

Article

On New Generalized Dunkel Type Integral Inequalities with Applications

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Abstract: In this paper, by applying majorization theory, we study the Schur convexity of functions related to Dunkel integral inequality. We establish some new generalized Dunkel type integral inequalities and their applications to inequality theory.

Keywords: Dunkel type integral inequality; Schur-convexity; majorization theory; arithmetic mean-geometric mean (AM-GM) inequality

MSC: 26A51; 26E60; 26D15; 26D05; 34K38; 39B62



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1. Introduction and Preliminaries

Over the last half a century, rapid developments in inequality theory and its applications have contributed greatly to many branches of mathematics such as linear and nonlinear analysis, differential equations, finance, statistics, physics, fractional calculus, and so on; for more details, one can refer to [1–4] and the references therein.

The original Dunkel integral inequality can be stated as follows.

Theorem 1 (original Dunkel integral inequality; see [1–3,5,6]). *Let $f(x)$ be a continuous real-valued function on $[a, b]$ which is not identically zero and satisfies $0 \leq f(x) \leq M$ for all $x \in [a, b]$. Then*

$$0 < \left(\int_a^b f(x) dx \right)^2 - \left(\int_a^b f(x) \cos x dx \right)^2 - \left(\int_a^b f(x) \sin x dx \right)^2 \leq \frac{1}{12} M^2 (b - a)^2. \quad (1)$$

There are many ways to prove Dunkel integral inequality (see [1–3,5,6] and references therein). Some interesting proofs of Dunkel integral inequality are the probabilistic method (see, e.g., [1]), re-integral method (see [2,3]), and so on.

In fact, if $f(x)$ is a nonnegative continuous real-valued function on $[a, b]$ (here, f is allowed to be a zero function), then from (1) one deduces the following fascinating concise inequality:

$$\left(\int_a^b f(x) \cos x dx \right)^2 + \left(\int_a^b f(x) \sin x dx \right)^2 \leq \left(\int_a^b f(x) dx \right)^2. \quad (2)$$

In 1923, Professor Issai Schur first systematically studied the functions preserving the ordering of majorization. In Schur's honor, such functions are named to have "Schur-convexity". During the previous more than four decades, majorization theory and Schur-convexity have been applied widely in many areas of mathematics including integral inequality, stochastic matrices, rearrangement theory, analytic inequalities, information

theory, quantum correlations, quantum cryptography, combinatorial optimization, and other related fields (see, e.g., [7–12]).

Let us recall some basic definitions and notation that will be needed in this paper.

Definition 1 (see [4,8]). Let Ω be a nonempty subset of \mathbb{R}^n .

- (i) Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. x is said to be majorized by y (in symbols $x \prec y$) if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \dots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geq \dots \geq x_{[n]}$ and $y_{[1]} \geq \dots \geq y_{[n]}$ are rearrangements of x and y in a descending order;
- (ii) Ω is called convex if $\alpha x + \beta y \in \Omega$ for any $x, y \in \Omega$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$;
- (iii) Ω is called symmetric if $x \in \Omega$ implies $Px \in \Omega$ for every $n \times n$ permutation matrix P ;
- (iv) A function $\varphi : \Omega \rightarrow \mathbb{R}$ is called symmetric if for every permutation matrix P , $\varphi(Px) = \varphi(x)$ for all $x \in \Omega$;
- (v) A function $\varphi : \Omega \rightarrow \mathbb{R}$ is said to be Schur convex on Ω if $x \prec y$ on Ω implies $\varphi(x) \leq \varphi(y)$. φ is said to be Schur concave on Ω if and only if $-\varphi$ is Schur convex.

The paper is divided into five sections. In Sections 2 and 3, by applying majorization theory, we present some new generalized Dunkel type integral inequalities and new Dunkel (p)-type integral inequalities for $p \geq 2$. As applications of our new results, some new integral inequalities are established in Section 4. Finally, some summary and conclusions are given in Section 5.

2. Some Generalizations of Dunkel Integral Inequality

The following two known results are important for proving our new theorem.

Lemma 1 (see [4]). Let $a \leq b$. Let $u(t) := ta + (1-t)b$ and $v(t) := tb + (1-t)a$ for $\frac{1}{2} \leq t_1 \leq t_2 \leq 1$ or $0 \leq t_1 \leq t_2 \leq \frac{1}{2}$. Then

$$\left(\frac{a+b}{2}, \frac{a+b}{2}\right) \prec (u(t_2), v(t_2)) \prec (u(t_1), v(t_1)) \prec (a, b).$$

Lemma 2 (see [4,7]). Let $\Omega \subset \mathbb{R}^n$ be a nonempty convex set and has a nonempty interior set Ω° . Let $\varphi : \Omega \rightarrow \mathbb{R}$ be continuous on Ω and differentiable in Ω° . Then, φ is a Schur convex (resp. Schur concave) function, if and only if it is symmetric on Ω and

$$(x_1 - x_2) \left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2} \right) \geq 0 \quad (\text{resp. } \leq 0)$$

holds for any $x = (x_1, \dots, x_n) \in \Omega^\circ$.

