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Global Asymptotic Stability of Competitive Neural Networks with Reaction-Diffusion Terms and Mixed Delays

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Abstract: In this article, a new competitive neural network (CNN) with reaction-diffusion terms and mixed delays is proposed. Because this network system contains reaction-diffusion terms, it belongs to a partial differential system, which is different from the existing classic CNNs. First, taking into account the spatial diffusion effect, we introduce spatial diffusion for CNNs. Furthermore, since the time delay has an essential influence on the properties of the system, we introduce mixed delays including time-varying discrete delays and distributed delays for CNNs. By constructing suitable Lyapunov–Krasovskii functionals and virtue of the theories of delayed partial differential equations, we study the global asymptotic stability for the considered system. The effectiveness and correctness of the proposed CNN model with reaction-diffusion terms and mixed delays are verified by an example. Finally, some discussion and conclusions for recent developments of CNNs are given.

Keywords: competitive neural networks; global asymptotic stability; reaction-diffusion; delays

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1. Introduction

In 1996, Meyer-Bäse et al. [1] firstly introduced competitive neural networks (CNNs) with different time scales. Using a quadratic-type Lyapunov function for the flow of a CNN with different time scales as a global stability method, the authors studied the local stability behavior around individual equilibrium points. In the earlier networks, the pools of mutually inhibitory neurons with fixed synaptic connections were considered. In the CNNs, there are two types of state variables: the short-term memory (STM) state variables which describe the fast dynamics of the system, and the long-term memory (LTM) state variables which describe the slow dynamics of the system. Because CNNs can accurately reflect the state transformation of neurons, a large number of achievements have been made in the study of different types of CNNs in the recent decades. Nie et al. [2] studied the exact existence and dynamical behaviors of multiple equilibrium points for delayed competitive neural networks with a class of nondecreasing piecewise linear activation functions. Lu et al. [3,4] considered global exponential stability of delayed competitive neural networks with different time scales. Competitive neural networks with time-varying and distributed delays were studied in [5]. For more results about competitive neural networks, see, e.g., [6–9] and related references.

On the other hand, diffusion phenomena exist widely in a neural network system, especially, when neurons are shifting in asymmetric neural networks or when metabolites and proteins move from one tier to other levels, see [10–12]. Hence, the study of neural network needs to consider the changes of neurons in time and space at the same time. It is of great theoretical and practical value to study the neural network system with a diffusion term. Cao et al. [13] studied global exponential synchronization of delayed memristive neural networks with reaction-diffusion terms. In [14], the authors studied inverse optimal synchronization control of competitive neural networks with constant time delays by means of the drive–response idea and inverse optimality techniques. Xu et al. [15] investigated global asymptotic stability of fractional-order competitive neural networks with multiple time-varying-delay links. Zheng et al. [16] considered the fixed-time synchronization of discontinuous competitive neural networks. Dynamical behavior of reaction-diffusion neural networks and their synchronization was considered in [17]. For state estimation for delayed genetic regulatory networks with reaction-diffusion terms, see [18]; for stability and asymptotic stability problems for neural networks with reaction-diffusion terms, see [19,20]; and for oscillatory behaviors for neural networks with reaction-diffusion terms, see [21,22].

In this paper, we mainly deal with the global asymptotic stability for CNNs when there exist reaction-diffusion terms and mixed delays in CNNs. For classic CNNs (without reaction-diffusion terms), there exist many results, see, e.g., [3–5] and relevant references. For research into CNNs, the most important results can be found in [1]. The main research approaches for studying CNNs are the Lyapunov function method [1,4], fixed point theorem, and matrix theory [2]. The main limitation for the above methods is that they are not suitable for dealing with stability problems for CNNs with reaction-diffusion terms and mixed delays. For overcoming the above difficulties, we construct a new Lyapunov function for the considered model on the base of theories of delayed partial differential equations and Lyapunov stability. Up to now, to the best of our knowledge, there are no articles studying the stability problems of CNNs with reaction-diffusion terms and mixed delays. Inspired by the above reasons, we study the stability problems of the equilibrium point for a class of reaction-diffusion CNNs with time-varying delays and distributed delays. The main innovation points are summarized in the following three aspects:

(1) A new CNN model is introduced in this paper which extends some previous results; to our best knowledge, there exist few papers for studying this new CNN model, such as [3–6].
(2) The model in the present paper contains various types of time delays: time-varying delays, distributed delays, bounded delays, and unbounded delays. Time delay is one of the inherent parameters of the system, which has an important impact on the properties of the control system. The study of time delay has very important application value.
(3) A simple method for studying CNNs with reaction-diffusion terms and various types of delays is given. There is reason to believe that the method used in this paper can easily be used to study other types of dynamic systems.

