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Analysis of Adaptive Progressive Type-II Hybrid Censored Dagum Data with Applications

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Abstract: In life testing and reliability studies, obtaining whole data always takes a long time and lots of monetary and human resources. In this case, the experimenters prefer to gather data using censoring schemes that make a balance between the length of the test, the desired sample size, and the cost. Lately, an adaptive progressive type-II hybrid censoring scheme is suggested to enhance the efficiency of the statistical inference. By utilizing this scheme, this paper seeks to investigate classical and Bayesian estimations of the Dagum distribution. The maximum likelihood and Bayesian estimation methods are considered to estimate the distribution parameters and some reliability indices. The Bayesian estimation is developed under the assumption of independent gamma priors and by employing symmetric and asymmetric loss functions. Due to the tough form of the joint posterior distribution, the Markov chain Monte Carlo technique is implemented to gather samples from the full conditional distributions and in turn obtain the Bayes estimates. The approximate confidence intervals and the highest posterior density credible intervals are also obtained. The effectiveness of the various suggested methods is compared through a simulated study. The optimal progressive censoring plans are also shown, and number of optimality criteria are explored. To demonstrate the applicability of the suggested point and interval estimators, two real data sets are also examined. The outcomes of the simulation study and data analysis demonstrated that the proposed scheme is adaptable and very helpful in ending the experiment when the experimenter's primary concern is the number of failures.

Keywords: Dagum distribution; adaptive progressive type-II hybrid censoring; likelihood estimation; Bayesian estimation; optimum progressive censoring

MSC: 62F10; 62F15; 62N01; 62N02; 62N05

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1. Introduction

The Dagum distribution offered by Dagum [1] has an essential role in modeling income distributions that could be utilized instead of some popular models including log-normal and Pareto models. Recently, authors have also considered the Dagum distribution in the context of reliability and survival analysis due to its flexibility for modeling lifetime data; see for example Domma et al. [2] and Emam and Sultan [3]. Presume that X is a lifetime random variable of an experimental item follows the three-parameter Dagum distribution, denoted by $Dagum(\boldsymbol{\xi})$, where $\boldsymbol{\xi} = (\alpha, \beta, \theta)^\top$ is the vector of the unknown parameters, with scale parameter θ and shape parameters α and β. Hence, the related probability density function (PDF) and the cumulative distribution function (CDF) of X, are given by

$$f(x;\boldsymbol{\xi}) = \frac{\alpha\beta\theta}{x^{\beta+1}}(1+\theta x^{-\beta})^{-(\alpha+1)},\ x>0,\ \alpha,\beta,\theta>0, \tag{1}$$

and

$$F(x;\boldsymbol{\xi}) = (1+\theta x^{-\beta})^{-\alpha},\ x>0,\ \alpha,\beta,\theta>0, \tag{2}$$

respectively. One can see that the Dagum distribution can be considered as a mixture model in terms of inverse Weibull and generalized gamma models. Kleiber and Kotz [4] and Kleiber [5] furnished a detailed appraisal of the core of the Dagum model as well as its applications. Further, two reliability indices of the Dagum distribution can be considered as unknown parameters, namely, reliability function (RF) $R(\cdot)$ and hazard rate function (HRF) $h(\cdot)$ at distinct time t which can be provided, respectively, by

$$R(t;\boldsymbol{\xi}) = 1-(1+\theta t^{-\beta})^{-\alpha},\ t>0,\ \alpha,\beta,\theta>0, \tag{3}$$

and

$$h(t;\boldsymbol{\xi}) = \frac{\alpha\beta\theta(1+\theta t^{-\beta})^{-(\alpha+1)}}{t^{\beta+1}[1-(1+\theta t^{-\beta})^{-\alpha}]},\ t>0,\ \alpha,\beta,\theta>0. \tag{4}$$

The HRF of the Dagum distribution is either decreasing, upside-down, or a bathtub then upside-down bathtub. This appealing flexibility makes the HRF of the Dagum distribution meet appropriately even non-monotone HRF behaviors that are probable to be seen in a variety of domains. Different studies using the Dagum distribution have been achieved. Arif et al. [6] investigated the Bayesian estimation based on the Markov chain Monte Carlo (MCMC) technique. Naqash et al. [7] studied the Bayesian estimation of the scale parameter using different loss functions. Dey et al. [8] addressed different frequentist estimation methods for the unknown parameters. Alotaibi et al. [9] studied the Bayesian estimation using progressively type-I interval censored data. Kumari et al. [10] studied the classical and Bayesian estimation of the stress strength reliability using progressively type-II censored data.

Various censoring plans are known in the literature, which can be categorized into single-stage and multistage censoring schemes. Single-stage censoring schemes include type-I, type-II, and hybrid censoring. On the other hand, the most popular multistage censoring scheme is the progressive type-II censoring in which n units are placed on a test and m is a prefixed number of items to be failed with prefixed progressive censoring plan $R_1,\ldots,R_m$. At the time of the i^{th} failure $X_{i:m:n}$, $R_i, i=1,\ldots,m-1$ surviving units are randomly removed from the test. At the time of the last failure $X_{m:m:n}$, all the surviving units are removed. For further information about the progressive type-II censoring scheme, see Balakrishnan [11]. Kundu and Joarder [12] proposed a progressive type-I hybrid censoring scheme that has the same schematic representation as the progressive type-II censoring scheme but in this case, the test is stopped at $T^* = min(X_{m:m:n}, T)$, where T is a prefixed time.

The main drawback of this scheme is that the desired sample size is random and might turn out to be a very small number. As a consequence, the statistical deduction methods will be inadequate. To overpower this weakness, a more flexible censoring plan is proposed, namely an adaptive progressive type-II hybrid censoring (APT-II HC) scheme by Ng et al. [13]. In the APT-II HC, the experiment time is allowed to run over the time T and some values of $R_i, i=1,\ldots,m-1$ conceivably revised during the test. If $X_{m:m:m} < T$, the test stops at $X_{m:m:m}$ and we will retain the standard progressive type-II censoring. Otherwise, if $X_{D:m:n} < T < X_{D+1:m:n}$, where $D+1 < m$ and $X_{D:m:n}$ is the D^{th} failure time occur before time T, then we will not remove any surviving units from the test by placing $R_{D+1}, R_{D+2}, \cdots, R_{m-1} = 0$, and at the time of the last failure $X_{m:m:n}$, all the remaining units are removed, i.e., $R_m = n - m - \sum_{i=1}^{D} R_i$. This adaption guarantees the ending of the test when we gather the desired number of failures m, and the total test time will not be too outlying from the ideal time T. Suppose that $x_{1:m:n} < \cdots < x_{D:m:n} < T < x_{D+1:m:n} <$

$\ldots x_{m:m:n}$ are an observed APT-II HC sample from a continuous population with PDF $f(x)$ and CDF $F(x)$, then the likelihood function can be expressed as follows

$$L(\boldsymbol{\xi}) = C\prod_{i=1}^{m} f(x_{i:m:n}) \prod_{i=1}^{D} [1 - F(x_{i:m:n})]^{R_i} \, [1 - F(x_{m:m:n})]^{R_m}, \tag{5}$$

where C is a constant that is independent of the parameters. Many works have been performed based on the APT-II HC scheme. Hemmati and Khorram [14] addressed the estimation of the competing risks model for the exponential distribution. Al Sobhi and Soliman [15] investigated the estimation issues of the exponentiated Weibull distribution. Nassar et al. [16] studied the classical and Bayesian estimation methods for the Weibull distribution. Panahi and Moradi [17] considered some estimations method for the inverted exponentiated Rayleigh distribution. Elshahhat and Nassar [18] studied the Bayesian estimation for the Hjorth distribution. See also the work of Kohansal and Shoaee [19], Panahi and Asadi [20], Ahmad et al. [21], Du and Gui [22], Ateya et al. [23], Alotaibi et al. [24,25], and Nassar et al. [26]. Recently, Elshahhat and Nassar [27] extended the APT-II HC scheme to binomial random removals.

We can motivate this study via (1) the significance of the APT-II HC scheme in increasing the efficiency of the statistical inference by avoiding getting small observed sample sizes. (2) The flexibility of the Dagum distribution in modeling different types of data sets with different HRF shapes including decreasing, upside-down, or a bathtub then an upside-down bathtub. As a result, we can list our objectives in this study as:

1. To explore the maximum likelihood estimators (MLEs) of the unknown parameters including the reliability measures as well as the associated approximate confidence intervals (ACIs).
2. To investigate the Bayes estimators and the highest posterior density (HPD) credible intervals. The Bayes estimators are acquired by using the MCMC method and by employing two loss functions, namely, squared error (SE) and general entropy (GE) loss functions.
3. It is not possible to judge which procedure provides the best estimates theoretically. Therefore, an extensive simulation study is implemented to study the behavior of the different estimates and make the comparison achievable.
4. To construct a guideline for picking the most appropriate estimation procedure for the Dagum distribution based on APT-II HC.
5. To determine the optimal progressive sampling plane for APT-II HC scheme in the case of Dagum distribution.
6. Because the applicability of the proposed methods is an important issue. The proposed methods are applied to investigate two real data sets.

The remainder of the paper is arranged as follows: The MLEs and ACIs are discussed in Section 2. The Bayes estimators and HPD credible intervals are considered in Section 3. Section 4 displays the outcomes of the simulation study. In Section 5, we provide various methods for choosing the best censoring plan. Section 6 investigates two applications for real data. Finally, Section 7 concludes the paper.

2. Frequentist Inference

Assume that $x_{1:m:n} < \cdots < x_{D:m:n} < T < x_{D+1:m:n} < \ldots x_{m:m:n}$ are an APT-II HC sample of size m with $R_1, \ldots, R_D, 0, \ldots, 0, R_m$ taken from the Dagum distribution with PDF and CDF given, respectively, by (1) and (2). In this case, one can derive the likelihood function based on (1), (2), and (5), after ignoring the constant term, as follows

$$L(\boldsymbol{\xi}) = (\alpha\beta\theta)^m \prod_{i=1}^{m} \frac{(1+\theta x_i^{-\beta})^{-(\alpha+1)}}{x_i^{\beta+1}} \prod_{i=1}^{D} \left[1 - (1+\theta x_i^{-\beta})^{-\alpha}\right]^{R_i} \left[1 - (1+\theta x_m^{-\beta})^{-\alpha}\right]^{R_m}, \tag{6}$$

where $x_i = x_{i:m:n}$ for simplicity of notation. Practically, it is more convenient to work with the log-likelihood function rather than the likelihood function itself. Therefore, by taking the natural logarithm of the likelihood function in (6), the log-likelihood function can be written as

$$\begin{aligned}\ell(\boldsymbol{\xi}) \equiv \log L(\boldsymbol{\xi}) &= m\log(\alpha\beta\theta) - (\alpha+1)\sum_{i=1}^{m}\log(1+\theta x_i^{-\beta}) - (\beta+1)\sum_{i=1}^{m}\log(x_i) \\ &+ \sum_{i=1}^{D} R_i \log\Big[1-(1+\theta x_i^{-\beta})^{-\alpha}\Big] + R_m \log\Big[1-(1+\theta x_m^{-\beta})^{-\alpha}\Big]. \end{aligned} \tag{7}$$

Let $\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$ denote MLEs of the unknown parameters α, β, and θ, respectively. These estimators can be acquired by maximizing the objective function $\ell(\boldsymbol{\xi})$ with respect to α, β, and θ. An alternative approach to obtain the needed estimators is by solving the following three normal equations simultaneously

$$\frac{\partial \ell(\boldsymbol{\xi})}{\partial \alpha} = \frac{m}{\alpha} - \sum_{i=1}^{m}\log(v_i) + \sum_{i=1}^{D} R_i \frac{\log(v_i)}{v_i^{\alpha}(1-v_i^{-\alpha})} + \frac{R_m \log(v_m)}{v_m^{\alpha}(1-v_m^{-\alpha})} = 0, \tag{8}$$

$$\frac{\partial \ell(\boldsymbol{\xi})}{\partial \beta} = \frac{m}{\beta} + \theta(\alpha+1)\sum_{i=1}^{m}\frac{\log(x_i)}{x_i^{\beta} v_i} - \sum_{i=1}^{m}\log(x_i) - \theta\alpha\sum_{i=1}^{D}\frac{R_i \log(x_i)}{x_i^{\beta} v_i^{\alpha+1}(1-v_i^{\alpha})} - \frac{\theta\alpha R_m \log(x_m)}{x_m^{\beta} v_m^{\alpha+1}(1-v_m^{\alpha})} = 0 \tag{9}$$

and

$$\frac{\partial \ell(\boldsymbol{\xi})}{\partial \theta} = \frac{m}{\theta} - (\alpha+1)\sum_{i=1}^{m}\frac{1}{x_i^{\beta} v_i} + \alpha\sum_{i=1}^{D}\frac{R_i}{x_i^{\beta} v_i^{\alpha+1}(1-v_i^{\alpha})} + \frac{\alpha R_m}{x_m^{\beta} v_m^{\alpha+1}(1-v_m^{\alpha})} = 0, \tag{10}$$

where $v_i = (1+\theta x_i^{-\beta})$, $i = 1\ldots, m$. It is evident from the nonlinear equations in (8)–(10) that the MLEs of the unknown parameters α, β and θ can not be obtained in explicit expressions. To overcome this problem, some numerical techniques can be implemented to obtain the MLEs in this case. Once the MLEs $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\theta}$ are obtained, we can utilize the invariance property of the MLEs to estimate the RF and HRF at a distinct time t. Employing the invariance property, the MLEs of the RF and HRF can be obtained using (3) and (4) as follow

$$\hat{R}(t) = 1-(1+\hat{\theta}t^{-\hat{\beta}})^{-\hat{\alpha}} \quad \text{and} \quad \hat{h}(t) = \frac{\hat{\alpha}\hat{\beta}\hat{\theta}(1+\hat{\theta}t^{-\hat{\beta}})^{-(\hat{\alpha}+1)}}{t^{\hat{\beta}+1}\big[1-(1+\hat{\theta}t^{-\hat{\beta}})^{-\hat{\alpha}}\big]}.$$