Remark 1. It is worth noticing that Lemma 2 is equivalent to the following: φ is a Schur convex (resp. Schur concave) function, if and only if it is symmetric on Ω and

$$\frac{\partial \varphi}{\partial x_i} \geq \frac{\partial \varphi}{\partial x_{i+1}} \quad (\text{resp. } \leq 0), i = 1, 2, \dots, n-1.$$

for all $x \in D \cap \Omega$, where $D = \{x : x_1 \geq \dots \geq x_n\}$.

With the help of Lemmas 1 and 2, we can establish the following crucial result.

Theorem 2. Let I be an interval of \mathbb{R} . Assume that $f(x)$ and $g(x)$ are two nonnegative continuous real-valued functions on I , and $\kappa(x)$ and $\lambda(x)$ are two continuous real-valued functions on I . Define $L : I \times I \rightarrow \mathbb{R}$ by

$$L(a, b) = \int_a^b f(x)dx \int_a^b g(x)dx - \int_a^b f(x)\kappa(x)dx \int_a^b g(x)\kappa(x)dx - \int_a^b f(x)\lambda(x)dx \int_a^b g(x)\lambda(x)dx$$

for any $(a, b) \in I \times I$. Then the following holds:

- (i) If $\kappa(x) \cdot \kappa(b) + \lambda(x) \cdot \lambda(b) \leq 1$ and $\kappa(x) \cdot \kappa(a) + \lambda(x) \cdot \lambda(a) \leq 1$ for $x, a, b \in I$, then L is Schur convex on $I \times I$;
- (ii) If $\kappa(x) \cdot \kappa(b) + \lambda(x) \cdot \lambda(b) \geq 1$ and $\kappa(x) \cdot \kappa(a) + \lambda(x) \cdot \lambda(a) \geq 1$ for $x, a, b \in I$, then L is Schur concave on $I \times I$.

Proof. Obviously, $L(a, b)$ is a symmetric operator for $a, b \in I$. So, without loss of generality, we may assume that $b \geq a$. Since

$$\begin{aligned} \frac{\partial L}{\partial b} &= f(b) \int_a^b g(x)dx + g(b) \int_a^b f(x)dx \\ &\quad - f(b)\kappa(b) \int_a^b g(x)\kappa(x)dx - g(b)\kappa(b) \int_a^b f(x)\kappa(x)dx \\ &\quad - f(b)\lambda(b) \int_a^b g(x)\lambda(x)dx - g(b)\lambda(b) \int_a^b f(x)\lambda(x)dx \end{aligned}$$

and

$$\begin{aligned} \frac{\partial L}{\partial a} &= -f(a) \int_a^b g(x)dx - g(a) \int_a^b f(x)dx \\ &\quad + f(a)\kappa(a) \int_a^b g(x)\kappa(x)dx + g(a)\kappa(a) \int_a^b f(x)\kappa(x)dx \\ &\quad + f(a)\lambda(a) \int_a^b g(x)\lambda(x)dx + g(a)\lambda(a) \int_a^b f(x)\lambda(x)dx, \end{aligned}$$

we have

$$\begin{aligned} \Delta &:= (b - a) \left(\frac{\partial L}{\partial b} - \frac{\partial L}{\partial a} \right) \\ &= (f(a) + f(b)) \int_a^b g(x)dx + (g(a) + g(b)) \int_a^b f(x)dx \\ &\quad - (f(b)\kappa(b) + f(a)\kappa(a)) \int_a^b g(x)\kappa(x)dx - (f(b)\lambda(b) + f(a)\lambda(a)) \int_a^b g(x)\lambda(x)dx \\ &\quad - (g(b)\kappa(b) + g(a)\kappa(a)) \int_a^b f(x)\kappa(x)dx - (g(b)\lambda(b) + g(a)\lambda(a)) \int_a^b f(x)\lambda(x)dx \\ &= f(b) \int_a^b g(x)(1 - \kappa(b)\kappa(x) - \lambda(b)\lambda(x))dx + f(a) \int_a^b g(x)(1 - \kappa(a)\kappa(x) \\ &\quad - \lambda(a)\lambda(x))dx + g(b) \int_a^b f(x)(1 - \kappa(b)\kappa(x) - \lambda(b)\lambda(x))dx \\ &\quad + g(a) \int_a^b f(x)(1 - \kappa(a)\kappa(x) - \lambda(a)\lambda(x))dx. \end{aligned}$$

- (i) When $\kappa(x) \cdot \kappa(b) + \lambda(x) \cdot \lambda(b) \leq 1$ and $\kappa(x) \cdot \kappa(a) + \lambda(x) \cdot \lambda(a) \leq 1$, we have $\Delta \geq 0$. By Lemma 2, L is Schur convex on $I \times I$.

