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Fixed-Point Theorems in Fuzzy Normed Linear Spaces for Contractive Mappings with Applications to Dynamic-Programming

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Abstract: The aim of this paper is to provide new ways of dealing with dynamic programming using a context of newly proven results about fixed-point problems in linear spaces endowed with a fuzzy norm. In our paper, the general framework is set to fuzzy normed linear spaces as they are defined by Nădăban and Dzitac. When completeness is required, we will use the George and Veeramani (G-V) setup, which, for our purposes, we consider to be more suitable than Grabiec-completeness. As an important result of our work, we give an original proof for a version of Banach's fixed-point principle on this particular setup of fuzzy normed spaces, a variant of Jungck's fixed-point theorem in the same setup, and they are proved in G-V-complete fuzzy normed spaces, paving the way for future developments in various fields within this framework, where our application of dynamic programming makes a proper example. As the uniqueness of almost every dynamic programming problem is necessary, the fixed-point theorems represent an important tool in achieving that goal.

Keywords: dynamic programming; fixed-point theorems; fuzzy normed linear space; G-V-completeness; fuzzy continuous

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1. Introduction

The chaotic behavior of various complex systems, both natural and human made, has been mitigated using various methods for centuries. One of these methods works mostly in management problems and starts by using singular hypotheses, pursuing multi-stage decisions is dynamic programming. These decisions could be optimized later, leading to a model with wide applicability. However, this implies that a more complex problem has to be reduced to simpler, elementary cases, such that each case belongs to a known model with its own optimal solution. Models or recipes for quickly reaching the solution of a dynamic programming problem do not exist. We do, however, find similar assumptions and reasoning connected to the solution of a dynamic programming problem.

Richard Bellman [1] worked for the first time with this method of optimization, called by him "dynamic programming". It helps optimizing systems based on phases or sequences, using their mathematical representations. Systems of this type are frequently encountered in economic studies or in the development of advanced programs, such as those concerning cosmic navigation [2]. When encountering the need of the regularization of stocks, management of equipment, mining prospecting, and investment, as well as macroeconomic problems, such as the national planning, the sequential structure of the problems paves the way for the use of well-chosen methods to perform optimization calculations and clarify problems in introducing precise concepts such as decision criteria and appropriate policies.

Given the uniqueness of almost every dynamic programming problem, a multitude of methods and algorithms from various branches of mathematics was used to solve those problems. Of these, the Banach fixed-point principle stands out as being a remarkable tool for elegantly providing at least the solution's unicity of a dynamic programming problem [3].

Fixed-point theory and functional analysis is being used in more and more applications from various fields such as economy, physics and astronomy, game theory, and many others.

Fuzzy metric spaces were the general set up for the first fixed-point results. One of the pioneers in this field was M. Grabiec [4], who defined the fuzzy complete metric space (denoted G-complete) and extended this framework to reach the Banach contraction theorem. It is also worth mentioning the work of J.X. Fang [5], who extended the results pool in this direction, and A. George and P. Veeramani [6], who gave a modified definition of a Cauchy sequence, yielding a new notion of completitude in fuzzy spaces endowed with a metric (G-V completeness). Since all fuzzy metric spaces that are G-complete are also G-V complete, it makes sense to work on theorems referring to fixed-point properties in these fuzzy metric spaces which are G-V complete.

As important results of our work, some classic fixed-point theorems will be proved in the setting of G-V complete fuzzy normed spaces, and the application of dynamic programming will use this new type of Banach's contraction principle on G-V complete fuzzy normed spaces.

In our paper, the general framework is set to fuzzy normed linear spaces as they are defined in [7], and, where completeness is required, we will use the G-V setup.

The results obtained are established in a different context than in other papers that deal with fuzzy fixed-point theory. More precisely, we work with the A. George and P. Veeramani definitions of completeness, while other authors consider the definition given by M. Grabiec [8–10]. Another important difference is the way fuzzy normed spaces are defined. We utilize the one given by S. Nadaban and I. Dzitac in 2014, which induces a Kramosil–Michálek type metric [11]. In other articles [5,12,13], different definitions of fuzzy normed spaces are given, in which the fuzzy norm generates a fuzzy metric of the type of George and Veeramani.

The structure of the work is: after the preliminary section, in Section 3 we generalize, extend, and obtain new fixed-point theorems similar to those that can be found in [14–24]. The first result obtained is Banach's Contraction Principle in FNLSs of Nădăban-Dzitac type, using completeness in the George–Veeramani sense. An important original result is Theorem 2, which presents in the context of Nădăban–Dzitac type FNLSs, a result obtained by Jungck. More precisely, it offers a version of Jungck's theorem in this FNLSs, which is induced by a fuzzy metric that is more general than the one given in the case of the fuzzy metric of Grabiec type (see Theorem 5.4.12 [12]). In Theorem 5 the property of continuous dependence of fixed-points on a parameter is presented. In Section 4, we present new ways of dealing with dynamic programming using the obtained fixed-point theorems. Section 5 is dedicated to the conclusions.

2. Preliminaries

For completion purposes, we will present here how the above mentioned notions are defined.

For $\alpha, \beta \in [0,1]$, we denote $\alpha \wedge \beta = min(\alpha, \beta)$.

