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A More Accurate Half-Discrete Hilbert-Type Inequality Involving One upper Limit Function and One Partial Sum

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Abstract: In this paper, by virtue of the symmetry principle, we construct proper weight coefficients and use them to establish a more accurate half-discrete Hilbert-type inequality involving one upper limit function and one partial sum. Then, we prove the new inequality with the help of the Euler–Maclaurin summation formula and Abel's partial summation formula. Finally, we illustrate how the obtained results can generate some new half-discrete Hilbert-type inequalities.

Keywords: weight coefficient; Euler–Maclaurin summation formula; Abel's partial summation formula; half-discrete Hilbert-type inequality; upper limit function

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1. Introduction

The celebrated Hardy–Hilbert's inequality reads as:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\Big(\sum_{m=1}^{\infty}a_m^p\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{\infty}b_n^q\Big)^{\frac{1}{q}}, \tag{1}$$

where $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_m, b_n \geq 0$, $0 < \sum_{m=1}^{\infty} a_m^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible (see [1], Theorem 315).

A more accurate form of (1) was provided in ([1], Theorem 323), as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n-1} < \frac{\pi}{\sin(\pi/p)}\Big(\sum_{m=1}^{\infty}a_m^p\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{\infty}b_n^q\Big)^{\frac{1}{q}}. \tag{2}$$

In 2006, by introducing parameters $\lambda_i \in (0,2](i=1,2), \lambda_1+\lambda_2=\lambda \in (0,4]$, an extension of (1) was provided by [2] as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1,\lambda_2)\Big[\sum_{m=1}^{\infty}m^{p(1-\lambda_1)-1}a_m^p\Big]^{\frac{1}{p}}\Big[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q\Big]^{\frac{1}{q}}, \tag{3}$$

where the constant factor $B(\lambda_1,\lambda_2)$ is the best possible, and the beta function is defined as:

$$B(u,v) = \int_0^{\infty}\frac{t^{u-1}}{(1+t)^{u+v}}dt\,(u,v>0).$$

Obviously, when $\lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, inequality (2) reduces to (1); when $p = q = 2$, $\lambda_1 = \lambda_2 = \frac{\lambda}{2}$, inequality (2) reduces to the inequality presented by Yang in [3].

Recently, applying inequality (3) and Abel's summation by parts formula, Adiyasuren et al. [4] gave a new inequality with the kernel $\frac{1}{(m+n)^{\lambda}}$ involving two partial sums. Inequality (1), with its integral analogues, is playing an important role in analysis and its applications (see [5–8]).

In 1934, a half-discrete Hilbert-type inequality was given as follows ([1], Theorem 351): assuming that $K(t)\,(t>0)$ is a decreasing function, $0<\phi(s):=\int_0^\infty K(t)t^{s-1}dt<\infty$, $a_n\geq 0$, such that $0<\sum_{n=1}^\infty a_n^p<\infty$, we have:

$$\int_0^\infty x^{p-2}(\sum_{n=1}^\infty K(nx)a_n)^p dx<\phi^p(\frac{1}{q})\sum_{n=1}^\infty a_n^p. \tag{4}$$

In 2016, Hong et al. [9] considered some equivalent statements of the extensions of (1) with the best possible constant factor related to several parameters. Some extensions of inequality (4) were given by [10–15]. Recently, Yang et al. [16,17] gave reverse half-discrete Hardy-Hilbert's inequalities and dealt with their equivalent statements of the best possible constant factor related to several parameters.

In this article, following the method of [2,4,9], in the light of the symmetry principle, we construct proper weight coefficients and use them to establish a more accurate half-discrete Hilbert-type inequality involving one upper limit function and one partial sum. Subsequently, we prove this new inequality by means of the Hermite–Hadamard inequality, Euler–Maclaurin summation formula and Abel's partial summation formula. As an extension of the obtained results, the equivalent statements of the best possible constant factor related to several parameters are discussed. It is shown that some new half-discrete Hilbert-type inequalities can be derived from the special cases of our main results.

2. Some Lemmas

In what follows, we suppose that $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\eta\in[0,\frac{1}{4}]$, $\lambda\in(0,2]$, $\lambda_1\in(0,\lambda+1)$, $\lambda_2\in(0,\frac{1}{2}]\cap(0,\lambda+1)$, $\hat{\lambda}_1:=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}$, $\hat{\lambda}_2:=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}$. We also assume that $f(x)(\geq 0)$ is a Lebesgue integrable function in any interval $(0,b]\,(b>0)$, and define the upper limit function $F(x):=\int_0^x f(t)dt\,(x\geq 0)$ with the partial sums as follows:

$$A_n:=\sum_{k=1}^n a_k(a_n\geq 0, n\in \mathrm{N}:=\{1,2,\cdots\}),$$

which satisfies $F(x)=o(e^{tx})$, $A_n=o(e^{t(n-\eta)})$ $(t>0;x,n\to\infty)$:

$$0<\int_0^\infty x^{-p\hat{\lambda}_1-1}F^p(x)<\infty \text{ and } 0<\sum_{n=1}^\infty (n-\eta)^{-q\hat{\lambda}_2-1}A_n^q<\infty. \tag{5}$$

Lemma 1. *(i) Let $(-1)^i\frac{d^i}{dt^i}g(t)>0, t\in[m,\infty)\,(m\in \mathrm{N})$ with $g^{(i)}(\infty)=0$ $(i=0,1,2,3)$, and let $P_i(t), B_i\,(i\in \mathrm{N})$ be the Bernoulli functions and the Bernoulli numbers of i-order. Then, we have* ([5]):

$$\int_m^\infty P_{2q-1}(t)g(t)dt=-\varepsilon_q\frac{B_{2q}}{2q}g(m)(0<\varepsilon_q<1;q\in \mathrm{N}). \tag{6}$$

