symmetry

MDPI

Article

# $p$-Adic $q$-Twisted Dedekind-Type Sums

**Abdelmejid Bayad [1,*] and Yilmaz Simsek [2]**

1 Laboratoire de Mathématiques et Modélisation d'Évry(LAMME), CNRS (UMR 8071), Université Paris-Saclay, 91037 Evry, France
2 Department of Mathematics, Faculty of Arts and Science, University of Akdeniz, 07058 Antalya, Turkey; ysimsek@akdeniz.edu.tr
* Correspondence: abdelmejid.bayad@univ-evry.fr

**Abstract:** The main purpose of this paper is to define $p$-adic and $q$-Dedekind type sums. Using the Volkenborn integral and the Teichmüller character representations of the Bernoulli polynomials, we give reciprocity law of these sums. These sums and their reciprocity law generalized some of the classical $p$-adic Dedekind sums and their reciprocity law. It is to be noted that the Dedekind reciprocity laws, is a fine study of the existing symmetry relations between the finite sums, considered in our study, and their symmetries through permutations of initial parameters.

**Keywords:** twisted $(h,q)$-Bernoulli polynomials; $p$-adic $q$-Dedekind sums; reciprocity law; Volkenborn integral; the Teichmüller character

**MSC:** 11F20; 11S40; 11S80; 30B40; 44A20; 11B68



**Citation:** Bayad, A.; Simsek, Y. $p$-Adic $q$-Twisted Dedekind-Type Sums. *Symmetry* **2021**, *13*, 1756. https://doi.org/10.3390/sym13091756

Academic Editor: José Carlos R. Alcantud

Received: 16 June 2021
Accepted: 26 July 2021
Published: 20 September 2021



## 1. Introduction

For a positive integer $h$ and an integer $k$, the classical Dedekind sum is defined as

$$s(h,k)=\sum_{a=1}^{k-1}\left(\left(\frac{a}{k}\right)\right)\left(\left(\frac{ha}{k}\right)\right),$$

where $((x))=x-[x]-\frac{1}{2}$, if $x\notin\mathbb{Z}$, $((x))=0$, $x\in\mathbb{Z}$, and $[x]$ denotes the greatest integer not exceeding $x$. Dedekind [1] introduced this sum in connection with the transformation formula for, the well-known modular form of weight $\frac{1}{2}$, the Dedekind $\eta$-function given by

$$\eta(\tau)=e^{\frac{\pi i\tau}{12}}\prod_{n=1}^{\infty}\left(1-e^{2\pi in\tau}\right).$$

More precisely, we quote from [2] [p. 52, Theorem 3.4] the $\eta$-transformation formula.

**Theorem 1.** *If* $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})$, $c>0$, *and* $\operatorname{Im}\tau>0$, *we have*

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)=\varepsilon(a,b,c,d)\{-i(c\tau+d)\}^{1/2}\eta(\tau)$$

*where*

$$\varepsilon(a,b,c,d)=\exp\left\{\pi i\left(\frac{a+d}{12c}+s(-d,c)\right)\right\}.$$

From this transformation formula, for $h$ and $k$ are coprime integers, Dedekind deduced his reciprocity formula

$$s(h,k)+s(k,h)=-\frac{1}{4}+\frac{1}{12}\left(\frac{h}{k}+\frac{1}{kh}+\frac{k}{h}\right). \tag{1}$$

In 1950, Apostol [3] generalized $s(h,k)$ by defining

$$s_m(h,k)=\sum_{a=0}^{k-1}\frac{a}{k}\bar{B}_m\left(\frac{ha}{k}\right)$$

where $\bar{B}_m(x)$ is the $m$-th Bernoulli function defined by

$$\bar{B}_m(x)=B_m(\{x\}) \text{ for } m>1 \text{ and } \bar{B}_1(x)=((x)).$$

For odd values of $m$, these higher-order Dedekind sums enjoy a reciprocity law, first proved by Apostol [3]:

$$(m+1)(hk^m s_m(h,k)+kh^m s_m(k,h))=mB_{m+1}+\sum_{j=0}^{m+1}\binom{m+1}{j}(-1)^j B_j B_{m+1-j}h^j k^{m+1-j}. \tag{2}$$

In 1953, Carlitz [4] generalized $s_m(h,k)$ by defining the higher order Dedekind sums as

$$s_m^{(r)}(h,k)=\sum_{a=0}^{k-1}\bar{B}_{m-r+1}\left(\frac{ha}{k}\right)\bar{B}_r\left(\frac{a}{k}\right),$$

and proved their reciprocity laws similar to (2).

