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Some Fixed Point Results in E-Metric Spaces and an Application

Xinjie Shi [1], Pinhong Long [1,*], Huaping Huang [2] and Fengjun Li [1]

1 School of Mathematics and Statistics, Ningxia University, Yinchuan 750021, China
2 School of Mathematics and Statistics, Chongqing Three Gorges University, Wanzhou 404020, China
* Correspondence: longph@nxu.edu.cn

Abstract: The purpose of this paper is to study several fixed point problems in E-metric spaces. Mainly, we show the existence and uniqueness of fixed points for two contractive mappings, including Ćirić type contraction and α-ψ type contraction in E-metric spaces. Furthermore, we provide examples to support the accuracy of our results and present an application of our solution to a class of differential equations.

Keywords: E-metric space; Ćirić type contraction; α-ψ type contraction; e-Cauchy sequence; fixed point

MSC: 47H09; 47H10

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1. Introduction

The fixed point theory is a beautiful mixture of analysis, topology, and geometry. Over several decades the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena. In particular, fixed point theory has been applied to cope with the solutions to problems in functional equations, ordinary differential equations, integral equations, fractional equations, and more (see [1–9]). It has been applied in such diverse fields as biology, chemistry, economics, engineering, game theory, physics, and logic programming. One of the most celebrated fixed point theorems is the Banach contraction mapping principle (see [10]) or Banach fixed point theorem, which is stated as follows.

Let (X, ρ) be a complete metric space. Suppose that the mapping $f : X \to X$ is a Banach-type contraction, i.e., it satisfies

$$\rho(f\xi, f\eta) \leq \lambda\rho(\xi, \eta),$$

for all $\xi, \eta \in X$, where $\lambda \in [0, 1)$ is a constant. Then, the mapping f has a unique fixed point in X.

This principle has subsequently been developed further, including the presentation of the iteration sequence. In 1975, Kramosil and Michalek [11] considered fuzzy metric space, which is a generalization of typical metric space, and extended the relevant topological concepts, leading to a great many applications in different areas; readers may refer to [9] and the references therein. In 2007, Huang and Zhang [12] introduced cone metric space, which greatly generalizes metric space. Moreover, they obtained fixed point theorems for Banach-type contraction, Kannan-type contraction, and Chatterjea-type contraction. Afterwards, a large number of fixed point results in cone metric spaces were presented (see [13–15]). In 2015, cone metric properties were combined with fuzzy sets in metric space to deduce a new space called fuzzy cone metric space. This developmental contribution was established by Oner et al. [16], who discussed topological properties and studied fixed point results with applications under certain conditions in such spaces. Utilizing this concept, several different authors (see [8]) have considered various mappings, such as compatible and weakly compatible mappings, coupled contractive type mappings, quasi-contraction mappings, and rational contraction mappings, along with their applications, to study the existence of solutions for a number of different integral equations in fuzzy cone metric spaces.

In 2012 Rawashdeh et al. [17] defined an ordered space called E-metric space, which is similar to cone metric space, and proved that the contractive sequence is a Cauchy sequence in E-metric spaces. In 2013, Pales and Petre [18] introduced the concept of strict positivity in Riesz spaces and presented a multi-valued nonlinear fixed point theorem in E-metric spaces, generalizing the fixed point theorems obtained by Wegrzyk [1], Cevik and Altun [19], Critescu [20], and Matkowski [21]. In 2019, Huang [7] used semi-interior points in cones to generalize the fixed point theorems of Hardy–Rogers type contraction in E-metric spaces.

At present, there are few research results on fixed point theorems in E-metric spaces. In this paper, we obtain the existence and uniqueness of fixed points for Ćirić-type contraction [22] in E-metric spaces. In addition, we demonstrate the existence and uniqueness of fixed points for α-ψ-type contraction in E-metric spaces. We consider these to be new results, as thus far there have been no fixed point results presented for Ćirić-type contraction in E-metric spaces. In addition, it is well known that E-metric spaces greatly generalize metric spaces, cone metric spaces, and certain other spaces. From this viewpoint, our fixed point results in E-metric spaces have profound and far-reaching significance. Furthermore, for the sake of application, we provide the solutions to a class of differential equations.

2. Preliminaries

In this paper, without special explanations, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}^*$, and $\mathbb{N} = \mathbb{N}^* \cup \{0\}$ denote the set of all real numbers, the set of all nonnegative real numbers, the set of positive integers, and the set of all nonnegative integers, respectively.

In this section, we recall several basic concepts which are needed in the following sections.

Definition 1 ([12])**.** *Suppose that E is a Banach space, θ_E is the zero element of E, and P is a non-empty closed subset of E. If:*

(1) $P \neq \{\theta_E\}$;

(2) $\alpha, \beta \in \mathbb{R}^+ \Rightarrow \alpha P + \beta P \subseteq P$;

(3) $P \bigcap (-P) = \{\theta_E\}$,

then P is called a geometrical cone in E (in short, a cone). If $intP \neq \emptyset$, then P is said to be a solid cone, that is, $intP$ denotes the set of all interior points of P.

We say that "$\preceq$" and "$\ll$" are two partial orders in E if

$$\xi, \eta \in E \text{ and } \xi \preceq \eta \Leftrightarrow \eta - \xi \in P,$$

and

$$\xi, \eta \in E \text{ and } \xi \ll \eta \Leftrightarrow \eta - \xi \in \mathrm{int}P.$$

If there is a constant $M > 0$ such that $\theta_E \preceq \xi \preceq \eta$ implies

$$\|\xi\| \leq M\|\eta\|, \text{ for all } \xi, \eta \in P,$$

then P is called a normal cone in E (see [12]), where the least constant satisfying the above inequality is called the normal constant of P.

As an example, take $P = [0, +\infty)$ and $E = \mathbb{R}$; then, P is a cone in E, as it satisfies Definition 1, where $\xi \preceq \eta$ ("$\preceq$" is exactly "$\leq$") if and only if $\eta - \xi \in P$.

Definition 2 ([12])**.** *Let E be a Banach space, θ_E be the zero element of E, and E^+ be a non-empty closed convex subset of E. Then, E^+ is called a positive cone if:*

(1) $\xi \in E^+, \alpha \geq 0 \Rightarrow \alpha\xi \in E^+$;

(2) $\xi \in E^+, -\xi \in E^+ \Rightarrow \xi = \theta_E$.

Let $\xi_0 \in E^+$. If there exists $\alpha > 0$ such that $\xi_0 - \alpha U_+ \subseteq E^+$, then ξ_0 is called a semi-interior point in E^+ (see [23]). Denote

$$U = \{\xi \in E : \|\xi\| \leq 1\}$$

as the closed unit ball of E and

$$U_+ = U \cap E^+$$

as the positive part of U.