- (ii) When $\kappa(x) \cdot \kappa(b) + \lambda(x) \cdot \lambda(b) \geq 1$ and $\kappa(x) \cdot \kappa(a) + \lambda(x) \cdot \lambda(a) \geq 1$, we have $\Delta \leq 0$. By Lemma 2, L is Schur concave on $I \times I$.

The proof is completed. \square

We now present the following generalized Dunkel type integral inequality which is one of the main results of this paper.

Theorem 3. Let I be an interval of \mathbb{R} . Assume that $f(x)$ and $g(x)$ are two nonnegative continuous real-valued functions on I , and $\kappa(x)$ and $\lambda(x)$ are two continuous real-valued functions on I . Let $u(t) := ta + (1 - t)b$ and $v(t) := tb + (1 - t)a$, for $\frac{1}{2} \leq t \leq 1$. Then the following holds:

- (i) If $\kappa(x) \cdot \kappa(b) + \lambda(x) \cdot \lambda(b) \leq 1$ and $\kappa(x) \cdot \kappa(a) + \lambda(x) \cdot \lambda(a) \leq 1$ for $x, a, b \in I$, then

$$\begin{aligned} & \int_a^b f(x)\kappa(x)dx \int_a^b g(x)\kappa(x)dx + \int_a^b f(x)\lambda(x)dx \int_a^b g(x)\lambda(x)dx \\ & \leq \int_a^b f(x)dx \int_a^b g(x)dx - \int_{u(t)}^{v(t)} f(x)dx \int_{u(t)}^{v(t)} g(x)dx \\ & \quad + \int_{u(t)}^{v(t)} f(x)\kappa(x)dx \int_{u(t)}^{v(t)} g(x)\kappa(x)dx + \int_{u(t)}^{v(t)} f(x)\lambda(x)dx \int_{u(t)}^{v(t)} g(x)\lambda(x)dx \\ & \leq \int_a^b f(x)dx \int_a^b g(x)dx; \end{aligned}$$

- (ii) If $\kappa(x) \cdot \kappa(b) + \lambda(x) \cdot \lambda(b) \geq 1$ and $\kappa(x) \cdot \kappa(a) + \lambda(x) \cdot \lambda(a) \geq 1$ for $x, a, b \in I$, then

$$\begin{aligned} & \int_a^b f(x)\kappa(x)dx \int_a^b g(x)\kappa(x)dx + \int_a^b f(x)\lambda(x)dx \int_a^b g(x)\lambda(x)dx \\ & \geq \int_a^b f(x)dx \int_a^b g(x)dx - \int_{u(t)}^{v(t)} f(x)dx \int_{u(t)}^{v(t)} g(x)dx \\ & \quad + \int_{u(t)}^{v(t)} f(x)\kappa(x)dx \int_{u(t)}^{v(t)} g(x)\kappa(x)dx + \int_{u(t)}^{v(t)} f(x)\lambda(x)dx \int_{u(t)}^{v(t)} g(x)\lambda(x)dx \\ & \geq \int_a^b f(x)dx \int_a^b g(x)dx. \end{aligned}$$

Proof. We only show case (i) and a similar argument could be made for the case (ii). Define $L : I \times I \rightarrow \mathbb{R}$ by

$$\begin{aligned} L(a, b) &= \int_a^b f(x)dx \int_a^b g(x)dx \\ & \quad - \int_a^b f(x)\kappa(x)dx \int_a^b g(x)\kappa(x)dx - \int_a^b f(x)\lambda(x)dx \int_a^b g(x)\lambda(x)dx \end{aligned}$$

for any $(a, b) \in I \times I$. If $\kappa(x) \cdot \kappa(b) + \lambda(x) \cdot \lambda(b) \leq 1$ and $\kappa(x) \cdot \kappa(a) + \lambda(x) \cdot \lambda(a) \leq 1$ for $x, a, b \in I$, by applying Theorem 2 (i), we show that L is Schur convex on $I \times I$. On the other hand, by using Lemma 1, we get

$$L(a, b) \geq L(u(t), v(t)) \geq L\left(\frac{a+b}{2}, \frac{a+b}{2}\right) = 0.$$