Recently, many mathematicians study on $p$-adic Dedekind type sums.

Using Washington's definition of $p$-adic Hurwitz zeta functions [5], Rosen and Snyder [6] constructed a $p$-adic interpolation of Apostol's Dedekind sums $s_m(h,k)$ and established new reciprocity laws satisfied by these sums. Their method and techniques were generalized by Kudo [7] who defined a $p$-adic analogue of the higher order Dedekind sums $s_m^{(r)}(h,k)$. In fact, Kudo in terms of the Euler numbers, he defined continuous function which interpolates the Euler numbers and $p$-adic higher order Dedekind sums.

Using a $p$-adic $q$-integral invariant on $\mathbb{Z}_p$, Kim in [8–10] constructed a $p$-adic continuous function that provides a $p$-adic $q$-analogue of higher order Dedekind type DC sums.

In 2017, Hu and Min-Soo in [11] using Cohen-Tangedal-Young's theory on the $p$-adic Hurwitz zeta functions, they construct some analytic Dedekind sums on the $p$-adic complex plane $\mathbb{C}_p$. Their sums interpolate Carlitz's higher order Dedekind sums $p$-adically. They also proved a reciprocity relation for the special values of their $p$-adic Dedekind sums.

In this paper our motivation is the same of Rosen-Snyder [6], Kim [8,10] and Kudo [7], Hu and Min-Soo [11] and others. We shall consider $p$-adic $q$-analogue to the twisted Dedekind sums $s_m^{(r)}(h,k)$. For this, we define continuous functions on $\mathbb{Z}_p$, which involves $p$-adic twisted $q$-Dedekind type sums. We construct $p$-adic Dedekind sums, which are defined in Definition 2. We call them $p$-adic $q$-analogue Dedekind type sums $S_{p,\xi}^{(h)}(s;a,b:q)$, where the parameters $h$ is an integer number, $\xi \in \mathbb{T}_p$ and $s \in \mathbb{Z}_p$. The main result of this paper is to prove their Dedekind reciprocity laws.

## 2. Preliminaries and Definitions

In this section, we consider $p$-adic $(h,q)$-Dedekind type sums on $\mathbb{Z}_p$ and investigate some of their properties.

We need the following definitions and notations:

Let

$$[x] = [x : q] = \begin{cases} \frac{1-q^x}{1-q}, & q \neq 1 \\ x, & q = 1 \end{cases}.$$

If $q \in \mathbb{C}_p$, then

$$|1-q|_p < p^{-\frac{1}{p-1}}$$

so that

$$q^x = \exp(x \log q)$$

for $|x|_p \leqslant 1$.

If $q \in \mathbb{C}$, then we assume that $|q| < 1$.

According to [12,13], for each integer $N \geq 0$, $C_{p^N}$ denotes the multiplicative group of the primitive $p^N$th roots of unity in $\mathbb{C}_p^* = \mathbb{C}_p \setminus \{0\}$. Let

$$\mathbb{T}_p = \left\{ \xi \in \mathbb{C}_p : \xi^{p^N} = 1, \text{ for } N \geq 0 \right\} = \underset{N\geq 0}{\cup} C_{p^N}.$$

The dual of $\mathbb{Z}_p$, in the sense of $p$-adic Pontrjagin duality, is $\mathbb{T}_p = C_{p^\infty}$, the direct limit (under inclusion) of cyclic groups $C_{p^N}$ of order $p^N$ with $N \geq 0$, with the discrete topology. $\mathbb{T}_p$ admits a natural $\mathbb{Z}_p$-module structure which we shall write exponentially, *viz* $\xi^x$ for $\xi \in \mathbb{T}_p$ and $x \in \mathbb{Z}_p$. $\mathbb{T}_p$ can be embedded discretely in $\mathbb{C}_p$ as the multiplicative $p$-torsion subgroup, and we choose, for once and all, one such embedding. If $\xi \in \mathbb{T}_p$, then