In particular, for $q=1$, in view of $B_2=\frac{1}{6}$, we have:

$$-\frac{1}{12}g(m)<\int_m^\infty P_1(t)g(t)dt<0; \tag{7}$$

For $q=2$, in view of $B_4=-\frac{1}{30}$, it follows that:

$$0<\int_m^\infty P_3(t)g(t)dt<\frac{1}{120}g(m). \tag{8}$$

(ii) If $h(t)(>0) \in C^3[m,\infty), h^{(i)}(\infty) = 0$ $(i = 0,1,2,3)$, then we have the following Euler–Maclaurin summation formula:

$$\sum_{k=m}^{\infty} h(k) = \int_m^{\infty} h(t)dt + \frac{1}{2}h(m) + \int_m^{\infty} P_1(t)h'(t)dt, \tag{9}$$

where:

$$\int_m^{\infty} P_1(t)h'(t)dt = -\frac{1}{12}h'(m) + \frac{1}{6}\int_m^{\infty} P_3(t)h'''(t)dt. \tag{10}$$

Lemma 2. *Let $s \in (0,4]$,$s_2 \in (0,\frac{3}{2}] \cap (0,s)$,$k_s(s_i) := B(s_i, s-s_i)(i = 1,2)$, and let $\varpi(s_2,x)$ denote the following weight coefficient:*

$$\varpi(s_2,x) := x^{s-s_2}\sum_{n=1}^{\infty}\frac{(n-\eta)^{s_2-1}}{(x+n-\eta)^s}(x \in \mathrm{R}_+ := (0,\infty)) \tag{11}$$

Then, we have the following inequalities:

$$0 < k_s(s_2)(1 - O(\frac{1}{x^{s_2}})) < \varpi(s_2,x) < k_s(s_2), \tag{12}$$

where we indicate $O(\frac{1}{x^{s_2}}) := \frac{1}{k_s(s_2)}\int_0^{\frac{1-\eta}{x}} \frac{u^{s_2-1}}{(1+u)^s}du > 0$.

Proof. For fixed $x \in \mathrm{R}_+$, we define a function $g(x,t)$ by:

$$g(x,t) := \frac{(t-\eta)^{s_2-1}}{(x+t-\eta)^s}(t \in (\eta,\infty)),$$

which implies that $g(x,t) > 0$ $(t \in I_\eta)$ and $g \in C^\infty(I_\eta)$, where $I_\eta := (\eta,\infty)$. In the following, we consider two cases of $s_2 \in (0,1) \cap (0,s)$ and $s_2 \in [1,\frac{3}{2}] \cap (0,s)$ to prove inequalities (12).
(i) For $s_2 \in (0,1) \cap (0,s)$, since:

$$(-1)^i\frac{\partial^i}{\partial t^i}g(x,t) > 0(t > \eta; i = 0,1,2),$$

by the Hermite–Hadamard inequality, setting $u = \frac{t-\eta}{x}$, we have:

$$\varpi(s_2,x) = x^{s-s_2}\sum_{n=1}^{\infty} g(x,n) < x^{s-s_2}\int_{\frac{1}{2}}^{\infty} g(x,t)dt$$

$$= x^{s-s_2}\int_{\frac{1}{2}}^{\infty}\frac{(t-\eta)^{s_2-1}}{(x+t-\eta)^s}dt = \int_{\frac{\frac{1}{2}-\eta}{x}}^{\infty}\frac{u^{s_2-1}}{(1+u)^s}du$$
$$\leq \int_0^{\infty}\frac{u^{s_2-1}}{(1+u)^s}du = B(s_2, s-s_2) = k_s(s_2).$$

On the other hand, in view of the decreasingness property of series, setting $u = \frac{t-\eta}{x}$, we obtain:

$$\varpi(s_2,x) = x^{s-s_2}\sum_{n=1}^{\infty} g(x,n) > x^{s-s_2}\int_1^{\infty} g(x,t)dt$$

$$= \int_{\frac{1-\eta}{x}}^{\infty}\frac{u^{s_2-1}}{(1+u)^s}du = B(s_2, s-s_2) - \int_0^{\frac{1-\eta}{x}}\frac{u^{s_2-1}}{(1+u)^s}du$$
$$= k_s(s_2)(1 - O(\frac{1}{x^{s_2}})) > 0,$$

where $O(\frac{1}{x^{s_2}}) = \frac{1}{k_s(s_2)}\int_0^{\frac{1-\eta}{x}} \frac{u^{s_2-1}}{(1+u)^s}du > 0$, which satisfies:

$$0 < \int_0^{\frac{1-\eta}{x}} \frac{u^{s_2-1}}{(1+u)^s}du < \int_0^{\frac{1-\eta}{x}} u^{s_2-1}du = \frac{1}{s_2}(\frac{1-\eta}{x})^{s_2}(x \in \mathrm{R}_+).$$

In this case, we obtain (12).