Hence, we obtain

$$\begin{aligned}
 & \int_a^b f(x)dx \int_a^b g(x)dx - \int_a^b f(x)\kappa(x)dx \int_a^b g(x)\kappa(x)dx \\
 & - \int_a^b f(x)\lambda(x)dx \int_a^b g(x)\lambda(x)dx \\
 & \geq \int_{u(t)}^{v(t)} f(x)dx \int_{u(t)}^{v(t)} g(x)dx - \int_{u(t)}^{v(t)} f(x)\kappa(x)dx \int_{u(t)}^{v(t)} g(x)\kappa(x)dx \\
 & - \int_{u(t)}^{v(t)} f(x)\lambda(x)dx \int_{u(t)}^{v(t)} g(x)\lambda(x)dx \\
 & \geq 0,
 \end{aligned}$$

which implies

$$\begin{aligned}
 & \int_a^b f(x)\kappa(x)dx \int_a^b g(x)\kappa(x)dx + \int_a^b f(x)\lambda(x)dx \int_a^b g(x)\lambda(x)dx \\
 & \leq \int_a^b f(x)dx \int_a^b g(x)dx - \int_{u(t)}^{v(t)} f(x)dx \int_{u(t)}^{v(t)} g(x)dx \\
 & + \int_{u(t)}^{v(t)} f(x)\kappa(x)dx \int_{u(t)}^{v(t)} g(x)\kappa(x)dx + \int_{u(t)}^{v(t)} f(x)\lambda(x)dx \int_{u(t)}^{v(t)} g(x)\lambda(x)dx \\
 & \leq \int_a^b f(x)dx \int_a^b g(x)dx.
 \end{aligned}$$

The proof is completed. \square

As a direct consequence of Theorem 3, we can obtain the following generalized Dunkel integral inequality.

Theorem 4 (Generalized Dunkel integral inequality). *Let $f(x)$ and $g(x)$ be two nonnegative continuous real-valued functions on $[a, b]$ and m be any real number. Then*

$$\begin{aligned}
 & \left(\int_a^b f(x) \cos mx dx \right) \left(\int_a^b g(x) \cos mx dx \right) + \left(\int_a^b f(x) \sin mx dx \right) \left(\int_a^b g(x) \sin mx dx \right) \\
 & \leq \left(\int_a^b f(x) dx \right) \left(\int_a^b g(x) dx \right).
 \end{aligned} \quad (3)$$

Proof. In theorem 3, we take $I = [a, b]$, $\kappa(x) = \cos mx$, and $\lambda(x) = \sin mx$ for $x \in I$. Thus, $\kappa(x)$ and $\lambda(x)$ are two continuous real-valued functions on I . Clearly, we have

$$\begin{aligned}
 \kappa(x) \cdot \kappa(b) + \lambda(x) \cdot \lambda(b) &= \cos mx \cdot \cos mb + \sin mx \cdot \sin mb \\
 &= \cos m(b-x) \leq 1
 \end{aligned}$$

and

$$\begin{aligned}
 \kappa(x) \cdot \kappa(a) + \lambda(x) \cdot \lambda(a) &= \cos mx \cdot \cos ma + \sin mx \cdot \sin ma \\
 &= \cos m(a-x) \leq 1.
 \end{aligned}$$

Thus, all the assumptions of Theorem 3 (i) are satisfied. Therefore the desired conclusion follows immediately from Theorem 3. \square

The following generalized Dunkel integral inequality is an immediate consequence of Theorem 4.

Corollary 1 (Generalized Dunkel integral inequality). *Let $f(x)$ be a continuous nonnegative real-valued function on $[a, b]$ and m be any real number. Then*

$$\left(\int_a^b f(x) \cos mx dx\right)^2 + \left(\int_a^b f(x) \sin mx dx\right)^2 \leq \left(\int_a^b f(x) dx\right)^2. \tag{4}$$

Remark 2. *It is worth noticing that inequality (3) in Theorem 4 and inequality (4) in Corollary 1 are real generalizations of inequality (2).*

3. A New Dunkel (p)-Type Integral Inequality for $p \geq 2$

In this section, we will present a new Dunkel (p)-type integral inequality for $p \geq 2$. In order to prove our results, we need the following important auxiliary lemma.

Lemma 3. *Let $k \in \mathbb{N} \cup \{0\}$. Denote $I_k := [2k\pi, 2k\pi + \frac{\pi}{2}]$. Assume that $f(x)$ is a nonnegative continuous real-valued function on I_k . Define $M : I_k \times I_k \rightarrow \mathbb{R}$ by*

$$M(a, b) = \left[(b - a) \int_a^b f(x) \sin x dx \right]^p + \left[(b - a) \int_a^b f(x) \cos x dx \right]^p - \left[(b - a) \int_a^b f(x) dx \right]^p$$

for $(a, b) \in I_k \times I_k$. If $p \geq 2$, then M is Schur concave on $I_k \times I_k$.