Definition 1 ([7])**.** *Let $\mathcal{X}$ be a vector space over $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$). A fuzzy norm is a function $F : \mathcal{X} \times [0,\infty) \to [0,1]$ such that:*

(F1) $F(x,0) = 0$, for all $x \in \mathcal{X}$;

(F2) $[F(x,t) = 1$, for all $t > 0\,]$ iff $x = 0$;

(F3) $F(\lambda x, t) = F\left(x, \frac{t}{|\lambda|}\right)$, for all $x \in \mathcal{X}$, all $t \geq 0$, and all $\lambda \in \mathbb{K}^*$;

(F4) $F(x+y, t+s) \geq F(x,t) \wedge F(y,s)$, for any $x, y \in \mathcal{X}$, and any $t, s \geq 0$;

(F5) For any $x \in \mathcal{X}$, $F(x,\cdot)$ is left continuous and $\lim\limits_{t\to\infty} F(x,t) = 1$.

The triple $(\mathcal{X}, F, \wedge)$ is called a fuzzy normed linear space (shortly FNLS).

In [7], it is shown that $\mathcal{X}$ is a topological metrizable vector space with the topology given by
$\tau_F = \{\emptyset\} \cup \{\mathcal{D} \subset \mathcal{X} \ : \ x \in \mathcal{D} \textit{ if there exist } s > 0, \delta \in (0,1) \textit{ satisfying } S(x,\delta,s) \subseteq \mathcal{D}\}$,
with

$$S(x,\delta,s) = \{z \in \mathcal{X} \ : \ F(x-z,s) > 1-\delta\}.$$

A sequence (z_n) is convergent to $z \in \mathcal{X}$, denoted by $\lim\limits_{n\to\infty} z_n = z$ or $z_n \to z$, if $\lim\limits_{n\to\infty} F(z_n - z, s) = 1$, for all $s > 0$.

Throughout this paper, a definition of Cauchy sequence, inspired by A. George and P. Veeramani [6], is considered.

Definition 2 ([6])**.** *A sequence (z_n) in $(\mathcal{X}, F, \wedge)$ is called a G-V Cauchy sequence if*

$$(\forall)\delta \in (0,1), (\forall)s > 0, (\exists)n_0(\delta,s) \in \mathbb{N}^* : F(z_{n+p} - z_n, s) > 1-\delta, (\forall)n \geq n_0,$$

$(\forall)p \in \mathbb{N}^*$.

A fuzzy normed space $(\mathcal{X}, F, \wedge)$ is G-V complete if any G-V Cauchy sequence in $\mathcal{X}$ is convergent to a point from $\mathcal{X}$. A G-V complete fuzzy normed space is a fuzzy Banach space.

It is known (see [25]) that a function $S : (\mathcal{X}, F_1, \wedge) \to (Y, F_2, \wedge)$ is continuous in the fuzzy sense on $\mathcal{X}$, if S is continuous between $(\mathcal{X}, \tau_{F_1})$ and (Y, τ_{F_2}).

Definition 3 ([26])**.** *A function $S : (\mathcal{X}, F_1, \wedge) \to (Y, F_2, \wedge)$ is a fuzzy contraction if*

$$(\exists)M \in (0,1) : F_2(S(x) - S(y), t) \geq F_1\left(x - y, \frac{t}{M}\right), (\forall)t > 0, (\forall)x, y \in \mathcal{X}.$$

One can observe that a fuzzy contraction is continuous in the fuzzy sense.

3. Some Fixed-Point Results in Fuzzy Normed Spaces

Following the introduction of fuzzy metric spaces and fuzzy normed spaces, several mathematicians have developed fixed-point theories in such a context. This work consist of generalizations, extensions, or even new fixed-point theorems such as those which can be found in [14–24]. As we have already said, Grabiec [4] defined the complete fuzzy metric space and extended the Banach contraction theorem in fuzzy metric spaces. Our first result is obtaining Banach's contraction principle in FNLSs using G-V completeness, providing a new straightforward proof, using only the fuzzy norm's definition. We remark that the existence part from the next theorem is proved in a more general context (see Lemma 5.1.3 [12]).

Theorem 1. *Given $(\mathcal{X}, F, \wedge)$, a fuzzy Banach space, and $S : \mathcal{X} \to \mathcal{X}$ a fuzzy contraction, S has a unique fixed-point $y \in \mathcal{X}$ and $\lim\limits_{n\to\infty} S^n(x) = y, (\forall)x \in \mathcal{X}$.*

Proof. Let $x \in \mathcal{X}$. We will prove that $\{S^n(x)\}$ is a G-V Cauchy sequence, even if this proof is given in a more general context in [12]. By induction, we obtain that

$$F(S^{n+1}(x) - S^n(x), s) \geq F\left(S(x) - x, \frac{s}{M^n}\right), (\forall)s > 0, (\forall)n \in \mathbb{N}^*.$$