The rest of the article is organized as follows: In Section 2, a system description and some preliminaries are given. Section 3 gives main results for global asymptotic stability of CNNs. In Section 4, a numerical example is given to show the feasibility of the obtained results. Finally, some conclusions and discussions are drawn in Section 5.

2. Model Description and Preliminaries

Consider the following CNNs with mixed delays:

$$\begin{cases} STM : \dot{x}_k(t) & = -a_k x_k(t) + \sum_{d=1}^{m} b_{kd} f_d(x_d(t)) + \sum_{d=1}^{m} c_{kd} f_d(x_d(t-\tau_d(t))) \\ & \quad + \sum_{d=1}^{m} \tilde{c}_{kd} \int_{t-\gamma(t)}^{t} f_d(x_d(s))ds + B_k \sum_{d=1}^{i} y_{kd}(t)\omega_d + I_k(t) \\ LTM : \dot{y}_{kd}(t) & = -\alpha_k y_{kd}(t) + \omega_d \beta_k f_k(x_k(t)), \end{cases} \tag{1}$$

where $k = 1, 2, \cdots, m$, $x_k(t)$ denotes the state of the neuron current; $y_{kd}(t)$ denotes the synaptic transfer efficiency; ω_d is the external stimulus; $b_{kd}, c_{kd}, \tilde{c}_{kd}$ and B_k are connection weights; $I_k(t)$ denotes the external input; $f_d(\cdot)$ is the activation function; $\tau_d(t) \geq 0$ and $\gamma(t) \geq 0$ are time-varying delays, a_k, α_k and β_k are nonnegative constants. Let $s_k(t) = \sum_{d=1}^{i} y_{kd}(t)\omega_d = \omega^T y_k(t)$, where $\omega = (\omega_1, \cdots, \omega_i)^T$, $y_k(t) = (y_{k1}(t), \cdots, y_{ki}(t))^T$. Assume that $\omega^2 = \omega_1^2 + \cdots + \omega_i^2 = 1$. Then, system (1) can be rewritten by

$$\begin{cases} STM: \dot{x}_k(t) & = -a_k x_k(t) + \sum_{d=1}^{m} b_{kd} f_d(x_d(t)) + \sum_{d=1}^{m} c_{kd} f_d(x_d(t-\tau_d(t))) \\ & + \sum_{d=1}^{m} \tilde{c}_{kd} \int_{t-\gamma(t)}^{t} f_d(x_d(s))ds + B_k s_k(t) + I_k(t) \\ LTM: \dot{s}_k(t) & = -\alpha_k s_k(t) + \beta_k f_k(x_k(t)). \end{cases} \tag{2}$$

Based on the motivation of introducing spatial diffusion for CNNs, we consider the joint influences of spatial diffusion in system (2). Then,

$$\begin{cases} STM: \frac{\partial x_k(\delta,t)}{\partial t} & = \sum_{p=1}^{P} \frac{\partial}{\partial \delta_p}\left(d_{kp} \frac{\partial x_k(\delta,t)}{\partial \delta_p} \right) - a_k x_k(\delta,t) + \sum_{d=1}^{m} b_{kd} f_d(x_d(\delta,t)) \\ & + \sum_{d=1}^{m} c_{kd} f_d(x_d(\delta, t-\tau_d(t))) + \sum_{d=1}^{m} \tilde{c}_{kd} \int_{t-\gamma(t)}^{t} f_d(x_d(\delta,s))ds + B_k s_k(\delta,t) + I_k(t) \\ LTM: \frac{\partial s_k(\delta,t)}{\partial t} & = \sum_{p=1}^{P} \frac{\partial}{\partial \delta_p}\left(d_{kp}^{*} \frac{\partial s_k(\delta,t)}{\partial \delta_p} \right) - \alpha_k s_k(\delta,t) + \beta_k f_k(x_k(\delta,t)), \end{cases} \tag{3}$$

