symmetry

MDPI

Article

Some Results of New Subclasses for Bi-Univalent Functions Using Quasi-Subordination

Waggas Galib Atshan [1,*,†], Ibtihal Abdul Ridha Rahman [1,†] and Alina Alb Lupaş [2,†]

1 Department of Mathematics, University of Al-Qadisiyah, Diwaniyah 58001, Iraq; ma.post11@qu.edu.iq
2 Department of Mathematics and Computer Science, University of Oradea, 1 Universitatii Street, 410087 Oradea, Romania; alblupas@gmail.com
* Correspondence: waggas.galib@qu.edu.iq
† These authors contributed equally to this work.

Abstract: In this paper, we introduce new subclasses $\mathfrak{N}^{\mu,\alpha}_{\Sigma,b,c}(\lambda,\delta,\tau,\Phi)$ and $\mathfrak{K}^{\mu,\alpha}_{\Sigma,b,c}(\lambda,\delta,\eta,\Phi)$ of bi-univalent functions in the open unit disk U by using quasi-subordination conditions and determine estimates of the coefficients $|a_2|$ and $|a_3|$ for functions of these subclasses. We discuss the improved results for the associated classes involving many of the new and well-known consequences. We notice that there is symmetry in the results obtained for the new subclasses $\mathfrak{N}^{\mu,\alpha}_{\Sigma,b,c}(\lambda,\delta,\tau,\Phi)$ and $\mathfrak{K}^{\mu,\alpha}_{\Sigma,b,c}(\lambda,\delta,\eta,\Phi)$, as there is a symmetry for the estimations of the coefficients a_2 and a_3 for all the subclasses defind in our this paper.

Keywords: subordination; bi-univalent function; analytic function; quasi-subordination; hurwitz–lerch zeta function

Citation: Atshan, W.G.; Rahman, I.A.R.; Lupaş, A.A. Some Results of New Subclasses for Bi-Univalent Functions Using Quasi-Subordination. *Symmetry* **2021**, *13*, 1653. https://doi.org/10.3390/sym13091653

Academic Editor: Ioan Raşa

Received: 10 March 2021
Accepted: 1 April 2021
Published: 8 September 2021

1. Introduction

Let $\mathcal{H}$ be the class of analytic functions f defined in the open unit disk $U = \{z : |z| < 1\}$ and normalized by conditions $f(0) = 0$, $f'(0) = 1$. An analytic function $f \in \mathcal{H}$ has Taylor series expansion of the form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad z \in U. \tag{1}$$

The well-known Koebe-One Quarter Theorem [1] states that the image of the open unit disk U under each univalent function in a disk with the radius $\frac{1}{4}$. Thus, every univalent function f has an inverse f^{-1}, such that

$$f^{-1}(f(z)) = z, \quad z \in U,$$

and

$$f^{-1}(f(w)) = w, \quad |w| < r_0(f),\ r_0(f) \geq \frac{1}{4}.$$

Let Σ denote the class of all bi-univalent functions in U. Since f in Σ has the form (1), a computation shows that the inverse $g = f^{-1}$ has the following expansion

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right)w^4 + \ldots, \quad w \in U.$$

Let B be the class of all analytic and invertible univalent functions in the open unit disk, but the inverse function may not be defined on the entire disk U, for f in $\mathcal{H}$. An analytic function f is called bi-univalent in U if both f and f^{-1} are univalent in U.

The class of bi-univalent functions was introduced by Lewin [2] and proved that $|a_2| \geq 1.51$ for the function of the form (1). Subsequently, Brannan and Clunie [3] conjectured that $|a_2| \geq \sqrt{2}$. Later, Netanyahu, in [4], showed that $\max_{f\in\Sigma}|a_2| = \frac{4}{3}$. Several authors studied classes of bi-univalent analytic functions and found estimates of the coefficients estimate problem for each of the following Taylor–MacLaurin coefficients $|a_2|$ and $|a_3|$ for functions in these classes ([5–7]).

For functions $f, h \in \mathcal{H}$ of the form (1), respectively,

$$h(z) = z + \sum_{j=2}^{\infty} b_j z^j, \quad z \in U. \tag{2}$$

The convolution of the functions f and h denoted by $f(z) * h(z)$ is defined as

$$f(z) * h(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j, \quad z \in U.$$

Choi and Srivastava [8] found several interesting properties of Hurwitz–Lerch Zeta function $\phi(z, s, a)$ defined by

$$\phi(z, s, a) := \sum_{j=0}^{\infty} \frac{z^j}{(j+a)^s}, \tag{3}$$

$a\in \mathbb{C}\backslash\{0, -1, -2, \ldots\}$, $s\in \mathbb{C}$, $Re\ s > 1$ and $|z| = 1$.

In [9] Srivastava-Attiya introduced the following operator $\mathcal{D}_{\mu,b} : \mathcal{H} \longrightarrow \mathcal{H}$,

$$\mathcal{D}_{\mu,b}(z) = (1+b)^{\mu}\left[\phi(z, \mu, b) - b^{-\mu}\right],$$

which has the following form

$$\mathcal{D}_{\mu,b} f(z) = z + \sum_{j=2}^{\infty} \left(\frac{1+b}{j+b}\right)^{\mu} a_j z^j, \tag{4}$$

$b\in \mathbb{C}\backslash\{0, -1, -2, \ldots\}$, $\mu\in \mathbb{C}$, $z \in U$, $f \in \mathcal{H}$.

For $f \in \mathcal{H}$, Carlson and Shaffer [10] defined the following integral operator $\mathcal{T}_{\alpha} f(z)$ by

$$\mathcal{T}_{\alpha} f(z) = z + \sum_{j=2}^{\infty} \frac{(\alpha)_{j-1}}{(c)_{j-1}} a_j z^j. \tag{5}$$

Define the convolution (or Hadamard product) of the operators $\mathcal{D}_{\mu,b} f(z)$ and $\mathcal{T}_{\alpha} f(z)$,

$$\mathcal{N}_{\alpha,c}^{\mu,b} f(z) = \mathcal{D}_{\mu,b} f(z) * \mathcal{T}_{\alpha} f(z) = z + \sum_{j=2}^{\infty} \left(\frac{1+b}{j+b}\right)^{\mu} \left(\frac{(\alpha)_{j-1}}{(c)_{j-1}}\right) a_j z^j,$$

which can be written as

$$\mathcal{N}_{\alpha,c}^{\mu,b} f(z) = z + \sum_{j=2}^{\infty} \varphi_{j,\alpha} a_j z^j,$$

where $\varphi_{j,\alpha}= \left(\frac{1+b}{j+b}\right)^{\mu} \left(\frac{(\alpha)_{j-1}}{(c)_{j-1}}\right)$.