Let $p \in \mathbb{N}^*$. Observe that

$$(1-M)(1 + M + M^2 + \cdots M^{p-1}) = 1 - M^p < 1\,.$$

Thus

$$s(1-M)(1+M+M^2+\cdots M^{p-1}) = s(1-M^p) < s, (\forall)s > 0\,.$$

Hence, by using $(F3)$, it follows

$$F(S^n(x)-S^{n+p}(x),s) \geq F(S^n(x)-S^{n+p}(x),s(1-M^p)) =$$
$$= F(S^n(x)-S^{n+1}(x)+S^{n+1}(x)-S^{n+2}(x)+\cdots+S^{n+p-1}(x)-$$
$$-S^{n+p}(x),s(1-M)(1+M+M^2+\cdots M^{p-1})) \geq$$
$$\geq F(S^n(x)-S^{n+1}(x),s(1-M)) \wedge F(S^{n+1}(x)-S^{n+2}(x),s(1-M)M)\wedge$$
$$\wedge\cdots\wedge F(S^{n+p-1}(x)-S^{n+p}(x),s(1-M)M^{p-1}) \geq$$
$$\geq F\Big(S(x)-x,(1-M)\frac{s}{M^n}\Big) \wedge F\Big(S(x)-x,(1-M)\frac{s}{M^n}\Big)\wedge$$
$$\wedge\cdots\wedge F\Big(S(x)-x,(1-M)\frac{s}{M^n}\Big) = F\Big(S(x)-x,(1-M)\frac{s}{M^n}\Big)\,.$$

As $\frac{s}{M^n} \to \infty$, we have that $F\big(S(x)-x,(1-M)\frac{s}{M^n}\big) \to 1$. Thus

$$(\forall)\delta \in (0,1), (\forall)s > 0, (\exists)n_0(\delta,s) \in \mathbb{N}^*\;:\; F\Big(S(x)-x,(1-M)\frac{s}{M^n}\Big) > 1-\delta\,.$$

Hence $\{S^n(x)\}$ is a G-V Cauchy sequence. Due to the fact that $\mathcal{X}$ is a fuzzy Banach space, there exists $y \in \mathcal{X}$ such that $\lim_{n\to\infty} S^n(x) = y$. One can observe that

$$y = \lim_{n\to\infty} S^{n+1}(x) = \lim_{n\to\infty} S(S^n(x)) = S(y).$$

For proving the uniqueness, we assume that there exists $z \in \mathcal{X}$, $z \neq y$ with $S(z) = z$. As $z \neq y$, there exists $s > 0$ such that $F(y-z,s) = a < 1$. Then, for any $n \in \mathbb{N}^*$, we have

$$a = F(y-z,s) = F(S^n(y)-S^n(z),s) \geq F\Big(y-z,\frac{s}{M^n}\Big) \to 1.$$

Thus $a = 1$, contradicting our assumption. □

The next theorem generalizes the Banach's contraction principle (for $S = I$) but also represents an extension in the context of FNLSs of a result obtained by G. Jungck [27].

Theorem 2. *Consider $(\mathcal{X}, F, \wedge)$, a fuzzy Banach space where $x_n \to x$ implies $F(x_n,s) \to F(x,s)$, for all $s > 0$. Let $U, V : \mathcal{X} \to \mathcal{X}$ be two commuting operators satisfying:*

1. *$U(\mathcal{X}) \subseteq V(\mathcal{X})$;*
2. *$(\exists)M \in (0,1)\;:\; F(U(x)-U(y),s) \geq F(V(x)-V(y),\frac{s}{M}), (\forall)x,y \in \mathcal{X}, (\forall)s > 0$;*
3. *U or V is fuzzy continuous.*

Then, U and V have a unique common fixed point.

Proof. Let $x_0 \in \mathcal{X}$. As $U(\mathcal{X}) \subseteq V(\mathcal{X})$, there exists $x_1 \in \mathcal{X}\;:\; U(x_0) = V(x_1)$. For $x_1 \in \mathcal{X}$, as $U(\mathcal{X}) \subseteq V(\mathcal{X})$, there exists $x_2 \in \mathcal{X}$, such that $U(x_1) = V(x_2)$. We obtain, in this way, a sequence (x_n) with the property $U(x_{n-1}) = V(x_n)$.

Step 1. We prove that $\{U(x_n)\}$ is a G-V Cauchy sequence.

First, we note that

$$F(U(x_{n+1})-U(x_n),s) \geq F\Big(V(x_{n+1})-V(x_n),\frac{s}{M}\Big) =$$
$$= F\Big(U(x_n)-U(x_{n-1}),\frac{s}{M}\Big).$$

By inductive reasoning, we obtain

$$F(U(x_{n+1}) - U(x_n), s) \geq F\Big(U(x_1) - U(x_0), \frac{s}{M^n}\Big).$$

Let $p \in \mathbb{N}^*$. We have that:

$$F(U(x_{n+p}) - U(x_n), s) \geq F(U(x_n) - U(x_{n+p}), s(1 - M^p)) =$$

$$F(U(x_n) - U(x_{n+1}) + U(x_{n+1}) - U(x_{n+2}) + \cdots + U(x_{n+p-1}) -$$

$$-U(x_{n+p}), s(1 - M)(1 + M + M^2 + \cdots M^{p-1})) \geq$$

$$\geq F(U(x_n) - U(x_{n+1}), s(1 - M)) \wedge F(U(x_{n+1}) - U(x_{n+2}), s(1 - M)M) \wedge$$

$$\wedge \cdots \wedge F(U(x_{n+p-1}) - U(x_{n+p}), s(1 - M)M^{p-1}) \geq$$

$$\geq F\Big(U(x_1) - U(x_0), (1 - M)\frac{s}{M^n}\Big) \wedge F\Big(U(x_1) - U(x_0), (1 - M)\frac{s}{M^n}\Big) \wedge$$

$$\wedge \cdots \wedge F\Big(U(x_1) - U(x_0), (1 - M)\frac{s}{M^n}\Big) = F\Big(U(x_1) - U(x_0), (1 - M)\frac{s}{M^n}\Big).$$

As $\frac{s}{M^n} \to \infty$, we have that $F\big(U(x_1) - U(x_0), (1 - M)\frac{s}{M^n}\big) \to 1$. Thus $(\forall)\delta \in (0, 1), (\forall)s > 0, (\exists)n_0(\delta, s) \in \mathbb{N}^* \; : \; F\big(U(x_1) - U(x_0), (1 - M)\frac{s}{M^n}\big) > 1 - \delta$. So, $\{U(x_n)\}$ is G-V Cauchy sequence in $\mathcal{X}$ (a complete space). Therefore, $(\exists)z \in \mathcal{X}$ with the property $\lim\limits_{n\to\infty} U(x_n) = z$ and $U(x_{n-1}) = V(x_n) \to z$.