Definition 3 ([17])**.** *Let E be a real normed space with a norm $\|\cdot\|$. If the following conditions hold:*

(1) for all $\xi, \eta, \zeta \in X$, $\xi \preceq \eta \Rightarrow \xi + \zeta \preceq \eta + \zeta$;
(2) for any $\alpha \geq 0$, $\xi \in E$, $\theta_E \preceq \xi \Rightarrow \theta_E \preceq \alpha\xi$,
then E is called a real ordered vector space.

Definition 4 ([17])**.** *Let X be a nonempty set and E be a real normed space. The mapping $d^E : X \times X \to E$ is said to be an E-metric if, for all $\xi, \eta, \zeta \in X$, it satisfies*

(i) $\theta_E \preceq d^E(\xi, \eta)$, $d^E(\xi, \eta) = \theta_E \Leftrightarrow \xi = \eta$;
(ii) $d^E(\xi, \eta) = d^E(\eta, \xi)$;
(iii) $d^E(\xi, \eta) \preceq d^E(\xi, \zeta) + d^E(\zeta, \eta)$.
In this case, the pair (X, d^E) is called an E-metric space.

Remark 1. *With regard to the topology of E-metric spaces, especially for the properties of countability, Hausdorffness, and nets, readers may refer to [17,23].*

Both here and subsequently, we denote by $(E^+)^\ominus$ the set of all semi-interior points of E^+. We say $\lll$ is a partial order on E^+ if

$$\xi, \eta \in E^+, \ \ \xi \lll \eta \Leftrightarrow \eta - \xi \in (E^+)^\ominus.$$

Definition 5 ([24])**.** *Let (X, d^E) be an E-metric space, $\{\xi_n\}$ be a sequence in X, and $\xi \in X$, $(E^+)^\ominus \neq \emptyset$. We then say:*

(i) $\{\xi_n\}$ is e-convergent to ξ if for any $e \ggg \theta_E$, there exists $N \in \mathbb{N}^*$ such that $d^E(\xi_n, \xi) \lll e$ for all $n > N$. We denote $\xi_n \to \xi$ as $n \to \infty$;

(ii) $\{\xi_n\}$ is an e-Cauchy sequence if for any $e \ggg \theta_E$, there exists $N \in \mathbb{N}^*$ such that $d^E(\xi_n, \xi_m) \lll e$ for all $n, m > N$;

(iii) (X, d^E) is e-complete if every e-Cauchy sequence is e-convergent to some point in X.

Theorem 1 ([24])**.** *Suppose that (X, d^E) is an e-complete E-metric space and $(E^+)^\ominus \neq \emptyset$. If the mapping $f : X \to X$ satisfies*

$$d^E(f\xi, f\eta) \preceq \lambda d^E(\xi, \eta) \text{ for all } \ \xi, \eta \in X$$

where $\lambda \in [0, 1)$, then f has a unique fixed point in X.

Definition 6 ([7])**.** *A sequence $\{\xi_n\}$ in E^+ is said to be an e-sequence if for each $e \ggg \theta_E$ there exists $N \in \mathbb{N}^*$ such that $\xi_n \lll e$ for all $n > N$.*

Lemma 1 ([7])**.** *Let $\{\xi_n\}$ and $\{\eta_n\}$ be two sequences in E such that*

$$\xi_n \preceq \eta_n \ \text{ and } \ \eta_n \to \theta_E \ (n \to \infty).$$

Then, $\{\xi_n\}$ is an e-sequence.

Lemma 2 ([7]). *Let $\{\xi_n\}$ and $\{\eta_n\}$ be e-sequences in E and let $\alpha, \beta \geq 0$ be constants. Then, $\{\alpha\xi_n + \beta\eta_n\}$ is an e-sequence in E.*

Lemma 3 ([7]). *Let $x, y, z \in E$ and $x \preceq y \lll z$; then, $x \lll z$.*

Lemma 4 ([7]). *If $\theta_E \preceq u \lll e$ for any $e \ggg \theta_E$, then $u = \theta_E$.*

Lemma 5 ([15]). *If $0 \leq \lambda < 1$ is a constant and $\theta_E \preceq u \preceq \lambda u$, then $u = \theta_E$.*

Lemma 6 ([12]). *Let (X, d^E) be an E-metric space with a normal cone and let $\xi, \eta \in X$, $\{\xi_n\}$, $\{\eta_n\}$ be sequences in X such that $\xi_n \to \xi$ and $\eta_n \to \eta$, as $n \to \infty$. Then, $d^E(\xi_n, \eta_n) \to d^E(\xi, \eta)$, as $n \to \infty$.*

Lemma 7 ([7]). *Let $\xi, \eta \in E$ and $\xi \lll \eta + e$ for each $e \ggg \theta_E$; then, $\xi \lll \eta$.*

Definition 7 ([25]). *For a nonempty set X, let $\alpha : X \times X \to [0, +\infty)$ be a function and $f : X \to X$ be a mapping. Then, f is said to be an α-admissible function if, for any $\xi, \eta \in X$, it satisfies*

$$\alpha(\xi, \eta) \geq 1 \Rightarrow \alpha(f\xi, f\eta) \geq 1.$$

Definition 8 ([2]). *For a nonempty set X, $\{\xi_n\}$ is a sequence in X, $\xi \in X$. Suppose that $\alpha : X \times X \to [0, +\infty)$ is a function. Then, X is said to be α-regular if for any $n \in \mathbb{N}^*$ it satisfies*

$$\left.\begin{array}{l} \alpha(\xi_n, \xi_{n+1}) \geq 1 \\ \xi_n \to \xi \ (n \to \infty) \end{array}\right\} \Rightarrow \alpha(\xi_n, \xi) \geq 1.$$

Definition 9 ([26]). *Let X be a nonempty set, $s \geq 1$ be a constant, and $d : X \times X \to \mathbb{R}$ be a mapping. If, for any $\xi, \eta, \zeta \in X$, the following conditions hold:*
(i) $d(\xi, \eta) \geq 0$, $d(\xi, \eta) = 0 \Leftrightarrow \xi = \eta$;
(ii) $d(\xi, \eta) = d(\eta, \xi)$;
(iii) $d(\xi, \eta) \leq s[d(\xi, \zeta) + d(\zeta, \eta)]$,
then d is called a b-metric and the pair (X, d) is called a b-metric space.

Definition 10 ([27]). *Suppose that (X, d) is a b-metric space, $\{\xi_n\}$ is a sequence in X, and $\xi \in X$. We then say that:*
(1) $\{\xi_n\}$ is convergent to ξ if $\lim\limits_{n\to\infty} d(\xi_n, \xi) = 0$, i.e., $\lim\limits_{n\to\infty} \xi_n = \xi$ or $\xi_n \to \xi$ as $n \to \infty$;
(2) $\{\xi_n\}$ is a Cauchy sequence if $\lim\limits_{n,m\to\infty} d(\xi_n, \xi_m) = 0$;
(3) (X, d) is complete if every Cauchy sequence is convergent to some point in X.

