\sigma_i};q^2\right)_k} \\
&\times \frac{(1-aq^{2k-N})(aq^{1-N})_N(-1)^j(aq^{-N})^jq^{2kj-j(j+1)/2}}{\left[\left(\frac{a^2q^2}{\sigma_{i+1}};q^2\right)_{k-j}\right]\ldots\left[\left(\frac{a^2q^2}{\sigma_t};q^2\right)_{k-j}\right](\rho_1\sigma_1)^k\ldots(\rho_i\sigma_i)^k(\rho_{i+1}\sigma_{i+1})^{k-j}\ldots(\rho_t\sigma_t)^{k-j}} \\
&\times \begin{bmatrix} N \\ j \end{bmatrix}_q \frac{(aq^{1-N};q)_{2k-j-1}}{(aq^{1-N};q)_{2k-j+N}}\alpha_{k-j}^{(t)}(a^2,q^2). \qquad (14)
\end{aligned}$$

Proof. Step 1: Begin with Lemma 2 and replace $\rho_1 \to \rho_{i-1}$, $\rho_2 \to \sigma_{i-1}$, a by a^2q^{-2N} and $q \to q^2$, we have

$$\beta_L^{(i-2)}(a^2q^{-2N},q^2) = \sum_{j=0}^{L}\frac{(\rho_{i-1},\sigma_{i-1};q^2)_j\left(\frac{a^2q^{2-2N}}{\rho_{i-1}\sigma_{i-1}};q^2\right)_{L-j}\left(\frac{a^2q^{2-2N}}{\rho_{i-1}\sigma_{i-1}}\right)^j}{(q^2;q^2)_{L-j}\left(\frac{a^2q^{2-2N}}{\rho_{i-1}},\frac{a^2q^{2-2N}}{\sigma_{i-1}};q^2\right)_L}\beta_j^{(i-1)}(a^2q^{-2N},q^2)$$

and

$$\alpha_L^{(i-2)}(a^2q^{-2N};q^2) = \frac{(\rho_{i-1},\sigma_{i-1};q^2)_L(\frac{a^2q^{2-2N}}{\rho_{i-1}\sigma_{i-1}})^L}{(\frac{a^2q^{2-2N}}{\rho_{i-1}},\frac{a^2q^{2-2N}}{\sigma_{i-1}};q^2)_L}\alpha_L^{(i-1)}(a^2q^{-2N};q^2).$$

Let the Bailey chain be denoted by $(\alpha^{i-1},\beta^{i-1}) \to (\alpha^{i-2},\beta^{i-2}) \to \ldots \to (\alpha^0,\beta^0)$, where $\alpha^{(j)} = \{\alpha_m^{(j)}\}, \beta^{(j)} = \{\beta_m^{(j)}\}$, we have

$$\begin{aligned}\beta_m^{(0)}(a^2q^{-2N},q^2) = &\sum_{m\geq m_1\geq\cdots\geq m_{i-1}\geq 0}\frac{(\sigma_1;q^2)_{m_1}\ldots(\sigma_{i-1};q^2)_{m_{i-1}}}{\left[\left(\frac{a^2q^{2-2N}}{\rho_1};q^2\right)_m\right]\ldots\left[\left(\frac{a^2q^{2-2N}}{\rho_{i-1}};q^2\right)_{m_{i-2}}\right]}\\ &\times\frac{(\rho_1;q^2)_{m_1}\ldots(\rho_{i-1};q^2)_{m_{i-1}}\left(\frac{a^2q^{2-2N}}{\sigma_1\rho_1};q^2\right)_{m-m_1}\cdots\left(\frac{a^2q^{2-2N}}{\sigma_{i-1}\rho_{i-1}};q^2\right)_{m_{i-2}-m_{i-1}}}{[(q^2;q^2)_{m-m_1}]\ldots[(q^2;q^2)_{m_{i-2}-m_{i-1}}]}\\ &\times\frac{(a^2q^{2-2N})^{(m_1+\cdots+m_{i-1})}\beta_m^{(i-1)}(a^2q^{-2N},q^2)}{\left(\frac{a^2q^{2-2N}}{\sigma_1};q^2\right)_m\cdots\left(\frac{a^2q^{2-2N}}{\sigma_{i-1}};q^2\right)_{m_{i-2}}(\rho_1\sigma_1)^{m_1}\ldots(\rho_{i-1}\sigma_{i-1})^{m_{i-1}}}\end{aligned}$$

and

$$\begin{aligned}\alpha_k^{(0)}(a^2q^{-2N};q^2) = &\frac{(\rho_1;q^2)_k\ldots(\rho_{i-1};q^2)_k(\sigma_1;q^2)_k\ldots(\sigma_{i-1};q^2)_k}{\left[(\frac{a^2q^{2-2N}}{\rho_1};q^2)_k\right]\ldots\left[(\frac{a^2q^{2-2N}}{\rho_{i-1}};q^2)_k\right]}\\ &\times\frac{(a^2q^{2-2N})^{k(i-1)}\alpha_k^{(i-1)}(a^2q^{-2N};q^2)}{(\rho_1\sigma_1)^k\ldots(\rho_{i-1}\sigma_{i-1})^k\left[\left(\frac{a^2q^{2-2N}}{\sigma_1};q^2\right)_k\right]\ldots\left[\left(\frac{a^2q^{2-2N}}{\sigma_{i-1}};q^2\right)_k\right]}.\end{aligned}$$