Step 2. We note that $U(U(x_n)) \to U(z)$; $U(V(x_n)) \to U(z)$. Indeed, if U is fuzzy continuous, as $U(x_{n-1}) = V(x_n) \to z$, we obtain that $U(U(x_n)) \to U(z)$; $U(V(x_n)) \to U(z)$. If V is fuzzy continuous, as $U(x_n) \to z$, we obtain that $V(U(x_n)) \to V(z)$. Now, we have that

$$F(U(U(x_n)) - U(z), s) \geq F\Big(V(U(x_n)) - V(z), \frac{s}{M}\Big) \to 1.$$

Thus $U(U(x_n)) \to U(z)$. As

$$F(U(V(x_n)) - U(z), s) \geq F\Big(V(V(x_n)) - V(z), \frac{s}{M}\Big) \to 1,$$

we obtain that $U(V(x_n)) \to U(z)$.

Step 3. We will show that $U(z) = z$.

$$F(U(x_n) - U(U(x_n)), s) \geq F\Big(V(x_n) - V(U(x_n)), \frac{s}{M}\Big).$$

Since U and V commute, passing to the limit for $n \to \infty$, we obtain

$$F(z - U(z), s) \geq F\Big(z - U(z), \frac{s}{M}\Big).$$

By induction, we obtain that

$$F(z - U(z), s) \geq F\Big(z - U(z), \frac{s}{M^n}\Big).$$

Letting $n \to \infty$, we obtain $F(z - U(z), s) = 1, (\forall)s > 0$. Thus $U(z) = z$.

Step 4. We will show that $V(z) = z$.

As $U(\mathcal{X}) \subseteq V(\mathcal{X})$, it results there exists $v \in \mathcal{X} : V(v) = U(z)$. We have that

$$F(U(U(x_n)) - U(v)), s) \geq F\Big(V(U(x_n)) - V(v), \frac{s}{M}\Big).$$

Taking into account that U and V commute, for $n \to \infty$, it results in

$$F(z - U(v)), s) \geq F\Big(U(z) - V(v), \frac{s}{M}\Big) = 1.$$

Thus $U(v) = z$. Finally, $U(z) - V(z) = U(V(v)) - V(U(v)) = 0$. Therefore $U(z) = V(z) = z$.

Step 5. For uniqueness, consider $w \in \mathcal{X} \ : \ U(w) = V(w) = w$. We have that

$$F(z - w, s) = F(U(z) - U(w), s) \geq F\Big(V(z) - V(w), \frac{s}{M}\Big) = F\Big(z - w, \frac{s}{M}\Big).$$

Applying the inductive method, $F(z - w, s) = F\big(z - w, \frac{s}{M^n}\big) \to 1$. Thus $z = w$. $\square$

Corollary 1. *If $(\mathcal{X}, F, \wedge)$ is a fuzzy Banach space satisfying $x_n \to x$ implies $F(x_n, s) \to F(x, s)$, $(\forall)s > 0$, and $U, V : \mathcal{X} \to \mathcal{X}$ two commuting operators satisfying :*

1. $U(\mathcal{X}) \subseteq V(\mathcal{X})$;
2. $(\exists)N \in (0,1), (\exists)l \in \mathbb{N}^* \ : \ F(U^l(x) - U^l(y), s) \geq F\big(V(x) - V(y), \frac{s}{N}\big)$, $(\forall)x, y \in \mathcal{X}, (\forall)s > 0$;
3. *U or V is continuous,*

then both U and V have a unique common fixed-point.

Proof. It is obvious that U^l commutes with V and $U^l(\mathcal{X}) \subseteq U(\mathcal{X}) \subseteq V(\mathcal{X})$. By previous theorem, it exists $z \in \mathcal{X}$ with $U^l(z) = V(z) = z$. We have that $U^l(U(z)) = U(U^l(z)) = U(z)$. Thus, $U(z)$ is a fixed-point for U^l. In the same time $V(U(z)) = U(V(z)) = U(z)$. Thus, $U(z)$ is a fixed point for V. As z is unique, we have $U(z) = z$, so $U(z) = V(z) = z$. $\square$

The next theorem is an application of fuzzy Banach's contraction principle.

Theorem 3. *Given $(\mathcal{X}, F, \wedge)$, a fuzzy Banach space, where for any $x \neq 0$, $F(x, .)$ is strictly increasing, and $U : \mathcal{X} \to \mathcal{X}$ is fuzzy continuous with the property that $(\exists)\alpha, \beta \in (0,1)$:*

$$F(Ux - Uy, s) \geq F\Big(x - Ux, \frac{s}{\alpha}\Big) \wedge F\Big(y - Uy, \frac{s}{\beta}\Big), (\forall)x, y \in \mathcal{X}, (\forall)s > 0,$$

then U owns a unique fixed-point.