$$\phi_\xi : (\mathbb{Z}_p, +) \to (\mathbb{C}_p, .)$$

is the locally constant character, $x \to \xi^x$, which is locally analytic character if

$$\xi \in \{\xi \in \mathbb{C}_p : v_p(\xi - 1) > 0\},$$

where $v_p$ denotes the valuation. Then $\phi_\xi$ has continuation to a continuous group homomorphism from $(\mathbb{Z}_p, +)$ to $(\mathbb{C}_p, .)$ cf. ([12–15]), see also the references cited in each of these earlier works. If $\xi \in \mathbb{C}$, then we assume that $\xi$ is an $r$-th root of 1 with $r \in \mathbb{Z}^+$ the set of positive integers.

For $f \in UD(\mathbb{Z}_p, \mathbb{C}_p) = \{f \mid f : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\}$, the $p$-adic $q$-integral ($q$-*Volkenborn integral*) is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} q^x f(x), \tag{3}$$

where $\mu_q$ denotes $p$-adic $q$-*Haar distribution* which is defined by

$$\mu_q(a + p^N \mathbb{Z}_p) = \frac{q^a}{[p^N]_q}, N \in \mathbb{Z}^+.$$

**Definition 1.** *Let $h$, $a$ and $b$ be positive integers with $(a,b) = 1$, and let $p$ be an odd prime such that $p \nmid b$. For $\xi \in \mathbb{T}_p$, we define twisted $(h,q)$-Dedekind type sums as*

$$s_\xi^{(h)}(m, a, b : q) = \sum_{c=0}^{b-1} \frac{cq^{hc}\xi^c}{b} \int_{\mathbb{Z}_p} q^{hx} \phi_\xi(x) \left( x + \left\{ \frac{ca}{b} \right\} \right)^m d\mu_1(x), \tag{4}$$

*where $\{t\}$ denotes the fractional part of $t$.*

In Definition 1, we modify the $p$-adic $(h,q)$-higher order Dedekind type sums which are defined by Cenkci et al. [16]. Recently modified of the Dedekind type sums have been studied by many different methods (*cf.* [6,8,9,11,17–20]).

Using the integral representation (3), Simsek [21] defined the generating function of the twisted $(h, q)$-Bernoulli numbers as follows

$$\frac{(t + \log q^h)}{\xi q^h e^t - 1} = \sum_{n=0}^{\infty} B_{n,\xi}^{(h)}(q) \frac{t^n}{n!}, \tag{5}$$

where $|t + h \log q| < \pi$.

By applying the *umbral calculus* convention in the above equation, and the usual convention of symbolically replacing $(B_{\xi}^{(h)}(q))^n$ by $B_{n,\xi}^{(h)}(q)$, then we have

$$\begin{aligned} B_{0,\xi}^{(h)}(q) &= \frac{\log q^h}{\xi q^h - 1} \\ \xi q^h (B_{\xi}^{(h)}(q) + 1)^n - B_{n,\xi}^{(h)}(q) &= \delta_{1,n}, \ n \geq 1, \end{aligned} \tag{6}$$

where $\delta_{1,n}$ is denoted Kronecker symbol cf. ([21,22]).

Witt's type formula of the twisted $(h, q)$-Bernoulli numbers is given by Simsek [21] as follows

$$B_{n,\xi}^{(h)}(x, q) = \int_{\mathbb{Z}_p} \phi_{\xi}(t) q^{ht} x^n d\mu_1(t). \tag{7}$$

The twisted $(h, q)$-Bernoulli numbers of higher order are defined by means of the following generating function:

$$(\log q^h + t)^v \prod_{j=1}^{v} \frac{a_j}{(\xi q^h)^{a_j} e^{a_j t} - 1} = \sum_{n=0}^{\infty} \mathcal{B}_{n,\xi}^{(h,v)}(q \mid \overrightarrow{a}) \frac{t^n}{n!},$$

where $\overrightarrow{a} = (a_1, \cdots, a_v)$ and $\mid t + \log(\xi q^h) \mid < \min\left\{\mid \frac{2\pi}{a_1} \mid, \cdots, \mid \frac{2\pi}{a_v} \mid\right\}$ cf [18].