Proof. It is obvious that $M(a, b)$ is symmetric for a, b . Hence, without loss of generality, we may assume that $b \geq a$. By Corollary 1, we have

$$\begin{aligned} \frac{\partial M}{\partial b} &= p \left[(b - a) \int_a^b f(x) \sin x dx \right]^{p-1} \left[\int_a^b f(x) \sin x dx + (b - a)f(b) \sin b \right] \\ &+ p \left[(b - a) \int_a^b f(x) \cos x dx \right]^{p-1} \left[\int_a^b f(x) \cos x dx + (b - a)f(b) \cos b \right] \\ &- p \left[(b - a) \int_a^b f(x) dx \right]^{p-1} \left[\int_a^b f(x) dx + (b - a)f(b) \right] \\ &= p(b - a)^{p-1} \left[\int_a^b f(x) \sin x dx \right]^{p-2} \\ &\times \left[\left(\int_a^b f(x) \sin x dx \right)^2 + (b - a)f(b) \sin b \int_a^b f(x) \sin x dx \right] \\ &+ p(b - a)^{p-1} \left[\int_a^b f(x) \cos x dx \right]^{p-2} \\ &\times \left[\left(\int_a^b f(x) \cos x dx \right)^2 + (b - a)f(b) \cos b \int_a^b f(x) \cos x dx \right] \\ &- p(b - a)^{p-1} \left[\int_a^b f(x) dx \right]^{p-2} \left[\left(\int_a^b f(x) dx \right)^2 + (b - a)f(b) \int_a^b f(x) dx \right] \end{aligned}$$

$$\begin{aligned}
&\leq p(b-a)^{p-1} \left(\int_a^b f(x) dx \right)^{p-2} \\
&\times \left[\left(\int_a^b f(x) \sin x dx \right)^2 + \left(\int_a^b f(x) \cos x dx \right)^2 - \left(\int_a^b f(x) dx \right)^2 \right. \\
&+ (b-a)f(b) \int_a^b f(x) (\sin b \cdot \sin x + \cos b \cdot \cos x - 1) dx \left. \right] \\
&= p(b-a)^{p-1} \left(\int_a^b f(x) dx \right)^{p-2} \\
&\times \left[\left(\int_a^b f(x) \sin x dx \right)^2 + \left(\int_a^b f(x) \cos x dx \right)^2 - \left(\int_a^b f(x) dx \right)^2 \right. \\
&+ (b-a)f(b) \int_a^b f(x) (\cos(b-x) - 1) dx \left. \right] \\
&\leq 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial M}{\partial a} &= p \left[(b-a) \int_a^b f(x) \sin x dx \right]^{p-1} \left[- \int_a^b f(x) \sin x dx - (b-a)f(a) \sin a \right] \\
&+ p \left[(b-a) \int_a^b f(x) \cos x dx \right]^{p-1} \left[- \int_a^b f(x) \cos x dx - (b-a)f(a) \cos a \right] \\
&- p \left[(b-a) \int_a^b f(x) dx \right]^{p-1} \left[- \int_a^b f(x) dx - (b-a)f(a) \right] \\
&= -p(b-a)^{p-1} \left[\int_a^b f(x) \sin x dx \right]^{p-2} \\
&\times \left[\left(\int_a^b f(x) \sin x dx \right)^2 + (b-a)f(a) \sin a \int_a^b f(x) \sin x dx \right] \\
&- p(b-a)^{p-1} \left[\int_a^b f(x) \cos x dx \right]^{p-2} \\
&\times \left[\left(\int_a^b f(x) \cos x dx \right)^2 + (b-a)f(a) \cos a \int_a^b f(x) \cos x dx \right] \\
&+ p(b-a)^{p-1} \left[\int_a^b f(x) dx \right]^{p-2} \left[\left(\int_a^b f(x) dx \right)^2 + (b-a)f(a) \int_a^b f(x) dx \right] \\
&\geq -p(b-a)^{p-1} \left(\int_a^b f(x) dx \right)^{p-2} \\
&\times \left[\left(\int_a^b f(x) \sin x dx \right)^2 + \left(\int_a^b f(x) \cos x dx \right)^2 - \left(\int_a^b f(x) dx \right)^2 \right. \\
&+ (b-a)f(a) \int_a^b f(x) (\sin a \cdot \sin x + \cos a \cdot \cos x - 1) dx \left. \right]
\end{aligned}$$

$$\begin{aligned}
 &= -p(b-a)^{p-1} \left(\int_a^b f(x) dx \right)^{p-2} \\
 &\times \left[\left(\int_a^b f(x) \sin x dx \right)^2 + \left(\int_a^b f(x) \cos x dx \right)^2 - \left(\int_a^b f(x) dx \right)^2 \right. \\
 &\left. + (b-a)f(a) \int_a^b f(x)(\cos(a-x) - 1) dx \right] \\
 &\geq 0,
 \end{aligned}$$

which deduce

$$\Delta' := (b-a) \left(\frac{\partial M}{\partial b} - \frac{\partial M}{\partial a} \right) \leq 0.$$

By Lemma 2, M is Schur concave on $I_k \times I_k$. The proof is completed. \square

The following result is a new Dunkel (p)-type integral inequality for $p \geq 2$.