Step 2: By using Theorem 1 and taking $\rho_1^2 \to \rho_i$, $\rho_2^2 \to \sigma_i$, $a \to aq^{-N}$, we have that

$$\begin{aligned}\beta_m^{(i-1)}(a^2q^{-2N};q^2) = &\sum_{m_i\geq 0}^{m_{i-1}}\frac{(\rho_i;q^2)_{m_i}(\sigma_i;q^2)_{m_i}\left(\frac{a^2q^{2-2N}}{\rho_i\sigma_i};q^2\right)_{m_{i-1}-m_i}(-aq;q)_{2m_i}}{(a^2q^{2-2N}/\rho_i;q^2)_{m_{i-1}}(a^2q^{2-2N}/\sigma_i;q^2)_{m_{i-1}}(-aq^{1-N};q)_{2m_i}}\\ &\times\frac{\beta_m^{(i)}(a^2,q^2)}{(q^2;q^2)_{m_{i-1}-m_i}}\left(\frac{a^2q^{2-2N}}{\rho_i\sigma_i}\right)^{m_i}\end{aligned}$$

and

$$\begin{aligned}\alpha_k^{(i-1)}(a^2q^{-2N};q^2) = &(1-aq^{2k-N})(aq^{1-N};q)_N\frac{(\rho_i;q^2)_k(\sigma_i;q^2)_k(\frac{a^2q^{2-2N}}{\rho_i\sigma_i})^k}{(\frac{a^2q^{2-2N}}{\rho_i};q^2)_k(\frac{a^2q^{2-2N}}{\sigma_i};q^2)_k}\\ &\times\sum_{j=0}^{N}(-1)^j(aq^{-N})^jq^{2kj-j(j+1)/2}\frac{(aq^{1-N};q)_{2k-j-1}}{(aq^{1-N})_{2k-j+N}}\begin{bmatrix}N\\j\end{bmatrix}_q\alpha_{k-j}^{(i)}(a^2,q^2).\end{aligned}$$

Step 3: Alternately apply Lemma 2 $(t-i)$ times with $a \to a^2$ and $q \to q^2$, we have that

$$\beta_m^{(i)}(a^2,q^2) = \sum_{m_i\geq m_{i+1}\geq\cdots\geq m_t\geq 0}\frac{(\sigma_{i+1};q^2)_{m_{i+1}}\ldots(\sigma_t;q^2)_{m_t}}{\left[\left(\frac{a^2q^2}{\rho_{i+1}};q^2\right)_{m_i}\right]\ldots\left[\left(\frac{a^2q^2}{\rho_t};q^2\right)_{m_{t-1}}\right]}$$

$$\times \frac{(\rho_{i+1};q^2)_{m_{i+1}}\dots(\rho_t;q^2)_{m_t}\left(\frac{a^2q^2}{\sigma_{i+1}\rho_{i+1}}\right)_{m_i-m_{i-1}}\dots\left(\frac{a^2q^2}{\sigma_t\rho_t}\right)_{m_{t-1}-m_t}}{\left[(q^2;q^2)_{m_i-m_{i-1}}\right]\dots\left[(q^2;q^2)_{m_{t-1}-m_t}\right]}$$
$$\times \frac{(a^2q^2)^{(m_{i+1}+\dots+m_t)}\beta_m^{(t)}(a^2,q^2)}{\left(\frac{a^2q^2}{\sigma_i};q^2\right)_{m_i}\dots\left(\frac{a^2q^2}{\sigma_t};q^2\right)_{m_{t-1}}(\rho_{i+1}\sigma_{i+1})^{m_{i+1}}\dots(\rho_t\sigma_t)^{m_t}}$$

and

$$\alpha_{k-j}^{(i)}(a^2,q^2)=\frac{(\rho_{i+1};q^2)_{k-j}\dots(\rho_t;q^2)_{k-j}\left[(\sigma_{i+1};q^2)_{k-j}\right]\dots\left[(\sigma_t;q^2)_{k-j}\right]}{\left[\left(\frac{a^2q^2}{\rho_{i+1}};q^2\right)_{k-j}\right]\dots\left[\left(\frac{a^2q^2}{\rho_t};q^2\right)_{k-j}\right]}$$
$$\times \frac{(a^2q^2)^{(k-j)(t-i)}\alpha_{k-j}^{(t)}(a^2,q^2)}{\left[\left(\frac{a^2q^2}{\sigma_{i+1}};q^2\right)_{k-j}\right]\dots\left[\left(\frac{a^2q^2}{\sigma_t};q^2\right)_{k-j}\right](\rho_{i+1}\sigma_{i+1})^{k-j}\dots(\rho_t\sigma_{i+1}\sigma_t)^{k-j}}.$$

By making substitution $(\alpha^{(t)},\beta^{(t)})$ by $(\alpha^{(i)},\beta^{(i)})$ and also then substituting the result for $(\alpha^{(0)},\beta^{(0)})$, in conjunction with (1). Finally, replace a and q by a^2q^{-2N} and q^2, respectively, we can easily get the desired result. □

Inserting Bailey pairs in Theorem 3, we can derive some new identities. In this regard, we only choose the most elementary Bailey pair relative to a, which satisfies (1) and is given by(see [32] (p. 586, Equations (12.2.5) and (12.2.6))):

$$\begin{aligned}\beta_n &= \chi \qquad (n=0),\\ \alpha_n &= \frac{(a;q)_n}{(q;q)_n}\frac{(1-aq^{2n})}{(1-a)}(-1)^n q^{\binom{n}{2}}.\end{aligned} \tag{15}$$

Taking $N=1$ in Theorem 3, we get the following corollary.