(ii) For $s_2 \in [1, \frac{3}{2}] \cap (0, s)$, by (9), we have:

$$\begin{aligned}\sum_{n=1}^{\infty} g(x,n) &= \int_1^{\infty} g(x,t)dt + \tfrac{1}{2}g(x,1) + \int_1^{\infty} P_1(t)\tfrac{\partial}{\partial t}g(x,t)dt \\ &= \int_\eta^{\infty} g(x,t)dt - h(x),\end{aligned}$$

where $h(x)$ is defined by:

$$h(x) := \int_\eta^1 g(x,t)dt - \frac{1}{2}g(x,1) - \int_1^{\infty} P_1(t)\frac{\partial}{\partial t}g(x,t)dt.$$

We obtain $-\frac{1}{2}g(x,1) = \frac{-(1-\eta)^{s_2-1}}{2(x+1-\eta)^s}$, and then integrating by parts, it follows that:

$$\int_\eta^1 g(x,t)dt = \int_\eta^1 \frac{(t-\eta)^{s_2-1}}{(x+t-\eta)^s}dt = \frac{1}{s_2}\int_\eta^1 \frac{d(t-\eta)^{s_2}}{(x+t-\eta)^s} = \frac{1}{s_2}\frac{(t-\eta)^{s_2}}{(x+t-\eta)^s}|_\eta^1 + \frac{s}{s_2}\int_\eta^1 \frac{(t-\eta)^{s_2}dt}{(x+t-\eta)^{s+1}}$$

$$= \frac{1}{s_2}\frac{(1-\eta)^{s_2}}{(x+1-\eta)^s} + \frac{s}{s_2(s_2+1)}\int_\eta^1 \frac{1}{(x+1-\eta)^{s+1}}d(t-\eta)^{s_2+1}$$

$$> \frac{1}{s_2}\frac{(1-\eta)^{s_2}}{(x+1-\eta)^s} + \frac{s}{s_2(s_2+1)}\Big[\frac{(t-\eta)^{s_2+1}}{(x+t-\eta)^{s+1}}\Big]_\eta^1 + \frac{s(s+1)}{s_2(s_2+1)(x+1-\eta)^{s+2}}\int_\eta^1 (t-\eta)^{s_2+1}dt$$

$$= \frac{1}{s_2}\frac{(1-\eta)^{s_2}}{(x+1-\eta)^s} + \frac{s}{s_2(s_2+1)}\frac{(1-\eta)^{s_2+1}}{(x+1-\eta)^{s+1}} + \frac{s(s+1)(1-\eta)^{s_2+2}}{s_2(s_2+1)(s_2+2)(x+1-\eta)^{s+2}}.$$

We find:

$$\begin{aligned}-\tfrac{\partial}{\partial t}g(x,t) &= -\tfrac{(s_2-1)(t-\eta)^{s_2-2}}{(x+t-\eta)^s} + \tfrac{s(t-\eta)^{s_2-1}}{(x+t-\eta)^{s+1}} \\ &= \tfrac{(1-s_2)(t-\eta)^{s_2-2}}{(x+t-\eta)^s} + \tfrac{s(t-\eta)^{s_2-2}}{(x+t-\eta)^s} - \tfrac{sx(t-\eta)^{s_2-2}}{(x+t-\eta)^{s+1}} \\ &= \tfrac{(s+1-s_2)(t-\eta)^{s_2-2}}{(x+t-\eta)^s} - \tfrac{sx(t-\eta_2)^{s_2-2}}{(x+t-\eta)^{s+1}},\end{aligned}$$

additionally, for $s_2 \in [1, \frac{3}{2}] \cap (0, s)$, it follows that:

$$(-1)^i\frac{\partial^i}{\partial t^i}\Big[\frac{(t-\eta)^{s_2-2}}{(x+t-\eta)^s}\Big] > 0, (-1)^i\frac{\partial^i}{\partial t^i}\Big[\frac{(t-\eta)^{s_2-2}}{(x+t-\eta)^{s+1}}\Big] > 0(t > \eta; i = 0,1,2,3).$$

By (8), (9) and (10), setting $a := 1-\eta(\in [\frac{3}{4}, 1])$, we obtain:

$$\begin{aligned}&(s+1-s_2)\int_1^{\infty} P_1(t)\tfrac{(t-\eta)^{s_2-2}}{(x+t-\eta)^s}dt > -\tfrac{s+1-s_2}{12(x+1-\eta)^s}a^{s_2-2}, \\ &-xs\int_1^{\infty} P_1(t)\tfrac{(t-\eta)^{s_2-2}}{(x+t-\eta)^{s+1}}dt > \tfrac{xs}{12(x+1-\eta)^{s+1}}a^{s_2-2} - \tfrac{xs}{720}\big[\tfrac{(t-\eta)^{s_2-2}}{(x+t-\eta)^{s+1}}\big]''_{t=1} \\ &> \tfrac{(x+1-\eta)s-as}{12(x+1-\eta)^{s+1}}a^{s_2-2} - \tfrac{(x+1-\eta)s}{720}\big[\tfrac{(s+1)(s+2)a^{s_2-2}}{(x+1-\eta)^{s+3}} + \tfrac{2(s+1)(2-s_2)a^{s_2-3}}{(x+1-\eta)^{s+2}} + \tfrac{(2-s_2)(3-s_2)a^{s_2-4}}{(x+1-\eta)^{s+1}}\big]. \\ &= \tfrac{sa^{s_2-2}}{12(x+1-\eta)^s} - \tfrac{sa^{s_2-1}}{12(x+1-\eta)^{s+1}} - \tfrac{s}{720}\big[\tfrac{(s+1)(s+2)a^{s_2-2}}{(x+1-\eta)^{s+2}} + \tfrac{2(s+1)(2-s_2)a^{s_2-3}}{(x+1-\eta)^{s+1}} + \tfrac{(2-s_2)(3-s_2)a^{s_2-4}}{(x+1-\eta)^s}\big],\end{aligned}$$

and then we have:

$$h(x) > \frac{a^{s_2-4}}{(x+1-\eta)^s}h_1 + \frac{sa^{s_2-3}}{(x+1-\eta)^{s+1}}h_2 + \frac{s(s+1)a^{s_2-2}}{(x+1-\eta)^{s+2}}h_3,$$

where $h_i(i=1,2,3)$ are indicated as:

$$\begin{aligned} h_1 &:= \tfrac{a^4}{s_2} - \tfrac{a^3}{2} - \tfrac{(1-s_2)a^2}{12} - \tfrac{s(2-s_2)(3-s_2)}{720}, \\ h_2 &:= \tfrac{a^4}{s_2(s_2+1)} - \tfrac{a^2}{12} - \tfrac{(s+1)(2-s_2)}{360}, \end{aligned}$$

$$h_3 := \frac{a^4}{s_2(s_2+1)(s_2+2)} - \frac{s+2}{720}.$$

For $s \in (0,4], s_2 \in [1, \frac{3}{2}] \cap (0,s)$,$a \in [\frac{3}{4}, 1]$, we find:

$$h_1 > \frac{a^2}{12s_2}[s_2^2 - (6a+1)s_2 + 12a^2] - \frac{1}{90}.$$

In view of:

$$\frac{d}{da}[s_2^2 - (6a+1)s_2 + 12a^2] = 6(4a - s_2) \geq 6(4 \cdot \frac{3}{4} - \frac{3}{2}) > 0, \text{and}$$

$$\begin{aligned} &\tfrac{d}{ds_2}[s_2^2 - (6a+1)s_2 + 12a^2] = 2s_2 - (6a+1) \\ &\leq 2 \cdot \tfrac{3}{2} - (6 \cdot \tfrac{3}{4} + 1) = 3 - \tfrac{11}{2} < 0, \end{aligned}$$

we obtain:

$$h_1 \geq \frac{(3/4)^2}{12(3/2)}[(\frac{3}{2})^2 - (6 \cdot \frac{3}{4} + 1)\frac{3}{2} + 12(\frac{3}{4})^2] - \frac{1}{90}$$

$$h_2 > a^2(\frac{4a^2}{15} - \frac{1}{12}) - \frac{1}{72} \geq (\frac{3}{4})^2[\frac{4}{15}(\frac{3}{4})^2 - \frac{1}{12}] - \frac{1}{72} = \frac{3}{80} - \frac{1}{72} > 0,$$

$$h_3 \geq \frac{8a^4}{105} - \frac{6}{720} \geq \frac{8}{105}(\frac{3}{4})^4 - \frac{1}{120} = \frac{27}{1120} - \frac{1}{120} > 0,$$

and then we obtain $h(x) > 0$.

On the other hand, similar to the above, we have:

$$\begin{aligned} \sum_{n=1}^{\infty} g(x,n) &= \int_1^\infty g(x,t)dt + \tfrac{1}{2}g(x,1) + \int_1^\infty P_1(t)\tfrac{\partial}{\partial t}g(x,t)dt \\ &= \int_1^\infty g(x,t)dt + H(x), \end{aligned}$$

where $H(x)$ is indicated as:

$$H(x) := \frac{1}{2}g(x,1) + \int_1^\infty P_1(t)\frac{\partial}{\partial t}g(x,t)dt.$$

Thus, we obtain that $\frac{1}{2}g(x,1) = \frac{a^{s_2-1}}{2(x+1-\eta)^s}$ and:

$$\frac{\partial}{\partial t}g(x,t) = -\frac{(s+1-s_2)(t-\eta)^{s_2-2}}{(x+t-\eta)^s} + \frac{sx(t-\eta)^{s_2-2}}{(x+t-\eta)^{s+1}}.$$

For $s_2 \in (0, \frac{3}{2}] \cap (0,s), 0 < s \leq 4$, by (7), we obtain:

$$-(s+1-s_2)\int_1^\infty P_1(t)\frac{(t-\eta)^{s_2-2}}{(x+t-\eta)^s}dt > 0,$$

$$\begin{array}{l} xs\int_1^\infty P_1(t)\frac{(t-\eta)^{s_2-2}}{(x+t-\eta)^{s+1}}dt > \frac{-xs}{12(x+1-\eta)^{s+1}}a^{s_2-2} = \frac{-(x+1-\eta)s+as}{12(x+1-\eta)^{s+1}}a^{s_2-2} \\ = \frac{-s}{12(x+1-\eta)^s}a^{s_2-2} + \frac{s}{12(x+1-\eta)^{s+1}}a^{s_2-1} > \frac{-s}{12(x+1-\eta)^s}a^{s_2-2}. \end{array}$$

Hence, we have:

$$\begin{array}{rl} H(x) & > \frac{a^{s_2-1}}{2(x+1-\eta)^s} - \frac{sa^{s_2-2}}{12(x+1-\eta)^s} = (\frac{a}{2} - \frac{s}{12})\frac{a^{s_2-2}}{(x+1-\eta)^s} \\ & \geq (\frac{1}{2}\cdot\frac{3}{4} - \frac{4}{12})\frac{a^{s_2-2}}{(x+1-\eta)^s} = (\frac{3}{8} - \frac{1}{3})\frac{a^{s_2-2}}{(x+1-\eta)^s} > 0. \end{array}$$

Therefore, we obtain:

$$\int_1^\infty g(x,t)dt < \sum_{n=1}^\infty g(x,n) < \int_\eta^\infty g(x,t)dt(x>0)$$

In view of the results obtained in the case (i), we obtain (12). This completes the proof of lemma 2. □