where $d_{kp}, d_{kp}^{*} \geq 0$ are constants of diffusion effects, $\delta = (\delta_1, \delta_2, \cdots, \delta_P)^T \in \Omega \subset \mathbb{R}^P$, Ω is a bounded compact set with smooth boundary $\partial\Omega$ and $mes\Omega > 0$ in space $\mathbb{R}^P$. Means of other parameters in system (3) are similar to the corresponding ones in system (1). System (3) has the following initial values

$$\begin{cases} \frac{\partial x_k}{\partial n} = (\frac{\partial x_k}{\partial \delta_1}, \frac{\partial x_k}{\partial \delta_2}, \cdots \frac{\partial x_k}{\partial \delta_P})^T, \ k = 1, 2, \cdots, m, \\ \frac{\partial s_k}{\partial n} = (\frac{\partial s_k}{\partial \delta_1}, \frac{\partial s_k}{\partial \delta_2}, \cdots \frac{\partial s_k}{\partial \delta_P})^T, \ k = 1, 2, \cdots, m, \end{cases} \tag{4}$$

and

$$\begin{cases} x_k(\delta,s) = \phi_{xk}(\delta,s), \ s \in [-\tau, 0], \ k = 1, 2, \cdots, m, \\ s_k(\delta,s) = \phi_{sk}(\delta,s), \ s \in [-\tau, 0], \ k = 1, 2, \cdots, m, \end{cases} \tag{5}$$

where $\tau = \max_{t\in\mathbb{R}}\{\tau_d(t), \gamma(t)\}$ for $d = 1, 2, \cdots, m$, $\phi_{xk}(\delta,s)$ and $\phi_{sk}(\delta,s)$ are bounded and continuous on $\Omega \times [-\tau, 0]$.

Throughout this paper, we need the following assumptions:

(H_1) For each $i = 1, 2, \cdots, m$, $f_i: \mathbb{R} \to \mathbb{R}$ is bounded and satisfies the Lipschitz condition, i.e., there exists a constant $F_i > 0$ such that

$$|f_i(x) - f_i(y)| \leq F_i|x - y| \ for \ all \ x, y \in \mathbb{R}.$$

For convenience, some notations are given. For each $u = (u_1, u_2, \cdots, u_m)^T \in \mathbb{R}^m$, define 1-norm of u by $||u||_1 = \sum_{i=1}^{m} |u_i|$; for each $x = (x_1(\delta,t), x_2(\delta,t), \cdots, x_m(\delta,t))^T \in \mathbb{R}^m$, denote

$$||x_i(\delta,t)||_2 = \left(\int_{\Omega} |x_i(\delta,t)|^2 d\delta \right)^{\frac{1}{2}}, \ i = 1, 2, \cdots, m, \ \ t \in \mathbb{R}.$$

Definition 1. *Assume that $Y^* = (x^*, s^*)^T$ is an equilibrium point of system (3), where $x^* = (x_1^*, x_2^*, \cdots, x_m^*)^T$ and $s^* = (s_1^*, s_2^*, \cdots, s_m^*)^T$. We say that the equilibrium point Y^* is globally asymptotically stable, if there exists a constant $M \geq 1$ such that*

$$\sum_{i=1}^{m} ||x_i - x_i^*||_2 + \sum_{i=1}^{m} ||s_i - s_i^*||_2 \leq M\Big(||\phi_x - x^*||_2 + ||\phi_s - s^*||_2 \Big) \text{ for all } t \geq 0,$$

where

$$||\phi_x - x^*||_2 = \sup_{s\in[-\tau,0]} \sum_{i=1}^{m} ||\phi_{xi}(\delta,s) - x_i^*||_2, \ ||\phi_s - s^*||_2 = \sup_{s\in[-\tau,0]} \sum_{i=1}^{m} ||\phi_{si}(\delta,s) - s_i^*||_2.$$

3. Main Results

Theorem 1. *Suppose that assumption (H_1) holds. Then, the equilibrium point $Y^* = (x^*, s^*)^T$ of system (3) is globally asymptotically stable under the initial conditions (4) and (5), provided that*

$$-2a_k + |B_k| + |\beta_k|F_k + 2\sum_{d=1}^{m}\left(|b_{kd}|F_d + |c_{kd}|F_d + |\tilde{c}_{kd}|F_d\tau\right) < 0 \tag{6}$$

and

$$-2\alpha_k + |\beta_k|F_k < 0, \tag{7}$$

where $k = 1, 2, \cdots, m$.