In the year 1970, the concept of quasi-subordination was first mentioned in [11]. For two analytic functions g and f in U, we say that the function f is quasi-subordinate to g in U, if there are analytic functions ϕ and F, with $|\phi(z)| \leq 1$, $F(0) = 0$ and $|F(z)| < 1$, such that $f(z) = \phi(z) g(F(z))$, and denote this quasi-subordination by [12], as follows

$$f(z) \prec_q g(z), \quad z \in U. \tag{6}$$

Note that if $\phi(z) = 1$, then $f(z) = g(F(z))$, hence $f(z) \prec g(z)$ in U ([13]). Furthermore, if $F(z) = z$, then $f(z) = \phi(z)g(z)$ and this case f is majorized by g, written as $f(z) \ll g(z)$ in U.

Ma and Minda [14], using the method of subordination of defined and studied classes $S^*(\Phi)$ and $G^*(\Phi)$ of starlike functions. See also [15,16]

$$S^*(\Phi) = \left\{ f \in \mathcal{H} : \frac{zf'(z)}{f(z)} \prec \Phi(z), \quad z \in U \right\},$$

and

$$G^*(\Phi) = \left\{ f \in \mathcal{H} : 1 + \frac{zf''(z)}{f'(z)} \prec \Phi(z), \quad z \in U \right\},$$

where

$$\phi(z) = K_0 + K_1 z + K_2 z^2 + \dots, \quad z \in U. \tag{7}$$

Now, consider

$$\Phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots, \quad B_1 > 0, \tag{8}$$

an analytic and univalent function with a positive real part in U, symmetric with respect to the real axis and starlike with respect to $\Phi(0) = 1$ and $\Phi'(0) > 0$.

By ${S^*}_\Sigma(\Phi)$ and ${G^*}_\Sigma(\Phi)$ we denote the bi-starlike of Ma-Minda and bi-convex of the Ma–Minda type, respectively ([17,18]).

In [17,19] Brannan and Taha get initial coefficient bounds for subclasses of bi-univalent functions. Later, Srivastava et al. [20] introduced and investigated subclasses of bi-univalent functions and get bounds for the initial coefficients. Also, Ali et al. [21] get the coefficient bounds for bi-univalent Ma-Minda starlike and convex functions. Some more important results on coefficient inequalities can be found in [12,21–23].

Here, we discuss the improved results for the associated classes involving many of the new-known consequences.

We need the following Lemma to achieve the results.

Lemma 1 ([24])**.** *If $p \in P$, then $|p_i| \leq 2$ for each i, where P is the family of all analytic functions p, for which $Re\{p(z)\} > 0$, $z \in U$,where*

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots, \ z \in U.$$

2. The Subclass $\mathfrak{R}^{\mu,\alpha}_{\Sigma,B,C}(\lambda, \delta, \tau, \Phi)$

Definition 1. *A function $f \in \mathcal{H}$ is said to be in the class $\mathfrak{R}^{\mu,\alpha}_{\Sigma,b,c}(\lambda, \delta, \tau, \Phi)$, $0 \leq \lambda \leq 1$, $0 \leq \delta \leq 1$, and $\tau \in \mathbb{C} \backslash \{0\}$, if the following quasi-subordinations hold*

$$\frac{1}{\tau}\left[\left\{ \frac{z\left(\mathcal{N}^{\mu,b}_{\alpha,c} f(z)\right)' + z^2\left(\mathcal{N}^{\mu,b}_{\alpha,c} f(z)\right)''}{(1-\lambda)z + \lambda z\left(\mathcal{N}^{\mu,b}_{\alpha,c} f(z)\right)'} + \delta z\left(\mathcal{N}^{\mu,b}_{\alpha,c} f(z)\right)'' \right\} - 1\right] \prec_q \Phi(z) - 1, \tag{9}$$

$$\frac{1}{\tau}\left[\left\{ \frac{w\left(\mathcal{N}^{\mu,b}_{\alpha,c} g(w)\right)' + w^2\left(\mathcal{N}^{\mu,b}_{\alpha,c} g(w)\right)''}{(1-\lambda)w + \lambda w\left(\mathcal{N}^{\mu,b}_{\alpha,c} g(w)\right)'} + \delta w\left(\mathcal{N}^{\mu,b}_{\alpha,c} g(w)\right)'' \right\} - 1\right] \prec_q \Phi(w) - 1, \tag{10}$$

where g is the inverse function of f and $z, w \in U$.

For special values of parameters $\lambda, \delta, \tau, \mu$, we obtain new and well-known classes.

Remark 1. *For $\delta = 0$ and $\tau \in \mathbb{C}\backslash\{0\}$, $0 \le \lambda \le 1$, $0 \le \delta \le 1$, a function $f \in \Sigma$ defined by (1) is said to be in the class $\mathfrak{R}^{\mu,\alpha}_{\Sigma,b,c}(\lambda, \tau, \Phi)$, if the following quasi-subordination condition are satisfied*

$$\frac{1}{\tau}\left[\left\{\frac{z\left(\mathcal{N}^{\mu,b}_{\alpha,c} f(z)\right)' + z^2\left(\mathcal{N}^{\mu,b}_{\alpha,c} f(z)\right)''}{(1-\lambda)z + \lambda z\left(\mathcal{N}^{\mu,b}_{\alpha,c} f(z)\right)'}\right\} - 1\right] \prec_q \Phi(z) - 1,$$

and

$$\frac{1}{\tau}\left[\left\{\frac{w\left(\mathcal{N}^{\mu,b}_{\alpha,c} g(w)\right)' + w^2\left(\mathcal{N}^{\mu,b}_{\alpha,c} g(w)\right)''}{(1-\lambda)w + \lambda w\left(\mathcal{N}^{\mu,b}_{\alpha,c} g(w)\right)'}\right\} - 1\right] \prec_q \Phi(w) - 1,$$

where g is the inverse function of f and $z, w \in U$.