Aside from obtaining the point estimates of the unknown parameters α, β, and θ, it is also of interest to obtain the confidence intervals for these parameters. Here, we utilize the asymptotic properties of the MLEs to construct the ACIs of the unknown parameters as well as the reliability measures. It is known that based on the theory of large samples the asymptotic distribution of $\hat{\boldsymbol{\xi}}$, where $\hat{\boldsymbol{\xi}}$ is the MLE of $\boldsymbol{\xi}$, is normal distribution with mean $\boldsymbol{\xi}$ and variance–covariance matrix $\boldsymbol{I}^{-1}(\boldsymbol{\xi})$. Due to the complicated expressions of the Fisher information matrix, it is not easy to obtain such a variance–covariance matrix. In this case, we can consider $\boldsymbol{I}^{-1}(\hat{\boldsymbol{\xi}})$ to estimate $\boldsymbol{I}^{-1}(\boldsymbol{\xi})$, which can be acquired using the observed Fisher information matrix and given by

$$I^{-1}(\hat{\boldsymbol{\xi}}) = \begin{bmatrix} -J_{\alpha\alpha} & -J_{\alpha\beta} & -J_{\alpha\theta} \\ -J_{\beta\alpha} & -J_{\beta\beta} & -J_{\beta\theta} \\ -J_{\theta\alpha} & -J_{\theta\beta} & -J_{\theta\theta} \end{bmatrix}^{-1}_{(\alpha,\beta,\theta)=(\hat{\alpha},\hat{\beta},\hat{\theta})} = \begin{bmatrix} \widehat{var}(\hat{\alpha}) & \widehat{cov}(\hat{\alpha},\hat{\beta}) & \widehat{cov}(\hat{\alpha},\hat{\theta}) \\ & \widehat{var}(\hat{\beta}) & \widehat{cov}(\hat{\beta},\hat{\theta}) \\ & & \widehat{var}(\hat{\theta}) \end{bmatrix}, \tag{11}$$

where

$$J_{\alpha\alpha} = -\frac{m}{\alpha^2} - \sum_{i=1}^{D} R_i \frac{\log^2(v_i)}{v_i^{\alpha}(1-v_i^{-\alpha})^2} - R_m \frac{\log^2(v_m)}{v_m^{\alpha}(1-v_m^{-\alpha})^2},$$

$$J_{\beta\beta} = -\frac{m}{\beta^2} - \theta(\alpha+1)\sum_{i=1}^{m}\frac{\log^2(x_i)}{x_i^{\beta}v_i^2} - \theta\alpha\sum_{i=1}^{D}R_i\log(x_i)\phi_i - \theta\alpha R_m\log(x_m)\phi_m,$$

$$J_{\theta\theta} = -\frac{m}{\theta^2} + (\alpha+1)\sum_{i=1}^{m}\frac{1}{x_i^{2\beta}v_i^2} - \alpha\sum_{i=1}^{D}R_i\psi_i - \alpha R_m\psi_m,$$

$$J_{\alpha\beta} = \theta\sum_{i=1}^{m}\frac{\log(x_i)}{x_i^{\beta}v_i} - \theta\sum_{i=1}^{D}\frac{R_i\log(x_i)}{x_i^{\beta}v_i^{\alpha+1}(1-v_i^{-\alpha})}\left[1+\frac{\alpha\log(v_i)}{1-v_i^{-\alpha}}\right] - \frac{\theta R_m\log(x_m)}{x_m^{\beta}v_m^{\alpha+1}(1-v_m^{-\alpha})}\left[1+\frac{\alpha\log(v_m)}{1-v_m^{-\alpha}}\right],$$

$$J_{\alpha\theta} = \sum_{i=1}^{m}\frac{1}{x_i^{\beta}v_i} + \sum_{i=1}^{D}\frac{R_i}{x_i^{\beta}v_i^{\alpha+1}(1-v_i^{-\alpha})}\left[1-\frac{\alpha\log(v_i)}{1-v_i^{-\alpha}}\right] - \frac{R_m}{x_m^{\beta}v_m^{\alpha+1}(1-v_m^{-\alpha})}\left[1-\frac{\alpha\log(v_m)}{1-v_m^{-\alpha}}\right]$$

and

$$J_{\beta\theta} = (\alpha+1)\sum_{i=1}^{m}\frac{\log(x_i)}{x_i^{\beta}v_i^2} - \alpha\sum_{i=1}^{D}\frac{R_i\log(x_i)}{x_i^{\beta}}\left[\frac{1}{v_i^{\alpha+1}(1-v_i^{-\alpha})} - \theta\psi_i\right] - \frac{\alpha R_m\log(x_m)}{x_m^{\beta}}\left[\frac{1}{v_m^{\alpha+1}(1-v_m^{-\alpha})} - \theta\psi_m\right],$$

where $\phi_i = \frac{\log(x_i)\left\{\alpha\theta+[\theta(\alpha+1)-x_i^{-\beta}][v_i^{\alpha}-1]\right\}}{x_i^{2\beta}v_i^{2(\alpha+1)}(1-v_i^{-\alpha})^2}$ and $\psi_i = \frac{1+\alpha-v_i^{-\alpha}}{x_i^{2\beta}v_i^{\alpha+2}(1-v_i^{-\alpha})^2}$.

Presently, the $100(1-\varepsilon)\%$ ACIs of α, β, and θ can be obtained as follows

$$\hat{\alpha} \pm z_{\varepsilon/2}\sqrt{\widehat{var}(\hat{\alpha})},\ \hat{\beta} \pm z_{\varepsilon/2}\sqrt{\widehat{var}(\hat{\beta})} \text{ and } \hat{\theta} \pm z_{\varepsilon/2}\sqrt{\widehat{var}(\hat{\theta})},$$

where $\widehat{var}(\hat{\alpha}), \widehat{var}(\hat{\beta})$, and $\widehat{var}(\hat{\theta})$ are the values obtained from (11), respectively, and $z_{\varepsilon/2}$ is the upper $(\varepsilon/2)^{th}$ percentile point of the standard normal distribution.

In addition to this, to construct the ACIs of the RF and HRF we need to obtain the variance of their estimators $\hat{R}(t)$ and $\hat{h}(t)$. One of the most popular ways to approximate these variances is to apply the so-called delta method; see Greene [28] for more details. For example, to approximate the variance of $\hat{R}(t)$, the delta method stated that, under some regularity conditions, the distribution of the statistics $\hat{R}(t)$ can be approximated by the normal distribution with mean $R(t)$ and variance $\Delta_R I^{-1}(\xi)\Delta_R^{\top}$, where $\Delta_R = (\partial R(t)/\partial\alpha, \partial R(t)/\partial\beta, \partial R(t)/\partial\theta)$ with the following elements

$$\frac{\partial R(t)}{\partial\alpha} = \frac{\log(1+\theta t^{-\beta})}{(1+\theta t^{-\beta})^{\alpha}},\ \frac{\partial R(t)}{\partial\beta} = -\frac{\alpha\theta\log(t)}{t^{\beta}(1+\theta t^{-\beta})^{\alpha+1}},\ \text{and}\ \frac{\partial R(t)}{\partial\theta} = \frac{\alpha}{t^{\beta}(1+\theta t^{-\beta})^{\alpha+1}}.$$

Thus, one can obtain the approximate estimate of variance of $\hat{R}(t)$ as $\widehat{var}(\hat{R}) = (\Delta_R I^{-1}(\hat{\xi})\Delta_R^{\top})$, which is evaluated at the MLEs $\hat{\alpha}, \hat{\beta}$, and $\hat{\theta}$. Similarly, we can acquire the approximate estimate of variance of $\hat{h}(t)$. Let $\Delta_h = (\partial h(t)/\partial\alpha, \partial h(t)/\partial\beta, \partial h(t)/\partial\theta)$, where

$$\frac{\partial h(t)}{\partial\alpha} = -\frac{\beta\theta\left[\alpha\log(1+\theta t^{-\beta}) + (1+\theta t^{-\beta})^{-\alpha} - 1\right]}{t^{\beta+1}(1+\theta t^{-\beta})^{\alpha+1}[1-(1+\theta t^{-\beta})^{-\alpha}]^2},$$

$$\frac{\partial h(t)}{\partial\beta} = \frac{\alpha\theta\left\{1-\beta\log(t) + \frac{\beta\log(t)-1}{(1+\theta t^{-\beta})^{\alpha}} + \frac{\beta\theta\log(t)\left[1+\alpha-(1+\theta t^{-\beta})^{-\alpha}\right]}{t^{\beta}(1+\theta t^{-\beta})}\right\}}{t^{\beta+1}(1+\theta t^{-\beta})^{\alpha+1}[1-(1+\theta t^{-\beta})^{-\alpha}]^2}$$

and

$$\frac{\partial h(t)}{\partial\theta} = -\frac{\alpha\beta\left\{(1+\theta t^{-\beta})^{-\alpha} - 1 + \frac{\theta\left[1+\alpha-(1+\theta t^{-\beta})^{-\alpha}\right]}{t^{\beta}(1+\theta t^{-\beta})}\right\}}{t^{\beta+1}(1+\theta t^{-\beta})^{\alpha+1}[1-(1+\theta t^{-\beta})^{-\alpha}]^2}.$$

Hence, we can obtain the approximate estimate of variance of $\hat{h}(t)$ as $\widehat{var}(\hat{h}) = (\Delta_h I^{-1}(\hat{\xi})\Delta_h^{\top})$, which is evaluated at the MLEs of the unknown parameters. Using the mentioned results, the two-sided ACIs for $R(x)$ and $h(x)$ at the confidence level $100(1-\varepsilon)\%$ are expressed, respectively, as

$$\hat{R} \pm z_{\frac{\varepsilon}{2}} \sqrt{\widehat{var}(\hat{R})} \text{ and } \hat{h} \pm z_{\frac{\varepsilon}{2}} \sqrt{\widehat{var}(\hat{h})}.$$

3. Bayesian Inference

This section derives the Bayesian estimators for the unknown parameters α, β, and θ, as well as the $R(t)$ and $h(t)$. In addition to the point estimates, the HPD credible intervals are studied. In the statistical investigation, the Bayesian technique has influential benefits over the maximum likelihood method because it delivers a natural path of combining prior information about the unknown parameters with new data within a solid theoretical framework.. The Bayesian technique is particularly usable in dependability studies and numerous other disciplines where data availability is a key barrier. This analysis explores the Bayesian estimation beneath the premise that the unknown parameters are independent and have gamma distributions, i.e., $\alpha \sim Gamma(a_1, b_1)$, $\beta \sim Gamma(a_2, b_2)$, and $\theta \sim Gamma(a_3, b_3)$. Based on these assumptions, the joint prior distribution of α, β, and θ can be expressed as

$$\pi(\boldsymbol{\xi}) \propto \alpha^{a_1-1}\, \beta^{a_2-1}\, \theta^{a_3-1}\, e^{-(b_1\alpha+b_2\beta+b_3\theta)}, \tag{12}$$

where a_k and $b_k, k = 1,2,3$, are the hyper-parameters and are always greater than zero. Combining the sample information provided by the likelihood function with the prior knowledge about the unknown parameters presented through the joint prior distribution and by applying the Bayes theorem, one can derive the posterior distribution of the unknown parameters α, β, and θ. Therefore, from (6) and (12), the joint posterior distribution of α, β and θ takes the form

$$\begin{aligned} g(\boldsymbol{\xi}|\underline{x}) = A^{-1}\alpha^{m+a_1-1}\beta^{m+a_2-1}\theta^{m+a_3-1} \exp\Bigg\{ &-\alpha\left[\sum_{i=1}^{m}\log(1+\theta x_i^{-\beta})+b_1\right] - \sum_{i=1}^{m}\log(1+\theta x_i^{-\beta}) \\ &-\beta\left[\sum_{i=1}^{m}\log(x_i)+b_2\right] - b_3\theta\Bigg\} \prod_{i=1}^{D}\left[1-(1+\theta x_i^{-\beta})^{-\alpha}\right]^{R_i}\left[1-(1+\theta x_m^{-\beta})^{-\alpha}\right]^{R_m}, \end{aligned} \tag{13}$$

where $\underline{x} = (x_1, \ldots, x_m)$ and A is the normalized constant. The loss function plays a critical role in Bayesian estimation because it can be used to identify overestimation and underestimation in the investigation. Here, we take into account the SE and GE loss functions. The SE loss function is one of the most often used symmetric loss functions, whereas the GE loss function is asymmetric. It is well known that the Bayes estimator in the case of the SE loss function is the posterior mean where the overestimation and underestimation are treated equally. Conversely, the GE loss function delivers diverse importance for overestimation and underestimation. The GE loss function introduced by Calabria and Pulcini [29] and defined as

$$G(\tilde{\delta}, \delta) \propto \left(\frac{\tilde{\delta}}{\delta}\right)^{\kappa} - \kappa \log\left(\frac{\tilde{\delta}}{\delta}\right) - 1,$$

where $\tilde{\delta}$ is the estimator of δ and κ is a parameter that determines the degree of asymmetry. The Bayes estimator of δ using GE loss function is given by

$$\tilde{\delta}_{GE} = \left[E_{\delta}(\delta^{-\kappa})\right]^{-\frac{1}{\kappa}}, \tag{14}$$

provided that $E_{\delta}(\delta^{-\kappa})$ exists and is finite.