Proof. It is easy to see that bounded activation functions guarantee the existence of an equilibrium point for system (3). The uniqueness of the equilibrium point for system (3) can be obtained by the global asymptotic stability of the equilibrium point.

Assume that $(x_1(\delta,t), x_2(\delta,t), \cdots, x_m(\delta,t), s_1(\delta,t), s_2(\delta,t), \cdots, s_m(\delta,t))^T$ is any solution of the system (3). We rewrite system (3) as follows:

$$\begin{aligned}\frac{\partial(x_k - x_k^*)}{\partial t} &= \sum_{p=1}^{P}\frac{\partial}{\partial\delta_p}\left(d_{kp}\frac{\partial(x_k - x_k^*)}{\partial\delta_p}\right) - a_k(x_k - x_k^*)\\ &+ \sum_{d=1}^{m} b_{kd}[f_d(x_d) - f_d(x_d^*)] + \sum_{d=1}^{m} c_{kd}[f_d(x_d(\delta, t - \tau_d(t))) - f_d(x_d^*)]\\ &+ \sum_{d=1}^{m}\tilde{c}_{kd}\int_{t-\gamma(t)}^{t}[f_d(x_d(\delta,s)) - f_d(x_d^*)]ds + B_k(s_k - s_k^*)\end{aligned} \tag{8}$$

and

$$\begin{aligned}\frac{\partial(s_k - s_k^*)}{\partial t} &= \sum_{p=1}^{P}\frac{\partial}{\partial\delta_p}\left(d_{kp}^*\frac{(s_k - s_k^*)}{\partial\delta_p}\right)\\ &- \alpha_k(s_k - s_k^*) + \beta_k[f_k(x_k(\delta,t)) - f_k(x_k^*)].\end{aligned} \tag{9}$$

Multiplying both sides of (8) by $x_k - x_k^*$ and integrating them on Ω, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega(x_k - x_k^*)^2 d\delta &= \sum_{p=1}^{P}\int_\Omega(x_k - x_k^*)\frac{\partial}{\partial\delta_p}\left(d_{kp}\frac{\partial(x_k - x_k^*)}{\partial\delta_p}\right)d\delta - \int_\Omega a_k(x_k - x_k^*)^2 d\delta\\ &+ \sum_{d=1}^{m}\int_\Omega b_{kd}[f_d(x_d) - f_d(x_d^*)](x_k - x_k^*)d\delta\\ &+ \sum_{d=1}^{m}\int_\Omega c_{kd}[f_d(x_d(\delta, t - \tau_d(t))) - f_d(x_d^*)](x_k - x_k^*)d\delta\\ &+ \sum_{d=1}^{m}\int_\Omega(x_k - x_k^*)\left(\tilde{c}_{kd}\int_{t-\gamma(t)}^{t}[f_d(x_d(\delta,s)) - f_d(x_d^*)]ds\right)d\delta\\ &+ \int_\Omega B_k(s_k - s_k^*)(x_k - x_k^*)d\delta.\end{aligned} \tag{10}$$

From the boundary conditions (4) and (5), we have

$$\sum_{p=1}^{P}\int_\Omega(x_k - x_k^*)\frac{\partial}{\partial\delta_p}\left(d_{kp}\frac{\partial(x_k - x_k^*)}{\partial\delta_p}\right)d\delta = -\sum_{p=1}^{P}\int_\Omega d_{kp}\left(\frac{\partial(x_k - x_k^*)}{\partial\delta_p}\right)^2 d\delta \tag{11}$$

and

$$\sum_{p=1}^{P}\int_\Omega(s_k - s_k^*)\frac{\partial}{\partial\delta_p}\left(d_{kp}^*\frac{\partial(s_k - s_k^*)}{\partial\delta_p}\right)d\delta = -\sum_{p=1}^{P}\int_\Omega d_{kp}^*\left(\frac{\partial(s_k - s_k^*)}{\partial\delta_p}\right)^2 d\delta. \tag{12}$$