Proof. For $x \in \mathcal{X}$, $y = U(x)$, and $s > 0$, it follows:

$$F(Ux - U^2x, s) \geq F\Big(x - Ux, \frac{s}{\alpha}\Big) \wedge F\Big(Ux - U^2x, \frac{s}{\beta}\Big).$$

As $F\Big(Ux - U^2x, \frac{s}{\beta}\Big) > F(Ux - U^2x, s)$, for $Ux \neq U^2x$, it results that $F(Ux - U^2x, s) \geq F\big(x - Ux, \frac{s}{\alpha}\big)$. Using the same arguments such as in the proof of fuzzy Banach's contraction principle, we deduce that U has a fixed point.

Now, suppose that $z, w \in \mathcal{X}$ are two fixed-points for U. Then, it follows

$$F(Uz - Uw, s) \geq F(z - Uz, \frac{s}{\alpha}) \wedge F(w - Uw, \frac{s}{\beta})$$

for some $\alpha, \beta \in (0,1)$ and every $s > 0$, whence $F(z - w, s) = 1$, for all $s > 0$. Therefore, $z = w$, that completes the proof. $\square$

Theorem 4. *If $(\mathcal{X}, F, \wedge)$ is a fuzzy normed space such that $F(\cdot, s)$ is a continuous function for all $s > 0$ and $h : \mathcal{X} \to \mathcal{X}$ is fuzzy continuous such that it satisfies:*

1. $F(h(x_1) - h(x_2), s) > F(x_1 - x_2, s), (\forall)s > 0, (\forall)x_1, x_2 \in \mathcal{X}, x_1 \neq x_2$,
2. *The closure of $(h - I)(\mathcal{X})$ is compact and invariant to h,*

then h has a unique fixed point $\nu \in \overline{(h-I)(\mathcal{X})}$. Moreover, if for any $x_0 \in \overline{(h-I)(\mathcal{X})}$, the sequence $h^n(x_0)$ converges, then its limit is ν, where h^n represents the n-th iterate of h.

Proof. Fix $s>0$. From the hypothesis, the continuous function $x \mapsto F(h(x)-x,s)$ attains its maximum at some $\nu \in \overline{(h-I)(\mathcal{X})}$. If $h(\nu) \neq \nu$, we have

$$F(h(\nu)-\nu,s) = \max_{x \in \overline{(h-I)(\mathcal{X})}} F(h(x)-x,s) \geq F(h(h(\nu))-h(\nu),s) > F(h(\nu)-\nu,s),$$

which is impossible. Thus ν is a fixed-point for h.

If $x_1 \neq x_2$ are fixed-points of h, then

$$F(x_1-x_2,s) = F(h(x_1)-h(x_2),s) > F(x_1-x_2,s), (\forall)s>0,$$

which is impossible. Thus, $x_1 = x_2$. Therefore, ν is the unique fixed point of h and also it is a fixed point for h^n, for all $n \geq 2$.

Consider $x_0 \in \overline{(h-I)(\mathcal{X})}$ and $\lim\limits_{n\to\infty} h^n(x_0) = v$. Since h is fuzzy continuous, it results in $h(v) = \lim\limits_{n\to\infty} h^{n+1}(x_0) = v$, whence v is a fixed point for h. However, ν is the unique fixed point of h hence $v = \nu$. □

The following example proves the existence of a fuzzy continuous function on a fuzzy Banach space $(\mathcal{X}, F, \wedge)$, which verifies the hypothesis in Theorem 4, and this fuzzy continuous function has the property that $h-I$ is not surjective, contrary to the case where h is a fuzzy contraction.

Example 1. *In the space $(\mathcal{X}, F, \wedge)$, we consider*

$$F(z,s) = \begin{cases} \dfrac{s}{s+|z|} & \text{if } s>0, z \in \mathbb{R} \\ 0 & \text{if } s=0, z \in \mathbb{R} \end{cases}$$

Let $\rho : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$\rho(z) = \begin{cases} z - \dfrac{z^2}{z^2+1} & ; z \geq 0 \\ z + \dfrac{z^2}{z^2+1} & ; z < 0 \end{cases}$$

Then ρ is fuzzy continuous on $\mathbb{R}$, the closure of $(\rho - I)(\mathbb{R})$ is a compact set invariant to ρ and $F(\rho(z_1)-\rho(z_2),s) > F(z_1-z_2,s)$, for all $z_1, z_2 \in \mathbb{R}$, $z_1 \neq z_2$ and all $s>0$.

Indeed, it is easy to see that ρ is fuzzy continuous on $\mathbb{R}$, the closure of $(\rho - I)(\mathbb{R})$ is $[-1,1]$ and $\rho([-1,1]) = \left[-\dfrac{1}{2}, \dfrac{1}{2}\right] \subset [-1,1]$. Now, let $z_1, z_2 \geq 0, z_1 \neq z_2$ and $s>0$. We have

$$F(\rho(z_1)-\rho(z_2),s) = \frac{s}{s+|\rho(z_1)-\rho(z_2)|} = \frac{s}{s+|z_1 - \dfrac{z_1^2}{z_1^2+1} - z_2 + \dfrac{z_2^2}{z_2^2+1}|}$$

$$= \frac{s}{s+|(z_1-z_2)(1-\dfrac{z_1+z_2}{(z_1^2+1)(z_2^2+1)})|}.$$

Since $|1-\frac{z_1+z_2}{(z_1^2+1)(z_2^2+1)}|<1$, it results that $F(\rho(z_1)-\rho(z_2),s)>F(z_1-z_2,s)$. If $z_1,z_2<0, z_1\neq z_2$ and $s>0$, then

$$F(\rho(z_1)-\rho(z_2),s)=\frac{s}{s+|(z_1-z_2)(1+\frac{z_1+z_2}{(z_1^2+1)(z_2^2+1)})|}$$