It can be seen that when $\kappa = -1$, the Bayes estimator in (14) coincides with the Bayes estimator under the SE loss function. Now, let $\zeta(\boldsymbol{\xi})$ any function of the unknown parameters, then the Bayes estimators based on the SE and GE loss functions can be obtained directly from (13), respectively, as follow

$$\tilde{\zeta}_{SE}(\boldsymbol{\xi}) = \int_0^\infty \int_0^\infty \int_0^\infty \zeta(\boldsymbol{\xi})\, g(\boldsymbol{\xi}|\underline{x})\, d\alpha\, d\beta\, d\theta \tag{15}$$

and

$$\tilde{\zeta}_{GE}(\boldsymbol{\xi}) = \left[\int_0^\infty \int_0^\infty \int_0^\infty [\zeta(\boldsymbol{\xi})]^{-\kappa}\, g(\boldsymbol{\xi}|\underline{x})\, d\alpha\, d\beta\, d\theta\right]^{-\frac{1}{\kappa}}. \tag{16}$$

Clearly, calculating the Bayes estimators using (15) and (16) analytically are unattainable. As a result, we advocate employing the MCMC technique to obtain the Bayes estimates of α, β, and θ and the associated HPD credible intervals. To apply the MCMC technique, we should first derive the full conditional distributions of α, β, and θ. The required full conditional distributions can be given from (13) as follow

$$\begin{aligned} g(\alpha|\beta,\theta,\underline{x}) \quad &\propto \quad \alpha^{m+a_1-1} \exp\left\{-\alpha\left[\sum_{i=1}^{m} \log(1+\theta x_i^{-\beta}) + b_1\right]\right\} \\ &\times \quad \prod_{i=1}^{D}\left[1-(1+\theta x_i^{-\beta})^{-\alpha}\right]^{R_i}\left[1-(1+\theta x_m^{-\beta})^{-\alpha}\right]^{R_m}, \end{aligned} \tag{17}$$

$$\begin{aligned} g(\beta|\alpha,\theta,\underline{x}) \quad &\propto \quad \beta^{m+a_2-1} \exp\left\{-(\alpha+1)\left[\sum_{i=1}^{m} \log(1+\theta x_i^{-\beta})\right] - \beta\left[\sum_{i=1}^{m}\log(x_i) + b_2\right]\right\} \\ &\times \quad \prod_{i=1}^{D}\left[1-(1+\theta x_i^{-\beta})^{-\alpha}\right]^{R_i}\left[1-(1+\theta x_m^{-\beta})^{-\alpha}\right]^{R_m} \end{aligned} \tag{18}$$

and

$$\begin{aligned} g(\theta|\alpha,\beta,\underline{x}) \quad &\propto \quad \theta^{m+a_3-1} \exp\left\{-(\alpha+1)\left[\sum_{i=1}^{m} \log(1+\theta x_i^{-\beta})\right] - b_3\theta\right\} \\ &\times \quad \prod_{i=1}^{D}\left[1-(1+\theta x_i^{-\beta})^{-\alpha}\right]^{R_i}\left[1-(1+\theta x_m^{-\beta})^{-\alpha}\right]^{R_m}. \end{aligned} \tag{19}$$

Nevertheless, it is noticeable that the full conditional posterior distributions of α, β, and θ cannot be tended analytically to famous distributions. Consequently, it is not probable to generate samples straight by traditional techniques, whereas the plots of them indicate that they are equivalent to normal distribution. So, we need to induce the unknown parameters by employing Metropolis-–Hasting (MH) sampling. To involve the MH sampling, we assume the normal distribution as the proposal distribution to acquire the Bayesian estimates and to obtain the HPD credible intervals. The MH sampling functions as follows to generate samples from (17)–(19)

Step 1. Put $l = 1$.

Step 2. Set $\left(\alpha^{(0)}, \beta^{(0)}, \theta^{(0)}\right) = (\hat{\alpha}, \hat{\beta}, \hat{\theta})$.

Step 3. Generate $\alpha^{(l)}$ from the full conditional posterior distribution (17) using normal distribution, i.e., $N\left(\alpha^{(l-1)}, \widehat{var}(\alpha^{(l-1)})\right)$, and by applying the MH steps.

Step 4. Repeat step 3 to generate $\beta^{(l)}$ and $\theta^{(l)}$ from (18) and (19), respectively.

Step 5. Use the generated sample to compute $R^{(l)}(t)$ and $h^{(l)}(t)$ from (3) and (4), respectively.

Step 6. Set $l = l + 1$.

Step 7. Redo steps 3–6, B times to obtain

$$\left[\alpha^{(1)}, \beta^{(1)}, \theta^{(1)}, R^{(1)}(t), h^{(1)}(t)\right], \ldots, \left[\alpha^{(B)}, \beta^{(B)}, \theta^{(B)}, R^{(B)}(t), h^{(B)}(t)\right].$$

To assure convergence and to withdraw the affection of the choice of starting values, the first Q generated variates are scrapped. In this case, we have $\alpha^{(l)}, \beta^{(l)}, \theta^{(l)}, R^{(l)}(t)$, and $h^{(l)}(t), l = Q+1, \ldots, B$. Based on large B, the generated sample forms an approximate posterior sample which can be employed to obtain the Bayes estimates and the HPD credible intervals. Now, let ξ be the unknown parameter to be estimated. Then, the Bayes estimate of ξ based on the SE loss function can be obtained as

$$\tilde{\xi}_{SE} = \frac{1}{B-Q} \sum_{l=Q+1}^{B} \xi^{(l)}.$$

Similarly, the Bayes estimate of ξ based on the GE loss function can be computed as follows

$$\tilde{\xi}_{GE} = \left\{ \frac{1}{B-Q} \sum_{l=Q+1}^{B} [\xi^{(l)}]^{-\kappa} \right\}^{-\frac{1}{\kappa}}.$$

On the other hand, to compute the HPD credible intervals of α, β, θ, $R(t)$ and $h(t)$, say ξ, we order $\xi^{(l)}$, as $\xi^{(Q+1)} < \xi^{(Q+2)} < \cdots < \xi^{(B)}$. Then, the $100(1-\varepsilon)\%$ two-sided HPD credible interval of ξ becomes $\left[\xi^{(l^*)}, \xi^{(l^*+(1-\varepsilon)(M-Q))}\right]$, where $l^* = Q+1, Q+2, \ldots, B$ is specified such that

$$\xi^{(l^*+[(1-\varepsilon)(B-Q)])} - \xi^{(l^*)} = \min_{1 \leqslant l \leqslant \varepsilon(B-Q)} \left[\xi^{(l+[(1-\varepsilon)(B-Q)])} - \xi^{(l)}\right],$$

where $[\epsilon]$ denotes the largest integer less than or equal to ϵ. It is noteworthy to mention here that the results of Arif et al. [6] can be obtained as a special case of the results derived in this paper when $R_1 = R_2 = \cdots = R_m = 0$, with $T \to \infty$, which is the complete sample case.

4. Monte Carlo Simulation

In this section, Monte Carlo simulations are performed to know the performance of the proposed estimators developed in the previous sections of the parameters, reliability, and hazard functions based on an APT-II HC scheme. First, we describe the simulation design. Then, some discussions regarding the simulation outcomes are reported.

4.1. Simulation Design

This subsection is devoted to how to conduct the proposed numerical study. First, we simulate 1000 APT-II HC samples from Dagum$(0.4, 0.2, 0.1)$ based on various choices of T (threshold time), n (total sample size), m (effective sample size) and $\mathbf{R}$ (progressive censoring). Taking $t = 0.5$, the actual values of the reliability characteristics $R(t)$ and $h(t)$ are 0.043 and 0.371, respectively. Using T(=0.1, 0.5), n(=50,100) and m is specified as a percentage of n as $\frac{m}{n}$(=50, 80)%, the proposed numerical experiments are performed. In addition, for each n and m, different removal patterns of the progressive type-II censoring mechanism, where $\mathbf{R} = (5, 0, 0, 0, 5)$ is symbolized by $\mathbf{R}$ =(5.0*3.5) , are used as

$$\begin{aligned}
&\text{Scheme-1}: \quad \mathbf{R} = (n-m, 0^*(m-1)),\\
&\text{Scheme-2}: \begin{cases} \mathbf{R} = (0^*(\frac{m}{2}-1), n-m, 0^*(\frac{m}{2}));\ \text{if } m \text{ is even},\\ \mathbf{R} = (0^*(\frac{m-1}{2}), n-m, 0^*(\frac{m-1}{2}));\ \text{if } m \text{ is odd}, \end{cases}\\
&\text{Scheme-3}: \quad \mathbf{R} = (0^*(m-1), n-m).
\end{aligned}$$

To simulate APT-II HC samples of size m from a given sample of size n with given progressive censoring R_i, $i = 1, 2, ..., m$, do the following steps:

Step 1. Generate a conventional progressive type-II sample (X_i, R_i), $i = 1, 2, \ldots, m$, as

(a) Generate $w_1, w_2, \ldots, w_m$ from uniform $U(0,1)$ distribution.

(b) Put $p_i = w_i^{\left(i+\sum_{j=m-i+1}^{m} R_j\right)^{-1}}$, for $i = 1, 2, \ldots, m$.

(c) Set $u_i = 1 - p_m p_{m-1} \cdots p_{m-i+1}$ for $i = 1, 2, \ldots, m$. Hence, u_i, $i = 1, 2, \ldots, m$ is a simulated progressive type-II sample of size m from the uniform $U(0,1)$ distribution.

(d) Set $X_i = F^{-1}(u_i; \boldsymbol{\xi})$, $i = 1, 2, \ldots, m$, the progressive type-II from Dagum(α, β, θ) is generated.

Step 2. Determine D and discard X_i for $i = D + 2, \ldots, m$.

Step 3. Using truncated distribution $f(x)[1 - F(x_{D+1})]^{-1}$, generate the first-order statistics $X_{D+2}, \ldots, X_m$ of size $n - D - \sum_{j=1}^{D} R_j - 1$.

In frequentist investigation, from the 1000 APT-II HC samples, the MLEs (along their 95% ACIs) of α, β, θ, $R(t)$, and $h(t)$ are computed. In Bayesian analysis, to evaluate the effects of the priors, two informative sets of hyper-parameters are used; namely Prior-1: $(a_1, a_2, a_3) = (2, 1, 0.5)$, and $b_i = 5, i = 1, 2, 3$; Prior-2: $(a_1, a_2, a_3) = (4, 2, 1)$ and $b_i = 10$, $i = 1, 2, 3$. All hyper-parameter values associated with each unknown parameter are chosen in such a way that the prior average is equal to the expected value of the corresponding unknown parameter; see Kundu (2008). It is important to mention here that the frequentist methods may be better than the Bayes method because the latter is computationally more expensive if there is no prior information about the parameters of interest. A large 12,000 MCMC variates via MH sampler are generated and then the first 2000 variates are removed as burn-in period. Next, based on 10,000 MCMC variates, the average Bayes estimates of α, β, θ, $R(t)$, and $h(t)$ using the SE and GE (for $v(= -2, +2)$) loss functions as well as the associated 95% HPD intervals are computed. The point estimates of the unknown parameters α, β, θ, $R(t)$, and $h(t)$ (say ξ for short), are compared using root-mean-squared errors (RMSEs) and mean relative absolute biases (MRABs) given, respectively, as

$$\mathrm{RMSE}(\xi^{\diamond}) = \sqrt{\frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \left(\xi^{\diamond(i)} - \xi\right)^2} \quad \text{and} \quad \mathrm{MRAB}(\xi^{\diamond}) = \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \frac{1}{\xi} \left|\xi^{\diamond(i)} - \xi\right|,$$

where $\mathcal{N}$ is the number of replications and $\xi^{\diamond(i)}$ is the estimate of ξ at the j^{th} sample. In addition, the performance of the interval estimates is compared using their average confidence lengths (ACLs) and coverage percentages (CPs) delivered, respectively, by

$$\mathrm{ACL}_{(1-\varepsilon)\%}(\xi) = \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \left(\mathcal{U}_{\xi^{\diamond(i)}} - \mathcal{L}_{\xi^{\diamond(i)}}\right) \quad \text{and} \quad \mathrm{CP}_{(1-\varepsilon)\%}(\xi) = \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \mathbf{1}_{\left(\mathcal{L}_{\xi^{\diamond(i)}}; \mathcal{U}_{\xi^{\diamond(i)}}\right)}(\xi),$$

where $\mathbf{1}(\cdot)$ is the indicator function and $\mathcal{L}(\cdot)$ and $\mathcal{U}(\cdot)$ denote the lower and upper interval bounds, respectively.

All calculations implemented are performed using R 4.1.2 software by using two packages, namely (a) 'coda' (by Plummer et al. [30]) and (b) 'maxLik' (by Henningsen and Toomet [31]). These packages were also recommended by Elshahhat and Elemary [32]. Graphically, all simulation results of α, β, θ, $R(t)$, and $h(t)$ are displayed with heatmap plots in Figures 1–5, respectively, while all simulation outputs are provided as Supplementary Materials. In each heatmap, the 'x-lab' displays the proposed point (or interval) estimation methods while the 'y-lab' represents the given settings T, n, m, and $\mathbf{R}$, which are denoted by 'T-(n, m)-Scheme'. For instance; based on Prior-1 set (say P1), we have used the notation "SE-P1" for the Bayes estimates from the SE loss; the notations "GE1-P1" and "GE2-P1" for the Bayes estimates from the GE loss based on $\kappa = -2$ and $\kappa = +2$, respectively; "HPD-P1" denotes to HPD intervals. The color vector beside the heatmap represents the calculated values of the RMSEs, MRABs, ACLs, or CPs for each unknown parameter in each setting from lowest to highest value from yellow to red.