Theorem 5. Let $k \in \mathbb{N} \cup \{0\}$. Denote $I_k := [2k\pi, 2k\pi + \frac{\pi}{2}]$. Assume that $f(x)$ is a nonnegative continuous real-valued function on I_k . If $p \geq 2$ and $[a, b] \subseteq I_k$, then

$$\left(\int_a^b f(x) \cos x dx \right)^p + \left(\int_a^b f(x) \sin x dx \right)^p \leq \left(\int_a^b f(x) dx \right)^p.$$

Proof. Define $M : I_k \times I_k \rightarrow \mathbb{R}$ by

$$\begin{aligned}
 M(a, b) &= \left[(b-a) \int_a^b f(x) \sin x dx \right]^p \\
 &+ \left[(b-a) \int_a^b f(x) \cos x dx \right]^p - \left[(b-a) \int_a^b f(x) dx \right]^p
 \end{aligned}$$

for $(a, b) \in I_k \times I_k$. By Lemmas 1 and 3, we obtain

$$M(a, b) \leq M\left(\frac{a+b}{2}, \frac{a+b}{2}\right) = 0,$$

which means that

$$\left(\int_a^b f(x) \cos x dx \right)^p + \left(\int_a^b f(x) \sin x dx \right)^p \leq \left(\int_a^b f(x) dx \right)^p.$$

\square

The following result is immediate from Theorem 5.

Corollary 2. Let $n \in \mathbb{N}$. Let $k_1, k_2, \dots, k_n \in \mathbb{N} \cup \{0\}$. Assume that $f_i(x)$ is a nonnegative continuous real-valued function on $[2k_i\pi, 2k_i\pi + \frac{\pi}{2}]$ and $[a_i, b_i] \subseteq [2k_i\pi, 2k_i\pi + \frac{\pi}{2}]$ for any $1 \leq i \leq n$. If $p \geq 2$, then

$$\sum_{i=1}^n \left(\int_{a_i}^{b_i} f_i(x) \cos x dx \right)^p + \sum_{i=1}^n \left(\int_{a_i}^{b_i} f_i(x) \sin x dx \right)^p \leq \sum_{i=1}^n \left(\int_{a_i}^{b_i} f_i(x) dx \right)^p.$$

4. Some New Integral Inequalities

In this section, we will provide some new integral inequalities by applying our main results.

Lemma 4 (Bessel inequality; see [1]). Let $f(x)$ be a continuous or a piecewise continuous nonnegative function on $[0, 2\pi]$. The Fourier series of $f(x)$ is

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx),$$

where $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$, $a_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos mx dx$, and $b_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin mx dx$, for $m \in \mathbb{N}$. Then

$$\frac{a_0^2}{2} + \sum_{m=1}^n (a_m^2 + b_m^2) \leq \frac{1}{\pi} \int_0^{2\pi} f^2(x) dx.$$

Lemma 5 (see [1]). Let $f(x)$ be a nonnegative integrable concave function on $[a, b]$. If $p \geq 1$, then

$$\int_a^b f^p(x) dx \leq \frac{2^p}{(b-a)^{p-1}(p+1)} \left(\int_a^b f(x) dx \right)^p.$$

Theorem 6. Let $f(x)$ be a nonnegative continuous concave function on $[0, 2\pi]$. Then

$$\left(\int_0^{2\pi} f(x) \sin x dx \right)^2 + \left(\int_0^{2\pi} f(x) \cos x dx \right)^2 \leq \frac{4}{9\pi^2} \left(\int_0^{2\pi} f(x) dx \right)^2.$$

Proof. Using the notations in Lemma 4 and applying Theorem 4, we get

$$\sum_{m=1}^n (a_m^2 + b_m^2) \leq na_0^2. \tag{5}$$

By combining (5) with Bessel inequality (see Lemma 4), we obtain

$$\sum_{m=1}^n (a_m^2 + b_m^2) \leq 2n \left(\frac{1}{\pi} \int_0^{2\pi} f^2(x) dx - \sum_{m=1}^n (a_m^2 + b_m^2) \right)$$

which implies

$$\sum_{m=1}^n (a_m^2 + b_m^2) \leq \frac{2n}{(2n+1)\pi} \int_0^{2\pi} f^2(x) dx.$$