Theorem 2 ([5]). *Suppose that (X, d) is a complete b-metric space with the parameter $s > 1$, $\beta : [0, +\infty) \to [0, 1)$ is a function, $\psi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function, and $f : X \to X$ is an α-admissible function such that*

$$\alpha(\xi, \eta)\psi(s^\varepsilon d(f\xi, f\eta)) \leq \beta(\psi(M(\xi, \eta)))\psi(M(\xi, \eta)), \ \text{for all} \ \xi, \eta \in X,$$

where $M(\xi, \eta) = \max\{d(\xi, \eta), d(\xi, f\xi), d(\eta, f\eta)\}$, $\varepsilon > 0$, $\alpha : X \times X \to [0, +\infty)$ is a mapping. If there exists $\xi_0 \in X$ such that $\alpha(\xi_0, f\xi_0) \geq 1$, and one of the following conditions holds:
(1) f is continuous, or
(2) X is α-regular,
then f has a fixed point in X.

3. Main Results

First, motivated by Theorem 1, we aim to consider the existence and uniqueness of fixed points in E-metric space if the following Ćirić-type contractive condition is satisfied:

$$d^E(f\xi, f\eta) \preceq \lambda M_1(\xi, \eta),$$

where

$$M_1(\xi, \eta) \in \{d^E(\xi, \eta), d^E(\xi, f\xi), d^E(\xi, f\eta), d^E(\eta, f\xi), d^E(\eta, f\eta)\}. \tag{1}$$

Theorem 3. *Let (X, d^E) be an e-complete E-metric space and let $(E^+)^{\ominus} \neq \emptyset$ and P be a cone in E. If the mapping $f : X \to X$ satisfies the following Ćirić-type contractive condition:*

$$d^E(f\xi, f\eta) \preceq \lambda M_1(\xi, \eta), \ \textit{for all} \ \xi, \eta \in X, \tag{2}$$

where $\lambda \in [0, \frac{1}{2})$ and $M_1(\xi, \eta)$ are the same as in (1), then f has a unique fixed point in X.

Proof. Choose $\xi_0 \in X$ and construct the Picard iterative sequence $\{\xi_n\}$ by $\xi_1 = f\xi_0$, $\xi_2 = f\xi_1, \cdots, \xi_n = f\xi_{n-1}, \cdots$. If there exists $n_0 \in \mathbb{N}$ such that $\xi_{n_0+1} = f\xi_{n_0} = \xi_{n_0}$, then ξ_{n_0} is a fixed point of f. Thus, the proof is completed. Without loss of generality, we assume that $\xi_{n+1} \neq \xi_n$ for any $n \in \mathbb{N}$. Taking advantage of (2), we can conclude that

$$d^E(\xi_{n+1}, \xi_{n+2}) = d^E(f\xi_n, f\xi_{n+1}) \preceq \lambda M_1(\xi_n, \xi_{n+1}), \tag{3}$$

where

$$\begin{aligned} M_1(\xi_n, \xi_{n+1}) \in &\{d^E(\xi_n, \xi_{n+1}), d^E(\xi_n, f\xi_n), d^E(\xi_n, f\xi_{n+1}), \\ &d^E(\xi_{n+1}, f\xi_n), d^E(\xi_{n+1}, f\xi_{n+1})\} \\ =&\{d^E(\xi_n, \xi_{n+1}), d^E(\xi_n, \xi_{n+2}), \theta_E, d^E(\xi_{n+1}, \xi_{n+2})\}. \end{aligned}$$

We discuss (3) as follows:
(A) If $M_1(\xi_n, \xi_{n+1}) = d^E(\xi_n, \xi_{n+1})$, we have

$$d^E(\xi_{n+1}, \xi_{n+2}) \preceq \lambda d^E(\xi_n, \xi_{n+1}),$$

which follows that

$$\begin{aligned} d^E(\xi_n, \xi_{n+1}) &\preceq \lambda d^E(\xi_{n-1}, \xi_n) \preceq \lambda^2 d^E(\xi_{n-2}, \xi_{n-1}) \\ &\preceq \cdots \preceq \lambda^n d^E(\xi_0, \xi_1). \end{aligned} \tag{4}$$

Subsequently, according to (4) and Condition (iii) in Definition 4, for any $m, n \in \mathbb{N}^*$, $m > n$, we have

$$\begin{aligned} d^E(\xi_n, \xi_m) &\preceq d^E(\xi_n, \xi_{n+1}) + d^E(\xi_{n+1}, \xi_m) \\ &\preceq d^E(\xi_n, \xi_{n+1}) + d^E(\xi_{n+1}, \xi_{n+2}) + \cdots + d^E(\xi_{m-1}, \xi_m) \\ &\preceq \lambda^n d^E(\xi_0, \xi_1) + \lambda^{n+1} d^E(\xi_0, \xi_1) + \cdots + \lambda^{m-1} d^E(\xi_0, \xi_1) \\ &= \lambda^n (1 + \lambda + \lambda^2 + \cdots + \lambda^{m-n-1}) d^E(\xi_0, \xi_1) \\ &= \lambda^n \frac{1 - \lambda^{m-n}}{1 - \lambda} d^E(\xi_0, \xi_1) \\ &\preceq \frac{\lambda^n}{1 - \lambda} d^E(\xi_0, \xi_1) \to \theta_E \ (n \to \infty). \end{aligned} \tag{5}$$

Using (5) and Lemma 1, we can be sure that $\{\xi_n\}$ is an e-Cauchy sequence in X.

(B) If $M_1(\xi_n, \xi_{n+1}) = d^E(\xi_n, \xi_{n+2})$, we obtain

$$d^E(\xi_{n+1}, \xi_{n+2}) \preceq \lambda d^E(\xi_n, \xi_{n+2}) \preceq \lambda[d^E(\xi_n, \xi_{n+1}) + d^E(\xi_{n+1}, \xi_{n+2})],$$

and furthermore, we have

$$d^E(\xi_{n+1}, \xi_{n+2}) \preceq \frac{\lambda}{1-\lambda} d^E(\xi_n, \xi_{n+1}).$$

Take $k = \frac{\lambda}{1-\lambda}$, then $k \in [0, 1)$. Thus, from the proof of (A), we know that $\{\xi_n\}$ is an e-Cauchy sequence.

(C) If $M_1(\xi_n, \xi_{n+1}) = \theta_E$, then by combining (3) and Condition (i) in Definition 4 we have $d^E(\xi_{n+1}, \xi_{n+2}) = \theta_E$, which contradicts our hypothesis.

(D) If $M_1(\xi_n, \xi_{n+1}) = d^E(\xi_{n+1}, \xi_{n+2})$, then

$$d^E(\xi_{n+1}, \xi_{n+2}) \preceq \lambda d^E(\xi_{n+1}, \xi_{n+2}),$$

which means that

$$(1-\lambda) d^E(\xi_{n+1}, \xi_{n+2}) \preceq \theta_E.$$

On account of $\lambda \in [0, \frac{1}{2})$, $d^E(\xi_{n+1}, \xi_{n+2}) = \theta_E$. This result conflicts with our hypothesis.