Remark 2. *For $\mu = 0$, $b = 0$, $c = 0$, $\alpha = 0$, $\tau \in \mathbb{C}\backslash\{0\}$, a function $f \in \Sigma$ defined by (1) is said to be in the class $\mathfrak{R}_{\Sigma}(\lambda, \delta, \tau, \Phi)$, if the following quasi-subordination conditions are satisfied*

$$\frac{1}{\tau}\left[\left\{\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} + \delta z f''(z)\right\} - 1\right] \prec_q \Phi(z) - 1,$$

and

$$\frac{1}{\tau}\left[\left\{\frac{wg'(w) + w^2 g''(w)}{(1-\lambda)w + \lambda w g'(w)} + \delta w g''(w)\right\} - 1\right] \prec_q \Phi(w) - 1,$$

where g is the inverse function of f and $z, w \in U$.

Next, we find estimates for the coefficients $|a_2|$ and $|a_3|$ for the functions in class $\mathfrak{R}^{\mu,\alpha}_{\Sigma,b,c}(\lambda, \delta, \tau, \Phi)$.

Theorem 1. *If f given by (1) belongs to the subclass $\mathfrak{R}^{\mu,\alpha}_{\Sigma,b,c}(\lambda, \delta, \tau, \Phi)$, then*

$$|a_2| \le \frac{\tau|h_0|B_1\sqrt{B_1}}{3\tau(3-\lambda+6\delta)h_0 B_1^2 \varphi_{3,\alpha} - 4\left[\lambda\tau(2-\lambda)h_0 B_1^2 + (2-\lambda+\delta)^2(B_2 - B_1)\right]\varphi_{2,\alpha}^2}, \tag{11}$$

and

$$|a_3| \le \frac{\tau(|h_0| + |h_1|)B_1}{6(3-\lambda+6\delta)\varphi_{3,\alpha}} + \frac{\tau^2 B_1^2 h_0{}^2}{4(2-\lambda+\delta)^2 \varphi_{2,\alpha}^2}, \quad B_1 > 1, \tag{12}$$

where

$$\varphi_{2,\alpha} = \left(\frac{1+b}{2+b}\right)^{\mu}\left(\frac{(\alpha)_{2-1}}{(c)_{2-1}}\right), \text{ and } \varphi_{3,\alpha} = \left(\frac{1+b}{3+b}\right)^{\mu}\left(\frac{(\alpha)_{3-1}}{(c)_{3-1}}\right).$$

Proof. Since $f \in \mathfrak{R}^{\mu,\alpha}_{\Sigma,b,c}(\lambda, \delta, \tau, \Phi)$, then there exist analytic functions ϕ, F in U and ϕ, $F : U \to U$, with $|\phi(z)| \le 1$, such that $\phi(0) = F(0) = 0$ and $|F(z)| < 1$, satisfied

$$\frac{1}{\tau}\left[\left\{\frac{z\left(\mathcal{N}^{\mu,b}_{\alpha,c} f(z)\right)' + z^2\left(\mathcal{N}^{\mu,b}_{\alpha,c} f(z)\right)''}{(1-\lambda)z + \lambda z\left(\mathcal{N}^{\mu,b}_{\alpha,c} f(z)\right)'} + \delta z\left(\mathcal{N}^{\mu,b}_{\alpha,c} f(z)\right)''\right\} - 1\right] = \phi(z)(\Phi(F(z)) - 1, \tag{13}$$

and

$$\frac{1}{\tau}\left[\left\{\frac{w\left(\mathcal{N}^{\mu,b}_{\alpha,c} g(w)\right)' + w^2\left(\mathcal{N}^{\mu,b}_{\alpha,c} g(w)\right)''}{(1-\lambda)w + \lambda w\left(\mathcal{N}^{\mu,b}_{\alpha,c} g(w)\right)'} + \delta w\left(\mathcal{N}^{\mu,b}_{\alpha,c} g(w)\right)''\right\} - 1\right] = \phi(w)(\Phi(F(w)) - 1, \tag{14}$$

where g is the inverse function of f and $z, w \in U$.

Define the functions u and v by

$$u(z)=\frac{1+F(z)}{1-F(z)}=1+u_1z+u_2z^2+u_3z^3\ldots, \tag{15}$$

and

$$v(w)=\frac{1+\phi(w)}{1-\phi(w)}=1+v_1w+v_2w^2+v_3w^3+\ldots, \tag{16}$$

or equivalently,

$$F(z)=\frac{u(z)-1}{u(z)+1}=\frac{1}{2}[u_1z+\left(u_2-\frac{u_1^2}{2}\right)z^2+\ldots, \tag{17}$$

and

$$\phi(w)=\frac{v(w)-1}{v(w)+1}=\frac{1}{2}[v_1w+\left(v_2-\frac{v_1^2}{2}\right)w^2+\ldots. \tag{18}$$

Using (17) and (18) in (13) and (14), we obtain

$$\frac{1}{\tau}\left[\left\{\frac{z\left(\mathcal{N}_{\alpha,c}^{\mu,b}f(z)\right)'+z^2\left(\mathcal{N}_{\alpha,c}^{\mu,b}f(z)\right)''}{(1-\lambda)z+\lambda z\left(\mathcal{N}_{\alpha,c}^{\mu,b}f(z)\right)'}+\delta z\left(\mathcal{N}_{\alpha,c}^{\mu,b}f(z)\right)''\right\}-1\right]$$
$$=\phi(z)\left(\Phi\left(\frac{u(z)-1}{u(z)+1}\right)\right)-1, \tag{19}$$

and

$$\frac{1}{\tau}\left[\left\{\frac{w\left(\mathcal{N}_{\alpha,c}^{\mu,b}g(w)\right)'+w^2\left(\mathcal{N}_{\alpha,c}^{\mu,b}g(w)\right)''}{(1-\lambda)w+\lambda w\left(\mathcal{N}_{\alpha,c}^{\mu,b}g(w)\right)'}+\delta w\left(\mathcal{N}_{\alpha,c}^{\mu,b}g(w)\right)''\right\}-1\right]$$
$$=\phi(w)\left(\Phi\left(\frac{v(w)-1}{v(w)+1}\right)\right)-1. \tag{20}$$

We can write

$$\phi(z)\left(\Phi\left(\frac{u(z)-1}{u(z)+1}\right)\right)=\frac{1}{2}h_0B_1u_1z+\left\{\frac{1}{2}h_1B_1u_1+\frac{1}{2}h_0B_1\left(u_2-\frac{u_1^2}{2}\right)+\frac{1}{4}h_0B_2u_1^2\right\}z^2+\ldots, \tag{21}$$

and

$$\phi(w)\left(\Phi\left(\frac{v(w)-1}{v(w)+1}\right)\right)=\frac{1}{2}h_0B_1v_1w+\left\{\frac{1}{2}h_1B_1v_1+\frac{1}{2}h_0B_1\left(v_2-\frac{v_1^2}{2}\right)+\frac{1}{4}h_0B_2v_1^2\right\}w^2+\ldots. \tag{22}$$