By substituting $v = 1$ into the above, we have

$$\mathcal{B}_{n,\xi}^{(h,1)}(q \mid a_1) = a_1 B_{n,\xi^{a_1}}^{(h)}(q^{a_1})$$

(*cf.* [21]). Setting $v = 2$ in the above, we have

$$\mathcal{B}_{n,\xi}^{(h,2)}(q \mid (a_1, a_2)) = \left({}^1 B_{j,\xi^{a_1}}^{(h)}(q^{a_1}) a_1 + {}^2 B_{l,\xi^{a_2}}^{(h)}(q^{a_2}) a_2\right)^n, \tag{8}$$

where

$$\left({}^i B_{\xi^a}^{(h)}(q^a)\right)^j = B_{j,\xi^a}^{(h)}(q^a)$$

and

$$\left({}^i B_{\xi^a}^{(h)}(q^a)\right)^j \left({}^l B_{\xi^a}^{(h)}(q^a)\right)^d \neq B_{j+d,\xi^a}^{(h)}(q^a)$$

if $i \neq l$ (*cf.* [18]). Using (4), we obtain

$$s_{\xi}^{(h)}(m, a, b : q) = \sum_{j=0}^{b-1} \frac{j q^{hj} \xi^j}{b} \sum_{c=0}^{m} \binom{m}{c} \left\{\frac{ja}{b}\right\}^{m-c} \int_{\mathbb{Z}_p} q^{hx} \phi_{\xi}(x) x^c d\mu_1(x).$$

Combining the above equation with (7), we obtain

$$s_{\xi}^{(h)}(m, a, b : q) = \sum_{j=0}^{b-1} \frac{j q^{hj} \xi^j}{b} B_{m,\xi}^{(h)}\left(\left\{\frac{ja}{b}\right\}, q\right),$$

(*cf.* [18]).

Let $< a >$ denote the principal unit associated with $a$, which is the unique unit given by the decomposition

$$< a >= w^{-1}(a)a = \frac{a}{w(a)},$$

where

$$a \in \mathbb{Z}_p^* = \mathbb{Z}_p \setminus p\mathbb{Z}_p = \left\{x : |x|_p = 1\right\},$$

and $w$ denotes the Teichmüller character (or function), which is defined as follows:

$$w : \mathbb{Z}_p \to \mathbb{Q}_p$$

For $a, b \in \mathbb{Z}_p$,

$$w(a) = \underbrace{\lim_{n\to\infty}}_{p-adically} a^{p^n}, \tag{9}$$

$$w(ab) = w(a)w(b),$$

and

$$|w(a+b) - w(a) - w(a)|_p < 1$$

(*cf* [23], see also [5,6]).

Some properties of the Teichmüller character are given by

$$|a - w(a)|_p < 1,$$

$$(w(a))^p = w(a)$$

and the well-known Teichmüller expansion of $a$ is given by

$$a = \sum_{j\geq 0} w_j(a)p^j,$$

where

$$w_{j+1}(a) = \left(a - \sum_{m=0}^{j} w_m(a)p^m\right)p^{-(j+1)},$$

$w_0(a) = w(a)$ (*cf* [23], see also [5,6]).

Using (9), a few values of $w(a)$ are give as follows:

$w(0) = 0$, $w(1) = 1$, $w(5) = -1$, $w\left(\frac{1}{5}\right) = -1$, $w_1\left(\frac{1}{5}\right) = w\left(\frac{2}{5}\right) = 1$, $w_2\left(\frac{1}{5}\right) = w_2\left(-\frac{2}{5}\right) = 1$, and so on (*cf* [23]).

Let $a$ and $b$ be positive integers such that $(p, a) = 1$ and $p|b$. We extend

$$b^{m-1} B_{k,\xi}^{(h)}\left(\frac{a}{b}, q\right).$$

Thus, we define

$$\mathcal{T}_\xi^{(h)}(s; j, b : q) = \omega^{-1}(j)\frac{\langle j\rangle^s}{b}\sum_{k=0}^{\infty}\binom{s}{k}\left(\frac{b}{j}\right)^k B_{k,\xi}^{(h)}(q), \tag{10}$$

where $s \in \mathbb{Z}_p$, and

$$\binom{s}{k} = \frac{s(s-1)(s-2)...(s-k+1)}{k!}.$$

## 3. Statement of Main Result

We define the following $p$-adic $q$-twisted Dedekind sum as follows.