Corollary 3. *Let α, β be sequences satisfying Equation (1), then we have*

$$\beta_m^{(0)}(a^2q^{-2},q^2)=\sum_{m\geq m_1\geq\dots\geq m_i\geq\dots\geq m_t\geq 0}\frac{(\rho_1;q^2)_{m_1}\dots(\rho_t;q^2)_{m_t}}{\left[\left(\frac{a^2}{\rho_1};q^2\right)_m\right]\dots\left[\left(\frac{a^2}{\rho_i};q^2\right)_{m_{i-1}}\right]}$$
$$\times \frac{(\sigma_1;q^2)_{m_1}\dots(\sigma_t;q^2)_{m_t}}{\left[\left(\frac{a^2q^2}{\rho_{i+1}};q^2\right)_{m_i}\right]\dots\left[\left(\frac{a^2q^2}{\rho_t};q^2\right)_{m_{t-1}}\right]}\frac{\left(\frac{a^2}{\sigma_1\rho_1}\right)_{m-m_1}\dots\left(\frac{a^2}{\sigma_i\rho_i}\right)_{m_{i-1}-m_i}}{\left[\left(\frac{a^2}{\sigma_1};q^2\right)_m\right]\dots\left[\left(\frac{a^2}{\sigma_i};q^2\right)_{m_{i-1}}\right]}$$
$$\times \frac{\left(\frac{a^2q^2}{\sigma_{i+1}\rho_{i+1}};q^2\right)_{m_i-m_{i-1}}\dots\left(\frac{a^2q^2}{\sigma_t\rho_t};q^2\right)_{m_{t-1}-m_t}}{\left[\left(\frac{a^2q^2}{\sigma_{i+1}};q^2\right)_{m_i}\right]\dots\left[\left(\frac{a^2q^2}{\sigma_t};q^2\right)_{m_{t-1}}\right]}\frac{(a^2)^{(m_1+\dots+m_i)}(a^2q^2)^{(m_{i+1}+\dots+m_t)}}{\left[(q^2;q^2)_{m-m_1}\right]\dots\left[(q^2;q^2)_{m_{t-1}-m_t}\right]}$$
$$\times \frac{(-aq;q)_{2m_i}\beta_m^{(t)}(a^2,q^2)}{(-a;q)_{2m_i}(\rho_1\sigma_1)^{m_1}\dots(\rho_i\sigma_i)^{m_i}(\rho_{i+1}\sigma_{i+1})^{m_{i+1}}\dots(\rho_t\sigma_t)^{m_t}}$$
$$=\sum_{k=0}^{m}\frac{1}{(q^2;q^2)_{m-k}(a^2;q^2)_{m+k}}\frac{(\rho_1;q^2)_k\dots(\rho_i;q^2)_k(\sigma_1;q^2)_k\dots(\sigma_i;q^2)_k}{(\frac{a^2}{\rho_1};q^2)_k\dots(\frac{a^2}{\rho_i};q^2)_k(\frac{a^2}{\sigma_1};q^2)_k\dots\left(\frac{a^2}{\sigma_i};q^2\right)_k}$$

$$\times \frac{(1-a)a^{2ki}}{(\rho_1\sigma_1)^k\dots(\rho_i\sigma_i)^k}\Bigg[\frac{(\rho_{i+1};q^2)_k\dots(\rho_t;q^2)_k(\sigma_{i+1};q^2)_k\dots(\sigma_t;q^2)_k}{\left[\left(\frac{a^2q^2}{\rho_{i+1}};q^2\right)_k\right]\dots\left[\left(\frac{a^2q^2}{\rho_t};q^2\right)_k\right]\left(\frac{a^2q^2}{\sigma_{i+1}};q^2\right)_k\dots\left(\frac{a^2q^2}{\sigma_t};q^2\right)_k}$$
$$\times \frac{(a^2q^2)^{(k)(t-i)}\alpha_k^{(t)}(a^2,q^2)}{(\rho_{i+1}\sigma_{i+1})^k\dots(\rho_t\sigma_t)^k(1-aq^{2k})}$$
$$-aq^{2k-2}\frac{(\rho_{i+1};q^2)_{k-1}\dots(\rho_t;q^2)_{k-1}(\sigma_{i+1};q^2)_{k-1}\dots(\sigma_t;q^2)_{k-1}}{\left[\left(\frac{a^2q^2}{\rho_{i+1}};q^2\right)_{k-1}\right]\dots\left[\left(\frac{a^2q^2}{\rho_t};q^2\right)_{k-1}\right]\left(\frac{a^2q^2}{\sigma_{i+1}};q^2\right)_{k-1}\dots\left(\frac{a^2q^2}{\sigma_t};q^2\right)_{k-1}}$$
$$\times \frac{(a^2q^2)^{(k-1)(t-i)}A_{k-1}^{(t)}(a^2,q^2)}{(\rho_{i+1}\sigma_{i+1})^{k-1}\dots(\rho_t\sigma_t)^{k-1}(1-aq^{2k-2})}\Bigg]. \tag{16}$$

From Theorem 3 and Corollary 3, we derive a number of q-multisums new forms with multiple variables. The special example is Theorem 4. Next, we use Corollary 3 to prove Theorem 4.

Theorem 4. *Let $a,\rho_1,\rho_2,\sigma_1,\sigma_2$ be indeterminate and $\left|\frac{a^4q^{2+2n}}{\rho_1\rho_2\sigma_1\sigma_2}\right| < 1$. We have the following transformation:*

$$ {}_7\phi_6\left[\begin{matrix} a^2,-aq^2,\rho_1,\sigma_1,\rho_2,\sigma_2,q^{-2n} \\ -a,a^2/\rho_1,a^2/\sigma_1,a^2q^2/\rho_2,a^2q^2/\sigma_2,a^2q^{2n}\end{matrix};q^2,\frac{a^4q^{2+2n}}{\rho_1\rho_2\sigma_1\sigma_2}\right]$$
$$-\frac{(1-\rho_1)(1-\sigma_1)(1-q^{-2n})}{(1-a^2/\rho_1)(1-a^2/\sigma_1)(1-a^2q^{2n})}\left[\frac{a^3q^{2n}}{\rho_1\sigma_1}\right]$$
$$\times{}_7\phi_6\left[\begin{matrix} a^2,-aq^2,q^2\rho_1,q^2\sigma_1,\rho_2,\sigma_2,q^{-2n+2} \\ -a,a^2q^2/\rho_1,a^2q^2/\sigma_1,a^2q^2/\rho_2,a^2q^2/\sigma_2,a^2q^{2n+2}\end{matrix};q,\frac{a^4q^{2+2n}}{\rho_1\rho_2\sigma_1\sigma_2}\right]$$
$$=\frac{(a^2;q^2)_n(\frac{a^2}{\rho_1\sigma_1};q^2)_n}{(\frac{a^2}{\rho_1};q^2)_n(\frac{a^2}{\sigma_1};q^2)_n}{}_5\phi_4\left[\begin{matrix}\rho_1,\sigma_1,a^2q^2/\rho_2\sigma_2,-aq^2,q^{-2n} \\ a^2q^2/\rho_2,a^2q^2/\sigma_2,-a,\rho_1\sigma_1q^{2-2n}/a^2\end{matrix};q^2,q^2\right]. \tag{17}$$