Since

$$\frac{1}{\tau}\left[\left\{\frac{z\left(\mathcal{N}_{\alpha,c}^{\mu,b}f(z)\right)'+z^2\left(\mathcal{N}_{\alpha,c}^{\mu,b}f(z)\right)''}{(1-\lambda)z+\lambda z\left(\mathcal{N}_{\alpha,c}^{\mu,b}f(z)\right)'}+\delta z\left(\mathcal{N}_{\alpha,c}^{\mu,b}f(z)\right)''\right\}-1\right]=$$

$$\frac{1}{\tau}\left[2(2-\lambda+\delta)\varphi_{2,\alpha}a_2z+\left(3(3-\lambda+6\delta)\varphi_{3,\alpha}a_3-4\lambda(2-\lambda)\varphi_{2,\alpha}^2a_2^2\right)z^2+\ldots\right], \tag{23}$$

and

$$\frac{1}{\tau}\left[\left\{\frac{w\left(\mathcal{N}_{\alpha,c}^{\mu,b}g(w)\right)'+w^2\left(\mathcal{N}_{\alpha,c}^{\mu,b}g(w)\right)''}{(1-\lambda)w+\lambda w\left(\mathcal{N}_{\alpha,c}^{\mu,b}g(w)\right)'}+\delta w\left(\mathcal{N}_{\alpha,c}^{\mu,b}g(w)\right)''\right\}-1\right]=$$

$$\frac{1}{\tau}\Big[-2(2-\lambda+\delta)\varphi_{2,\alpha}a_2w+\Big(3(3-\lambda+6\delta)\varphi_{3,\alpha}\Big(2a_2^2-a_3\Big)-4\lambda(2-\lambda)\varphi_{2,\alpha}^2a_2^2\Big)w^2+\dots\Big], \tag{24}$$

putting (21) and (23) in (19) and putting (22) and (24) in (20) and equating coefficients in both sides, we get

$$\frac{2}{\tau}(2-\lambda+\delta)\varphi_{2,\alpha}a_2=\frac{1}{2}h_0B_1u_1, \tag{25}$$

$$\frac{1}{\tau}\Big[3(3-\lambda+6\delta)\varphi_{3,\alpha}a_3-4\lambda(2-\lambda)\varphi_{2,\alpha}^2a_2^2\Big]=\frac{1}{2}h_1B_1u_1+\frac{1}{2}h_0B_1\left(u_2-\frac{u_1^2}{2}\right)+\frac{1}{4}h_0B_2u_1^2, \tag{26}$$

and

$$\frac{-2}{\tau}(2-\lambda+\delta)\varphi_{2,\alpha}a_2=\frac{1}{2}h_0B_1v_1, \tag{27}$$

$$\begin{aligned}\frac{1}{\tau}\Big[-2(2-\lambda+\delta)\varphi_{2,\alpha}a_2+\Big(3(3-\lambda+6\delta)\varphi_{3,\alpha}\Big(2a_2^2-a_3\Big)-4\lambda(2-\lambda)\varphi_{2,\alpha}^2a_2^2\Big)\Big]\\=\frac{1}{2}h_1B_1v_1+\frac{1}{2}h_0B_1\left(v_2-\frac{v_1^2}{2}\right)+\frac{1}{4}h_0B_2v_1^2.\end{aligned} \tag{28}$$

From (25) and (27), we obtain

$$a_2=\frac{\tau h_0B_1u_1}{4(2-\lambda+\delta)\varphi_{2,\alpha}}=-\frac{\tau h_0B_1v_1}{4(2-\lambda+\delta)\varphi_{2,\alpha}}. \tag{29}$$

It follows that

$$u_1=-v_1, \tag{30}$$

and

$$32(2-\lambda+\delta)^2\varphi_{2,\alpha}^2a_2^2=\tau^2{h_0}^2{B_1}^2(u_1^2+v_1^2). \tag{31}$$

Adding (26) and (28), by using (30) and (31) ,we have

$$\begin{aligned}8\Big[3\tau h_0B_1^2(3-\lambda+6\delta)\varphi_{3,\alpha}a_2^2-4\lambda\tau h_0B_1^2(2-\lambda)\varphi_{2,\alpha}^2\Big]a_2^2=\\2\tau^2h_0^2B_1^3(u_2+v_2)+32(2-\lambda+\delta)^2(B_2-B_1)\varphi_{2,\alpha}^2a_2^2,\end{aligned} \tag{32}$$

which implies

$$a_2^2=\frac{2\tau^2h_0^2B_1^3(u_2+v_2)}{8\Big\{3\tau(3-\lambda+6\delta)h_0B_1^2\varphi_{3,\alpha}-4\Big[\lambda\tau(2-\lambda)h_0B_1^2+(2-\lambda+\delta)^2(B_2-B_1)\Big]\varphi_{2,\alpha}^2\Big\}}. \tag{33}$$

Applying Lemma 1 in (33), we get (11).

Now, in order to find the bound of the coefficient $|a_3|$, by subtracting (26) and (28) we get,

$$\frac{4}{\tau}\Big[6(3-\lambda+6\delta)\varphi_{3,\alpha}a_3-6(3-\lambda+6\delta)\varphi_{3,\alpha}a_2^2\Big]=2h_1B_1u_1+h_0B_1(u_2-v_2). \tag{34}$$

By substituting (28) from (26), further computation using (30) and (31), we obtain

$$a_3=\frac{2\tau h_1B_1u_1}{24(3-\lambda+6\delta)\varphi_{3,\alpha}}+\frac{\tau h_0B_1(u_2-v_2)}{24(3-\lambda+6\delta)\varphi_{3,\alpha}}+\frac{\tau^2h_0^2B_1^2(u_1^2+v_1^2)}{32(2-\lambda+\delta)^2\varphi_{2,\alpha}^2}. \tag{35}$$

Applying Lemma 1 in (35), we get (12). The proof is complete. □

Taking $\delta=0$ in Theorem 1, we obtain the following corollary.