**Definition 2.** *Let a and b be relatively prime positive integers such that $p \nmid b$. Let $s \in \mathbb{Z}_p$. The p-adic Dedekind sum is defined by*

$$S_{p,\xi}^{(h)}(s;a,b:q) = \sum_{c=0}^{b-1} cq^{hc}\xi^c \mathcal{T}_\xi^{(h)}(s;ac,b:q).$$

The main result of this paper is the following theorem.

**Theorem 2.** *Let a and b relatively prime positive integers $p \equiv 1 \pmod{ab}$. Then*

$$ab^n S_{p,\xi}^{(h)}(s;a,b:q) + ba^n S_{p,\xi}^{(h)}(s;b,a:q) +$$
$$\tfrac{2\log q^h}{n+1}\Big(ab^{n+1}S_{p,\xi}^{(h)}(s+1;a,b:q) + ba^{n+1}S_{p,\xi}^{(h)}(s+1;b,a:q)\Big)+$$
$$\tfrac{\left(\log q^h\right)^2}{(n+1)(n+2)}\Big(ab^{n+2}S_{p,\xi}^{(h)}(s+2;a,b:q) + ba^{n+2}S_{p,\xi}^{(h)}(s+2;b,a;q)\Big)$$

*is a continuous function of s on $\mathbb{Z}_p$ and for all positive integer n such that $n+1 \equiv 0 \pmod{(p-1)}$ this reciprocity law reduces to the reciprocity law for higher-order q-Dedekind type sum which is given by authors [18]:*

$$\begin{aligned}
& ab^n S_{n,\xi}^{(h)}(a,b:q) + ba^n S_{n,\xi}^{(h)}(b,a:q) \\
& + \frac{2\log q^h}{n+1}\Big(ab^{n+1}S_{n+1,\xi}^{(h)}(a,b:q) + ba^{n+1}S_{n+1,\xi}^{(h)}(b,a:q)\Big) \\
& + \frac{\left(\log q^h\right)^2}{(n+1)(n+2)}\Big(ab^{n+2}S_{n+2,\xi}^{(h)}(a,b:q) + ba^{n+2}S_{n+2,\xi}^{(h)}(b,a:q)\Big) \\
= \; & (1-p^{n-1})\frac{n\Big(B_{n+2,\xi}^{(h,2)}(q \mid (a,b)) + B_{n+1,\xi}^{(h)}(q)\Big)}{n+1} \\
& + (1-p^{n-1})ab\Big(nB_{n,\xi}^{(h)}(q) + \Big(\log\Big(q^{\frac{h}{n+1}}\Big)\Big)B_{n+1,\xi}^{(h)}(q)\Big) \\
& + (1-p^{n-1})\log q^{\frac{h_1}{n+2}} B_{n+2,\xi}^{(h)}(q),
\end{aligned}$$

*where $B_{n,\xi}^{(h)}(q)$ and $B_{n+2,\xi}^{(h,2)}(q \mid (a,b))$ are given respectively by (5) and (8).*

The above theorem is a generalization of the Rosen and Snyder' [6] $p$-adic Dedekind sums.

**Proof.** Let

$$\mathcal{G}_{m,\xi}^{(h)}(j;b:q) = \frac{\omega^{-1}(j)}{b}\int_{\mathbb{Z}_p} q^{hx}\xi^x(bx+j)^m d\mu_1(x).$$