Proof. Taking $t=2$, $i=1$ and applying (15) in Corollary 3, we can find that

$$\frac{(a^2;q^2)_m(\frac{a^2}{\rho_1\sigma_1};q^2)_m}{(\frac{a^2}{\rho_1};q^2)_m(\frac{a^2}{\sigma_1};q^2)_m}\sum_{m\geq m_1\geq 0}\frac{(\rho_1,\sigma_1,\frac{a^2q^2}{\rho_2\sigma_2},-aq^2,q^{-2m};q^2)_{m_1}}{\left(\frac{a^2q^2}{\rho_2},\frac{a^2q^2}{\sigma_2},q^2,-a,\frac{q^{2-2m}\rho_1\sigma_1}{a^2};q^2\right)_{m_1}}q^{2m}$$
$$=\sum_{k=0}^{m}\frac{(\rho_1,\rho_2;q^2)_k(\sigma_1,\sigma_2;q^2)_k(q^{-2m};q^2)_k}{(\frac{a^2}{\rho_1};q^2)_k(\frac{a^2q^2}{\rho_2};q^2)_k(\frac{a^2}{\sigma_1};q^2)_k\left(\frac{a^2q^2}{\sigma_2};q^2\right)_k}\frac{(a^2;q^2)_k}{(a^2q^{2m};q^2)_k(q^2;q^2)_k}$$
$$\times\frac{(1+aq^{2k})}{(1+a)}\left(\frac{a^4q^{2+2m}}{\rho_1\rho_2\sigma_1\sigma_2}\right)^k-\frac{(1-\rho_1)(1-\sigma_1)(1-q^{-2m})}{(1-\frac{a^2}{\rho_1})(1-\frac{a^2}{\sigma_1})(1-a^2q^{2m})}\frac{a^3q^{2m}}{\rho_1\sigma_1}$$
$$\times\sum_{k=1}^{m-1}\frac{(\rho_1q^2,\rho_2;q^2)_{k-1}(\sigma_1q^2,\sigma_2;q^2)_{k-1}(q^{-2m+2};q^2)_{k-1}}{(\frac{a^2q^2}{\rho_1};q^2)_{k-1}(\frac{a^2q^2}{\rho_2};q^2)_{k-1}(\frac{a^2q^2}{\sigma_1};q^2)_{k-1}\left(\frac{a^2q^2}{\sigma_2};q^2\right)_{k-1}}$$
$$\times\frac{(a^2;q^2)_{k-1}}{(a^2q^{2m+2};q^2)_{k-1}(q^2;q^2)_{k-1}}\frac{(1+aq^{2k-2})}{(1+a)}\left(\frac{a^4q^{2+2m}}{\rho_1\rho_2\sigma_1\sigma_2}\right)^{k-1}.$$

□

Likewise, we also have the corresponding conclusions to Theorem 2.

Theorem 5. *Let α, β be sequences satisfying Equation (1) with $b = aq^N$ where N is non negative, then we have*

$$\sum_{m\geq m_1\geq\cdots\geq m_i\geq\cdots\geq m_t\geq 0} \frac{(\rho_1;q^2)_{m_1}\ldots(\rho_t;q^2)_{m_t}}{\left[\left(\frac{a^2q^{2-2N}}{\rho_1};q^2\right)_m\right]\ldots\left[\left(\frac{a^2q^{2-2N}}{\rho_{i-1}};q^2\right)_{m_{i-2}}\right]}$$
$$\times\frac{(\sigma_1;q^2)_{m_1}\ldots(\sigma_t;q^2)_{m_t}}{\left[\left(\frac{a^2q^2}{\rho_i};q^2\right)_{m_{i-1}}\right]\ldots\left[\left(\frac{a^2q^2}{\rho_t};q^2\right)_{m_{t-1}}\right]}\frac{\left(\frac{a^2q^{2-2N}}{\sigma_1\rho_1}\right)_{m-m_1}\ldots\left(\frac{a^2q^{2-2N}}{\sigma_{i-1}\rho_{i-1}}\right)_{m_{i-2}-m_{i-1}}}{\left[\left(\frac{a^2q^{2-2N}}{\sigma_1};q^2\right)_m\right]\ldots\left[\left(\frac{a^2q^{2-2N}}{\sigma_{i-1}};q^2\right)_{m_{i-2}}\right]}$$
$$\times\frac{\left(\frac{a^2q^2}{\sigma_i\rho_i}\right)_{m_{i-1}-m_i}\ldots\left(\frac{a^2q^2}{\sigma_t\rho_t}\right)_{m_{t-1}-m_t}}{\left[\left(\frac{a^2q^2}{\sigma_i};q^2\right)_{m_{i-1}}\right]\ldots\left[\left(\frac{a^2q^2}{\sigma_t};q^2\right)_{m_{t-1}}\right]}\frac{(a^2q^{2-2N})^{(m_1+\cdots+m_{i-1})}(a^2q^2)^{(m_i+\cdots+m_t)}}{[(q^2;q^2)_{m-m_1}]\ldots[(q^2;q^2)_{m_{t-1}-m_t}]}$$
$$\times\frac{(-aq;q)_{2m_{i-1}}\beta_m^{(t)}(a^2,q^2)}{(-aq^{1-N};q)_{2m_{i-1}}(\rho_1\sigma_1)^{m_1}\ldots(\rho_{i-1}\sigma_{i-1})^{m_{i-1}}(\rho_i\sigma_i)^{m_i}\ldots(\rho_t\sigma_t)^{m_t}}$$
$$=\sum_{k=0}^{m}\sum_{j=0}^{N}\frac{1}{(q^2;q^2)_{m-k}(a^2q^{2-2N};q^2)_{m+k}}\frac{(\rho_1;q^2)_k\ldots(\rho_{i-1};q^2)_k(\rho_i;q^2)_{k-j}\ldots(\rho_t;q^2)_{k-j}}{\left[(\frac{a^2q^{2-2N}}{\rho_1};q^2)_k\right]\ldots\left[(\frac{a^2q^{2-2N}}{\rho_{i-1}};q^2)_k\right]}$$
$$\times\frac{(\sigma_1;q^2)_k\ldots(\sigma_{i-1};q^2)_k(\sigma_i;q^2)_{k-j}\ldots(\sigma_t;q^2)_{k-j}}{\left[\left(\frac{a^2q^2}{\rho_i};q^2\right)_{k-j}\right]\ldots\left[\left(\frac{a^2q^2}{\rho_t};q^2\right)_{k-j}\right]}\frac{(a^2q^{2-2N})^{k(i-1)}(a^2q^2)^{(k-j)(t-i+1)}}{\left(\frac{a^2q^{2-2N}}{\sigma_1};q^2\right)_k\ldots\left(\frac{a^2q^{2-2N}}{\sigma_{i-1}};q^2\right)_k}$$