In summary, we claim that $\{\xi_n\}$ is an e-Cauchy sequence. Because (X, d^E) is an e-complete E-metric space, there exists $\xi \in X$ such that $\xi_n \to \xi$ as $n \to \infty$, which is to say that $\{d^E(\xi_n, \xi)\}$ is an e-sequence in E.

In the following, we prove that f has a fixed point.

Combining (2) and Condition (iii) in Definition 4, we conclude that

$$\begin{aligned} d^E(f\xi, \xi) &\preceq d^E(f\xi, \xi_n) + d^E(\xi_n, \xi) \\ &= d^E(f\xi, f\xi_{n-1}) + d^E(\xi_n, \xi) \\ &\preceq \lambda M_1(\xi, \xi_{n-1}) + d^E(\xi_n, \xi), \end{aligned} \tag{6}$$

where

$$M_1(\xi, \xi_{n-1}) \in \{d^E(\xi, \xi_{n-1}), d^E(\xi, f\xi), d^E(\xi, f\xi_{n-1}), d^E(\xi_{n-1}, f\xi), d^E(\xi_{n-1}, f\xi_{n-1})\}.$$

In the following, we divide the above into five cases.

(i) If $M_1(\xi, \xi_{n-1}) = d^E(\xi, \xi_{n-1})$, then by (6), we have

$$d^E(f\xi, \xi) \preceq \lambda d^E(\xi, \xi_{n-1}) + d^E(\xi_n, \xi).$$

Making the most of Lemma 2 and the fact that $\{d^E(\xi_n, \xi)\}$ is an e-sequence, we deduce that $\{\lambda d^E(\xi, \xi_{n-1}) + d^E(\xi_n, \xi)\}$ is an e-sequence. Hence, from Lemmas 3 and 4, it is obvious that $d^E(f\xi, \xi) = \theta_E$, i.e., $f\xi = \xi$. That is, ξ is a fixed point of f.

(ii) If $M_1(\xi, \xi_{n-1}) = d^E(\xi, f\xi)$, then from (6), we have

$$d^E(f\xi, \xi) \preceq \lambda d^E(\xi, f\xi) + d^E(\xi_n, \xi),$$

from which it follows that

$$(1-\lambda) d^E(f\xi, \xi) \preceq d^E(\xi_n, \xi).$$

Because $\{d^E(\xi_n, \xi)\}$ is an e-sequence, from Lemmas 3 and 4 we have $(1-\lambda) d^E(f\xi, \xi) = \theta_E$. Therefore, $d^E(f\xi, \xi) = \theta_E$, i.e., $f\xi = \xi$. That is, ξ is a fixed point of f.

(iii) If $M_1(\xi, \xi_{n-1}) = d^E(\xi, f\xi_{n-1})$, then from (6), we can speculate that

$$d^E(f\xi, \xi) \preceq \lambda d^E(\xi, f\xi_{n-1}) + d^E(\xi_n, \xi) = (\lambda + 1) d^E(\xi_n, \xi).$$

Because $\{d^E(\xi_n, \xi)\}$ is an e-sequence, from Lemma 2, it follows that $\{(\lambda+1)d^E(\xi_n, \xi)\}$ is an e-sequence. Accordingly, based on Lemmas 3 and 4, we claim that $d^E(f\xi, \xi) = \theta_E$, i.e., $f\xi = \xi$. That is, ξ is a fixed point of f.

(iv) If $M_1(\xi, \xi_{n-1}) = d^E(\xi_{n-1}, f\xi)$, then by (6) we arrive at

$$d^E(f\xi, \xi) \preceq \lambda d^E(\xi_{n-1}, f\xi) + d^E(\xi_n, \xi) \preceq \lambda d^E(\xi_{n-1}, \xi) + \lambda d^E(\xi, f\xi) + d^E(\xi_n, \xi),$$

which means that

$$(1-\lambda)d^E(f\xi, \xi) \preceq \lambda d^E(\xi_{n-1}, \xi) + d^E(\xi_n, \xi).$$

Because $\{d^E(\xi_n, \xi)\}$ is an e-sequence, from Lemma 2 it is easy to see that $\{\lambda d^E(\xi_{n-1}, \xi) + d^E(\xi_n, \xi)\}$ is an e-sequence. Consequently, from Lemmas 3 and 4 we have $(1-\lambda)d^E(f\xi, \xi) = \theta_E$. Thus, $d^E(f\xi, \xi) = \theta_E$, i.e., $f\xi = \xi$. That is, ξ is a fixed point of f.

(v) If $M_1(\xi, \xi_{n-1}) = d^E(\xi_{n-1}, f\xi_{n-1})$, then from (6) we obtain

$$d^E(f\xi, \xi) \preceq \lambda d^E(\xi_{n-1}, f\xi_{n-1}) + d^E(\xi_n, \xi) = \lambda d^E(\xi_{n-1}, \xi_n) + d^E(\xi_n, \xi).$$

Note $\{\xi_n\}$ is an e-Cauchy sequence, implying that $\{d^E(\xi_{n-1}, \xi_n)\}$ is an e-sequence as well. Because $\{d^E(\xi_n, \xi)\}$ is an e-sequence, per Lemma 2 it is valid that $\{\lambda d^E(\xi_{n-1}, \xi_n) + d^E(\xi_n, \xi)\}$ is an e-sequence. Now, via Lemmas 3 and 4, we have $d^E(f\xi, \xi) = \theta_E$, i.e., $f\xi = \xi$. Thus, ξ is a fixed point of f.

Finally, we prove that f has only one fixed point. To this end, suppose that ξ and η are two fixed points of f. According to (2), we have

$$d^E(\xi, \eta) = d^E(f\xi, f\eta) \preceq \lambda M_1(\xi, \eta), \tag{7}$$

where

$$\begin{aligned} M_1(\xi, \eta) &\in \{d^E(\xi, \eta), d^E(\xi, f\xi), d^E(\xi, f\eta), d^E(\eta, f\xi), d^E(\eta, f\eta)\} \\ &= \{d^E(\xi, \eta), \theta_E\}. \end{aligned}$$

We discuss two cases concerning (7) as follows:

(A_1) If $M_1(\xi, \eta) = d^E(\xi, \eta)$, then

$$d^E(\xi, \eta) \preceq \lambda d^E(\xi, \eta).$$

In view of $\lambda \in [0, \frac{1}{2})$ and Lemma 5, we have $d^E(\xi, \eta) = \theta_E$. Hence, $\xi = \eta$.

(A_2) If $M_1(\xi, \eta) = \theta_E$, then

$$d^E(\xi, \eta) \preceq \theta_E.$$

Making use of Condition (i) in Definition 4, we infer that $d^E(\xi, \eta) = \theta_E$. Thus, $\xi = \eta$. □

From the proof of Theorem 3, we reach the following conclusion.