Since

$$\left|\binom{s}{k}\right|_p \le 1, \left|\frac{b}{j}\right|_p < 1$$

and

$$\left|B_{k,\xi}^{(h)}(q)\right|_p \le 1,$$

the sum

$$\sum_{k=0}^{\infty}\binom{s}{k}\left(\frac{b}{j}\right)^k B_{k,\xi}^{(h)}(q)$$

converges to a continuous function of $s$ in $\mathbb{Z}_p$. Substituting $s = m$ in (10), we have

$$\mathcal{T}_\xi^{(h)}(m;j,b:q) = \mathcal{G}_{m,\xi}^{(h)}(j;b:q).$$

If $m+1 \equiv 0 \pmod{(p-1)}$, then we have

$$\mathcal{T}_{\xi}^{(h)}(m;j,b:q) = \frac{1}{b}\int_{\mathbb{Z}_p} q^{hx}\phi_{\xi}(x)(bx+j)^m d\mu_1(x) = b^{m-1}\int_{\mathbb{Z}_p} q^{hx}\phi_{\xi}(x)\left(x+\frac{j}{b}\right)^m d\mu_1(x).$$

By Witt's type formula of the twisted $(h,q)$-Bernoulli polynomials (see [21]), we have

$$\mathcal{T}_{\xi}^{(h)}(m;j,b:q) = b^{m-1}B_{m,\xi}^{(h)}\left(\frac{j}{b},q\right),$$

which is continuous $p$-adic extension of $b^{m-1}B_{m,\xi}^{(h)}\left(\frac{j}{b},q\right)$. By using the above integral equation, we have

$$N^{m-1}\int_{\mathbb{Z}_p}\phi_{\xi}(x)q^{hx}\left(x+\frac{j}{N}\right)^m d\mu_1(x) = \sum_{c=0}^{b-1}\xi^c q^{hc}(pN)^{m-1}\int_{\mathbb{Z}_p}\phi_{\xi^p}(x)q^{hpx}(x+\frac{j+cN}{Np})^m d\mu_1(x).$$

From the above, we obtain

$$N^{m-1}B_{m,\xi}^{(h)}\left(\frac{j}{b},q\right) = (pN)^{m-1}\sum_{c=0}^{p-1}\xi^c q^{hc}B_{m,\xi^p}^{(h)}\left(\frac{j+cN}{Np},q^p\right).$$

Each term in the above sum is interpolated except for the one term for which $a+jN \equiv 0 \pmod{p}$. Let $(p^{-1}a)_N$ denote the integer $x$ such that $0 \le x < N$ and $px \equiv a \pmod{p}$. The exceptional term may be written as

$$(pN)^{m-1}B_{m,\xi^p}^{(h)}\left(\frac{(p^{-1}j)_N}{N},q^p\right).$$

Thus, we may $p$-adically interpolate

$$N^{m-1}B_{m,\xi}^{(h)}\left(\frac{j}{b},q\right) - (pN)^{m-1}B_{m,\xi^p}^{(h)}\left(\frac{(p^{-1}j)_N}{N},q^p\right)$$

and

$$N^{m-1}B_{m,\xi}^{(h)}\left(\frac{j}{b},q\right) - B_{m,\xi^p}^{(h)}\left(\frac{(p^{-1}j)_N}{N},q^p\right).$$

The $p$-adic function is given by

$$\mathcal{T}_{\xi}^{(h)}(s;j,b:q) = \sum_{\substack{c=0 \\ j+cb \not\equiv 0 \pmod{p}}}^{p-1} \mathcal{T}_{\xi^{Np}}^{(h)}\left(s;(j+cb)_{pb},pb:q^{bp}\right)$$

so that

$$\begin{aligned}\mathcal{T}_{\xi}^{(h)}(m;j,b:q) &= b^{m-1}\int_{\mathbb{Z}_p}\phi_{\xi}(x)q^{hx}\left(x+\frac{j}{b}\right)^m d\mu_1(x) \\ &\quad -(pb)^{m-1}\int_{\mathbb{Z}_p}\phi_{\xi^{pb}}(x)(q^{hpb})^x\left(x+\frac{(p^{-1}j)_b}{b}\right)^m d\mu_1(x).\end{aligned} \tag{11}$$

Thus, we have

$$\mathcal{T}_{\xi}^{(h)}(m;j,b:q) = b^{m-1}B_{m,\xi}^{(h)}\left(\frac{j}{b},q\right) - (pb)^{m-1}B_{m,\xi^{bp}}^{(h)}\left(\frac{(p^{-1}j)_b}{b},q^{bp}\right)$$

where $(p^{-1}a)_N$ denotes the integer $x$ such that $0 \leq x < N$ and $px \equiv a \pmod p$ and $m$ is integer with $m+1 \equiv 0 \pmod{(p-1)}$.