As $|1+\frac{z_1+z_2}{(z_1^2+1)(z_2^2+1)}|<1$, it follows that $F(\rho(z_1)-\rho(z_2),s)>F(z_1-z_2,s)$. Finally, for $z_1\geq 0, z_2<0$ and $s>0$, we have

$$F(\rho(z_1)-\rho(z_2),s)=\frac{s}{s+|z_1-\frac{z_1^2}{z_1^2+1}-z_2-\frac{z_2^2}{z_2^2+1}|}.$$

Since $2y-\frac{y^2}{y^2+1}>0, (\forall)y>0$, we deduce $|z_1-z_2|>|z_1-\frac{z_1^2}{z_1^2+1}-z_2+\frac{z_2^2}{z_2^2+1}|$, whence $F(\rho(z_1)-\rho(z_2),s)>F(z_1-z_2,s)$.

In the following theorem, the property of the continuous dependence of fixed points with respect to a parameter is presented.

Theorem 5. *Let $(\mathcal{X},F,\wedge)$ be a fuzzy Banach space, where $F(x,.)$ is strictly increasing for any $x\neq 0$, and let ψ be a Hausdorff topological space. If $h:\mathcal{X}\times\psi\longrightarrow\mathcal{X}$ has the properties:*

1. *the mapping $h_x:\psi\longrightarrow\mathcal{X}$, $h_x(y)=h(x,y)$ is continuous for all $x\in\mathcal{X}$,*
2. *there exists $\alpha\in(0,\frac{1}{2})$ such that $F(h(x_1,y)-h(x_2,y),s)\geq F(x_1-x_2,\frac{s}{\alpha})$ for any $x_1,x_2\in\mathcal{X}$, any $y\in\psi$ and any $s>0$,*

then, for all $y\in\psi$, the mapping $h_y:\mathcal{X}\longrightarrow\mathcal{X}, h_y(x)=h(x,y)$ has a unique fixed point x_y^*, and the function $\phi:\psi\longrightarrow\mathcal{X},\phi(y)=x_y^*$ is continuous.

Proof. Since, for each $y\in\psi, h_y$ is a fuzzy contraction, it follows from Theorem 1 that h_y has a unique fixed point x_y^*. Consider $y_0,y\in\psi,s>0$. We have

$$\begin{aligned}F(\phi(y)-\phi(y_0),s)&=F(x_y^*-x_{y_0}^*,s)=F(h_y(x_y^*)-h_{y_0}(x_{y_0}^*),s)=\\&=F(h_y(x_y^*)-h_y(x_{y_0}^*)+h_y(x_{y_0}^*)-h_{y_0}(x_{y_0}^*),s)\geq\\&\geq F(h_y(x_y^*)-h_y(x_{y_0}^*),\frac{s}{2})\wedge F(h_y(x_{y_0}^*)-h_{y_0}(x_{y_0}^*),\frac{s}{2})\geq\\&\geq F(x_y^*-x_{y_0}^*,\frac{s}{2\alpha})\wedge F(h_y(x_{y_0}^*)-h_{y_0}(x_{y_0}^*),\frac{s}{2}).\end{aligned}$$

Now, taking into account the hypothesis about the fuzzy norm F, we deduce that $F(\phi(y)-\phi(y_0),s)\geq F(h_y(x_{y_0}^*)-h_{y_0}(x_{y_0}^*),\frac{s}{2})$, for any $y,y_0\in\psi$ and any $s>0$. We prove that ϕ is continuous. Let $y_0\in\psi$. As $h_{x_{y_0}^*}$ is continuous at y_0, it follows that, for any $\delta\in(0,1)$ and for every $s>0$, there exists a neighborhood V of y_0 with the property $F(h_y(x_{y_0}^*)-h_{y_0}(x_{y_0}^*),s)=F(h_{x_{y_0}^*}(y)-h_{x_{y_0}^*}(y_0),s)>1-\delta$, for each $y\in V$, whence, for any $\delta\in(0,1)$ and for every $s>0$, there exists a neighborhood V of y_0, such that $F(\phi(y)-\phi(y_0),s)>1-\delta$, for each $y\in V$. Therefore, ϕ is continuous at y_0, whence ϕ is continuous on ψ. □

Theorem 6. *Let $(\mathcal{X},F,\wedge)$ be a fuzzy Banach space, where, for any $u\neq 0$, $F(u,.)$ is strictly increasing. If $\phi:\mathcal{X}\to\mathcal{X}$ is fuzzy continuous, such that $(\exists)a,b\in(0,1),a+b<1:$*

$$F(\phi u-\phi v,s)\geq F\Big(u-\phi v,\frac{s}{a}\Big)\wedge F\Big(v-\phi u,\frac{s}{b}\Big),(\forall)u,v\in\mathcal{X},(\forall)s>0,$$

then ϕ has a fixed-point.