Lemma 3. *Let $s \in (0,4]$,$s_1 \in (0,s)$, $s_2 \in (0,\frac{3}{2}] \cap (0,s)$.Then, we have the following more accurate half-discrete Hardy–Hilbert inequality:*

$$I = \int_0^\infty \sum_{n=1}^\infty \frac{a_n f(x)}{(x+n-\eta)^s}dx \leq (k_s(s_2))^{\frac{1}{p}}(k_s(s_1))^{\frac{1}{q}}$$

$$\times\left\{\int_0^\infty x^{p[1-(\frac{s-s_2}{p}+\frac{s_1}{q})]-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^\infty (n-\eta)^{q[1-(\frac{s-s_1}{q}+\frac{s_2}{p})]-1}a_n^q\right\}^{\frac{1}{q}}. \quad (13)$$

Proof. For $s_1 \in (0,s)$, setting $u = \frac{x}{n-\eta}$, we have the following expression of the weight coefficient:

$$\omega(s_1,n) := (n-\eta)^{s-s_1}\int_0^\infty \frac{x^{s_1-1}}{(x+n-\eta)^s}dx = \int_0^\infty \frac{u^{s_1-1}}{(u+1)^s}du = k_s(s_1)(n \in \mathrm{N}). \quad (14)$$

By using Hölder's inequality [18], we obtain:

$$\begin{array}{rl} I & = \int_0^\infty \sum\limits_{n=1}^\infty \frac{1}{(x+n-\eta)^s}[\frac{x^{(1-s)1/q}}{(n-\eta)^{(1-s_2)/p}}f(x)][\frac{(n-\eta)^{(1-s_2)/p}}{x^{(1-s)1/q}}a_n]dx \\ & \leq [\int_0^\infty \sum\limits_{n=1}^\infty \frac{1}{(x+n-\eta)^s}\frac{x^{(1-s1)(p-1)}}{(n-\eta)^{1-s_2}}f^p(x)dx]\frac{1}{p} \\ & \times[\sum\limits_{n=1}^\infty \int_0^\infty \frac{1}{(x+n-\eta)^s}\frac{(n-\eta)^{(1-s_2)(q-1)}}{x^{1-s1}}dxa_n^q]^{\frac{1}{q}} \\ & = \left\{\int_0^\infty \varpi(s_2,x)x^{p[1-(\frac{s-s_2}{p}+\frac{s1}{q})]-1}f^p(x)dx\right\}\frac{1}{p} \\ & \times\left\{\sum\limits_{n=1}^\infty \omega(s,1n)(n-\eta)^{q[1-(\frac{s-s_1}{q}+\frac{s_2}{p})]-1}a_n^q\right\}\frac{1}{q}. \end{array}$$

Then, by (12) and (14), we derive inequality (13). The Lemma 3 is proved. □

Remark 1. *In (13), for $s = \lambda+2 \in (2,4]$, $\lambda \in (0,2]$, $s_1 = \lambda_1+1 \in (1,s)$, $\lambda_1 \in (0,\lambda+1)$,*

$$s_2 = \lambda_2+1 \in (1,\frac{3}{2}], \lambda_2 \in (0,\frac{1}{2}] \cap (0,\lambda+1)$$

Replacing $f(x)(resp. a_n) by F(x)(resp. A_n)$, *in view of Lemma 3 and (5), we have:*

$$\int_0^\infty \sum_{n=1}^\infty \frac{A_n}{(x+n-\eta)^{\lambda+2}} F(x)dx < (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}$$

$$\times [\int_0^\infty x^{-p\hat{\lambda}_1-1} F^p(x)dx]^{\frac{1}{p}} [\sum_{n=1}^\infty (n-\eta)^{-q\hat{\lambda}_2-1} A_n^q]^{\frac{1}{q}}. \tag{15}$$

Lemma 4. *For* $t > 0$, *we have:*

$$\int_0^\infty e^{-tx} f(x)dx = t\int_0^\infty e^{-tx} F(x)dx, \tag{16}$$

$$\sum_{n=1}^\infty e^{-t(n-\eta)} a_n \le t \sum_{n=1}^\infty e^{-t(n-\eta)} A_n. \tag{17}$$

Proof. Integration by parts, in view of $F(0) = 0, F(x) = o(e^{tx})(t > 0; x \to \infty)$, it follows that:

$$\int_0^\infty \sum_{n=1}^\infty \frac{A_n}{(x+n-\eta)^{\lambda+2}} F(x)dx < (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}$$

$$= \lim_{x\to\infty} e^{-tx} F(x) + t\int_0^\infty e^{-tx} F(x)dx = t\int_0^\infty e^{-tx} F(x)dx,$$

and then (16) follows.

In view of $A_n e^{-t(n-\eta)} = o(1)(n \to \infty)$, by Abel's summation by parts formula, we obtain:

$$\sum_{n=1}^\infty e^{-t(n-\eta)} a_n = \lim_{n\to\infty} A_n e^{-t(n-\eta)} + \sum_{n=1}^\infty A_n[e^{-t(n-\eta)} - e^{-t(n-\eta+1)}]$$
$$= \sum_{n=1}^\infty A_n[e^{-t(n-\eta)} - e^{-t(n-\eta+1)}] = (1-e^{-t}) \sum_{n=1}^\infty e^{-t(n-\eta)} A_n.$$