Corollary 1. *Let f given by (1) belongs to the class $\mathfrak{R}^{\mu,\alpha}_{\Sigma,b,c}(\lambda,0,\tau,\Phi)$. Then*

$$|a_2| \leq \frac{\tau|h_0|B_1\sqrt{B_1}}{3\tau(3-\lambda)h_0B_1^2\varphi_{3,\alpha}-4(2-\lambda)\left[\lambda\tau h_0B_1^2+(2-\lambda)(B_2-B_1)\right]\varphi_{2,\alpha}^2},$$

and

$$|a_3| \leq \frac{\tau(|h_0|+|h_1|)B_1}{6(3-\lambda)\varphi_{3,\alpha}}+\frac{\tau^2B_1^2h_0{}^2}{4(2-\lambda)^2\varphi_{2,\alpha}^2}, \quad B_1>1.$$

For $\lambda=1$, we obtain

Corollary 2. *Let f, given by (1,) belong to the class $\mathfrak{R}^{\mu,\alpha}_{\Sigma,b,c}(1,\delta,\tau,\Phi)$, and $\tau\in\mathbb{C}\backslash\{0\}$. Then*

$$|a_2| \leq \frac{\tau|h_0|B_1\sqrt{B_1}}{6(1+3\delta)\tau h_0B_1^2\varphi_{3,\alpha}-\left[4\tau h_0B_1^2+4(1+\delta)^2(B_2-B_1)\right]\varphi_{2,\alpha}^2},$$

and

$$|a_3| \leq \frac{\tau(|h_0|+|h_1|)B_1}{12(1+3\delta)\varphi_{3,\alpha}}+\frac{\tau^2B_1^2h_0{}^2}{4(1+\delta)^2\varphi_{2,\alpha}^2}, \quad B_1>1.$$

Corollary 3. *Let f given by (1) belong to the class $\mathfrak{R}_\Sigma(\lambda,\delta,\tau,\Phi)$, and $\tau\in\mathbb{C}\backslash\{0\}$, where $0\leq\lambda\leq1$, $0\leq\delta\leq1$. Then*

$$|a_2| \leq \frac{\tau|h_0|B_1\sqrt{B_1}}{3\tau(3-\lambda+6\delta)h_0B_1^2-4\lambda\tau(2-\lambda)h_0B_1^2-4(2-\lambda+\delta)^2(B_2-B_1)},$$

and

$$|a_3| \leq \frac{\tau(|h_0|+|h_1|)B_1}{6(3-\lambda+6\delta)}+\frac{\tau^2B_1^2h_0{}^2}{4(2-\lambda+\delta)^2}, \quad B_1>1.$$

3. The Subclass $\mathfrak{K}^{\mu,\alpha}_{\Sigma,b,c}(\lambda,\delta,\eta,\Phi)$

Definition 2. *A functions $f\in\mathcal{H}$ is said to be in the class $\mathfrak{K}^{\mu,\alpha}_{\Sigma,b,c}(\lambda,\delta,\eta,\Phi)$, $\eta\geq1$, $\lambda\geq0$, and $\delta\in\mathbb{C}\backslash\{0\}$, if it satisfies the following quasi-subordination*

$$\frac{1}{\delta}\left[\left\{(1-\eta)\frac{z\left(\mathcal{N}^{\mu,b}_{\alpha,c}f(z)\right)'}{\mathcal{N}^{\mu,b}_{\alpha,c}f(z)}+\eta\left(\mathcal{N}^{\mu,b}_{\alpha,c}f(z)\right)'+\lambda z\left(\mathcal{N}^{\mu,b}_{\alpha,c}f(z)\right)''\right\}-1\right]\prec_q\Phi(z)-1, \quad (36)$$

and

$$\frac{1}{\delta}\left[\left\{(1-\eta)\frac{w\left(\mathcal{N}^{\mu,b}_{\alpha,c}g(w)\right)'}{\mathcal{N}^{\mu,b}_{\alpha,c}g(w)}+\eta\left(\mathcal{N}^{\mu,b}_{\alpha,c}g(w)\right)'+\lambda w\left(\mathcal{N}^{\mu,b}_{\alpha,c}g(w)\right)''\right\}-1\right]\prec_q\Phi(w)-1, \quad (37)$$

where g is the inverse function of f and $z,w\in U$.

For special values of parameters η,δ,λ,μ, we obtain new and well-known classes.

Remark 3. *For $\eta=1$ and $\delta\in\mathbb{C}\backslash\{0\}$, $\lambda\geq0$, a function $f\in\Sigma$ defined in (1) is said to be in the class $\mathfrak{K}^{\mu,\alpha}_{\Sigma,b,c}(\lambda,\delta,\eta,\Phi)$, if the following quasi-subordination conditions are satisfied:*

$$\frac{1}{\delta}\left[\left\{\left(\mathcal{N}^{\mu,b}_{\alpha,c}f(z)\right)'+\lambda z\left(\mathcal{N}^{\mu,b}_{\alpha,c}f(z)\right)''\right\}-1\right]\prec_q\Phi(z)-1,$$

and

$$\frac{1}{\delta}\left[\left\{\left(\mathcal{N}_{\alpha,c}^{\mu,b}g(w)\right)' + \lambda w\left(\mathcal{N}_{\alpha,c}^{\mu,b}g(w)\right)''\right\} - 1\right] \prec_q \Phi(w) - 1,$$

where g is the inverse function of f and $z, w \in U$.

Remark 4. *For $\mu = 0$, $b = 0$, $c = 0$, $\alpha = 0$ and $\delta \in \mathbb{C}\backslash\{0\}$, $\eta \geq 1$, $\lambda \geq 0$, a function $f \in \Sigma$ defined in (1) is said to be in the class $\mathfrak{K}_{\Sigma}(\lambda, \delta, \eta, \Phi)$, if the following quasi-subordination conditions are satisfied:*

$$\frac{1}{\delta}\left[\left\{(1-\eta)\frac{zf'(z)}{f(z)} + \eta f'(z) + \lambda z f''(z)\right\} - 1\right] \prec_q \Phi(z) - 1,$$

and

$$\frac{1}{\delta}\left[\left\{(1-\eta)\frac{wg'(w)}{g(w)} + \eta g'(w) + \lambda w g''(w)\right\} - 1\right] \prec_q \Phi(w) - 1,$$

where g is the inverse function of f and $z, w \in U$.