Proof. Suppose that $a \leq b$. Let $u \in \mathcal{X}$ and $v = \phi u$. For $s > 0$, we have:

$$F(\phi u - \phi^2 u, s) \geq F\left(u - \phi^2 u, \frac{s}{a}\right) \wedge F\left(\phi u - \phi u, \frac{s}{b}\right) = F\left(u - \phi^2 u, \frac{s}{a}\right) =$$

$$= F\left((u - \phi u) + (\phi u - \phi^2 u), \frac{s}{a+b} + \frac{s}{\frac{a(a+b)}{b}}\right) \geq$$

$$\geq F\left(u - \phi u, \frac{s}{a+b}\right) \wedge F\left(\phi u - \phi^2 u, \frac{s}{\frac{a(a+b)}{b}}\right).$$

From $a + b < 1$ and the hypothesis on F, it results
$F(\phi u - \phi^2 u, s) < F\left(\phi u - \phi^2 u, \frac{s}{\frac{a(a+b)}{b}}\right)$, for $\phi u \neq \phi^2 u$.
Thus, we deduce $F(\phi u - \phi^2 u, s) \geq F\left(u - \phi u, \frac{s}{a+b}\right), (\forall) s > 0$.
Similarly, for $b \leq a$, we obtain $F(\phi u - \phi^2 u, s) \geq F\left(u - \phi u, \frac{s}{a+b}\right), (\forall) s > 0$.
Such as in the proof of Theorem 3, it results that ϕ has a fixed point. □

A similar result to the one given in Theorem 3 is pointed out.

Theorem 7. *If, in a fuzzy Banach space, $(\mathcal{X}, F, \wedge)$, $F(u, .)$ is strictly increasing, for any $u \neq 0$, and if $\phi : \mathcal{X} \to \mathcal{X}$ is fuzzy continuous such that $(\exists) a, b, c \in (0, 1)$:*

$$F(\phi u - \phi v, s) \geq F\left(u - v, \frac{s}{a}\right) \wedge F\left(u - \phi u, \frac{s}{b}\right) \wedge F\left(v - \phi v, \frac{s}{c}\right),$$

$(\forall) u, v \in \mathcal{X}, (\forall) s > 0$, *then ϕ has a unique fixed point.*

Proof. Consider $u \in \mathcal{X}$ and $v = \phi u$. For $s > 0$, we have

$$F(\phi u - \phi^2 u, s) \geq F\left(u - \phi u, \frac{s}{a}\right) \wedge F\left(u - \phi u, \frac{s}{b}\right) \wedge F\left(\phi u - \phi^2 u, \frac{s}{c}\right).$$

Since $F\left(\phi u - \phi^2 u, \frac{s}{c}\right) > F(\phi u - \phi^2 u, s)$, for $\phi u \neq \phi^2 u$, it results that

$$F(\phi u - \phi^2 u, s) \geq F\left(u - \phi u, \min\left\{\frac{s}{a}, \frac{s}{b}\right\}\right).$$

Similar to the proof of Theorem 3, we deduce that ϕ has a fixed point $\nu \in \mathcal{X}$. Now, if we assume that $\phi\nu = \nu$ and $\phi\mu = \mu$, it follows that

$$F(\nu - \mu, s) = F(\phi\nu - \phi\mu, s) \geq F\left(\nu - \mu, \frac{s}{a}\right) \wedge 1 \wedge 1 = F\left(\nu - \mu, \frac{s}{a}\right), (\forall) s > 0.$$

Thus, $F(\nu - \mu, s) \geq F\left(\nu - \mu, \frac{s}{a^n}\right), (\forall) s > 0, (\forall) n \in \mathbb{N}^*$. Passing the limit for $n \to \infty$, it results that $F(\nu - \mu, s) = 1, (\forall) s > 0$. Therefore, $\nu = \mu$. □

4. An Application of Fuzzy Banach Fixed-Point Principle to Dynamic Programming

There are a great number of processes that exist when ordering large amounts of goods in order to satisfy an unknown demand. Consider a process where the storage of a single type of goods is requested. After an order is placed and fulfilled, a request for that good is produced. This request is possibly satisfied, but, if demand is bigger than the goods in stock, a penalty cost is applied. Here, there are several constants and positive functions involved, such as: k-the minimum order cost to elevate the stock level, $p(z)$-the order cost of z units of goods for avoiding an excess z of the order amount in relation to stock, also known as "penalty cost". Our purpose is to determine an order strategy for minimizing the overall probable cost by the means of solving a differential equation.

In dynamic programming, we come across functional equations of the form:

$$h(u) = \max_{v} H(u, v, h(G(u, v))) \tag{1}$$

with u being the state variable, v the decision variable, and h the optimum function.

In the study of these equations, we can use fixed-point theorems. As an example, we will investigate the following equation, known as the "equation the optimal distribution of supplies":

$$h(u) = \inf_{v \geq u} H(u, v, h), \tag{2}$$

where

$$H(u, v, h) = a(u - v) + k \int_{v}^{\infty} p(\lambda - v) g(\lambda) d\lambda + kh(0) \int_{v}^{\infty} g(\lambda) d\lambda + k \int_{0}^{v} h(v - \lambda) g(\lambda) d\lambda. \tag{3}$$

Here, a, p, g are given functions, $k \in \mathbb{R}^{*}{}_{+}$, and h is the unknown function.