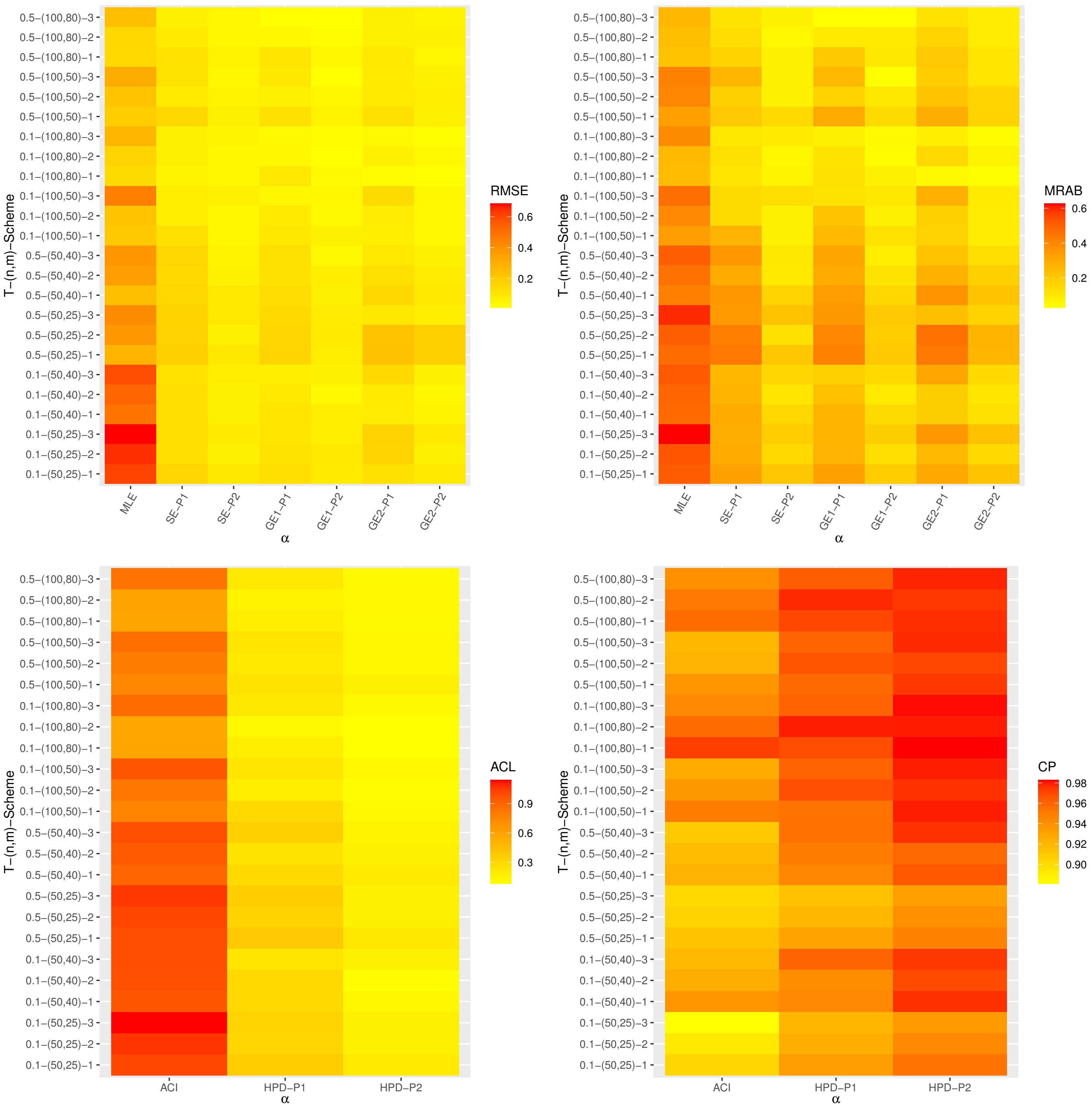

Figure 1. Heatmap for the estimation results of α.

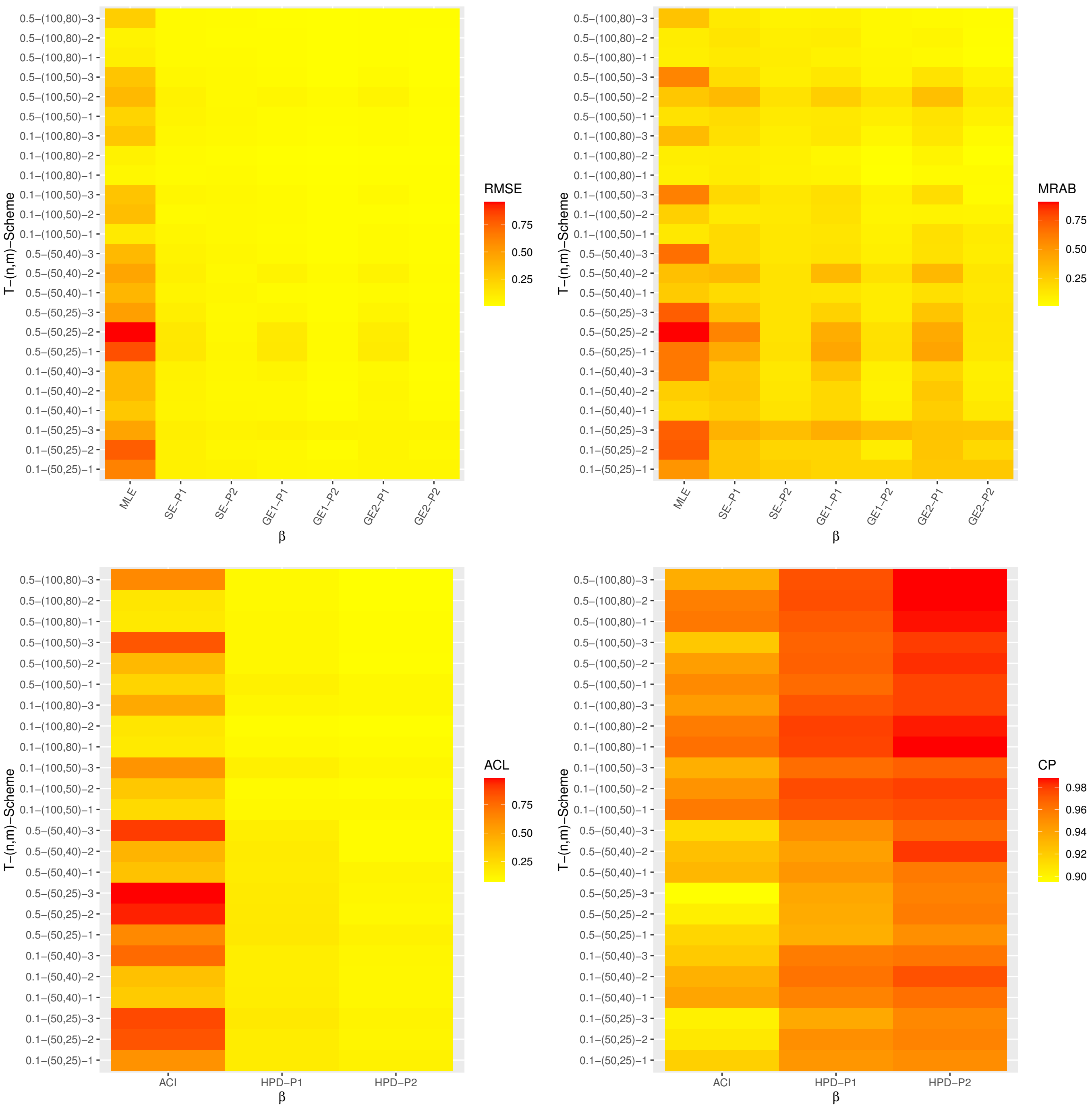

Figure 2. Heatmap for the estimation results of β.

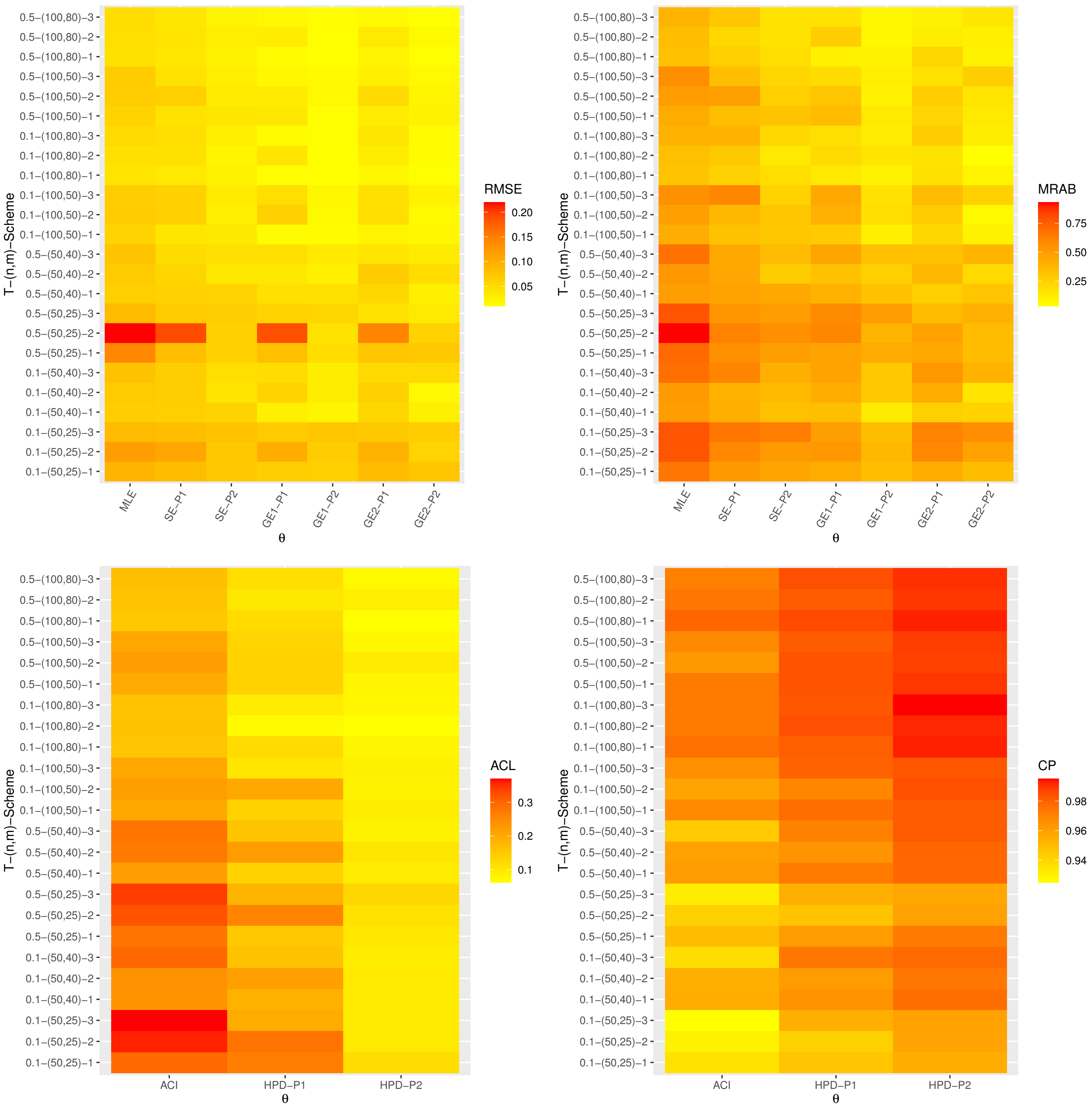

Figure 3. Heatmap for the estimation results of θ.

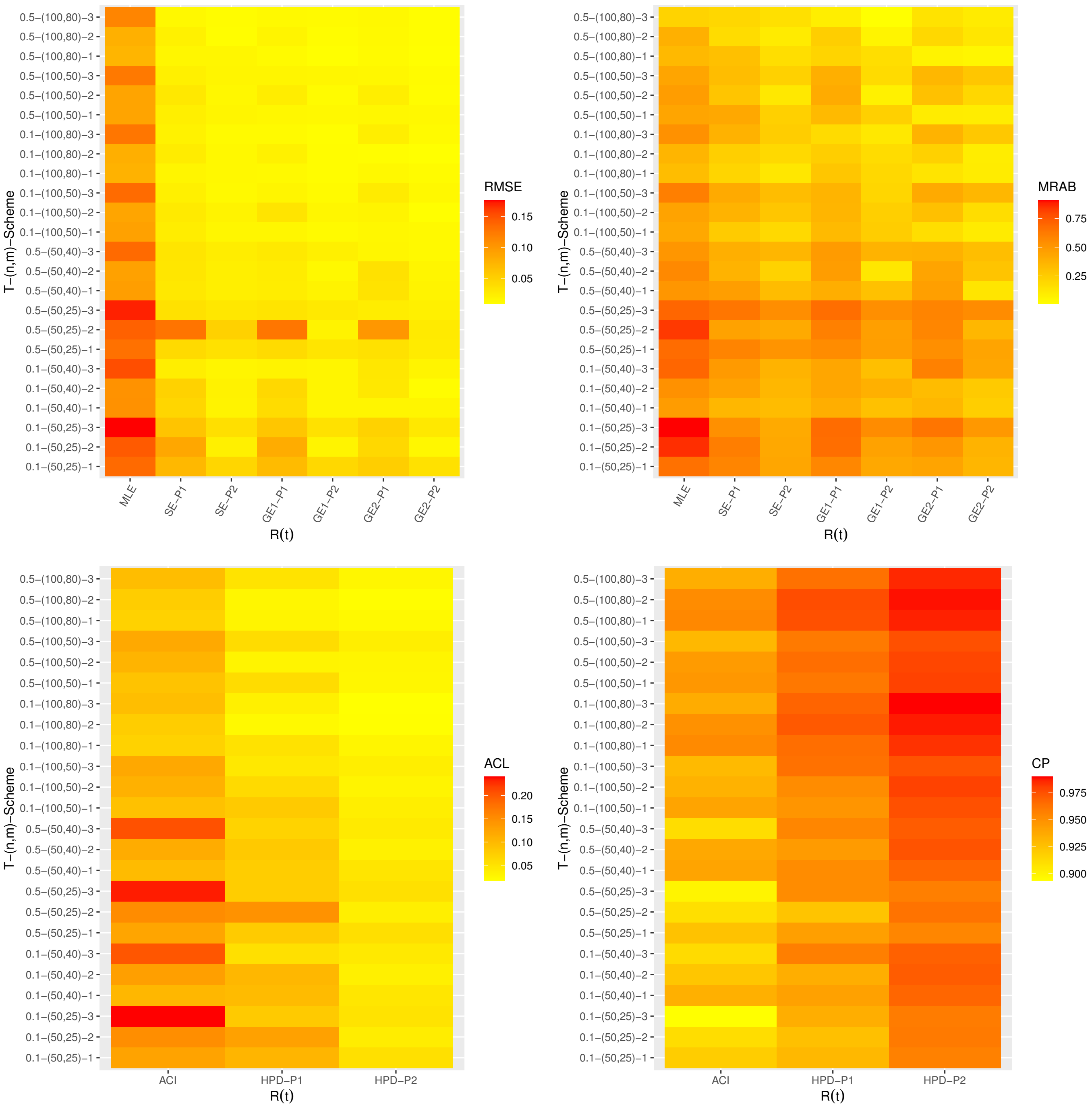

Figure 4. Heatmap for the estimation results of $R(t)$.