Remark 5. *For $\alpha = 0$ and $\delta \in \mathbb{C}\backslash\{0\}$, $\eta \geq 1$, $\lambda \geq 0$, a function $f \in \Sigma$ defined in (1) is said to be in the class $\mathfrak{K}_{\Sigma,b,c}^{\mu,\alpha}(\lambda, \delta, \eta, \Phi)$, if the following quasi-subordination conditions are satisfied:*

$$\frac{1}{\delta}\left[\left\{(1-\eta)\frac{z\left(\mathcal{N}_{\alpha,c}^{\mu,b}f(z)\right)'}{\mathcal{N}_{\alpha,c}^{\mu,b}f(z)} + \eta\left(\mathcal{N}_{\alpha,c}^{\mu,b}f(z)\right)'\right\} - 1\right] \prec_q \Phi(z) - 1,$$

and

$$\frac{1}{\delta}\left[\left\{(1-\eta)\frac{w\left(\mathcal{N}_{\alpha,c}^{\mu,b}g(w)\right)'}{\mathcal{N}_{\alpha,c}^{\mu,b}g(w)} + \eta\left(\mathcal{N}_{\alpha,c}^{\mu,b}g(w)\right)'\right\} - 1\right] \prec_q \Phi(w) - 1,$$

where g is the inverse function of f and $z, w \in U$.

Next, we find estimates of the coefficients $|a_2|$ and $|a_3|$ for the functions in class $\mathfrak{K}_{\Sigma,b,c}^{\mu,\alpha}(\lambda, \delta, \eta, \Phi)$.

Theorem 2. *If f given by (1) belong to the subclass $\mathfrak{K}_{\Sigma,b,c}^{\mu,\alpha}(\lambda, \delta, \eta, \Phi)$, then*

$$|a_2| \leq \min\left\{\frac{\delta|h_0|B_1}{(1+\eta+2\lambda)\varphi_{2,\alpha}}, \sqrt{\frac{\delta|h_0|(B_1+|B_2-B_1|)}{(2+\eta+6\lambda)\varphi_{3,\alpha}-(1-\eta)\varphi_{2,\alpha}^2}}\right\}, \tag{38}$$

and

$$|a_3| \leq \min\left\{\frac{\delta(h_1B_1+h_0B_1)}{2(2+\eta+6\lambda)\varphi_{3,\alpha}} + \frac{\delta|h_0|(B_1+|B_2-B_1|)}{(2+\eta+6\lambda)\varphi_{3,\alpha}-(1-\eta)\varphi_{2,\alpha}^2}, \right. \tag{39}$$

$$\left. \frac{\delta(h_1B_1+h_0B_1)}{2(2+\eta+6\lambda)\varphi_{3,\alpha}} + \frac{\delta^2h_0^2B_1^2}{(1+\eta+2\lambda)^2\varphi_{2,\alpha}^2}\right\},\ B_1 > 1.$$

Proof. If $f \in \mathfrak{K}_{\Sigma,b,c}^{\mu,\alpha}(\lambda, \delta, \eta, \Phi)$ and $g = f^{-1}$, then there are analytic functions ϕ, F in U and $\phi, F : U \to U$, with $|\phi(z)| \leq 1$, such that $\phi(0) = F(0) = 0$ and $|F(z)| < 1$, satisfied

$$\frac{1}{\delta}\left[\left\{(1-\eta)\frac{zf'(z)}{f(z)} + \eta f'(z) + \lambda z f''(z)\right\} - 1\right] = \phi(z)(\Phi(F(z)) - 1, \tag{40}$$

and

$$\frac{1}{\delta}\left[\left\{(1-\eta)\frac{wg'(w)}{g(w)} + \eta g'(w) + \lambda w g''(w)\right\} - 1\right] = \phi(w)(\Phi(F(w)) - 1. \tag{41}$$

Define the function $u(z)$ and $v(w)$ by (15) and (16) respectively.
Proceeding similarly as in Theorem 1, we obtain

$$\frac{1}{\delta}\left[\left\{(1-\eta)\frac{zf'(z)}{f(z)}+\eta f'(z)+\lambda zf''(z)\right\}-1\right]=\phi(z)\left(\Phi\left(\frac{u(z)-1}{u(z)+1}\right)\right)-1, \tag{42}$$

and

$$\frac{1}{\delta}\left[\left\{(1-\eta)\frac{wg'(w)}{g(w)}+\eta g'(w)+\lambda wg''(w)\right\}-1\right]=\phi(w)\left(\Phi\left(\frac{v(w)-1}{v(w)+1}\right)\right)-1, \tag{43}$$

since

$$\frac{1}{\delta}\left[\left\{(1-\eta)\frac{zf'(z)}{f(z)}+\eta f'(z)'+\lambda zf''(z)\right\}-1\right]=$$
$$\frac{1}{\delta}\left[(1+\eta+2\lambda)\varphi_{2,\alpha}a_2z+(2+\eta+6\lambda)a_3\varphi_{3,\alpha}z^2-(1-\eta)a_2^2\varphi_{2,\alpha}^2z^2\right], \tag{44}$$

and

$$\frac{1}{\delta}\left[\left\{(1-\eta)\frac{wg'(w)}{g(w)}+\eta g'(w)+\lambda wg''(w)\right\}-1\right]=$$
$$\frac{1}{\delta}\left[-(1+\eta+2\lambda)\varphi_{2,\alpha}a_2w+(2+\eta+6\lambda\alpha)\left(2a_2^2-a_3\right)\varphi_{3,\alpha}w^2-(1-\eta)a_2^2\varphi_{2,\alpha}^2w^2\right]. \tag{45}$$

Comparing the coefficients of (44) with (21) and (45) with (22), then we have

$$\frac{1}{\delta}(1+\eta+2\lambda)\varphi_{2,\alpha}a_2=\frac{1}{2}h_0B_1u_1, \tag{46}$$

$$\frac{1}{\delta}\left[(2+\eta+6\lambda)\varphi_{3,\alpha}a_3-(1-\eta)\varphi_{2,\alpha}^2a_2^2\right]=\frac{1}{2}h_1B_1u_1+\frac{1}{2}h_0B_1\left(u_2-\frac{u_1^2}{2}\right)+\frac{1}{4}h_0B_2u_1^2, \tag{47}$$

and

$$\frac{1}{\delta}(-(1+\eta+2\lambda)\varphi_{2,\alpha}a_2)=\frac{1}{2}h_0B_1v_1, \tag{48}$$

$$\frac{1}{\delta}\left[(2+\eta+6\lambda)\left(2a_2^2-a_3\right)\varphi_{3,\alpha}-(1-\eta)\varphi_{2,\alpha}^2a_2^2\right]=\frac{1}{2}h_1B_1v_1+\frac{1}{2}h_0B_1\left(v_2-\frac{v_1^2}{2}\right)+\frac{1}{4}h_0B_2v_1^2. \tag{49}$$