Let $n = 1$. By applying Lemma 5, we obtain

$$\begin{aligned} \left(\int_0^{2\pi} f(x) \sin x dx \right)^2 + \left(\int_0^{2\pi} f(x) \cos x dx \right)^2 &\leq \frac{2}{3\pi} \int_0^{2\pi} f^2(x) dx \\ &\leq \frac{4}{9\pi^2} \left(\int_0^{2\pi} f(x) dx \right)^2. \end{aligned}$$

The proof is completed. \square

Theorem 7. Let $f(x)$ be a nonnegative continuous function on $[a, b]$. If $0 < a \leq x \leq b < \frac{\pi}{2}$, then

$$\begin{aligned} &\left(\int_a^b f(x) \tan x dx \right)^2 + \left(\int_a^b f(x) \cot x dx \right)^2 \\ &\geq 2 \left(\int_a^b f(x) dx \right)^2 + \left(\int_{u(t)}^{v(t)} f(x) \tan x dx \right)^2 + \left(\int_{u(t)}^{v(t)} f(x) \cot x dx \right)^2 - 2 \left(\int_{u(t)}^{v(t)} f(x) dx \right)^2 \\ &\geq 2 \left(\int_a^b f(x) dx \right)^2, \end{aligned}$$

where $u(t) = ta + (1 - t)b$ and $v(t) = tb + (1 - t)a$, for $\frac{1}{2} \leq t \leq 1$.

Proof. Let $\kappa(x) = \frac{\sqrt{2}}{2} \tan x$ and $\lambda(x) = \frac{\sqrt{2}}{2} \cot x$ for $x \in [a, b]$. By the arithmetic mean-geometric mean (AM-GM) inequality, we have

$$\begin{aligned} \kappa(x) \cdot \kappa(b) + \lambda(x) \cdot \lambda(b) &= \frac{1}{2} \tan x \cdot \tan b + \frac{1}{2} \cot x \cdot \cot b \\ &\geq (\tan x \cdot \tan b \cdot \cot x \cdot \cot b)^{\frac{1}{2}} = 1. \end{aligned}$$

In the same way, we also have $\kappa(x) \cdot \kappa(a) + \lambda(x) \cdot \lambda(a) \geq 1$. By Theorem 2 (ii), we obtain

$$\begin{aligned} &\left(\int_a^b f(x) \frac{\sqrt{2} \tan x}{2} dx \right)^2 + \left(\int_a^b f(x) \frac{\sqrt{2} \cot x}{2} dx \right)^2 \\ &\geq \left(\int_a^b f(x) dx \right)^2 + \left(\int_{u(t)}^{v(t)} f(x) \frac{\sqrt{2} \tan x}{2} dx \right)^2 \\ &\quad + \left(\int_{u(t)}^{v(t)} f(x) \frac{\sqrt{2} \cot x}{2} dx \right)^2 - \left(\int_{u(t)}^{v(t)} f(x) dx \right)^2 \\ &\geq \left(\int_a^b f(x) dx \right)^2, \end{aligned}$$

which deduces

$$\begin{aligned} &\left(\int_a^b f(x) \tan x dx \right)^2 + \left(\int_a^b f(x) \cot x dx \right)^2 \\ &\geq 2 \left(\int_a^b f(x) dx \right)^2 + \left(\int_{u(t)}^{v(t)} f(x) \tan x dx \right)^2 \\ &\quad + \left(\int_{u(t)}^{v(t)} f(x) \cot x dx \right)^2 - 2 \left(\int_{u(t)}^{v(t)} f(x) dx \right)^2 \\ &\geq 2 \left(\int_a^b f(x) dx \right)^2. \end{aligned}$$

The proof is completed. \square

Theorem 8. Let $0 \leq a < b \leq 1$ and $f(x)$ be a nonnegative continuous function on $[a, b]$. If $\beta \geq \frac{1}{2}$, then

$$\left(\int_a^b x^\beta f(x) dx \right)^2 + \left(\int_a^b (1 - x)^\beta f(x) dx \right)^2 \leq \left(\int_a^b f(x) dx \right)^2. \tag{6}$$

Proof. Let $\kappa(x) = x^\beta$ and $\lambda(x) = (1 - x)^\beta$ for $x \in [a, b]$. For any $x \in [a, b]$, since $\beta \geq \frac{1}{2}$, we have

$$\begin{aligned} \kappa(x) \cdot \kappa(b) + \lambda(x) \cdot \lambda(b) &= x^\beta \cdot b^\beta + (1 - x)^\beta \cdot (1 - b)^\beta \\ &\leq x^{\frac{1}{2}} \cdot b^{\frac{1}{2}} + (1 - x)^{\frac{1}{2}} \cdot (1 - b)^{\frac{1}{2}} \\ &\leq \frac{x + b + 1 - x + 1 - b}{2} \\ &= 1. \end{aligned}$$