$$\times\frac{(1-aq^{2k-N})(aq^{1-N};q)_N(-1)^j(aq^{-N})^jq^{2kj-j(j+1)/2}}{\left[\left(\frac{a^2q^2}{\sigma_i};q^2\right)_{k-j}\right]\ldots\left[\left(\frac{a^2q^2}{\sigma_t};q^2\right)_{k-j}\right](\rho_1\sigma_1)^k\ldots(\rho_{i-1}\sigma_{i-1})^k(\rho_i\sigma_i)^{k-j}\ldots(\rho_t\sigma_t)^{k-j}}$$
$$\times\begin{bmatrix}N\\j\end{bmatrix}_q\frac{(aq^{1-N};q)_{2k-j-1}}{(aq^{1-N};q)_{2k-j+N}}\alpha_{k-j}^{(t)}(a^2,q^2). \tag{18}$$

Proof. The proof is quite similar to Theorem 3, so left for the reader. □

In similar, we set $N = 1$ in the above theorem. we have the following result.

Corollary 4. *Let α, β be sequences satisfying Equation (1), then we have*

$$\sum_{m\geq m_1\geq\cdots\geq m_i\geq\cdots\geq m_t\geq 0}\frac{(\rho_1;q^2)_{m_1}\ldots(\rho_t;q^2)_{m_t}}{\left[\left(\frac{a^2}{\rho_1};q^2\right)_m\right]\ldots\left[\left(\frac{a^2}{\rho_{i-1}};q^2\right)_{m_{i-2}}\right]}$$
$$\times\frac{(\sigma_1;q^2)_{m_1}\ldots(\sigma_t;q^2)_{m_t}}{\left[\left(\frac{a^2q^2}{\rho_i};q^2\right)_{m_{i-1}}\right]\ldots\left[\left(\frac{a^2q^2}{\rho_t};q^2\right)_{m_{t-1}}\right]}\frac{\left(\frac{a^2}{\sigma_1\rho_1};q^2\right)_{m-m_1}\ldots\left(\frac{a^2}{\sigma_{i-1}\rho_{i-1}};q^2\right)_{m_{i-2}-m_{i-1}}}{\left[\left(\frac{a^2}{\sigma_1};q^2\right)_m\right]\ldots\left[\left(\frac{a^2}{\sigma_{i-1}};q^2\right)_{m_{i-2}}\right]}$$
$$\times\frac{\left(\frac{a^2q^2}{\sigma_i\rho_i};q^2\right)_{m_{i-1}-m_i}\ldots\left(\frac{a^2q^2}{\sigma_t\rho_t};q^2\right)_{m_{t-1}-m_t}}{\left[\left(\frac{a^2q^2}{\sigma_i};q^2\right)_{m_{i-1}}\right]\ldots\left[\left(\frac{a^2q^2}{\sigma_t};q^2\right)_{m_{t-1}}\right]}\frac{(a^2)^{(m_1+\cdots+m_{i-1})}(a^2q^2)^{(m_i+\cdots+m_t)}}{[(q^2;q^2)_{m-m_1}]\ldots[(q^2;q^2)_{m_{t-1}-m_t}]}$$
$$\times\frac{(-aq;q)_{2m_{i-1}}B_m^{(t)}(a^2,q^2)}{(-a;q)_{2m_{i-1}}(\rho_1\sigma_1)^{m_1}\ldots(\rho_{i-1}\sigma_{i-1})^{m_{i-1}}(\rho_i\sigma_i)^{m_i}\ldots(\rho_t\sigma_t)^{m_t}}$$

$$= \sum_{k=0}^{m} \frac{1}{(q^2;q^2)_{m-k}(a^2;q^2)_{m+k}} \frac{(\rho_1;q^2)_k \dots (\rho_{i-1};q^2)_k(\sigma_1;q^2)_k \dots (\sigma_{i-1};q^2)_k}{(\frac{a^2}{\rho_1};q^2)_k \dots (\frac{a^2}{\rho_{i-1}};q^2)_k(\frac{a^2}{\sigma_1};q^2)_k \dots \left(\frac{a^2}{\sigma_{i-1}};q^2\right)_k}$$
$$\times \frac{(1-a)a^{2k(i-1)}}{(\rho_1\sigma_1)^k \dots (\rho_{i-1}\sigma_{i-1})^k} \left[\frac{(\rho_i;q^2)_k \dots (\rho_t;q^2)_k(\sigma_i;q^2)_k \dots (\sigma_t;q^2)_k}{\left[\left(\frac{a^2q^2}{\rho_i};q^2\right)_k\right] \dots \left[\left(\frac{a^2q^2}{\rho_t};q^2\right)_k\right] \left(\frac{a^2q^2}{\sigma_i};q^2\right)_k \dots \left(\frac{a^2q^2}{\sigma_t};q^2\right)_k} \right.$$
$$\times \frac{(a^2q^2)^{k(t-i+1)}\alpha_k^{(t)}(a^2,q^2)}{(\rho_i\sigma_i)^k \dots (\rho_t\sigma_t)^k(1-aq^{2k})} - aq^{2k-2}$$
$$\times \frac{(\rho_i;q^2)_{k-1} \dots (\rho_t;q^2)_{k-1}(\sigma_i;q^2)_{k-1} \dots (\sigma_t;q^2)_{k-1}}{\left[\left(\frac{a^2q^2}{\rho_i};q^2\right)_{k-1}\right] \dots \left[\left(\frac{a^2q^2}{\rho_t};q^2\right)_{k-1}\right] \left(\frac{a^2q^2}{\sigma_i};q^2\right)_{k-1} \dots \left(\frac{a^2q^2}{\sigma_t};q^2\right)_{k-1}}$$
$$\left. \times \frac{(a^2q^2)^{(k-1)(t-i+1)}\alpha_{k-1}^{(t)}(a^2,q^2)}{(\rho_i\sigma_i)^{k-1} \dots (\rho_t\sigma_t)^{k-1}(1-aq^{2k-2})} \right]. \tag{19}$$