From (46) and (48), we find that

$$u_1=-v_1, \tag{50}$$

and

$$8(1+\eta+2\lambda)^2\varphi_{2,\alpha}^2a_2^2=\delta^2h_0^2B_1^2\left(u_1^2+v_1^2\right). \tag{51}$$

Adding (47) and (49), we obtain

$$\frac{8}{\delta}\left[(2+\eta+6\lambda)\varphi_{3,\alpha}-(1-\eta)\varphi_{2,\alpha}^2\right]a_2^2=2h_0B_1(u_2+v_2)+h_0\left(B_2-B_1\left(u_1^2+v_1^2\right)\right), \tag{52}$$

which implies that

$$a_2^2=\frac{2\delta h_0B_1(u_2+v_2)+\delta h_0(B_2-B_1(u_1^2+v_1^2))}{8\left[(2+\eta+6\lambda)\varphi_{3,\alpha}-(1-\eta)\varphi_{2,\alpha}^2\right]}. \tag{53}$$

Applying Lemma 1 for the coefficients u_1, u_2, v_1 and v_2, it follows from (51) and (53),

$$|a_2| \leq \frac{\delta h_0 B_1}{(1+\eta+2\lambda)\varphi_{2,\alpha}}, \quad \text{and} \quad |a_2| \leq \sqrt{\frac{\delta|h_0|(B_1+(B_2-B_1))}{\left[(2+\eta+6\lambda)\varphi_{3,\alpha}-(1-\eta)\varphi_{2,\alpha}^2\right]}},$$

which yields the desired estimate on $|a_2|$ as asserted in (38).

Now, to find the bound of the coefficient $|a_3|$, by subtracting relations (47) and (49), we get

$$\frac{8}{\delta}\left[(2+\eta+6\lambda)\left(a_3-a_2^2\right)\varphi_{3,\alpha}\right] = 2h_0B_1u_1 + h_0B_1(u_2-v_2). \tag{54}$$

Upon substituting the value of a_2^2 from (51), (53) and putting (54) respectively, it follows that

$$|a_3| \leq \frac{\delta(2h_0B_1u_1+h_0B_1(u_2-v_2))}{8(2+\eta+6\lambda)\varphi_{3,\alpha}} + \frac{\delta^2h_0^2B_1^2(u_1^2+v_1^2)}{8(1+\eta+2\lambda)^2\varphi_{2,\alpha}^2}, \tag{55}$$

and

$$|a_3| \leq \frac{\delta(2h_0B_1u_1+h_0B_1(u_2-v_2))}{8(2+\eta+6\lambda)\varphi_{3,\alpha}} + \frac{2\delta h_0B_1(u_2+v_2)+\delta h_0\left(B_2-B_1\left(u_1^2+v_1^2\right)\right)}{8\left[(2+\eta+6\lambda)\varphi_{3,\alpha}-(1-\eta)\varphi_{2,\alpha}^2\right]}. \tag{56}$$

Applying Lemma 1 for the coefficients u_1, u_2, v_1 and v_2, we get

$$|a_3| \leq \frac{\delta(h_1B_1+h_0B_1)}{2(2+\eta+6\lambda)\varphi_{3,\alpha}} + \frac{\delta^2h_0^2B_1^2}{(1+\eta+2\lambda)^2\varphi_{2,\alpha}^2}, \tag{57}$$

and

$$|a_3| \leq \frac{\delta(h_1B_1+h_0B_1)}{2(2+\eta+6\lambda)\varphi_{3,\alpha}} + \frac{\delta|h_0|(B_1+|B_2-B_1|)}{(2+\eta+6\lambda)\varphi_{3,\alpha}-(1-\eta)\varphi_{2,\alpha}^2}, \tag{58}$$

which yields the desired estimate on $|a_3|$, as asserted in (39).

This completes the proof of Theorem 2. □

Taking $\eta = 1$ in Theorem 2 we obtain the following corollary.

Corollary 4. *Let f given by (1) belongs to the class $\mathfrak{K}_{\Sigma,b,c}^{\mu,\alpha}(\lambda,\delta,1,\Phi)$. Then*

$$|a_2| \leq \min\left\{\frac{\delta|h_0|B_1}{2(1+\lambda)\varphi_{2,\alpha}}, \sqrt{\frac{\delta|h_0|(B_1+|B_2-B_1|)}{3(1+2\lambda)\varphi_{3,\alpha}}}\right\},$$

and

$$|a_3| \leq \min\left\{\frac{\delta(h_1B_1+h_0B_1)}{6(1+2\lambda)\varphi_{3,\alpha}} + \frac{\delta|h_0|(B_1+|B_2-B_1|)}{3(1+2\lambda)\varphi_{3,\alpha}}, \frac{\delta(h_1B+h_0B_1)}{6(1+2\lambda)\varphi_{3,\alpha}} + \frac{\delta^2h_0^2B_1^2}{(2+2\alpha)^2\varphi_{2,\alpha}^2}\right\},$$

$B_1 > 1$.

Corollary 5. *Let f given by (1) belongs to the class $\mathfrak{K}_{\Sigma}(\lambda,\delta,\eta,\Phi)$, where $\delta \in \mathbb{C}\backslash\{0\}$, $\eta \geq 1$, $\lambda \geq 0$. Then*

$$|a_2| \leq \min\left\{\frac{\delta|h_0|B_1}{(1+\eta+2\lambda)}, \sqrt{\frac{\delta|h_0|(B_1+|B_2-B_1|)}{(2+\eta+6\lambda)-(1-\eta)}}\right\},$$

and

$$|a_3| \le \min\left\{\frac{\delta(h_1B_1+h_0B_1)}{2(2+\eta+6\lambda)}+\frac{\delta|h_0|(B_1+|B_2-B_1|)}{(2+\eta+6\lambda)-(1-\eta)}, \frac{\delta(h_1B_1+h_0B_1)}{2(2+\eta+6\lambda\alpha)}+\frac{\delta^2h_0^2B_1^2}{(1+\eta+2\lambda)^2}\right\},$$

$B_1 > 1$.