In the same way, we can also show that $\kappa(x) \cdot \kappa(a) + \lambda(x) \cdot \lambda(a) \leq 1$ for $x \in [a, b]$. Therefore, the desired inequality (6) follows immediately from Theorem 2 (i). \square

Theorem 9. Let $p \geq 1$ and $0 \leq a < b \leq 1$. Assume that $f(x)$ is a nonnegative continuous function on $[a, b]$. If $f(x)$ is decreasing and $xf(x)$ is increasing, or $f(x)$ is increasing and $(1-x)f(x)$ is decreasing, then

$$2m^p \left(\int_a^b f(x) dx \right)^2 \leq \left(\int_a^b x^p f(x) dx \right)^2 + \left(\int_a^b (1-x)^p f(x) dx \right)^2 \leq \left(\int_a^b f(x) dx \right)^2, \quad (7)$$

where $m := \min\{a(1-a), b(1-b)\}$.

Proof. From Theorem 8, we know that the right side of the desired inequality (7) holds. Next, we verify that the left side of desired inequality (7) also holds. By the AM-GM inequality, we have

$$\left(\int_a^b x^p f(x) dx \right)^2 + \left(\int_a^b (1-x)^p f(x) dx \right)^2 \geq 2 \int_a^b x^p f(x) dx \int_a^b (1-x)^p f(x) dx.$$

Let $\varphi(x) = x^p f(x)$ and $\mu(x) = (1-x)^p f(x)$ for $x \in [a, b]$. Thus, we get

$$\varphi'(x) = x^{p-1}(pf(x) + xf'(x))$$

and

$$\mu'(x) = (1-x)^{p-1}[-pf(x) + (1-x)f'(x)].$$

Since $p \geq 1$, if $f(x)$ is decreasing and $xf(x)$ is increasing, we obtain $\varphi'(x) \geq 0$ and $\mu'(x) \leq 0$. Similarly, if $f(x)$ is increasing and $(1-x)f(x)$ is decreasing, we also have $\varphi'(x) \geq 0$ and $\mu'(x) \leq 0$. By the Chebyshev inequality, we have

$$\begin{aligned} \left(\int_a^b x^p f(x) dx \right)^2 + \left(\int_a^b (1-x)^p f(x) dx \right)^2 &\geq 2 \int_a^b x^p f(x) dx \int_a^b (1-x)^p f(x) dx \\ &\geq 2(b-a) \int_a^b [x(1-x)]^p f(x) dx. \end{aligned}$$

Since $h(x) = x(1-x)$ is concave, h attains its minimum value $h(a)$ or $h(b)$. Due to $m = \min\{a(1-a), b(1-b)\}$, we obtain

$$\begin{aligned} &\left(\int_a^b x^p f(x) dx \right)^2 + \left(\int_a^b (1-x)^p f(x) dx \right)^2 \\ &\geq 2(b-a) \int_a^b [x(1-x)]^p f(x) dx \\ &\geq 2(b-a)m^p \int_a^b f^2(x) dx \\ &\geq 2m^p \left(\int_a^b f(x) dx \right)^2. \end{aligned}$$

The proof is completed. \square

5. Conclusions

In this paper, we establish the following two important main results for the generalized Dunkel type integral inequality:

- (Generalized Dunkel integral inequality; see Theorem 4.)

Let $f(x)$ and $g(x)$ be two nonnegative continuous real-valued functions on $[a, b]$ and m be any real number. Then

$$\left(\int_a^b f(x) \cos mx dx\right) \left(\int_a^b g(x) \cos mx dx\right) + \left(\int_a^b f(x) \sin mx dx\right) \left(\int_a^b g(x) \sin mx dx\right) \leq \left(\int_a^b f(x) dx\right) \left(\int_a^b g(x) dx\right).$$

- (Dunkel (p)-type integral inequality for $p \geq 2$; see Theorem 5.)

Let $k \in \mathbb{N} \cup \{0\}$. Denote $I_k := [2k\pi, 2k\pi + \frac{\pi}{2}]$. Assume that $f(x)$ is a nonnegative continuous real-valued function on I_k . If $p \geq 2$ and $[a, b] \subseteq I_k$, then

$$\left(\int_a^b f(x) \cos x dx\right)^p + \left(\int_a^b f(x) \sin x dx\right)^p \leq \left(\int_a^b f(x) dx\right)^p.$$

As applications of our new results, some new integral inequalities are presented in Section 4.

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