By (11), we interpolate $b^m s^{(h)}_{m,\xi}(a,b:q)$ where $p|b$ and $(ac,p) = 1$ for each $c = 0,1,\cdots,b-1$;

$$b^m s^{(h)}_{m,\xi}(a,b:q) = \sum_{c=0}^{b-1} cq^{hc}\xi^c \mathcal{T}^{(h)}_{\xi}(m;(ma)_b,b:q),$$

for all $m+1 \equiv 0 \pmod{(p-1)}$, where $(\alpha_n)$ denotes the integer $x$ such that $0 \leq x < n$ and $x \equiv \alpha \pmod n$. If $p|b$, then $s^{(h)}_{m,\xi}(a,b:q)$ is interpolated. However, $s^{(h)}_{m,\xi}(b,a:q)$ is not interpolated. Thus, if $p \nmid b$, then the reciprocity law of $s^{(h)}_{m,\xi}(a,b:q)$ is obtained from the $p$-adic interpolation of $s^{(h)}_{m,\xi}(a,b:q)$. Consequently, if $p \nmid b$ and $p \nmid hc$ for any $c = 1,2,\cdots$, $b-1$, then

$$b^m s^{(h)}_{m,\xi}(a,b:q) - p^{m-1} b^m s^{(h)}_{m,\xi}((p^{-1}a)_b,b)$$

can be $p$-adically interpolated by

$$\sum_{c=0}^{b-1} cq^{hc}\xi^c \mathcal{T}^{(h)}_{\xi}(s;ac,b:q).$$

□

We end this paper by the following remark.

**Remark 1.** *If $p \equiv 1 \pmod b$, then we have*

$$S^{(h)}_{p,\xi}(m;a,b:q) = (1-p^{m-1}) s^{(h)}_{m,\xi}(a,b:q)$$

*for all $m$ such that $m+1 \equiv 0 \pmod{(p-1)}$. Additionally, one can observe that by substituting $\xi = v = 1$, $q \to 1$ into Theorem 2, then we obtain the result of Rosen and Snyder' [6] and Ota' [24]. For more details and information on p-Dedekind type sums cf. [6–8,10,16,20,25–27]; see also the references cited in each of these earlier works.*

**Author Contributions:** Writing—original draft preparation, A.B. and Y.S. Both authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** This paper was supported by the Scientific Research Project Administration Akdeniz University.

**Conflicts of Interest:** The authors declare no conflict of interest.

## Abbreviations

The following abbreviations are used in this manuscript:

| | |
|---|---|
| $\mathbb{C}$ | the field of complex numbers. |
| $p$ | an odd rational prime number. |
| $\mathbb{Z}_p$ | the ring of $p$-adic integers. |
| $\mathbb{Q}_p$ | the field of fractions of $\mathbb{Z}_p$. |
| $\mathbb{C}_p$ | the completion of a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}$. |
| $v_p$ | the $p$-adic valuation of $\mathbb{C}_p$ normalized so that $\lvert p\rvert_p = p^{-v_p(p)} = p^{-1}$. |
| $C_{p^N}$ | denotes the multiplicative group of the primitive $p^N th$ roots of unity in $\mathbb{C}^*_p$. |
| $\mathbb{T}_p$ | denotes the set $\bigcup_{N\geq 0} C_{p^N}$. |