Corollary 1. *Suppose that (X, d^E) is an e-complete E-metric space, $(E^+)^\ominus \neq \emptyset$ and P is a cone in E. If $f : X \to X$ is a mapping satisfying*

$$d^E(f\xi, f\eta) \preceq \lambda M_2(\xi, \eta), \quad \textit{for all} \ \ \xi, \eta \in X, \tag{8}$$

where $\lambda \in [0, 1)$ and $M_2(\xi, \eta) \in \{d^E(\xi, \eta), d^E(\xi, f\xi), d^E(\eta, f\eta)\}$, then f has a unique fixed point in X.

Example 1 ([24])**.** *Suppose that X_n is the subset of $\mathbb{R}^2$ equipped with the pointwise partial order including the unit disk, while P_n is the polygon of $\mathbb{R}^2$ with vertices*

$$(1, 0), (0, 1), (-n, n), (-1, 0), (0, -1), (n, -n).$$

We take a Minkowski functional (see [23]) with respect to P_n. We can define the norm $\|\cdot\|_n$ by

$$\|(\xi,\eta)\|_n = \begin{cases} |\xi| + |\eta|, & \text{if } \xi\eta \geq 0\,; \\ \max\{|\xi|,|\eta|\} - \frac{n-1}{n}\min\{|\xi|,|\eta|\}, & \text{if } \xi\eta < 0\,. \end{cases}$$

Take a sequence $\boldsymbol{\xi} = (\xi_n)_{n\in\mathbb{N}}$ in E, where

$$\xi_n = (\xi_1^n, \xi_2^n) \in X_n, \quad \|\boldsymbol{\xi}\|_n \leq m_{\boldsymbol{\xi}} \quad (\forall\, n \in \mathbb{N}^*),$$

and $m_{\boldsymbol{\xi}} > 0$, which depends on $\boldsymbol{\xi}$. Here, let E be an ordered space. We can define the cone by

$$P = \Big\{\boldsymbol{\xi} = (\xi_n) \in E : \xi_n \in \mathbb{R}_+^2, n \in \mathbb{N}^*\Big\}$$

equipped with the norm

$$\|\boldsymbol{\xi}\|_\infty = \sup_{n\in\mathbb{N}} \|\xi_n\|_n.$$

We assume that $X = P$, P is a subspace of E and $d^E(\xi,\eta) : X \times X \to E$ is a mapping defined by

$$d^E(\boldsymbol{\xi},\boldsymbol{\eta}) = (\|\boldsymbol{\xi}-\boldsymbol{\eta}\|_\infty, \|\boldsymbol{\xi}-\boldsymbol{\eta}\|_\infty).$$

Setting $f\boldsymbol{\xi} = \frac{\boldsymbol{\xi}}{2}$ and $\lambda = \frac{1}{2}$, we establish that

$$d^E(f\boldsymbol{\xi}, f\boldsymbol{\eta}) = d^E(\frac{1}{2}\boldsymbol{\xi}, \frac{1}{2}\boldsymbol{\eta}) = \frac{1}{2}(\|\boldsymbol{\xi}-\boldsymbol{\eta}\|_\infty, \|\boldsymbol{\xi}-\boldsymbol{\eta}\|_\infty) = \frac{1}{2}d^E(\boldsymbol{\xi},\boldsymbol{\eta}).$$

Because of $M_2(\boldsymbol{\xi},\boldsymbol{\eta}) \in \{d^E(\boldsymbol{\xi},\boldsymbol{\eta}), d^E(\boldsymbol{\xi},\frac{1}{2}\boldsymbol{\xi}), d^E(\boldsymbol{\eta},\frac{1}{2}\boldsymbol{\eta})\}$, we take $M_2(\boldsymbol{\xi},\boldsymbol{\eta}) = d^E(\boldsymbol{\xi},\boldsymbol{\eta})$. Then,

$$d^E(f\boldsymbol{\xi}, f\boldsymbol{\eta}) \preceq \frac{1}{2}M_2(\boldsymbol{\xi},\boldsymbol{\eta}).$$

That is to say, f satisfies the condition (8) in Corollary 1, meaning that f has a unique fixed point.

Example 2. *For Example 1, we have*

$$d^E(f\boldsymbol{\xi}, f\boldsymbol{\eta}) = d^E(\frac{1}{2}\boldsymbol{\xi}, \frac{1}{2}\boldsymbol{\eta}) = \frac{1}{2}(\|\boldsymbol{\xi}-\boldsymbol{\eta}\|_\infty, \|\boldsymbol{\xi}-\boldsymbol{\eta}\|_\infty) = \frac{1}{2}d^E(\boldsymbol{\xi},\boldsymbol{\eta}).$$

Put $M_1(\boldsymbol{\xi},\boldsymbol{\eta}) := d^E(\boldsymbol{\xi},\boldsymbol{\eta})$. Because $\lambda \in [0,\frac{1}{2})$ in Theorem 3, we know that Theorem 3 is unsuitable for Example 1.

Example 3. *Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|\xi\| = \|\xi\|_\infty + \|\xi'\|_\infty$. Put $P = \{\xi \in E : \xi(t) \geq 0, t \in [0,1]\}$ and $X = C^1_{\mathbb{R}}[0,1]$. Define $d^E : X \times X \to E$ by $d^E(\xi,\eta) = \|\xi-\eta\|\varphi$ for all $\xi,\eta \in X$, where $\varphi : [0,1] \to \mathbb{R}$ such that $\varphi(t) = \mathrm{e}^t$. Then, P is a non-normal cone (see [15]) and (X, d^E) is an e-complete E-metric space. Define a mapping $f : X \to X$ by*

$$f\xi = u(t)\xi(t) + v(t)\int_0^1 \xi(t)\mathrm{d}t,$$

where $u, v \in X$. Let $\lambda(t) = \lambda(u(t), v(t)) = \|u\| + \|v\|_\infty < \frac{1}{2}$. Note that

$$
\begin{aligned}
d^E(f\xi, f\eta) = \|f\xi - f\eta\|\mathrm{e}^t &= \Big(\max_{0\le t\le 1}|f\xi - f\eta| + \max_{0\le t\le 1}|(f\xi - f\eta)'|\Big)\mathrm{e}^t \\
&= \Big(\max_{0\le t\le 1}\Big|u(t)[\xi(t)-\eta(t)] + v(t)\int_0^1[\xi(t)-\eta(t)]\mathrm{d}t\Big| \\
&\quad + \max_{0\le t\le 1}\big|u'(t)[\xi(t)-\eta(t)] + u(t)[\xi'(t)-\eta'(t)]\big|\Big)\mathrm{e}^t \\
&\le \Big[(\|u\|_\infty + \|v\|_\infty + \|u'\|_\infty)\max_{0\le t\le 1}|\xi(t)-\eta(t)| \\
&\quad + \|u\|_\infty \max_{0\le t\le 1}|\xi'(t)-\eta'(t)|\Big]\mathrm{e}^t \\
&\le (\|u\| + \|v\|_\infty)(\|\xi-\eta\|_\infty + \|\xi'-\eta'\|_\infty)\mathrm{e}^t \\
&= \lambda(t)\|\xi-\eta\|\mathrm{e}^t = \lambda(t)M_1(\xi,\eta),
\end{aligned}
$$

where $M_1(\xi,\eta) = d^E(\xi,\eta)$. It is obvious that $\lambda \in P$ and f is a Ćirić-type contraction and not a Banach-type contraction. Thus, all conditions of Theorem 3 are satisfied. Then, using Theorem 3, it follows that f has a unique fixed point in X.