Since $1 - e^{-t} < t(t > 0)$, we have (17). The Lemma 4 is proved. □

3. Main Results

Theorem 1. *Let* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \eta \in [0, \frac{1}{4}], \lambda \in (0,2], \lambda_1 \in (0, \lambda+1), \lambda_2 \in (0, \frac{1}{2}] \cap (0, \lambda+1)$, $\hat{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}, \hat{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$. *Then, we have the following half-discrete Hilbert-type inequality:*

$$I := \int_0^\infty \sum_{n=1}^\infty \frac{a_n}{(x+n-\eta)^\lambda} f(x)dx < \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}$$

$$\times [\int_0^\infty x^{-p\hat{\lambda}_1-1} F^p(x)dx]^{\frac{1}{p}} [\sum_{n=1}^\infty (n-\eta)^{-q\hat{\lambda}2-1} A_n^q]^{\frac{1}{q}}. \tag{18}$$

In particular, for $\lambda_1 + \lambda_2 = \lambda(\in (0,2])(\lambda_1 \in (0,\lambda), \lambda_2 \in (0, \frac{1}{2}] \cap (0,\lambda))$, *we have:*

$$\int_0^\infty \sum_{n=1}^\infty \frac{a_n}{(x+n-\eta)^\lambda} f(x)dx < \lambda_1\lambda_2 B(\lambda_1, \lambda_2)$$

$$\times [\int_0^\infty x^{-p\lambda_1-1} F^p(x)dx]^{\frac{1}{p}} [\sum_{n=1}^\infty (n-\eta)^{-q\lambda_2-1} A_n^q]^{\frac{1}{q}}. \tag{19}$$

Proof. Since for $\lambda > 0$, we have:

$$\frac{1}{(x+n-\eta)^{\lambda}} = \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-(x+n-\eta)t}dt,$$

it follows that:

$$I = \frac{1}{\Gamma(\lambda)}\int_0^\infty \sum_{n=1}^\infty a_n f(x)\int_0^\infty t^{\lambda-1}e^{-(x+n-\eta)t}dtdx$$
$$= \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}\int_0^\infty e^{-xt}f(x)dx\sum_{n=1}^\infty e^{-(n-\eta)t}a_n dt$$
$$\le \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda+1}\int_0^\infty e^{-xt}F(x)dx\sum_{n=1}^\infty e^{-(n-\eta)t}A_n dt$$
$$= \frac{1}{\Gamma(\lambda)}\int_0^\infty \sum_{n=1}^\infty A_n F(x)\int_0^\infty t^{(\lambda+2)-1}e^{-(x+n-\eta)t}dtdx$$
$$= \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)}\int_0^\infty \sum_{n=1}^\infty \frac{A_n}{(x+n-\eta)^{\lambda+2}}F(x)dx$$

By applying (15), we obtain (18).

In particular, for $\lambda_1+\lambda_2=\lambda(\in(0,2])(\lambda_1\in(0,\lambda),\lambda_2\in(0,\frac{1}{2}]\cap(0,\lambda))$, one has:

$$k_{\lambda+2}(\lambda_2+1) = k_{\lambda+2}(\lambda_1+1) = B(\lambda_1+1,\lambda_2+1)$$
$$= \frac{\Gamma(\lambda_1+1)\Gamma(\lambda_2+1)}{\Gamma(\lambda+2)} = \frac{\lambda_1\lambda_2\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\lambda+2)} = \frac{\Gamma(\lambda)}{\Gamma(\lambda+2)}\lambda_1\lambda_2 B(\lambda_1,\lambda_2).$$

Hence, it follows from (18) that:

$$\int_0^\infty \sum_{n=1}^\infty \frac{a_n}{(x+n-\eta)^{\lambda}}f(x)dx < \lambda_1\lambda_2 B(\lambda_1,\lambda_2)$$
$$\times\left[\int_0^\infty x^{-p\lambda_1-1}F^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty (n-\eta)^{-q\lambda_2-1}A_n^q\right]^{\frac{1}{q}}, \tag{20}$$

which is the desired inequality (19). □

Remark 3. *Putting $\eta = 0$ in (20), we have:*

$$\int_0^\infty \sum_{n=1}^\infty \frac{a_n}{(x+n)^{\lambda}}f(x)dx < \lambda_1\lambda_2 B(\lambda_1,\lambda_2)$$
$$\times\left[\int_0^\infty x^{-p\lambda_1-1}F^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{-q\lambda_2-1}A_n^q\right]^{\frac{1}{q}}. \tag{21}$$

Namely, (18) given by Theorem 1 is a more accurate extension of (21) above. It should be noted that here the statement of "more accurate inequality" borrows from the statement mentioned at the beginning of the paper on the comparison between inequalities (1) and (2) described in the previous literature.

Theorem 2. *If $\lambda-\lambda_1 \le \frac{1}{2}$, then the following statements (i), (ii) and (iii), associated with Theorem 1, are equivalent:*

(i) $(k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}} \le k_{\lambda+2}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}+1)$;

(ii) $\lambda_1+\lambda_2=\lambda(\in(0,2])$,*where*$\lambda_1\in(0,\lambda),\lambda_2\in(0,\frac{1}{2}]\cap(0,\lambda)$;

(iii) The constant factor:

$$\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)}(k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}$$

in (19) is the best possible.

Proof. " (i)⇒(ii)". By using Hölder inequality with weight, we obtain:

$$k_{\lambda+2}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}+1)$$
$$=\int_0^\infty \frac{1}{(1+u)^{\lambda+2}}u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}}du=\int_0^\infty \frac{1}{(1+u)^{\lambda+2}}(u^{\frac{\lambda-\lambda_2}{p}})(u^{\frac{\lambda_1}{q}})du$$
$$\leq [\int_0^\infty \frac{1}{(1+u)^{\lambda+2}}u^{\lambda-\lambda_2}du]^{\frac{1}{p}}[\int_0^\infty \frac{1}{(1+u)^{\lambda+2}}u^{\lambda_1}du]^{\frac{1}{q}}$$
$$=[\int_0^\infty \frac{1}{(1+v)^{\lambda+2}}v^{(\lambda_2+1)-1}dv]^{\frac{1}{p}}[\int_0^\infty \frac{1}{(1+u)^{\lambda+2}}u^{(\lambda_1+1)-1}du]^{\frac{1}{q}}$$
$$=(k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}. \tag{22}$$

In view of inequality (i), we conclude that (22) keeps the form of equality.