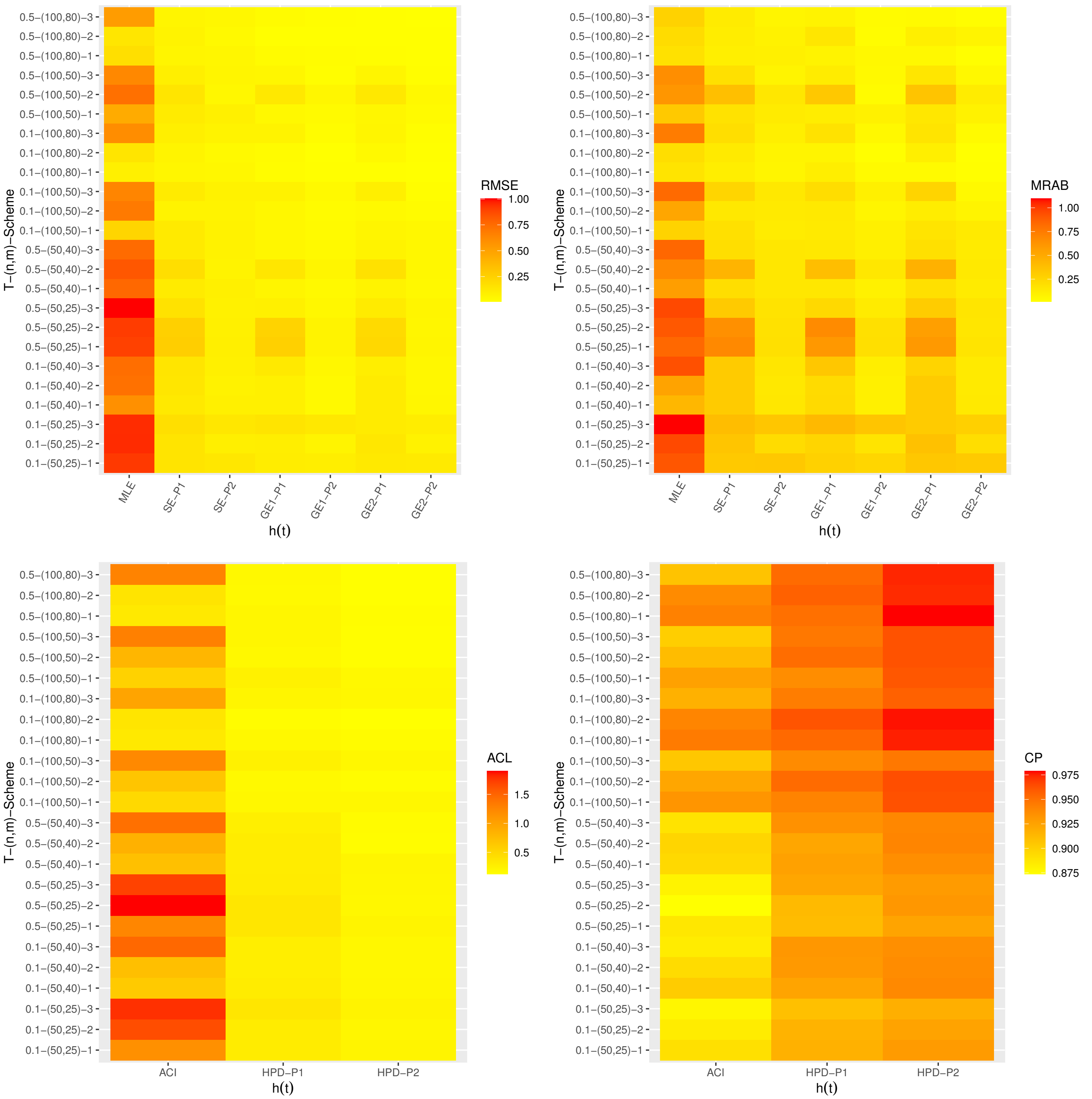

Figure 5. Heatmap for the estimation results of $h(t)$.

4.2. Simulation Discussions

Various appraisals of the performance of the proposed point and interval estimation methods are discussed in this subsection. From Figures 1–5, the following observations can be made:

- All calculated estimates have displayed satisfactory behavior in terms of minimum RMSEs, MRABs, and ACLs values, as well as in terms of highest CPs.
- As n increases, the offered estimates are pretty satisfactory. Identical behavior is observed when $\sum_{i=1}^{m} R_i$ (or $n - m$) lowers.

- As T increases, the RMSEs and MRABs for the MLEs of α, $R(t)$, and $h(t)$ decrease, while they increase for β and θ. Moreover, as T increases, the RMSEs and MRABs of the Bayes estimates of α, β, and $h(t)$ increase, while they decrease for θ and $R(t)$.
- As T increases, the ACLs of the ACIs of α, θ, and $R(t)$ decrease while they increase for β and $h(t)$. Further, when T increases, the ACLs of the HPD interval estimates of α, β, and $h(t)$ increase while they decrease for θ and $R(t)$. The opposite behavior is also observed in the case of the CPs for the ACI and HPD credible interval estimates of all unknown parameters.
- Since the Bayes estimates are expressed using the gamma density prior, the Bayes (point/interval) estimates using MH procedure perform better than the classical estimates in terms of the smallest RMSEs, MRABs, and ACLs as well as the highest CPs.
- It is also observed that the Bayes estimates based on Prior-2 are superior to Prior-1 for all unknown parameters. This is expected due to the fact that the variance of Prior-2 is smaller than Prior-1.
- It is noted that the CPs of the HPD intervals are almost closely (or greater) to the specified nominal level than the ACIs.
- It can be seen that the RMSEs, MRABs, ACLs, and CPs of α, β, θ, $R(t)$, and $h(t)$ are even good based on Scheme-1 than other schemes.
- It is known that the expected duration of an experiment based on Scheme-1 is greater than that of any other, thus the APT-II HC sample gathered under this scheme supplied more additional information about the unknown parameters than those obtained based on any other censoring scheme.
- Overall, the Bayes procedure via MH algorithm is advised to estimate the unknown parameters of Dagum distribution and its reliability characteristics under the APT-II HC plan.

5. Optimal Progressive Censoring Plan

Choosing the optimal censoring plans has earned a lot of awareness in the statistical literature. For specified n and m, probable censoring schemes refer to all $R_i, i = 1, \ldots, m$ mixtures such that $m + \sum_{i=1}^{m} R_i = n$ and picking the most suitable sample technique entails locating the progressive censoring scheme that delivers the most knowledge regarding the unknown parameters among all possible progressive censoring plans. For more details about optimal censoring plans, one can refer to Ng et al. [33] and Pradhan and Kundu [34]. In this study, we consider four optimality criteria that were widely used in the literature. Practically and as we mentioned before that we need to select the censoring scheme that provides us with the most information about the parameters. Table 1 furnishes some typically employed optimal criteria to aid us in choosing the most suitable progressive censoring scheme.

Table 1. Some optimal censoring plan criteria.

Criterion	Method
I	Maximize trace$(\mathbf{I}_{3\times3}(\hat{\xi}))$
II	Minimize trace$(\mathbf{I}^{-1}_{3\times3}(\hat{\xi}))$
III	Minimize det$(\mathbf{I}^{-1}_{3\times3}(\hat{\xi}))$
IV	Minimize $\widehat{var}(\log(\hat{\chi}_q)), 0 < q < 1$

One can see from Table 1 that the criteria I, II, and III are looking for the progressive censoring scheme that maximize the observed Fisher information matrix, minimize the determinant of $\mathbf{I}^{-1}_{3\times3}(\hat{\xi})$, and minimize the trace of $\mathbf{I}^{-1}_{3\times3}(\hat{\xi})$, respectively. On the other hand, the criterion IV tries to minimize the variance of logarithmic MLE of the q^{th} quantile, denoted by $\widehat{var}(\log(\hat{\chi}_q))$, where

$$\log(\hat{\chi}_q) = -\frac{1}{\hat{\beta}}\log\left[\frac{q^{-\frac{1}{\hat{\alpha}}}-1}{\hat{\theta}}\right], 0 < q < 1,$$

where the delta method can be used to approximate the variance of $\log(\hat{\chi}_q)$. To pick the optimal progressive censoring plan, one should select the progressive censoring plan that gives the maximum value of criterion I and the smallest values of criteria II, III, and IV.

6. Real-Life Applications

To demonstrate how one can apply the proposed methodologies to a real-life situation, two applications using real-life data sets from chemistry and engineering areas are discussed in this section.

6.1. Coating Weights of Iron Sheets

In this application, from chemistry field, we shall provide a statistical analysis for the real coating weights of iron sheets obtained from the Aluminium Africa Limited (ALAF) industry, Tanzania, during January-March, 2018. To improve the quality of steel roofing, the coating process is one of the most processes used in this industry. Therefore, the ALAF industry uses the manufacturing technology of aluminum–zinc in the coating process. This data set consists of 72 observations on coating weight (in $\mathrm{gm/m^2}$) by chemical method on top-center side from the ALAF industry; see Table 2. This data set was first discussed by Rao and Mbwambo [35] and also analyzed by Fan and Gui [36] recently.

Table 2. Coating weight data of iron sheets from ALAF industry.

28.7	29.4	30.4	31.6	31.8	32.7	32.9	33.2	33.2	33.6	33.7	34.0	34.2	34.5	35.6
36.2	36.7	36.8	36.8	37.3	37.8	38.5	38.9	38.9	39.1	39.9	40.1	40.2	40.3	40.5
40.6	40.7	41.2	41.2	41.3	42.3	42.3	42.6	42.8	42.8	42.8	42.8	43.1	44.2	44.9
45.2	45.3	45.4	45.8	46.3	47.1	47.2	47.2	48.2	48.3	48.4	48.5	49.8	50.1	52.6
52.8	54.2	54.5	55.4	55.8	56.8	58.2	58.4	58.7	58.9	59.2	61.2			

To check whether the Dagum distribution is appropriate statistical distribution to fit the coating weight data set or not, the MLEs of the Dagum parameters α, β, and θ are calculated to carry out the Kolmogorov–Smirnov (K-S) distance and associated P-value. The values of $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\theta}$ (with their standard errors (St.Es)) are 3163.52 (3.0251), 4.85561 (0.0329), and 16,655.9 (1.1860), respectively. The K-S (P-value) is 0.109 (0.364). This result indicates that the Dagum distribution is a proper lifetime model to fit the coating weight data. Moreover, using the complete coating weight data set, the estimated/empirical RF of the Dagum distribution is displayed in Figure 6.

From the original data set, three different APT-II HC samples are generated with $m = 20$ and reported in Table 3. Based on the generated samples, the MLEs and Bayes estimates with their St.Es of α, β, θ, $R(t)$, and $h(t)$ (at distinct time $t = 50$) are computed and presented in Table 4. Additionally, the two bounds of 95% ACI/HPD intervals with their interval lengths (ILs) of the unknown parameters are also calculated and provided in Table 5. In order to develop the Bayes estimates, we assume that the hyper-parameters a_i and b_i for $i = 1,2,3$ of α, β, θ, $R(t)$, and $h(t)$ are not available. Therefore, to run our computations, the hyper-parameter values are selected to be 0.001. To run the MCMC algorithm, the classical estimates of α, β, and θ are taken to be the initial guesses. Tables 4 and 5 indicated that the proposed Bayes estimates perform better than the frequentist estimates in terms of lowest St.Es, as well as, the HPD interval estimates also perform satisfactorily compared to the ACI estimates in terms of shortest ILs.

Table 3. Three APT-II HC samples from coating weight data.

Sample	Scheme	*T(D)*	R_m	Data									
$\mathcal{S}_1$	$(52, 0^*19)$	65(20)	0	28.7	48.2	48.3	48.4	48.5	49.8	50.1	52.6	52.8	54.2
				54.5	55.4	55.8	56.8	58.2	58.4	58.7	58.9	59.2	61.2
$\mathcal{S}_2$	$(0^*7, 10^*5, 2, 0^*7)$	45(11)	12	28.7	29.4	30.4	31.6	31.8	32.7	32.9	33.2	36.8	40.5
				42.8	47.2	47.2	48.2	48.3	48.4	48.5	49.8	50.1	52.6
$\mathcal{S}_3$	$(0^*19, 52)$	35(14)	52	28.7	29.4	30.4	31.6	31.8	32.7	32.9	33.2	33.2	33.6
				33.7	34.0	34.2	34.5	35.6	36.2	36.7	36.8	36.8	37.3

Table 4. Point estimates (St.Es) of α, β, θ, $R(t)$, and $h(t)$ from coating weight data.