Stimulated by Theorem 2, we obtain the following theorem.

Theorem 4. *Let (X, d^E) be an e-complete E-metric space, P be a normal cone with normal constant M, $(E^+)^\ominus \neq \emptyset$, $\beta : [0,+\infty) \to [0,1)$ be a function, and $\psi : [0,+\infty) \to [0,+\infty)$ be a nondecreasing function. Suppose that $f : X \to X$ is an α-admissible function satisfying the following α-ψ type contractive condition:*

$$
\alpha(\xi,\eta)\psi(\|d^E(f\xi, f\eta)\|) \le \beta(\psi(M_3(\xi,\eta)))\psi(\lambda M_3(\xi,\eta)), \quad \text{for all } \xi,\eta \in X, \tag{9}
$$

where $M_3(\xi,\eta) \in \{\|d^E(\xi,\eta)\|, \|d^E(\xi, f\xi)\|, \|d^E(\eta, f\eta)\|\}$, and $\lambda \in [0,1)$ is a constant. If there exists $\xi_0 \in X$ such that $\alpha(\xi_0, f\xi_0) \ge 1$ and one of the following conditions is satisfied:

(I) f is continuous, or

(II) X is α-regular,

then f has a fixed point in X. Moreover, if the following condition is satisfied:

(III) for each $\xi,\eta \in X$, there exists a $\zeta \in X$ such that $\alpha(\zeta,\xi) \ge 1$ and $\alpha(\zeta,\eta) \ge 1$,

then f has a unique fixed point in X.

Proof. Based on the assumption that there exists $\xi_0 \in X$ such that $\alpha(\xi_0, f\xi_0) \ge 1$, we define an iterative sequence $\{\xi_n\}$ by letting $\xi_1 = f\xi_0$, $\xi_2 = f\xi_1$, $\cdots$, $\xi_n = f\xi_{n-1}$, $\cdots$. Because $\alpha(\xi_0,\xi_1) = \alpha(\xi_0, f\xi_0) \ge 1$ and f is an α-admissible function, we have $\alpha(\xi_1,\xi_2) = \alpha(f\xi_0, f\xi_1) \ge 1$. By induction, we obtain $\alpha(\xi_n,\xi_{n+1}) \ge 1$ for any $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $\xi_{n_0+1} = f\xi_{n_0} = \xi_{n_0}$, then ξ_{n_0} is a fixed point of f. Thus, the proof is completed. Now, suppose that $\xi_{n+1} \neq \xi_n$ for any $n \in \mathbb{N}$. Making use of (9), we have

$$
\begin{aligned}
\psi(\|d^E(\xi_{n+1},\xi_{n+2})\|) &\le \alpha(\xi_n,\xi_{n+1})\psi(\|d^E(f\xi_n, f\xi_{n+1})\|) \\
&\le \beta(\psi(M_3(\xi_n,\xi_{n+1})))\psi(\lambda M_3(\xi_n,\xi_{n+1})) \\
&\le \psi(\lambda M_3(\xi_n,\xi_{n+1})),
\end{aligned} \tag{10}
$$

where

$$
\begin{aligned}
M_3(\xi_n,\xi_{n+1}) &\in \{\|d^E(\xi_n,\xi_{n+1})\|, \|d^E(\xi_n, f\xi_n)\|, \|d^E(\xi_{n+1}, f\xi_{n+1})\|\} \\
&= \{\|d^E(\xi_n,\xi_{n+1})\|, \|d^E(\xi_{n+1},\xi_{n+2})\|\}.
\end{aligned}
$$

We consider (10) as follows:

(i) If $M_3(\xi_n, \xi_{n+1}) = \|d^E(\xi_n, \xi_{n+1})\|$, then

$$\psi(\|d^E(\xi_{n+1}, \xi_{n+2})\|) \leq \psi(\lambda \|d^E(\xi_n, \xi_{n+1})\|). \tag{11}$$

Because ψ is nondecreasing, from (11) we obtain

$$\|d^E(\xi_{n+1}, \xi_{n+2})\| \leq \lambda \|d^E(\xi_n, \xi_{n+1})\|,$$

which establishes that

$$\begin{aligned} \|d^E(\xi_n, \xi_{n+1})\| &\leq \lambda \|d^E(\xi_{n-1}, \xi_n)\| \leq \lambda^2 \|d^E(\xi_{n-2}, \xi_{n-1})\| \\ &\leq \cdots \leq \lambda^n \|d^E(\xi_0, \xi_1)\| \to 0 \ (n \to \infty). \end{aligned} \tag{12}$$

From (12) and Condition (i) in Definition 4, it follows that $d^E(\xi_n, \xi_{n+1}) \to \theta_E$ as $n \to \infty$. Thus, for any $m, n \in \mathbb{N}^*$, $m > n$, we have

$$d^E(\xi_n, \xi_m) \preceq d^E(\xi_n, \xi_{n+1}) + d^E(\xi_{n+1}, \xi_{n+2}) + \cdots + d^E(\xi_{m-1}, \xi_m).$$

Because P is a normal cone in E, this implies that

$$\begin{aligned} \|d^E(\xi_n, \xi_m)\| &\leq M \|d^E(\xi_n, \xi_{n+1}) + d^E(\xi_{n+1}, \xi_{n+2}) + \cdots + d^E(\xi_{m-1}, \xi_m)\| \\ &\leq M\Big(\|d^E(\xi_n, \xi_{n+1})\| + \|d^E(\xi_{n+1}, \xi_{n+2})\| + \cdots + \|d^E(\xi_{m-1}, \xi_m)\|\Big) \\ &\leq M(\lambda^n \|d^E(\xi_0, \xi_1)\| + \lambda^{n+1} \|d^E(\xi_0, \xi_1)\| + \cdots + \lambda^{m-1} \|d^E(\xi_0, \xi_1)\|) \\ &= M\lambda^n (1 + \lambda + \lambda^2 + \cdots + \lambda^{m-n-1}) \|d^E(\xi_0, \xi_1)\| \\ &= M \frac{\lambda^n (1 - \lambda^{m-n})}{1 - \lambda} \|d^E(\xi_0, \xi_1)\| \to 0 \ (n \to \infty). \end{aligned} \tag{13}$$

Note that (13) means $\theta_E \preceq d^E(\xi_n, \xi_m) \to \theta_E$ as $n \to \infty$. As a consequence, per Lemma 1 we can confirm that $\{d^E(\xi_n, \xi_m)\}$ is an e-sequence. In other words, $\{\xi_n\}$ is an e-Cauchy sequence. Because (X, d^E) is e-complete, there exists $\xi \in X$ such that $\xi_n \to \xi$ as $n \to \infty$.