We observe that (22) keeps the form of equality if and only if there exist constants A and B, such that they are not both zero and $Au^{\lambda-\lambda_2}=Bu^{\lambda_1}$ *a.e.* in R_+ (see [18]). Assuming that $A\neq 0$, we have $u^{\lambda-\lambda_2-\lambda_1}=\frac{B}{A}$ *a.e.* in R_+, and then $\lambda-\lambda_2-\lambda_1=0$, namely, $\lambda_1+\lambda_2=\lambda(\in(0,2])$, where, $\lambda_1\in(0,\lambda),\lambda_2\in(0,\frac{1}{2}]\cap(0,\lambda)$.

"(ii)⇒(iii)". For $\lambda_1+\lambda_2=\lambda(\in(0,2])$, $\lambda_1\in(0,\lambda),\lambda_2\in(0,\frac{1}{2}]\cap(0,\lambda)$, (19) reduces to (20). For any $0<\varepsilon<\min\{p\lambda_1,q\lambda_2\}$, we set:

$$\widetilde{f}(x):=\begin{cases}0, 0<x<1,\\ x^{\lambda_1-\frac{\varepsilon}{p}-1}, x\geq 1\end{cases}, \widetilde{a}_n:=n^{\lambda_2-\frac{\varepsilon}{q}-1}(n\in\mathrm{N})$$

Then, it follows that:

$$\widetilde{F}(x):=\int_0^x \widetilde{f}(t)dt\leq\begin{cases}0, 0<x<1,\\ \frac{1}{\lambda_1-\frac{\varepsilon}{p}}x^{\lambda_1-\frac{\varepsilon}{p}}, x\geq 1\end{cases},$$
$$\widetilde{A}_n:=\sum_{k=1}^n \widetilde{a}_k=\sum_{k=1}^n k^{\lambda_2-\frac{\varepsilon}{q}-1}<\int_0^n t^{\lambda_2-\frac{\varepsilon}{q}-1}dt=\frac{1}{\lambda_2-\frac{\varepsilon}{q}}n^{\lambda_2-\frac{\varepsilon}{q}}(n\in\mathrm{N}).$$

If there exists a positive constant $M\leq\lambda_1\lambda_2B(\lambda_1,\lambda_2)$ such that (20) is valid when replacing $\lambda_1\lambda_2B(\lambda_1,\lambda_2)$ by M, then, in particular for $\eta=0$, by substitution of $f(x)=\widetilde{f}(x), a_n=\widetilde{a}_n$, $F(x)=\widetilde{F}(x)$ and $A_n=\widetilde{A}_n$ in (21), we have:

$$\widetilde{I}:=\int_0^\infty\sum_{n=1}^\infty \frac{\widetilde{a}_n\widetilde{f}(x)}{(x+n)^\lambda}dx< M(\int_0^\infty x^{-p\lambda_1-1}\widetilde{F}^p(x)dx)^{\frac{1}{p}}(\sum_{n=1}^\infty n^{-q\lambda_2-1}\widetilde{A}_n^q)^{\frac{1}{q}}. \tag{23}$$

In the following, we show that $\lambda_1\lambda_2B(\lambda_1,\lambda_2)\leq M$, and then $M=\lambda_1\lambda_2B(\lambda_1,\lambda_2)$ is the best possible constant factor in (20).

By (23) and the decreasingness property of series, we obtain:

$$\widetilde{I}<M\frac{1}{\lambda_1-\frac{\varepsilon}{p}}(\int_1^\infty x^{-p\lambda_1-1}x^{p\lambda_1-\varepsilon}dx)^{\frac{1}{p}}\frac{1}{\lambda_2-\frac{\varepsilon}{q}}(\sum_{n=1}^\infty n^{-q\lambda_2-1}n^{q\lambda_2-\varepsilon})^{\frac{1}{q}}$$
$$=M(\frac{1}{\lambda_1-\frac{\varepsilon}{p}})(\frac{1}{\lambda_2-\frac{\varepsilon}{q}})(\int_1^\infty x^{-\varepsilon-1}dx)^{\frac{1}{p}}(1+\sum_{n=2}^\infty n^{-\varepsilon-1})^{\frac{1}{q}}$$
$$<M(\frac{1}{\lambda_1-\frac{\varepsilon}{p}})(\frac{1}{\lambda_2-\frac{\varepsilon}{q}})(\int_1^\infty x^{-\varepsilon-1}dx)^{\frac{1}{p}}(1+\int_1^\infty y^{-\varepsilon-1}dy)^{\frac{1}{q}}$$
$$=\frac{M}{\varepsilon}(\frac{1}{\lambda_1-\frac{\varepsilon}{p}})(\frac{1}{\lambda_2-\frac{\varepsilon}{q}})(\varepsilon+1)^{\frac{1}{q}}.$$