Inserting (15) into the above corollary, we can get the following:

Corollary 5. *Let $a, \rho_1, \rho_2, \sigma_1, \sigma_2, \rho_3, \sigma_3$, be indeterminate and $\left|\frac{a^6q^{4+2n}}{\rho_1\rho_2\sigma_1\sigma_2}\right| < 1$. We have the following transformation:*

$$ {}_9\phi_8\left[\begin{matrix} a^2, -aq^2, \rho_1, \sigma_1, \rho_2, \sigma_2, \rho_3, \sigma_3, q^{-2n} \\ -a, a^2/\rho_1, a^2/\sigma_1, a^2q^2/\rho_2, a^2q^2/\sigma_2, a^2q^2/\rho_3, a^2q^2/\sigma_3, a^2q^{2n} \end{matrix} ; q^2, \frac{a^6q^{4+2n}}{\rho_1\rho_2\sigma_1\sigma_2\rho_3\sigma_3}\right]$$
$$- \frac{(1-\rho_1)(1-\sigma_1)(1-q^{-2n})}{(1-a^2/\rho_1)(1-a^2/\sigma_1)(1-a^2q^{2n})} \left[\frac{a^3q^{2n}}{\rho_1\sigma_1}\right]$$
$$\times {}_9\phi_8\left[\begin{matrix} a^2, -aq^2, q^2\rho_1, q^2\sigma_1, \rho_2, \sigma_2, \rho_3, \sigma_3, q^{-2n+2} \\ -a, a^2q^2/\rho_1, a^2q^2/\sigma_1, a^2q^2/\rho_2, a^2q^2/\sigma_2, a^2q^{2n+2} \end{matrix} ; q, \frac{a^6q^{4+2n}}{\rho_1\rho_2\sigma_1\sigma_2\rho_3\sigma_3}\right]$$
$$= \frac{(a^2;q^2)_m(a^2/\rho_1\sigma_1;q^2)_m}{(a^2/\rho_1;q^2)_m(a^2/\sigma_1;q^2)_m} \sum_{m \geq m_1 \geq 0} \frac{(\rho_1, \sigma_1, q^{-2m}, a^2q^2/\rho_2\sigma_2, -aq^2;q^2)_{m_1}}{(q^2, \rho_1\sigma_1q^{-2m}/a^2, a^2q^2/\rho_2.a^2q^2/\sigma_2, -a;q^2)_{m_1}} q^{2m_1}$$
$$ {}_4\phi_3\left[\begin{matrix} \rho_2, \sigma_2, a^2q^2/\rho_3\sigma_3, q^{-2m_1} \\ a^2q^2/\rho_3, a^2q^2/\sigma_3, \rho_2\sigma_2q^{-2m_1}/a^2 \end{matrix} ; q^2, q^2\right].$$

Taking $a^2q^2 = \rho_3\sigma_3$ in the above transformation, we can get (17).

In fact, we have many new identities like [26] (Theorems 5.1 and 5.2) rely on special cases of Theorem 3. For example, ρ_i, σ_i are taking different values and inserting other Bailey pair as a starting point.

5. Mock Theta Function

Many authors pointed out the number of Bailey pairs of great significance in the study of mock theta functions [9,20,30,31,33]. Iterating the Bailey pairs provides a virtual source of mixed mock modular forms. Lovejoy has described the relations of the indefinite theta series and mock theta functions. From the Appell–Lerch series, he proved there is a relation in the multisums and mock theta functions, and for more detailed study, we may refer to [30,31]. Similarly, we can also get the analogous results to those that were proved by Lovejoy [30,31]. In this paper, we just take an example in the following section.

To obtain the new Bailey chain, we need a new Bailey pair at hand.

Lemma 5. *The sequences α_{L_k} and β_{L_k} form a Bailey pair relative to q, where*

$$\alpha_{L_k} = \frac{q^{((m+l)/2)L_k^2+((m+l-2)/2)L_k}(1-q^{2L_k+1})}{(1-q)} \sum_{|j| \leq L_k} (-1)^j q^{-\frac{(m+l)j^2}{2}}$$

and

$$\beta_{L_k} = \sum_{L \geq L_k \cdots \geq L_1 \geq 0} \frac{(-q;q)_{2L_k}(-q^2;-q^2)_{2L_{k-1}} \dots (-q^{(m+l)/4};q^{(m+l)/4})_{L_1}}{(q^2;q^2)_{L-L_k}(q^4;q^4)_{L_k-L_{k-1}} \dots (q^{(m+l)/2};q^{(m+l)/2})_{L_2-L_1}} \frac{q^{L_k+2L_{k-1}+\cdots+(m+l)L_1/4}(-1)^{L_1}}{(q^{m+l};q^{m+l})_{L_1}}.$$

Proof. We consider the Bailey pair relative to q (see [9]) as:

$$\beta_{L_k} = \frac{(-1)^{L_k}}{(q^2;q^2)_{L_k}}, \tag{20}$$

and

$$\alpha_{L_k} = \frac{q^{L_k^2}(1-q^{2L_k+1})}{(1-q)} \sum_{|j| \leq L_k} (-1)^j q^{-j^2}. \tag{21}$$