(ii) If $M_3(\xi_n, \xi_{n+1}) = \|d^E(\xi_{n+1}, \xi_{n+2})\|$, then

$$\psi(\|d^E(\xi_{n+1}, \xi_{n+2})\|) \leq \psi(\lambda \|d^E(\xi_{n+1}, \xi_{n+2})\|). \tag{14}$$

Since ψ is nondecreasing, from (14) we obtain

$$\|d^E(\xi_{n+1}, \xi_{n+2})\| \leq \lambda \|d^E(\xi_{n+1}, \xi_{n+2})\|.$$

Owing to $\lambda \in [0, 1)$, we know that $\|d^E(\xi_{n+1}, \xi_{n+2})\| = 0$, i.e., $d^E(\xi_{n+1}, \xi_{n+2}) = \theta_E$. It is obvious that $\xi_{n+1} = \xi_{n+2}$, which conflicts with the previous hypothesis.

Next, we prove that ξ is a fixed point of f.

(I) If f is continuous, then

$$\xi = \lim_{n \to \infty} \xi_{n+1} = \lim_{n \to \infty} f\xi_n = f\Big(\lim_{n \to \infty} \xi_n\Big) = f\xi,$$

i.e., ξ is a fixed point of f.

(II) If X is α-regular, then from (9) we have

$$\begin{aligned} \psi(\|d^E(f\xi_n, f\xi)\|) &\leq \alpha(\xi_n, \xi) \psi(\|d^E(f\xi_n, f\xi)\|) \\ &\leq \beta(\psi(M_3(\xi_n, \xi))) \psi(\lambda M_3(\xi_n, \xi)) \\ &\leq \psi(\lambda M_3(\xi_n, \xi)). \end{aligned} \tag{15}$$

Since ψ is nondecreasing, via (15) we obtain

$$\|d^E(f\xi_n, f\xi)\| \leq \lambda M_3(\xi_n, \xi),$$

where

$$M_3(\xi_n, \xi) \in \{\|d^E(\xi_n, \xi)\|, \|d^E(\xi_n, \xi_{n+1})\|, \|d^E(\xi, f\xi)\|\}.$$

We can then discuss the above as follows:

(i) If $M_3(\xi_n, \xi) = \|d^E(\xi_n, \xi)\|$, then

$$\|d^E(\xi_{n+1}, f\xi)\| = \|d^E(f\xi_n, f\xi)\| \leq \lambda \|d^E(\xi_n, \xi)\|. \tag{16}$$

Passing to the limit from both sides of (16) and noting that $\xi_n \to \xi$ as $n \to \infty$ and P is a normal cone, from Lemma 6 we have $\|d^E(\xi, f\xi)\| = 0$. Thus, $\xi = f\xi$.

(ii) If $M_3(\xi_n, \xi) = \|d^E(\xi_n, \xi_{n+1})\|$, we note that

$$d^E(\xi_n, \xi_{n+1}) \preceq d^E(\xi_n, \xi) + d^E(\xi, \xi_{n+1}),$$

it then immediately follows from the normality of the cone that

$$\|d^E(\xi_n, \xi_{n+1})\| \leq M\|d^E(\xi_n, \xi) + d^E(\xi, \xi_{n+1})\|,$$

therefore,

$$\begin{aligned} \|d^E(\xi_{n+1}, f\xi)\| = \|d^E(f\xi_n, f\xi)\| &\leq \lambda \|d^E(\xi_n, \xi_{n+1})\| \\ &\leq \lambda M\Big(\|d^E(\xi_n, \xi)\| + \|d^E(\xi, \xi_{n+1})\|\Big). \end{aligned} \tag{17}$$

Passing to the limit from both sides of (17) and noting that $\xi_n \to \xi$, as $n \to \infty$ and P is a normal cone, per Lemma 6 we have $\|d^E(\xi, f\xi)\| = 0$. Thus, $\xi = f\xi$.

(iii) If $M_3(\xi_n, \xi) = \|d^E(\xi, f\xi)\|$, then

$$\|d^E(\xi_{n+1}, f\xi)\| = \|d^E(f\xi_n, f\xi)\| \leq \lambda \|d^E(\xi, f\xi)\|. \tag{18}$$

Passing to the limit from both sides of (18) and noting that $\xi_n \to \xi$, as $n \to \infty$ and P is a normal cone, per Lemma 6 we can claim that $\|d^E(\xi, f\xi)\| \leq \lambda \|d^E(\xi, f\xi)\|$. In view of $\lambda \in [0, 1)$, we have $\|d^E(\xi, f\xi)\| = 0$. Hence, $\xi = f\xi$. That is to say, f has a fixed point $\xi \in X$.

Assume that Condition (III) is satisfied. If f has two fixed points ξ, η, then per (III) there exists a ζ in X such that

$$\alpha(\zeta, \xi) \geq 1, \qquad \alpha(\zeta, \eta) \geq 1. \tag{19}$$

Due to (19) and the α-admissibility of f, for any $n \in \mathbb{N}^*$ we can obtain

$$\alpha(f^n\zeta, \xi) \geq 1, \qquad \alpha(f^n\zeta, \eta) \geq 1. \tag{20}$$

As a consequence of (9) and (20), it is easy to see that

$$\begin{aligned} \psi(\|f^{n+1}\zeta, f\xi)\|) &\leq \alpha(f^n\zeta, \xi)\psi(\|d^E(f^{n+1}\zeta, f\xi)\|) \\ &\leq \beta(\psi(M_3(f^n\zeta, \xi)))\psi(\lambda M_3(f^n\zeta, \xi)) \\ &\leq \psi(\lambda M_3(f^n\zeta, \xi)). \end{aligned} \tag{21}$$

Because ψ is nondecreasing, from (21) we obtain

$$\|d^E(\zeta_{n+1}, \xi)\| = \|d^E(f^{n+1}\zeta, f\xi)\| \leq \lambda M_3(f^n\zeta, \xi), \tag{22}$$

where

$$M_3(f^n\zeta,\xi) \in \{\|d^E(f^n\zeta,\xi)\|, \|d^E(f^n\zeta,f^{n+1}\zeta)\|, \|d^E(\xi,f\xi)\|\}$$
$$= \{\|d^E(\zeta_n,\xi)\|, \|d^E(\zeta_n,\zeta_{n+1})\|, 0\}.$$

Finally, we can show that

$$\lim_{n\to\infty} \zeta_n = \xi. \tag{23}$$

To this end, we discuss the following:

(i) If $M_3(f^n\zeta,\xi) = \|d^E(\zeta_n,\xi)\|$, then from (22) we have

$$\|d^E(\zeta_{n+1},\xi)\| \leq \lambda\|d^E(\zeta_n,\xi)\| \leq \lambda^2\|d^E(\zeta_{n-1},\xi)\| \leq \cdots \leq \lambda^n\|d^E(\zeta,\xi)\|. \tag{24}$$

On account of $0 \leq \lambda < 1$, if we take the limit as $n \to \infty$ from both sides of (24), we have (23).