$\mathcal{S}_i$	Par.	MLE	SE	GE		
$\kappa \rightarrow$				−3	−0.03	+3
$\mathcal{S}_1$	α	1252.3 (0.86 $\times$ 10^{+1})	1252.2 (3.90 $\times$ 10^{-4})	1252.2 (6.29 $\times$ 10^{-2})	1252.3 (6.29 $\times$ 10^{-2})	1252.26 (6.29 $\times$ 10^{-2})
	β	3.9852 (5.82 $\times$ 10^{-2})	3.9653 (2.29 $\times$ 10^{-4})	3.9658 (1.93 $\times$ 10^{-2})	3.9650 (2.01 $\times$ 10^{-2})	3.96423 (2.09 $\times$ 10^{-2})
	θ	3972.5 (0.19 $\times$ 10^{+1})	3972.4 (4.93 $\times$ 10^{-4})	3972.4 (6.44 $\times$ 10^{-2})	3972.4 (6.44 $\times$ 10^{-2})	3972.38 (6.43 $\times$ 10^{-2})
	$R(50)$	0.5697 (8.25 $\times$ 10^{-2})	0.5985 (3.24 $\times$ 10^{-4})	0.6055 (3.57 $\times$ 10^{-2})	0.6054 (2.53 $\times$ 10^{-2})	0.58368 (1.40 $\times$ 10^{-2})
	$h(50)$	0.0507 (6.29 $\times$ 10^{-3})	0.0484 (2.55 $\times$ 10^{-5})	0.0489 (1.84 $\times$ 10^{-3})	0.0481 (2.64 $\times$ 10^{-3})	0.04724 (3.51 $\times$ 10^{-3})
$\mathcal{S}_2$	α	367.75 (0.88 $\times$ 10^{+1})	367.69 (4.00 $\times$ 10^{-4})	367.68 (6.22 $\times$ 10^{-2})	367.69 (6.22 $\times$ 10^{-2})	367.69 (6.23 $\times$ 10^{-2})
	β	2.8438 (3.63 $\times$ 10^{-2})	2.8333 (1.62 $\times$ 10^{-4})	2.8336 (1.02 $\times$ 10^{-2})	2.8331 (1.07 $\times$ 10^{-2})	2.8325 (1.13 $\times$ 10^{-2})
	θ	165.34 (0.11 $\times$ 10^{+1})	165.28 (4.02 $\times$ 10^{-4})	165.27 (6.36 $\times$ 10^{-2})	165.27 (6.37 $\times$ 10^{-2})	165.27 (6.37 $\times$ 10^{-2})
	$R(50)$	0.5194 (5.11 $\times$ 10^{-2})	0.6065 (2.30 $\times$ 10^{-4})	0.6099 (1.86 $\times$ 10^{-2})	0.6048 (1.34 $\times$ 10^{-2})	0.5993 (7.85 $\times$ 10^{-3})
	$h(50)$	0.0351 (2.97 $\times$ 10^{-3})	0.0342 (1.36 $\times$ 10^{-5})	0.0344 (7.16 $\times$ 10^{-4})	0.0341 (1.04 $\times$ 10^{-3})	0.0338 (1.37 $\times$ 10^{-3})
$\mathcal{S}_3$	α	7887.5 (0.12 $\times$ 10^{+1})	7887.4 (4.03 $\times$ 10^{-4})	7887.4 (6.55 $\times$ 10^{-2})	7887.4 (6.54 $\times$ 10^{-2})	7887.4 (6.55 $\times$ 10^{-2})
	β	4.5696 (6.69 $\times$ 10^{-2})	4.5453 (2.45 $\times$ 10^{-4})	4.5458 (2.37 $\times$ 10^{-2})	4.5450 (2.45 $\times$ 10^{-2})	4.5442 (2.53 $\times$ 10^{-2})
	θ	1060.1 (0.17 $\times$ 10^{+1})	1060.1 (3.93 $\times$ 10^{-4})	1060.1 (6.32 $\times$ 10^{-2})	1060.1 (6.31 $\times$ 10^{-2})	1060.1 (6.31 $\times$ 10^{-2})
	$R(50)$	0.1342 (3.27 $\times$ 10^{-2})	0.1486 (1.30 $\times$ 10^{-4})	0.1532 (1.89 $\times$ 10^{-2})	0.1464 (1.22 $\times$ 10^{-2})	0.1394 (5.22 $\times$ 10^{-3})
	$h(50)$	0.0850 (2.89 $\times$ 10^{-3})	0.0838 (1.11 $\times$ 10^{-5})	0.0838 (1.12 $\times$ 10^{-3})	0.0837 (1.20 $\times$ 10^{-3})	0.0836 (1.29 $\times$ 10^{-3})

Table 5. Interval estimates [ILs] of α, β, θ, $R(t)$, and $h(t)$ from coating weight data.

$\mathcal{S}_i$	Par.	ACI	HPD
$\mathcal{S}_1$	α	1252.3 (0.86 $\times$ 10^{+1})	1252.2 (3.90 $\times$ 10^{-4})
	β	3.9852 (5.82 $\times$ 10^{-2})	3.9653 (2.29 $\times$ 10^{-4})
	θ	3972.5 (0.19 $\times$ 10^{+1})	3972.4 (4.93 $\times$ 10^{-4})
	$R(50)$	0.5697 (8.25 $\times$ 10^{-2})	0.5985 (3.24 $\times$ 10^{-4})
	$h(50)$	0.0507 (6.29 $\times$ 10^{-3})	0.0484 (2.55 $\times$ 10^{-5})
$\mathcal{S}_2$	α	367.75 (0.88 $\times$ 10^{+1})	367.69 (4.00 $\times$ 10^{-4})
	β	2.8438 (3.63 $\times$ 10^{-2})	2.8333 (1.62 $\times$ 10^{-4})
	θ	165.34 (0.11 $\times$ 10^{+1})	165.28 (4.02 $\times$ 10^{-4})
	$R(50)$	0.5194 (5.11 $\times$ 10^{-2})	0.6065 (2.30 $\times$ 10^{-4})
	$h(50)$	0.0351 (2.97 $\times$ 10^{-3})	0.0342 (1.36 $\times$ 10^{-5})
$\mathcal{S}_3$	α	7887.5 (0.12 $\times$ 10^{+1})	7887.4 (4.03 $\times$ 10^{-4})
	β	4.5696 (6.69 $\times$ 10^{-2})	4.5453 (2.45 $\times$ 10^{-4})
	θ	1060.1 (0.17 $\times$ 10^{+1})	1060.1 (3.93 $\times$ 10^{-4})
	$R(50)$	0.1342 (3.27 $\times$ 10^{-2})	0.1486 (1.30 $\times$ 10^{-4})
	$h(50)$	0.0850 (2.89 $\times$ 10^{-3})	0.0838 (1.11 $\times$ 10^{-5})

To show that the simulated MCMC samples are converged well, based on $\mathcal{S}_1$ as an example, the trace plots based on 40,000 chain values of α, β, θ, $R(t)$, and $h(t)$ are shown in Figure 7. Each trace plot represents the arithmetic sample mean (with solid (—) horizontal line) and two bounds of 95% HPD intervals (with dashed (- - -) horizontal line). It shows that the proposed MCMC algorithm converges well and the burn-in period has an appropriate size to ignore the effect of the starting guesses. Furthermore, based on $\mathcal{S}_1$ as an example, the approximated marginal density functions with their frequencies using Gaussian kernel of α, β, θ, $R(t)$, and $h(t)$ are displayed in Figure 8. It indicates that the simulated marginal posterior estimates of all the unknown parameters are fairly symmetrical. Furthermore, based on $\mathcal{S}_1$ as an example, some general statistics for the MCMC outputs of α, β, θ,

$R(t)$, and $h(t)$ after burn-in, namely: mean, mode, mode, quartiles (Q_1, Q_2, Q_3), standard deviation (St.D), and skewness (Skew.) are also computed and presented in Table 6. Other MCMC plots based on samples $\mathcal{S}_2$ and $\mathcal{S}_3$ of α, β, θ, $R(t)$, and $h(t)$ are plotted and reported in the Supplementary File for brevity. From Table 3, based on the four optimum criteria declared in Section 5, the problem of selecting the best (optimal) progressive censoring plan is discussed. The results of the different criteria are displayed in Table 7. It provides that the censoring scheme used in sample $\mathcal{S}_1$ is the optimum plan based on the given criteria II, the censoring scheme used in sample $\mathcal{S}_2$ is the optimum plan based on criterion I and III, and the censoring scheme used in sample $\mathcal{S}_3$ is the optimum plan based on criterion IV.

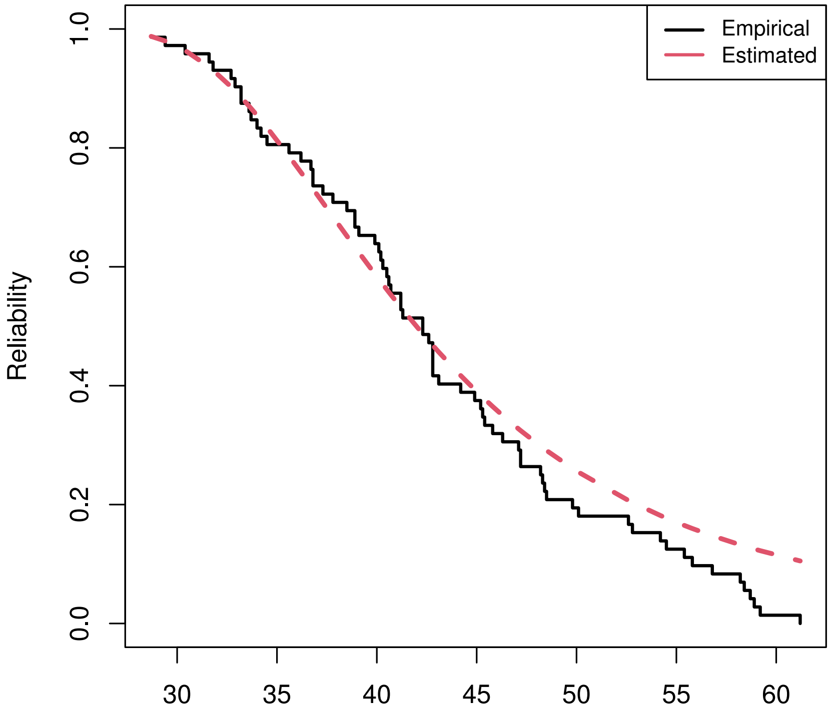

Figure 6. Plot of estimated/empirical Dagum reliability function from coating weight data.

Table 6. General MCMC statistics of α, β, θ, $R(t)$, and $h(t)$ from coating weight data.

$\mathcal{S}_i$	Par.	Mean	Mode	Q_1	Q_2	Q_3	St.D	Skew.
$\mathcal{S}_1$	α	1252.261	1252.068	1252.209	1252.260	1252.313	0.078036	0.0244751
	β	3.965291	3.907221	3.933091	3.963511	3.995504	0.045946	0.1275803
	θ	3972.392	3972.312	3972.338	3972.391	3972.445	0.078888	0.0201716
	$R(50)$	0.598523	0.681286	0.555022	0.600587	0.644260	0.064847	−0.0823151
	$h(50)$	0.048374	0.041782	0.044874	0.048364	0.051859	0.005102	−0.0849275
$\mathcal{S}_2$	α	367.6869	367.4750	367.6337	367.6883	367.7404	0.080099	−0.0316677
	β	2.833264	2.812148	2.811808	2.855040	2.855040	0.032354	0.1253853
	θ	165.2781	165.1830	165.2245	165.2788	165.3317	0.080479	−0.0474391
	$R(50)$	0.606512	0.636220	0.575109	0.608162	0.637130	0.046027	−0.0984429
	$h(50)$	0.034195	0.032474	0.032423	0.034146	0.036066	0.002712	−0.0080582
$\mathcal{S}_3$	α	7887.434	7887.308	7887.379	7887.434	7887.489	0.080521	0.0109896
	β	4.545287	4.553952	4.512051	4.543179	4.577726	0.049014	0.1251338
	θ	1060.076	1060.059	1060.024	1060.075	1060.129	0.078500	0.0402300
	$R(50)$	0.148638	0.130289	0.130289	0.147680	0.165139	0.026099	0.3562451
	$h(50)$	0.083788	0.084280	0.082341	0.083795	0.085312	0.002221	−0.1591992

Table 7. Optimal progressive censoring mechanisms from coating weight data.

Sample	Criteria					
	I	II	III	IV		
$q \rightarrow$				0.3	0.6	0.9
$\mathcal{S}_1$	295.8274	78.24976	0.931561	6.509199	11.16921	29.74584
$\mathcal{S}_2$	783.6007	78.78461	0.114358	4.650042	9.907341	38.79528
$\mathcal{S}_3$	223.3804	422.2169	88.67801	2.526266	4.086085	9.771803

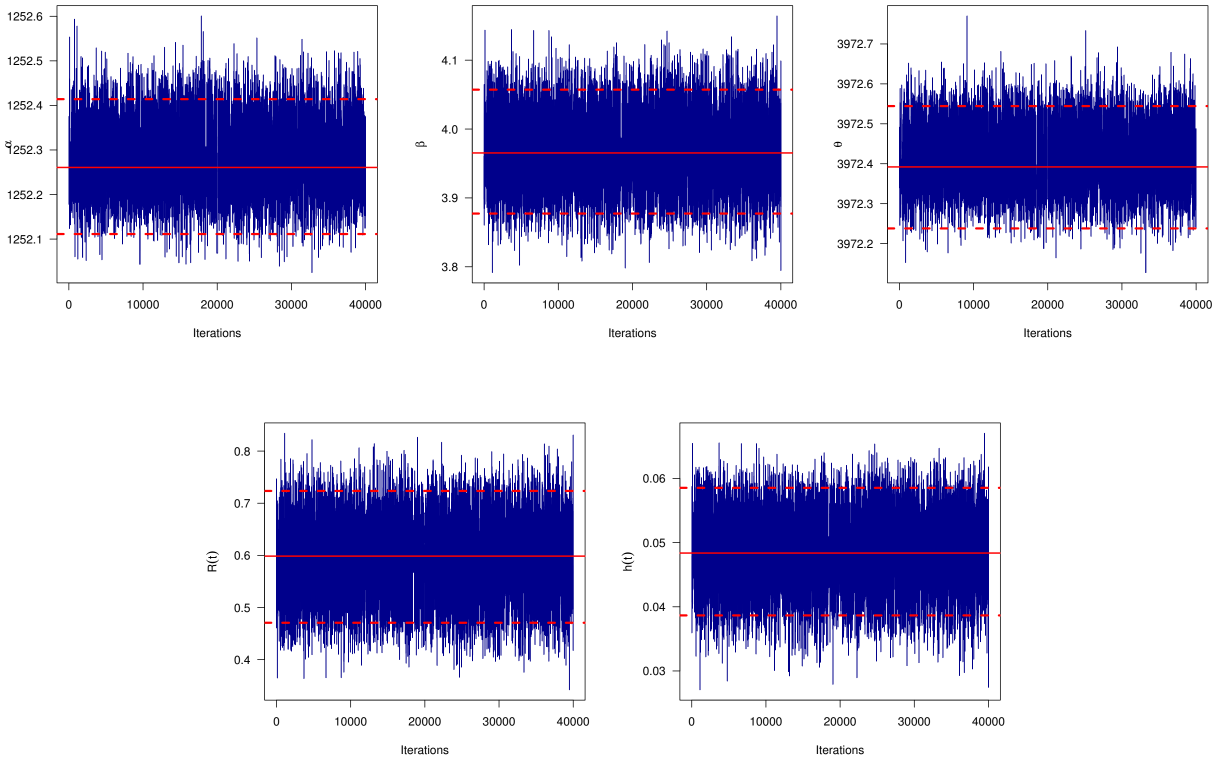

Figure 7. Trace plots of α, β, θ, $R(t)$, and $h(t)$ using $\mathcal{S}_1$ from coating weight data.