Iterating the above Bailey pair and using the following Bailey pair:

$$\alpha'_{L_k} = \frac{(1+q)}{(1+q^{2L_k+1})} q^{L_k} \alpha(q^2)$$

and

$$\beta'_{L_k} = \sum_{L_k \geq k} \frac{(-q;q)_{2k} q^k}{(q^2;q^2)_{L_k-k}} \beta_k(q^2).$$

After some simplification, we can give two sequences.

$$\alpha^{(k)}_{L_k} = \frac{q^{2^k L_k^2 + (2^k-1)L_k}(1-q^{2L_k+1})}{(1-q)} \sum_{|j| \leq L_k} (-1)^j q^{-2^k j^2}$$

and

$$\beta^{(k)}_{L_k} = \sum_{L \geq L_k \cdots \geq L_1 \geq 0} \frac{(-q;q)_{2L_k}(-q^2;-q^2)_{2L_{k-1}} \dots (-q^{2^{k-1}};q^{2^{k-1}})_{L_1}}{(q^2;q^2)_{L-L_k}(q^4;q^4)_{L_k-L_{k-1}} \dots (q^{2^k};q^{2^k})_{L_2-L_1}} \frac{q^{L_k+2L_{k-1}+\cdots+2^{k-1}L_1}(-1)^{L_1}}{(q^{2^{k+1}};q^{2^{k+1}})_{L_1}}.$$

By taking $2^{k+1} = m+l$ in the above sequence, we get the required result. This procedure is the same as given in [30] (Proposition 4.1). Therefore, we just take $2^{k+1} = m+l$ and reduce one time iteration in [30] (Proposition 4.1). □

Now, we begin to state and prove Theorem 6.

Theorem 6. *For $k \geq 1$ and $m, l \geq 0$, we have*

$$\begin{aligned}
&(q^2;q^2)_\infty \sum_{L \geq 0} \sum_{L \geq L_k \cdots \geq L_1 \geq 0} \frac{(-q^2;q^2)_{2L_k}(-q^4;-q^4)_{2L_{k-1}} \dots (-q^{(m+l)/2};q^{(m+l)/2})_{L_1}}{(q^4;q^4)_{L-L_k}(q^8;q^8)_{L_k-L_{k-1}} \dots (q^{m+l};q^{m+l})_{L_2-L_1}} \\
&\quad \times \frac{q^{2L_k+4L_{k-1}+\cdots+(m+l)L_1/2+2L^2}(-1)^{L_1}(1+q^{2L+1})}{(q^{2m+2l};q^{2m+2l})_{L_1}} \\
&= f_{1,m+l+1,1}(q^{m+l},q^{m+l},q^4) + q^{2m+2l} f_{1,m+l+1,1}(q^{3m+3l+4},q^{3m+3l+4},q^4) \\
&\quad + q f_{1,m+l+1,1}(q^{m+l+2},q^{m+l+2},q^4) + q^{2m+2l+3} f_{1,m+l+1,1}(q^{3m+3l+6},q^{3m+3l+6},q^4).
\end{aligned}$$

Proof. Step 1: Note that if $a = 1$ and $\rho_1 = \rho_2 = q^{1/2}$ in Corollary 1. We have $\alpha'_0 = \alpha_0$ and for $L \geq 1$,

$$\beta'_L = \frac{(1+q^{2L+1})}{(1+q)}\beta_L$$

and

$$\alpha'_L = \frac{(1-q)}{(1-q^{2L+1})}\alpha_L - q^{2L-1}\frac{(1-q)}{(1-q^{2L-1})}\alpha_{L-1}.$$

Step 2: Apply this to the Bailey pair relative to q^2 in Lemma 5, we have

$$\beta_L(q^2) = \beta_{L_m+1} = \sum_{L\geq L_k\cdots\geq L_1\geq 0} \frac{(-q^2;q^2)_{2L_k}(-q^4;-q^4)_{2L_{k-1}}\cdots(-q^{(m+l)/2};q^{(m+l)/2})_{L_1}}{(q^4;q^4)_{L-L_k}(q^8;q^8)_{L_k-L_{k-1}}\cdots(q^{m+l};q^{m+l})_{L_2-L_1}} \times \frac{q^{2L_k+4L_{k-1}+\cdots+(m+l)L_1/2}(-1)^{L_1}(1+q^{2L+1})}{(q^{2m+2l};q^{2m+2l})_{L_1}(1+q)}$$

and

$$\begin{aligned}\alpha_L(q^2) \quad = \quad & \frac{q^{(m+l)L^2+(m+l-2)L}(1+q^{2L+1})}{(1+q)}\sum_{j=-L}^{L}(-1)^j q^{-(m+l)j^2} \\ & -\chi(L\neq 0)\frac{(1+q^{2L-1})}{(1+q)}q^{(m+l)L^2-(m+l)L+1}\sum_{j=-L}^{L}(-1)^j q^{-(m+l)j^2}.\end{aligned}$$

Step 3: The Bailey pair relative to q^2 substitute into Bailey transformation (1) with $\rho_1, \rho_2 \to \infty$, we have

$$\begin{aligned}(q^2;q^2)_\infty \sum_{L\geq 0}\sum_{L\geq L_k\cdots\geq L_1\geq 0} & \frac{(-q^2;q^2)_{2L_k}(-q^4;-q^4)_{2L_{k-1}}\cdots(-q^{(m+l)/2};q^{(m+l)/2})_{L_1}}{(q^4;q^4)_{L-L_k}(q^8;q^8)_{L_k-L_{k-1}}\cdots(q^{m+l};q^{m+l})_{L_2-L_1}} \\ & \times\frac{q^{2L_k+4L_{k-1}+\cdots+(m+l)L_1/2+2L^2}(-1)^{L_1}(1+q^{2L+1})}{(q^{2m+2l};q^{2m+2l})_{L_1}(1+q)}, \\ = \sum_{L\geq 0} & \frac{q^{2L^2+(m+l)L^2+(m+l-2)L}(1+q^{2L+1})}{(1+q)}\sum_{j=-L}^{L}(-1)^j q^{-(m+l)j^2} \\ & -\chi(L\neq 0)\frac{(1+q^{2L-1})}{(1+q)}q^{(m+l)L^2-(m+l)L+1+2L^2}\sum_{j=-L}^{L}(-1)^j q^{-(m+l)j^2}.\end{aligned}$$