(ii) If $M_3(f^n\zeta,\xi) = \|d^E(\zeta_n,\zeta_{n+1})\|$, then per (22) we have

$$\|d^E(\zeta_{n+1},\xi)\| \leq \lambda\|d^E(\zeta_n,\zeta_{n+1})\|. \tag{25}$$

Via the above proof, it is not hard to verify that $\{\zeta_n\}$ is an e-Cauchy sequence, meaning that $\lim_{n\to\infty} \|d^E(\zeta_n,\zeta_{n+1})\| = 0$. Thus, from (25) we have (23).

(iii) If $M_3(f^n\zeta,\xi) = 0$, then from (22) we can obtain

$$\|d^E(\zeta_{n+1},\xi)\| = 0,$$

which implies (23).

Similar to the proof of (23), using (9) and (20) we can easily show that

$$\lim_{n\to\infty} \zeta_n = \eta. \tag{26}$$

By combining (23) and (26), we can claim that $\xi = \eta$. $\square$

Remark 2. *In Theorem 4, we prove the fixed point results for α-ψ type contraction in E-metric space, followed by Theorem 2.1 in [5] and Theorem 2.9 in [3], obtaining the fixed point theorem in ordered vector spaces.*

4. Application

In this section, we use Theorem 3, to consider the first-order periodic boundary problem

$$\begin{cases} \dfrac{\mathrm{d}\xi}{\mathrm{d}t} = G(t,\xi(t)), \\ \xi(0) = c, \end{cases} \tag{27}$$

where $G : [-h,h] \times [c-\delta, c+\delta]$ is a continuous function and $c, h, \delta > 0$ are constants.

Theorem 5. *Regarding the boundary problem in (27), suppose that the function G satisfies the local Lipschitz condition, i.e., for any $|\mu| \leq h$, $\nu_1, \nu_2 \in [c-\delta, c+\delta]$, we have*

$$|G(\mu,\nu_1) - G(\mu,\nu_2)| \leq L|\nu_1 - \nu_2|,$$

where $h < \min\{\frac{\delta}{M}, \frac{1}{2L}\}$, $M = \max\limits_{[-h,h]\times[c-\delta,c+\delta]} |G(\mu,\nu)|$. Then, Equation (27) has a unique solution.

Proof. Let $X = E = C([-h,h])$ be the set of all continuous functions on the closed interval $[-h,h]$. Put $P = \{\xi \in E : \xi \geq 0\}$; then, $(E^+)^\ominus \neq \emptyset$. Define a mapping $d^E : X \times X \to E$ as

$$d^E(\xi,\eta) = g(t) \max_{-h \leq t \leq h} |\xi(t) - \eta(t)|,$$

where $g : [-h,h] \to \mathbb{R}$ is a function such that $g(t) = \mathrm{e}^t$. Clearly, (X, d^E) is an e-complete E-metric space.

It is easy to see that (27) is equivalent to the integral equation

$$\xi(t) = c + \int_0^t G(\tau, \xi(\tau))\mathrm{d}\tau.$$

Define a mapping $f : C([-h,h]) \to \mathbb{R}$ by

$$f\xi(t) = c + \int_0^t G(\tau, \xi(\tau))\mathrm{d}\tau. \tag{28}$$

Let

$$\xi(t), \eta(t) \in B(c, \delta g) := \{\varphi(t) \in E : d^E(c,\varphi) \leq \delta g\},$$

then

$$\begin{aligned} d^E(f\xi, f\eta) &= g(t) \max_{-h \leq t \leq h} \left| \int_0^t G(\tau,\xi(\tau))\mathrm{d}\tau - \int_0^t G(\tau,\eta(\tau))\mathrm{d}\tau \right| \\ &= g(t) \max_{-h \leq t \leq h} \left| \int_0^t [G(\tau,\xi(\tau)) - G(\tau,\eta(\tau))]\mathrm{d}\tau \right| \\ &\leq hg(t) \max_{-h \leq \tau \leq h} |G(\tau,\xi(\tau)) - G(\tau,\eta(\tau))| \\ &\leq hLg(t) \max_{-h \leq \tau \leq h} |\xi(\tau) - \eta(\tau)| \\ &= hLd^E(\xi,\eta). \end{aligned} \tag{29}$$

Per (29), the inequality (2) from Theorem 3 holds (where $\lambda = hL \in [0, \frac{1}{2})$). Note that

$$d^E(f\xi, c) = g(t) \max_{-h \leq t \leq h} \left| \int_0^t G(\tau,\xi(\tau))\mathrm{d}\tau \right| \leq hg \max_{-h \leq \tau \leq h} |G(\tau,\xi(\tau))| \leq hMg \leq \delta g,$$

thus, $f : B(c,\delta g) \to B(c,\delta g)$ is a self-mapping.

Here, we show that $(B(c,\delta g), d^E)$ is an e-complete E-metric space. First, assume that $\{\xi_n\}$ is an e-Cauchy sequence in $B(c,\delta g)$; then, $\{\xi_n\}$ is an e-Cauchy sequence in X. Because (X, d^E) is e-complete, there exists $\xi \in X$ satisfying $\xi_n \to \xi$ $(n \to \infty)$. Thus, for any $e \ggg \theta_E$, there is $N \in \mathbb{N}^*$ such that for all $n > N$ we have $d(\xi_n, \xi) \lll e$. Accordingly, by virtue of Lemma 7 and

$$d(c,\xi) \leq d(\xi_n, c) + d(\xi_n, \xi) \leq \delta g + e,$$

we arrive at $d(c,\xi) \leq \delta g$, which implies that $\xi \in B(c,\delta g)$, that is to say, $(B(c,\delta g), d^E)$ is e-complete.

Based on the above statement, all conditions of Theorem 3 hold. Therefore, f has a unique fixed point $\xi(t) \in B(c,\delta g)$, that is, the integral Equation (28) has a unique solution. Therefore, the differential Equation (27) has a unique solution. This ends the proof. □

5. Conclusions

In this paper, we study two kinds of contraction, namely, Ćirić-type contraction and α-ψ-type contraction. We consider the existence and uniqueness of fixed points for the contractions in the framework of E-metric spaces. We provide three examples to support the superiority of our obtained results. As an application, we investigate the existence and uniqueness of a solution to a first-order periodic boundary problem. In summary, our results are original, meaningful, and valuable in the context of the existing literature. We hope that our new results can be applied to fields such as nonlinear analysis, mathematical physics, and other related fields in the future.

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