Corollary 6. *Let f given by (1) belongs to the class $\mathfrak{K}^{\mu,\alpha}_{\Sigma,b,c}(0,\delta,\eta,\Phi)$. Then*

$$|a_2| \le \min\left\{\frac{\delta|h_0|B_1}{(1+\eta)\varphi_{2,\alpha}}, \sqrt{\frac{\delta|h_0|(B_1+|B_2-B_1|)}{(2+\eta)\varphi_{3,\alpha}-(1-\eta)\varphi_{2,\alpha}^2}}\right\},$$

and

$$|a_3| \le \min\left\{\frac{\delta(h_1B_1+h_0B_1)}{2(2+\eta)\varphi_{3,\alpha}}+\frac{\delta|h_0|(B_1+|B_2-B_1|)}{(2+\eta)\varphi_{3,\alpha}-(1-\eta)\varphi_{2,\alpha}^2},\right.$$
$$\left.\frac{\delta(h_1B_1+h_0B_1)}{2(2+\eta)\varphi_{3,\alpha}}+\frac{\delta^2h_0^2B_1^2}{(1+\eta)^2\varphi_{2,\alpha}^2}\right\},\ B_1>1.$$

4. Discussion

We introduce new subclasses $\mathfrak{R}^{\mu,\alpha}_{\Sigma,b,c}(\lambda,\delta,\tau,\Phi)$ and $\mathfrak{K}_{\Sigma}(\lambda,\delta,\eta,\Phi)$ of bi-univalent functions in the open unit disk U by using quasi-subordination conditions and determine estimates of the coefficients $|a_2|$ and $|a_3|$ for functions of these subclasses. We obtained two new theorems with some new special cases for our new subclasses, and these results are different from the previous results for the other authors. Additionally, we discuss the improved results for the associated classes involving many of the new and well-known consequences. The results contained in the paper could inspire ideas for continuing the study, and we opened some windows for authors to generalize our new subclasses to obtain some new results in bi-univalent function theory.

Author Contributions: Conceptualization, methodology, software, validation, formal analysis, resources, by A.A.L., W.G.A. and I.A.R.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Duren, P.L. Univalent Functions. *Grundlehren der Mathematischen Wissenschaften*; Band 259; Springer: Berlin, Germany, 1983.
2. Lewin, M. On a coefficient problem for bi-univalent functions. *Proc. Am. Math. Soc.* **1967**, *18*, 63–68. [CrossRef]
3. Brannan, D.A.; Clunie, J.G. Aspects of Contemporary Complex Analysis. In Proceedings of the NATO Advanced Study Institute, University of Durham, Durham, UK, 1–20 July 1979; Academic Press: New York, NY, USA; London, UK, 1980.
4. Netanyahu, E. The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$. *Arch. Rational Mech. Anal.* **1969**, *32*, 100–112.
5. Al-Ameedee, S.A.; Atshan W.G.; Al-Maamori, F.A. Coefficients estimates of bi-univalent functions defined by new subclass function. *J. Phys. Conf. Ser.* **2020**, *1530*, 012105. [CrossRef]
6. Al-Ameedee, S.A.; Atshan, W.G.; Al-Maamori, F.A. Second Hankel determinant for certain subclasses of bi- univalent functions. *J. Phys. Conf. Ser.* **2020**, *1664*, 012044. [CrossRef]
7. Atshan, W.G.; Badawi, E.I. Results on coefficients estimates for subclasses of analytic and bi-univalent functions. *J. Phys. Conf. Ser.* **2019**, *1294*, 033025
8. Choi, J.; Srivastava, H.M. Certain Families of Series Associated with the Hurwitz-Lerch Zeta Function. *Appl. Math. Comput.* **2005**, *170*, 399–409. [CrossRef]

9. Srivastava, H.M.; Attiya, A.A. An Integral Operator Associated with the Hurwitz-Lerch Zeta Function and Differential Subordination. *Integral Transform. Spec. Funct.* **2007**, *18*, 207–216. [CrossRef]
10. Carlson, B.C.; Shaffer, D.B. Starlike and prestarlike hypergeometric functions. *SIAM J. Math. Anal.* **1984**, *15*, 737–746. [CrossRef]
11. Robertson, M.S. Quasi-subordination and coefficient conjecture. *Bull. Am. Math. Soc.* **1970**, *76*, 1–9. [CrossRef]
12. Kanas, S.; Darwish, H.E. Fekete-Szego problem for starlike and convex functions of complex order. *Appl. Math. Lett.* **2010**, *23*, 777–782. [CrossRef]
13. Mohd, M.H.; Darus, M. Fekete Szego problems for Quasi-subordination classes. *Abst. Appl. Anal.* **2012**, *2012*, 192956.
14. Ma, W.C.; Minda, D. A unified treatment of some special classes of univalent functions. In Proceedings of the Conference on Complex Analysis, Tianjin, China, 19–23 June 1992; pp. 157–169.
15. Atshan, W.G., Battor, A.H.; Abass, A.F. On third-order differential subordination results for univalent analytic functions involving an operator. *J. Phys. Conf. Ser.* **2020**, *1664*, 012041.
16. Atshan, W.G.; Hadi, R.A. Some differential subordination and superordination results of p-valent functions defined by differential operator, *J. Phys. Conf. Ser.* **2020**, *1664*, 012043.
17. Brannan, D.A.; Taha, T.S. On some classes of bi-univalent functions. *Studia Univ. Babeş-Bolyai Math.* **1986**, *31*, 70–77.
18. Taha, T.S. Topics in Univalent Function Theory. Ph.D. Thesis, University of London, London, UK, 1981.
19. Brannan, D.A.; Taha, T.S. On some classes of bi-univalent functions. In *Mathematical Analysis and Its Applications*; Mazhar, S.M., Hamoui, A., Faour, N.S., Eds.; KFAS Proceedings Series; Pergamon Press, Elsevier Science Limited: Oxford, UK, 1988; Volume 3, pp. 53–60. see also *Studia Univ. Babe-Bolyai Math.* 31(2) (1986) 70–77.
20. Srivastava, H.M.; Mishra, A.K.; Gochhayat, P. Certain subclasses analytic and bi-univalent functions. *Appl. Math. Lett.* **2010**, *23*, 1188–1192. [CrossRef]
21. Ali, R.; Ravichandran, V.; Seenivasgan, S. Coefficient bounds for p-valent functions. *Appl. Math. Comput* **2007**, *187*, 35–46. [CrossRef]
22. Atshan, W.G.; Yalcin, S.; Hadi, R.A. Coefficients estimates for special subclasses of k-fold symmetric bi-univalent functions. *Math. Appl.* **2020**, *9*, 83–90. [CrossRef]
23. Yalcin, S.; Atshan, W.G.; Hassan, H.Z. Coefficients assessment for certain subclasses of bi-univalent functions related with Quasi-subordination. *Publ. Institut Math.* **2020**, *108*, 155–162. [CrossRef]
24. Pommerenke, C. *Univalent Functions*; Vandenhoeck & Ruprecht: Göttingen, Germany, 1975.