Multiply by $(1+q)$ on both sides of the above identities, the right side of the above formula can be written as:

$$\begin{aligned}&\sum_{L\geq 0}\sum_{j=-L}^{L} q^{(m+l+2)L^2+(m+l-2)L}(-1)^j q^{-(m+l)j^2} - \sum_{L\geq 1}\sum_{j=-L}^{L-1} q^{(m+l+2)L^2-(m+l-2)L}(-1)^j q^{-(m+l)j^2} \\ &+\sum_{L\geq 0}\sum_{j=-L}^{L} q^{(m+l+2)L^2+(m+l-2)L+2L+1}(-1)^j q^{-(m+l)j^2} - \sum_{L\geq 1}\sum_{j=-L}^{L-1} q^{(m+l+2)L^2-(m+l)L+1}(-1)^j q^{-(m+l)j^2}.\end{aligned}$$

Step 4: We set $n = (r+s)/2$ and $j = (r-s)/2$ in the first sum and the third sum, $n = -(r+s)/2$ and $j = (r-s)/2$ in the second sum and the fourth sum, respectively, we have

$$(q^2;q^2)_\infty \sum_{L\geq 0} \sum_{L\geq L_k\cdots\geq L_1\geq 0} \frac{(-q^2;q^2)_{2L_k}(-q^4;-q^4)_{2L_{k-1}}\dots(-q^{(m+l)/2};q^{(m+l)/2})_{L_1}}{(q^4;q^4)_{L-L_k}(q^8;q^8)_{L_k-L_{k-1}}\dots(q^{m+l};q^{m+l})_{L_2-L_1}}$$
$$\times \frac{q^{2L_k+4L_{k-1}+\dots+(m+l)L_1/2+2L^2}(-1)^{L_1}(1+q^{2L+1})}{(q^{2m+2l};q^{2m+2l})_{L_1}}$$
$$= \left(\sum_{\substack{r,s\geq 0\\ r\equiv s(mod2)}} - \sum_{\substack{r,s<0\\ r\equiv s(mod2)}} \right)(-1)^{(r-s)/2}q^{r^2/2+s^2/2+(m+l+1)rs+r(m/2+l/2-1)+s(m/2+l/2-1)}$$
$$+q\left(\sum_{\substack{r,s\geq 0\\ r\equiv s(mod2)}} - \sum_{\substack{r,s<0\\ r\equiv s(mod2)}} \right)(-1)^{(r-s)/2}q^{r^2/2+s^2/2+(m+l+1)rs+r(m/2+l/2)+s(m/2+l/2)}.$$

Step 5: Replacing (r,s) by $(2r,2s)$ and $(2r+1,2s+1)$ in the above equation, we have

$$(q^2;q^2)_\infty \sum_{L\geq 0} \sum_{L\geq L_k\cdots\geq L_1\geq 0} \frac{(-q^2;q^2)_{2L_k}(-q^4;-q^4)_{2L_{k-1}}\dots(-q^{(m+l)/2};q^{(m+l)/2})_{L_1}}{(q^4;q^4)_{L-L_k}(q^8;q^8)_{L_k-L_{k-1}}\dots(q^{m+l};q^{m+l})_{L_2-L_1}}$$
$$\frac{q^{2L_k+4L_{k-1}+\dots+(m+l)L_1/2+2L^2}(-1)^{L_1}(1+q^{2L+1})}{(q^{2m+2l};q^{2m+2l})_{L_1}},$$

$$= \left(\sum_{r,s\geq 0} - \sum_{r,s<0} \right)(-1)^{r+s}q^{2r^2+2s^2+4(m+l+1)rs+r(m+l-2)+s(m+l-2)}$$
$$+q^{2m+2l}\left(\sum_{r,s\geq 0} - \sum_{r,s<0} \right)(-1)^{r+s}q^{2r^2+2s^2+4(m+l+1)rs+r(3m+3l+2)+s(3m+3l+2)}$$
$$+q\left(\sum_{r,s\geq 0} - \sum_{r,s<0} \right)(-1)^{r+s}q^{2r^2+2s^2+4(m+l+1)rs+r(m+l)+s(m+l)}$$
$$+q^{2m+2l+3}\left(\sum_{r,s\geq 0} - \sum_{r,s<0} \right)(-1)^{r+s}q^{2r^2+2s^2+4(m+l+1)rs+r(3m+3l+4)+s(3m+3l+4)}.$$

Which complete the proof of Theorem 6. □

According to Lovejoy [31] (Theorem 5.2), we see that the series in Theorem 6 are mock theta functions. As he said, there are many forms between the multisums and classical mock theta functions. Therefore, we can use the different Bailey lattices and different ways to get the kind of mock theta functions.

6. Concluding Remarks and Observations

Here in our present investigation, we have developed two new Bailey lattices. We have also described a number of q-multisums new forms with multiple variables for the basic hypergeometric series, which arise as consequences of these two new Bailey lattices. As applications, two new transformations for basic hypergeometric by using the unit Bailey pair have been derived systematically. Besides it, we have used this Bailey lattice and got some kind of mock theta functions. We have also highlighted some known and new consequences of our main results.

Finally, we would like to highlight that in [34] (Section 8), it is pointed out that some preliminary work has been done toward understanding the combinatorial significance of certain modular q-hypergeomatric multisums constructed using change of base lemmas for Bailey pairs. It is also believed that the q-series, Bailey lattices, q-multisums identities for the basic hypergeometric series and mock theta functions, which we have studied in this paper, as well as the various related recent works cited here, will provide motivation and inspiration for further studies on the topics that are dealt with and investigated in this paper. Moreover, the Bailey results which we have derived in this paper will indeed apply also to the work of Jia and Zeng [25] to produce certain new Bailey lemmas and Mock theta functions.

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