6.2. Electronic Components

In this application, we use a real-life data set from the engineering field taken from Lawless [37]. This data set describes the failure times (in minutes) for a sample of fifteen electronic components in an accelerated life test as: 1.4, 5.1, 6.3, 10.8, 12.1, 18.5, 19.7, 22.2, 23, 30.6, 37.3, 46.3, 53.9, 59.8, 66.2. The MLEs of the unknown parameters are $\hat{\alpha} = 0.5657(0.2563)$, $\hat{\beta} = 2.1834(0.4787)$, and $\hat{\theta} = 1882.5(3839.5)$. In addition, the K-S ($p$-value) is 0.107 (0.988). This result shows that the Dagum distribution fits the electronic components data set quite well. Furthermore, based on the electronic components data, the estimated/empirical RF of the Dagum distribution is displayed in Figure 9.

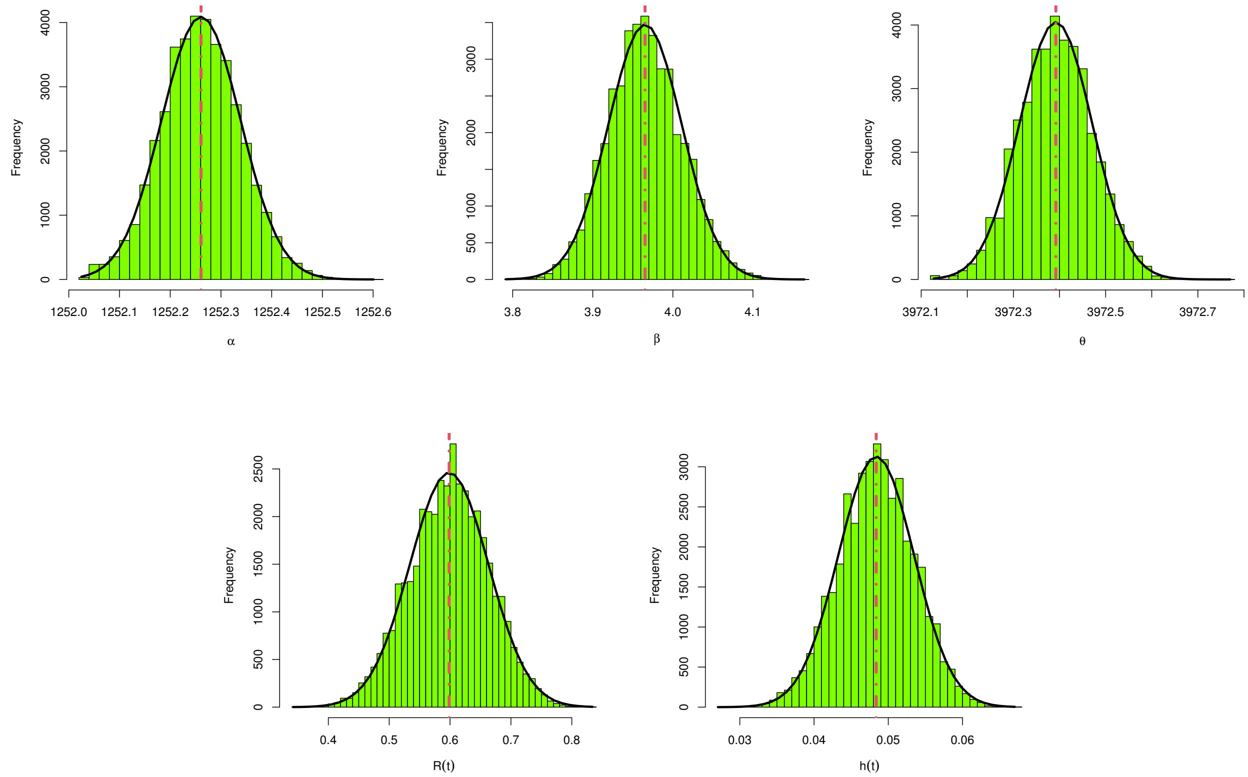

Figure 8. Histograms of α, β, θ, $R(t)$, and $h(t)$ using $\mathcal{S}_1$ from coating weight data.

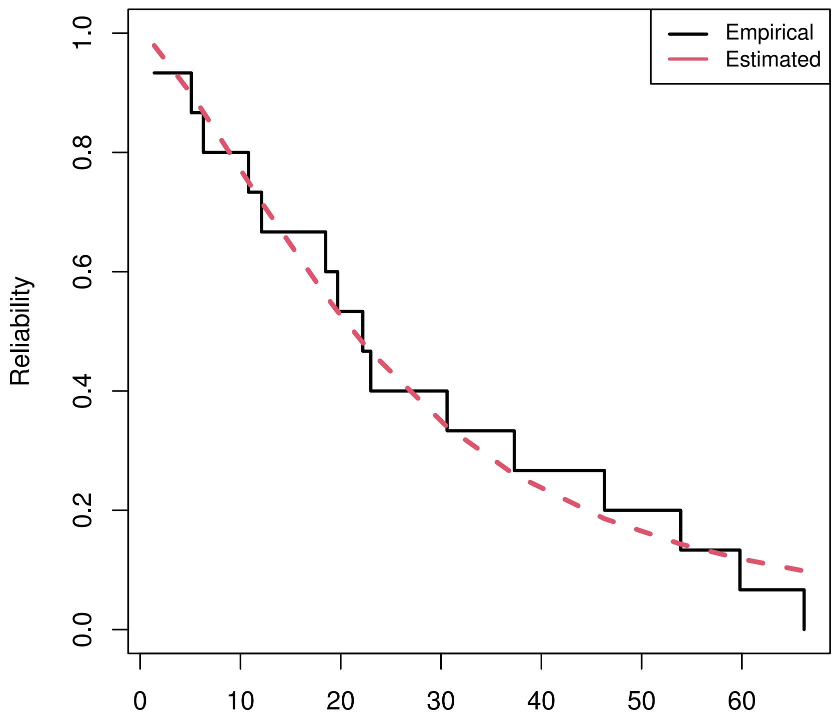

Figure 9. Plot of estimated/empirical Dagum reliability function from electronic components data.

From the entire electronic components data set, by employing various censoring schemes, three APT-II HC samples with $m = 10$ are generated and provided in Table 8. The different estimates of α, β, θ, $R(t)$, and $h(t)$ are calculated and reported in Tables 9 and 10, respectively.

The estimates of $R(t)$ and $h(t)$ are evaluated at distinct time point $t = 20$. Again, by running the MCMC algorithm 50,000 times and discarding the first 10,000 estimates as burn-in, the Bayes estimates are obtained using the SE and GE (for $\kappa(= -3, -0.03, +3)$) loss functions.

It can be seen, from Tables 9 and 10, that in terms of the lowest St.Es, the symmetric (or asymmetric) Bayes estimates of all unknown parameters perform better than the frequentist estimates. Moreover, in terms of the shortest interval width, the HPD interval estimates perform better than the ACIs.

Utilizing the simulated 40,000 MCMC variates of α, β, θ, $R(t)$, and $h(t)$, their trace and histograms plots based on APT-II HC samples obtained from the electronic components data are plotted and displayed in Figures 10 and 11, respectively. Figure 10 proves that the MCMC technique converges very well. In addition, Figure 11 shows that the distributions of the MCMC estimates of all unknown parameters are almost symmetric. Briefly, based on the simulated 40,000 MCMC variates of α, β, θ, $R(t)$, and $h(t)$ from $\mathcal{S}_1$ (as an example), some vital statistics called are calculated and listed in Table 11. In addition, the trace and histogram plots of the same unknown parameters based on samples $\mathcal{S}_2$ and $\mathcal{S}_3$ are also displayed in the Supplementary File.

In addition, using the optimum criteria reported in Section 5, the optimal progressive censoring mechanism is discussed. From the generated APT-II HC samples in Table 8, all optimum criteria are evaluated and presented in Table 12. It shows that the progressive type-II censoring plan used in $\mathcal{S}_2$ is the optimum censoring than other competing schemes based on criteria I, II, and III while the progressive type-II censoring plan used in $\mathcal{S}_3$ is the optimum censoring than others based on criterion IV for all specific percentile points.

Table 8. Three APT-II HC samples from electronic components data.

Sample	Scheme	$T(D)$	R_m	Data									
$\mathcal{S}_1$	(5,0*9)	25(4)	0	1.4	19.7	22.2	23	30.6	37.3	46.3	53.9	59.8	66.2
$\mathcal{S}_2$	(0*2,1*5,0*3)	20(5)	2	1.4	5.1	6.3	12.1	19.7	23	30.6	37.3	46.3	53.9
$\mathcal{S}_3$	(0*9,5)	15(5)	5	1.4	5.1	6.3	10.8	12.1	18.5	19.7	22.2	23	30.6

Table 9. Point estimates (St.Es) of α, β, θ, $R(t)$, and $h(t)$ from electronic components data.

$\mathcal{S}_i$	Par.	MLE	SE	GE		
$\kappa \rightarrow$				−3	−0.03	+3
$\mathcal{S}_1$	α	0.6295 (2.22×10^{-1})	0.5344 (4.29×10^{-4})	0.5478 (8.17×10^{-2})	0.5275 (1.02×10^{-1})	0.5042 (1.25×10^{-1})
	β	2.2722 (1.81×10^{-1})	2.1870 (4.15×10^{-4})	2.1901 (8.20×10^{-2})	2.1854 (8.67×10^{-2})	2.1807 (9.15×10^{-2})
	θ	4610.6 ($0.11 \times 10^{+2}$)	4610.5 (4.89×10^{-4})	4610.4 (9.74×10^{-2})	4610.5 (9.74×10^{-2})	4610.5 (9.73×10^{-2})
	$R(20)$	0.6797 (1.13×10^{-1})	0.6546 (3.37×10^{-4})	0.6614 (1.83×10^{-2})	0.6511 (2.86×10^{-2})	0.6393 (4.04×10^{-2})
	$h(20)$	0.0282 (9.95×10^{-3})	0.0264 (2.58×10^{-5})	0.0274 (7.49×10^{-4})	0.0259 (2.23×10^{-3})	0.0244 (3.75×10^{-3})
$\mathcal{S}_2$	α	0.5528 (1.71×10^{-1})	0.4689 (3.93×10^{-4})	0.4818 (7.10×10^{-2})	0.4624 (9.01×10^{-2})	0.4414 (1.11×10^{-1})
	β	1.9586 (1.92×10^{-1})	1.8670 (4.38×10^{-4})	1.8711 (8.75×10^{-2})	1.8650 (9.36×10^{-2})	1.8587 (9.99×10^{-2})
	θ	1613.8 ($0.84 \times 10^{+1}$)	1613.7 (5.27×10^{-4})	1613.7 (1.04×10^{-1})	1613.7 (1.04×10^{-1})	1613.7 (1.04×10^{-1})
	$R(20)$	0.6129 (1.05×10^{-1})	0.5924 (3.39×10^{-4})	0.6000 (1.29×10^{-2})	0.5886 (2.43×10^{-2})	0.5755 (3.74×10^{-2})
	$h(20)$	0.0281 (1.02×10^{-2})	0.0256 (2.42×10^{-5})	0.0265 (1.56×10^{-3})	0.0251 (2.91×10^{-3})	0.0237 (4.30×10^{-3})
$\mathcal{S}_3$	α	0.6112 (1.94×10^{-1})	0.5231 (4.05×10^{-4})	0.5353 (7.59×10^{-2})	0.5168 (9.44×10^{-2})	0.4959 (1.15×10^{-1})
	β	2.0253 (2.07×10^{-1})	1.9327 (4.29×10^{-4})	1.9365 (8.88×10^{-2})	1.9308 (9.45×10^{-2})	1.9250 (1.00×10^{-1})
	θ	1042.9 ($0.11 \times 10^{+2}$)	1042.7 (5.22×10^{-4})	1042.8 (1.03×10^{-1})	1042.8 (1.03×10^{-1})	1042.8 (1.03×10^{-1})
	$R(20)$	0.5281 (1.10×10^{-1})	0.5228 (3.36×10^{-4})	0.5313 (3.21×10^{-3})	0.5186 (9.53×10^{-3})	0.5046 (2.35×10^{-2})
	$h(20)$	0.0391 (1.44×10^{-2})	0.0349 (3.11×10^{-5})	0.0360 (3.09×10^{-3})	0.0344 (4.72×10^{-3})	0.0327 (6.41×10^{-3})

Table 10. Interval estimates [ILs] of α, β, θ, $R(t)$, and $h(t)$ from electronic components data.