## References


1. Dedekind, R. Erläuterungen zu Den Fragmenten XXVIII. In *B. Riemann's Gesammelte Mathematische Werke*, 2nd ed.; Teubner: Leipzig, Germany, 1892; pp. 466–478.
2. Apostol, T.M. *Modular Functions and Dirichlet Series in Number Theory*, 2nd ed.; Grad. Texts in Math.; Springer: New York, NY, USA, 1990; Volume 41.
3. Apostol, T.M. Generalized Dedekind sums and transformation formulae of certain Lambert series. *Duke Math. J.* **1950**, *17*, 147–157. [CrossRef]
4. Carlitz, L. Some sums analogous to Dedekind sums. *Duke Math. J.* **1953**, *20*, 161–171. [CrossRef]
5. Washington, L. A note onp-adic $L$-Functions. *J. Number Theory* **1976**, *8*, 245–250. [CrossRef]
6. Rosen, K.H.; Snyder, W.M. $p$-adic Dedekind sums. *J. Reine Angew. Math.* **1985**, *361*, 23–26.
7. Kudo, A. On $p$-adic Dedekind Sums. *Nagoya Math. J.* **1996**, *144*, 155–170. [CrossRef]
8. Kim, T. A note on $p$-adic $q$-Dedekind sums. *C. R. Acad. Bulgare Sci.* **2001**, *54*, 37–42.
9. Kim, T. Note on Dedekind type DC sums. *Adv. Stud. Contemp. Math.* **2009**, *18*, 249–260.
10. Kim, T. Note on $q$-Dedekind type sums related to $q$-Euler polynomials. *arXiv* **2012**, arXiv:0907.5095v1.
11. Hu, S.; Kim, M.-S. The $p$-adic analytic Dedekind sums. *J. Number Theory* **2017**, *171*, 112–127. [CrossRef]
12. Woodcock, C.F. Convolutions on the ring of $p$-adic integers. *J. Lond. Math. Soc.* **1979**, *20*, 101–108. [CrossRef]
13. Shratani, K.; Yokoyama, S. An Application of $p$-adic convolutions. *Mem. Fac. Sci. Kyushu Univ. Ser. A Math.* **1982**, *36*, 73–83. [CrossRef]
14. Kim, M.-S.; Son, J.-W. Analytic properties of the $q$-Volkenborn integral on the ring of $p$-adic integers. *Bull. Korean Math. Soc.* **2007**, *44*, 1–12. [CrossRef]
15. Kim, T. An analogue of Bernoulli numbers and their congruences. *Rep. Fac. Sci. Engrg. Saga Univ. Math.* **1994**, *22*, 21–26.
16. Cenkci, M.; Can, M.; Kurt, V.; Simsek, Y. Twisted Dedekind type sums associated with Barnes' type multiple Frobenius-Euler $l$-Functions. *Adv. Stud. Contemp. Math.* **2009**, *18*, 135–160.
17. Araci, S.; Şen, E.; Acikgoz, M. A Note on the Modified $q$-Dedekind Sums. *arXiv* **2013**, arXiv:1212.5837.
18. Bayad, A.; Simsek, Y. Multiple Dedekind type sums and their related zeta functions. *Mathematics* **2020**, preprint.
19. Bayad, A.; Simsek, Y. Values of twisted Barnes zeta functions at negative integers. *Russ. J. Math. Phys.* **2013**, *20*, 129–137. [CrossRef]
20. Snyder, C. $p$-adic interpolation of Dedekind sums. *Bull. Austral. Math. Soc.* **1988**, *38*, 293–301. [CrossRef]
21. Simsek, Y. Twisted $(h,q)$-Bernoulli numbers and polynomials related to twisted $(h,q)$-zeta function and $L$-function. *J. Math. Anal. Appl.* **2006**, *324*, 790–804. [CrossRef]
22. Simsek, Y. Twisted $p$-adic $(h,q)$-$L$-functions. *Comput. Math. Appl.* **2010**, *59*, 2097–2110. [CrossRef]
23. Baker, A. An Introduction to $p$-Adic Numbers and $p$-Adic Analysis. AMS Open Math Notes. Available online: https://www.ams.org/open-math-notes/files/course-material/OMN-202003-110818-1-Course_notes-v1.pdf (accessed on 14 June 2021).
24. Ota, K. Derivatives of Dedekind sums and their reciprocity law. *J. Number Theory* **2003**, *98*, 280–309. [CrossRef]
25. Kozuka, K. Multiple Dedekind sums attached to Dirichlet characters. *Proc. Jpn. Acad.* **2007**, *83*, 104–107. [CrossRef]
26. Simsek, Y. $q$-Dedekind type sums related to $q$-zeta function and basic $L$-series. *J. Math. Anal. Appl.* **2006**, *318*, 333–351. [CrossRef]
27. Simsek, Y. $p$-adic $q$-higher-order Hardy-type sums. *J. Korean Math. Soc.* **2006**, *43*, 111–131. [CrossRef]