By (11) (for $\eta = 0$), setting $\widetilde{\lambda}_2 = \lambda_2 - \frac{\varepsilon}{q} \in (0, \frac{1}{2}) \cap (0, \lambda)(0 < \widetilde{\lambda}_1 = \lambda_1 + \frac{\varepsilon}{q} < \lambda)$, we obtain:

$$\begin{aligned}\widetilde{I} &= \int_1^\infty [x^{(\lambda_1 + \frac{\varepsilon}{q})} \sum_{n=1}^\infty \tfrac{1}{(x+n)^\lambda} n^{(\lambda_2 - \frac{\varepsilon}{q})-1}] x^{-\varepsilon-1} dx \\ &= \int_1^\infty \varpi(\widetilde{\lambda}_2, x) x^{-\varepsilon-1} dx > B(\widetilde{\lambda}_1, \widetilde{\lambda}_2) \int_1^\infty [1 - O(\tfrac{1}{x^{\widetilde{\lambda}_2}})] x^{-\varepsilon-1} dx \\ &= B(\widetilde{\lambda}_1, \widetilde{\lambda}_2)[\int_1^\infty x^{-\varepsilon-1} dx - \int_1^\infty O(\tfrac{1}{x^{\lambda_2 + \frac{\varepsilon}{p} + 1}}) dx] \\ &= \tfrac{1}{\varepsilon} B(\lambda_1 + \tfrac{\varepsilon}{q}, \lambda_2 - \tfrac{\varepsilon}{q})(1 - \varepsilon O(1)).\end{aligned}$$

Then, in virtue of the above results, we have:

$$B(\lambda_1 + \frac{\varepsilon}{q}, \lambda_2 - \frac{\varepsilon}{q})(1 - \varepsilon O(1)) < \varepsilon \widetilde{I} < M(\frac{1}{\lambda_1 - \frac{\varepsilon}{p}})(\frac{1}{\lambda_2 - \frac{\varepsilon}{q}})(\varepsilon + 1)^{\frac{1}{q}}.$$

Putting $\varepsilon \to 0^+$, in view of the continuity of the beta function, we obtain $\lambda_1 \lambda_2 B(\lambda_1, \lambda_2) \leq M$. Hence, $M = \lambda_1 \lambda_2 B(\lambda_1, \lambda_2)$ is the best possible constant factor in (20).

"(iii)⇒(i)". Since $\lambda - \lambda_1 \leq \frac{1}{2}$, for $\hat{\lambda}_1 = \frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q}, \hat{\lambda}_2 = \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p}$, we find:

$$\hat{\lambda}_1 + \hat{\lambda}_2 = \frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q} + \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p} = \lambda, 0 < \hat{\lambda}_1, \hat{\lambda}_2 < \frac{\lambda}{p} + \frac{\lambda}{q} = \lambda,$$

$\hat{\lambda}_2 \leq \frac{1/2}{p} + \frac{1/2}{q} = \frac{1}{2}$, and $\hat{\lambda}_1 \hat{\lambda}_2 B(\hat{\lambda}_1, \hat{\lambda}_2) \in \mathrm{R}_+$.

If the constant factor $\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} (k_{\lambda+2}(\lambda_2 + 1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1 + 1))^{\frac{1}{q}}$ in (19) is the best possible, then by (21) (for $\lambda_i = \hat{\lambda}_i (i = 1, 2)$), we have:

$$\begin{aligned}&\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} (k_{\lambda+2}(\lambda_2 + 1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1 + 1))^{\frac{1}{q}} \\ &\leq \hat{\lambda}_1 \hat{\lambda}_2 B(\hat{\lambda}_1, \hat{\lambda}_2) = \tfrac{\Gamma(\lambda+2)}{\Gamma(\lambda)} k_{\lambda+2}(\hat{\lambda}_1 + 1) \\ &= \tfrac{\Gamma(\lambda+2)}{\Gamma(\lambda)} k_{\lambda+2}(\tfrac{\lambda - \lambda_2}{p} + \tfrac{\lambda_1}{q} + 1)(\in \mathrm{R}_+),\end{aligned}$$

namely, statement (i) is valid.

Hence, the statements (i), (ii) and (iii) are equivalent. This completes the proof of Theorem 2. □

Remark 4. *Putting $\eta = \frac{1}{4}$ in (20), we acquire:*

$$\int_0^\infty \sum_{n=1}^\infty \frac{a_n}{(x + n - \frac{1}{4})^\lambda} f(x) dx < \lambda_1 \lambda_2 B(\lambda_1, \lambda_2) \times [\int_0^\infty x^{-p\lambda_1 - 1} F^p(x) dx]^{\frac{1}{p}} [\sum_{n=1}^\infty (n - \frac{1}{4})^{-q\lambda_2 - 1} A_n^q]^{\frac{1}{q}} \quad (24)$$

In particular, for $\lambda = 1, \lambda_1 = \lambda_2 = \frac{1}{2}$, we have the following Hilbert-type inequality with the best possible constant factor $\frac{\pi}{4}$

$$\int_0^\infty \sum_{n=1}^\infty \frac{a_n}{x + n - \frac{1}{4}} f(x) dx < \frac{\pi}{4} [\int_0^\infty x^{-\frac{p}{2} - 1} F^p(x) dx]^{\frac{1}{p}} [\sum_{n=1}^\infty (n - \frac{1}{4})^{-\frac{q}{2} - 1} A_n^q]^{\frac{1}{q}}. \quad (25)$$

4. Conclusions

In this paper, based on the weight coefficients and the idea of introducing parameters, by applying Hermite–Hadamard inequality, the Euler–Maclaurin summation formula and Abel's summation by parts formula, a more accurate half-discrete Hilbert-type inequality involving one upper limit function as well as one partial sum is given in Theorem 1. The equivalent statements of the best possible constant factor related to several parameters are considered in Theorem 2. As applications of the main results, some new inequalities are proposed in Remarks 3 and 4. Our results would provide a significant supplement to the study of half-discrete Hilbert-type inequality.

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