$\mathcal{S}_i$	Par.	ACI	HPD
$\mathcal{S}_1$	α	(0.19442, 1.06465) [0.8702]	(0.37625, 0.70950) [0.3333]
	β	(1.91708, 2.62725) [0.7102]	(2.03085, 2.34927) [0.3184]
	θ	(4587.33, 4633.86) [46.524]	(4610.31, 4610.69) [0.3792]
	$R(20)$	(0.45801, 0.90134) [0.4433]	(0.51581, 0.77537) [0.2596]
	$h(20)$	(0.00866, 0.04770) [0.0390]	(0.01653, 0.03615) [0.0196]
$\mathcal{S}_2$	α	(0.21862, 0.88699) [0.6684]	(0.32547, 0.62131) [0.2958]
	β	(1.58223, 2.33503) [0.7528]	(1.69247, 2.03307) [0.3406]
	θ	(1597.35, 1630.24) [32.885]	(1613.46, 1613.88) [0.4166]
	$R(20)$	(0.40652, 0.81928) [0.4128]	(0.45554, 0.72107) [0.2655]
	$h(20)$	(0.00814, 0.04796) [0.0398]	(0.01667, 0.03514) [0.0185]
$\mathcal{S}_3$	α	(0.23019, 0.99228) [0.7621]	(0.36245, 0.67443) [0.3120]
	β	(1.61933, 2.43135) [0.8120]	(1.77332, 2.10441) [0.3311]
	θ	(1019.61, 1066.12) [46.509]	(1042.57, 1042.97) [0.3992]
	$R(20)$	(0.31206, 0.74414) [0.4321]	(0.38452, 0.64753) [0.2630]
	$h(20)$	(0.01096, 0.06728) [0.0563]	(0.02360, 0.04768) [0.0241]

Table 11. Vital MCMC statistics of α, β, θ, $R(t)$, and $h(t)$ from electronic components data.

$\mathcal{S}_i$	Par.	Mean	Mode	Q_1	Q_2	Q_3	St.D	Skew.
$\mathcal{S}_1$	α	0.534376	0.377818	0.476551	0.534183	0.534183	0.085719	0.029283
	β	2.186972	2.089641	2.129470	2.184992	2.242985	0.083015	0.070771
	θ	4610.496	4610.396	4610.428	4610.496	4610.562	0.097799	0.083025
	$R(20)$	0.654609	0.578008	0.610999	0.657592	0.704351	0.067400	−0.318981
	$h(20)$	0.026437	0.025883	0.022767	0.026031	0.029745	0.005158	0.410748
$\mathcal{S}_2$	α	0.468871	0.351668	0.413957	0.467575	0.520369	0.078599	0.168398
	β	1.867038	1.731206	1.810875	1.867141	1.926307	0.087674	−0.025261
	θ	1613.691	1613.458	1613.627	1613.695	1613.761	0.105405	−0.171534
	$R(20)$	0.592439	0.555406	0.549464	0.594472	0.639593	0.067723	−0.163227
	$h(20)$	0.025580	0.021936	0.022190	0.025155	0.028747	0.004841	0.400102
$\mathcal{S}_3$	α	0.523056	0.439083	0.467178	0.522073	0.576318	0.080907	0.078726
	β	1.932677	1.945098	1.876070	1.931637	1.990185	0.085892	0.059419
	θ	1042.765	1042.471	1042.693	1042.769	1042.837	0.104342	−0.095635
	$R(20)$	0.522836	0.460191	0.477698	0.522569	0.568159	0.067118	−0.000699
	$h(20)$	0.034938	0.037790	0.030653	0.034713	0.038802	0.006223	0.369621

Table 12. Optimal progressive censoring mechanisms from electronic components data.

Sample	Criteria					
	I	II	III	IV		
$q \rightarrow$				0.3	0.6	0.9
$\mathcal{S}_1$	69.09318	140.9431	0.167297	33.07438	92.52722	700.5803
$\mathcal{S}_2$	84.60432	70.44392	0.054867	26.06951	113.1988	1389.363
$\mathcal{S}_3$	75.13989	140.8521	0.151209	13.96922	56.30787	658.1227

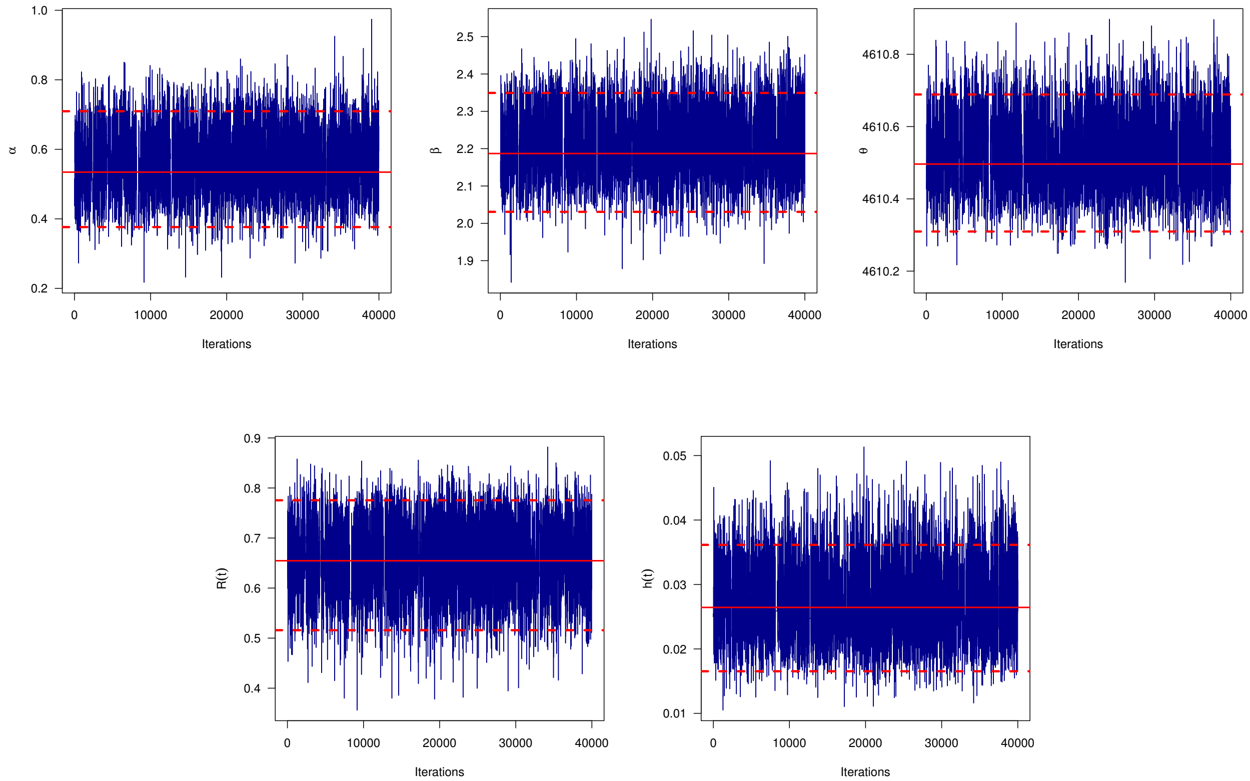

Figure 10. Trace plots of α, β, θ, $R(t)$, and $h(t)$ using $\mathcal{S}_1$ from electronic components data.

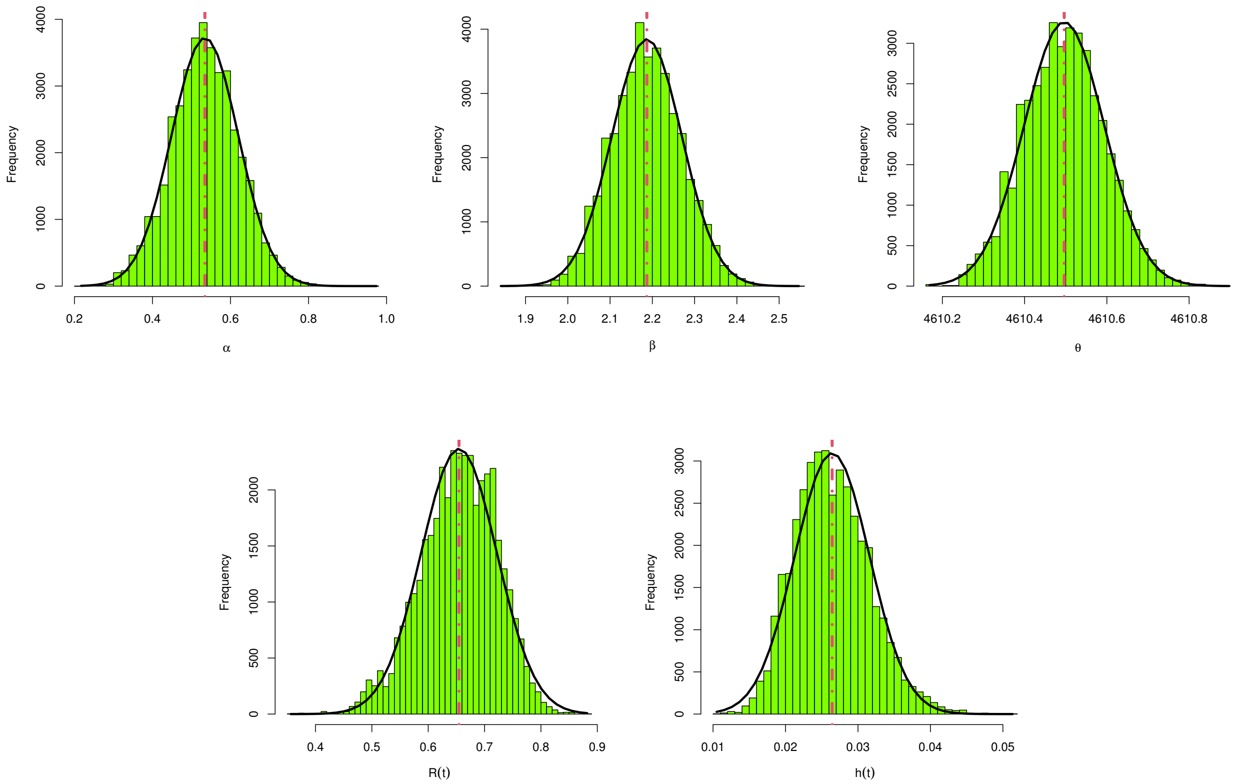

Figure 11. Histograms of α, β, θ, $R(t)$, and $h(t)$ using $\mathcal{S}_1$ from electronic components data.

7. Conclusions

In this study, based on an adaptive progressive type-II hybrid censoring scheme, we have attained the maximum likelihood and Bayes estimators of the unknown parameters, reliability, and hazard rate functions of the Dagum distribution. The Markov chain Monte Carlo approach is used to obtain the Bayes estimators based on squared error and general entropy loss functions. For the unknown parameters, reliability, and hazard rate functions, the approximative confidence intervals are obtained based on the asymptotic normality of the maximum likelihood estimators. In addition, the highest posterior density credible intervals are acquired. The optimal progressive censoring plans are shown and some optimality criteria are explored. A simulation study is used to examine the effectiveness of the various point and intervals estimators while taking various sample sizes and censoring strategies into account. The results of the simulation showed that the Bayesian approach offers estimates that are more accurate than the maximum likelihood approach. To demonstrate how the suggested estimators perform in real-world situations, we examined two actual data sets for coating weights of iron sheets and electronic components. The analysis showed that the Dagum distribution is a good choice to model these data and the Bayesian estimation method is advised to estimate the unknown parameters in the presence of adaptive progressively type-II hybrid censored Dagum data. For further research, the estimation of the reliability characteristics of the proposed model can be investigated by utilizing another estimation methods including the maximum product of spacing estimation method which may be a good alternative to the maximum likelihood method. Further, the methods developed in this paper can be extended to include the competing risks model or accelerated life tests.

Supplementary Materials: The following supporting information can be downloaded at: https://www.mdpi.com/article/10.3390/sym14102146/s1, Table S1: The APE (1st column), RMSEs (2nd column) and MRABs (3rd column) of α; Table S2: The APE (1st column), RMSEs (2nd column) and MRABs (3rd column) of β; Table S3: The APE (1st column), RMSEs (2nd column) and MRABs (3rd column) of θ; Table S4: The APE (1st column), RMSEs (2nd column) and MRABs (3rd column) of $R(t)$; Table S5: The APE (1st column), RMSEs (2nd column) and MRABs (3rd column) of $h(t)$; Table S6: The ACLs (1st column) and CPs (2nd column) of 95% ACI/HPD intervals of α; Table S7: The ACLs (1st column) and CPs (2nd column) of 95% ACI/HPD intervals of β; Table S8: The ACLs (1st column) and CPs (2nd column) of 95% ACI/HPD intervals of θ; Table S9: The ACLs (1st column) and CPs (2nd column) of 95% ACI/HPD intervals of $R(t)$; Table S10: The ACLs (1st column) and CPs (2nd column) of 95% ACI/HPD intervals of $h(t)$; Figure S1: Trace plots of α, β, θ, $R(t)$ and $h(t)$ using $\mathcal{S}_2$ from coating weight data; Figure S2: Trace plots of α, β, θ, $R(t)$ and $h(t)$ using $\mathcal{S}_3$ from coating weight data; Figure S3: Histograms of α, β, θ, $R(t)$ and $h(t)$ using $\mathcal{S}_2$ from coating weight data; Figure S4: Histograms of α, β, θ, $R(t)$ and $h(t)$ using $\mathcal{S}_3$ from coating weight data; Figure S5: Trace plots of α, β, θ, $R(t)$ and $h(t)$ using $\mathcal{S}_2$ from electronic components data; Figure S6: Trace plots of α, β, θ, $R(t)$ and $h(t)$ using $\mathcal{S}_3$ from electronic components data; Figure S7: Histograms of α, β, θ, $R(t)$ and $h(t)$ using $\mathcal{S}_2$ from electronic components data; Figure S8: Histograms of α, β, θ, $R(t)$ and $h(t)$ using $\mathcal{S}_3$ from electronic components data

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