We will search for the existence of the solution to this equation in a subset of $\mathcal{C}(\mathbb{R}_+)$ (the linear space of continuous functions on $\mathbb{R}_+$). For this purpose, we denote $\mathcal{BC}(\mathbb{R}_+)$(the subspace of bounded functions from $\mathcal{C}(\mathbb{R}_+)$).

We consider the Cebîşev norm:

$$\|h\| = \sup_{x \in \mathbb{R}_+} |h(x)|$$

and there the fuzzy norm $F(h, s) = \begin{cases} 0 & , \; s \leq 0 \\ \frac{s}{s + \|h\|} & , \; s > 0 \end{cases}.$

Then, $(\mathcal{BC}(\mathbb{R}_+), F, \wedge)$ forms a fuzzy normed space.

Now, we define the application $A : \mathcal{BC}(\mathbb{R}_+) \to \mathcal{BC}(\mathbb{R}_+)$ using:

$$(Ah)(u) = \inf_{v \geq u} H(u, v, h).$$

In order for $\mathcal{BC}(\mathbb{R}_+)$ to be an invariant subset for A, we will make the following assumptions on the hypothesis of the problem:

(i) $g \in \mathcal{L}^1(\mathbb{R}_+)$ and $\int_{0}^{\infty} g(\lambda) d\lambda = 1$;

(ii) $p \in \mathcal{C}(\mathbb{R}_+)$, $\int_{0}^{\infty} p(\lambda) g(\lambda) d\lambda < \infty$;

with these conditions, from (3) it results that $A : \mathcal{BC}(\mathbb{R}_+) \to \mathcal{BC}(\mathbb{R}_+)$ is correctly defined. It remains to verify the contraction condition for A. For that, consider the chain of relationships:

$$F(A(h_1) - A(h_2), s) = \frac{s}{s + \sup\limits_{u \in \mathbb{R}_+} (A(h_1(u)) - A(h_2(u)))} =$$

$$= \frac{s}{s + \sup\limits_{u \in \mathbb{R}_+} |\inf\limits_{v \geq u} H(u, v, h_1) - \inf\limits_{v \geq u} H(u, v, h_2)|} \geq$$

$$\geq \frac{s}{s + \sup\limits_{u \in \mathbb{R}_+} \sup\limits_{v \geq u} |H(u, v, h_1) - H(u, v, h_2)|} \geq$$

$$\frac{s}{s + k \cdot \sup\limits_{u \in \mathbb{R}_+} \sup\limits_{v \geq u} (|h_1(0) - h_2(0)| \cdot \int_{v}^{\infty} g(\lambda) d\lambda + \int_{0}^{v} |h_1(v - \lambda) - h_2(v - \lambda)| g(\lambda) d\lambda)} \geq$$

$$\frac{s}{s + k \cdot \sup_{u \in \mathbb{R}_+} \sup_{v \geq u} (|h_1(0) - h_2(0)| \cdot \int_v^\infty g(\lambda) d\lambda + \sup_{u \in \mathbb{R}_+} |h_1(v-u) - h_2(v-u)| \int_0^v g(\lambda) d\lambda)}$$

$$\geq \frac{s}{s + k \cdot \sup_{u \in \mathbb{R}_+} \sup_{v \geq u} (|h_1(0) - h_2(0)| \cdot \int_v^\infty g(\lambda) d\lambda + \sup_{u \in \mathbb{R}_+} |h_1(u) - h_2(u)| \int_0^v g(\lambda) d\lambda)} \geq$$

$$\geq \frac{s}{s + k \cdot \sup_{u \in \mathbb{R}_+} \sup_{v \geq u} (|h_1(u) - h_2(u)| \cdot (\int_v^\infty g(\lambda) d\lambda + \int_0^v g(\lambda) d\lambda))} \geq$$

$$\geq \frac{s}{s + k \cdot \|h_1 - h_2\|} = F(h_1 - h_2, \frac{s}{k}),$$

for all $s > 0$, $h_1, h_2 \in \mathcal{BC}(\mathbb{R}_+)$. The case where the fuzzy norm F is taken for $s \leq 0$ is trivial.

Hence, for $k \in \mathbb{R}^*{}_+$, we have

$$(F(A(h_1) - A(h_2)), s) \geq F(h_1 - h_2, \frac{s}{k}), (\forall) h_1, h_2 \in \mathcal{BC}(\mathbb{R}_+), (\forall) s \geq 0.$$

Now, if $k \in (0,1)$, the operator A becomes a fuzzy contraction, then, from the Banach fuzzy fixed-point principle, the Equation (2) has only one solution in $\mathcal{BC}(\mathbb{R}_+)$ that can be obtained by the successive approximation method, starting from any element from $\mathcal{BC}(\mathbb{R}_+)$.

5. Conclusions

We consider the above results relevant, especially within the branch of functional analysis in the framework of fuzzy spaces. Moreover, previously solved real-life problems and maybe some unsolved ones can be considered now in a different framework, using fixed-point theory on FNLSs. Such an endeavor was pursued in the above context by applying the fuzzy Banach fixed-point principle to a problem of dynamic programming. Following the work of A. George and P. Veeramani in the context of fuzzy metric spaces, we applied their definition for the Cauchy sequence in fuzzy normed spaces, which leads to another concept of completeness much more adequate than the Grabiec's one. In these fuzzy Banach spaces, we proved several theorems regarding fixed points occurring to contractive mappings, but the study will be continued in our following articles, and we dare to hope that other researchers will continue to investigate this new direction.

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