From (10), (11), assumption (H_1), and the Hölder integral inequality, we have

$$\begin{aligned}\frac{d||x_k - x_k^*||_2^2}{dt} \le& -2a_k||x_k - x_k^*||_2^2 + \sum_{d=1}^{m} |b_{kd}|F_d||x_d - x_d^*||_2^2 + \sum_{d=1}^{m} |b_{kd}|F_d||x_k - x_k^*||_2^2 \\ &+ \sum_{d=1}^{m} |c_{kd}|F_d||x_d - x_d^*||_2^2 + \sum_{d=1}^{m} |c_{kd}|F_d||x_k - x_k^*||_2^2 \\ &+ \sum_{d=1}^{m} |\tilde{c}_{kd}|F_d\tau||x_d - x_d^*||_2^2 + \sum_{d=1}^{m} |\tilde{c}_{kd}|F_d\tau||x_k - x_k^*||_2^2 \\ &+ |B_k|||x_k - x_k^*||_2^2 + |B_k|||s_k - s_k^*||_2^2 \\ =& \Big(-2a_k + |B_k| + \sum_{d=1}^{m} |b_{kd}|F_d + \sum_{d=1}^{m} |c_{kd}|F_d + \sum_{d=1}^{m} |\tilde{c}_{kd}|F_d\tau \Big)||x_k - x_k^*||_2^2 \\ &+ \sum_{d=1}^{m} \Big(|b_{kd}|F_d + |c_{kd}|F_d + |\tilde{c}_{kd}|F_d\tau \Big)||x_d - x_d^*||_2^2 + |B_k|||s_k - s_k^*||_2^2.\end{aligned} \tag{13}$$

Multiplying both sides of (9) by $s_k - s_k^*$ and integrating them on Ω, in view of (12) and assumption (H_1), we have

$$\begin{aligned}\frac{d||s_k - s_k^*||_2^2}{dt} &\le -2\alpha_k||s_k - s_k^*||_2^2 + |\beta_k|F_k|s_k - s_k^*||_2^2 + |\beta_k|F_k|x_k - x_k^*||_2^2 \\ &= \Big(-2\alpha_k + |\beta_k|F_k \Big)||s_k - s_k^*||_2^2 + |\beta_k|F_k|x_k - x_k^*||_2^2.\end{aligned} \tag{14}$$

Construct the following Lyapunov functional:

$$V(t) = \sum_{k=1}^{m} \Big(||x_k - x_k^*||_2^2 + ||s_k - s_k^*||_2^2 \Big). \tag{15}$$

Calculating the upper right Dini derivative $D^+V(t)$ of $V(t)$ along the solutions of system (3), it follows from (6), (7), (13), and (14) that

$$\begin{aligned}D^+V(t) \le& \sum_{k=1}^{m} \Big\{ -2a_k + |B_k| + \sum_{d=1}^{m} |b_{kd}|F_d + \sum_{d=1}^{m} |c_{kd}|F_d + \sum_{d=1}^{m} |\tilde{c}_{kd}|F_d\tau \\ &+ \sum_{d=1}^{m} \big(|b_{kd}|F_d + |c_{kd}|F_d + |\tilde{c}_{kd}|F_d\tau \big) + |\beta_k|F_k \Big\}||x_k - x_k^*||_2^2 \\ &+ \sum_{k=1}^{m} \Big(-2\alpha_k + |\beta_k|F_k + |B_k| \Big)||s_k - s_k^*||_2^2 \\ \le& \ 0.\end{aligned}$$

Hence, $V(t) \le V(0)$ for $t \ge 0$. Furthermore, by (3.10), we have

$$\begin{aligned}V(0) &= \sum_{k=1}^{m} \Big(||x_k(\delta,0) - x_k^*||_2^2 + ||s_k(\delta,0) - s_k^*||_2^2 \Big) \\ &\le \sup_{s\in[-\tau,0]} \sum_{k=1}^{m} \Big(||x_k(\delta,s) - x_k^*||_2^2 + ||s_k(\delta,s) - s_k^*||_2^2 \Big).\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{k=1}^{m} \Big(||x_k - x_k^*||_2 + ||s_k - s_k^*||_2 \Big) &\le \sup_{s\in[-\tau,0]} \sum_{k=1}^{m} \Big(||x_k(\delta,s) - x_k^*||_2 + ||s_k(\delta,s) - s_k^*||_2 \Big) \\ &= ||\phi_x - x^*||_2 + ||\phi_s - s^*||_2.\end{aligned}$$

This implies that the equilibrium point of system (3) is globally asymptotically stable. The proof is completed. □

In system (3), the distributed delay is bounded. If the distributed delay is unbounded, we consider the following system:

$$\begin{cases} STM : \frac{\partial x_k(\delta,t)}{\partial t} & = \sum_{p=1}^{P} \frac{\partial}{\partial \delta_p}\left(d_{kp}\frac{\partial x_k(\delta,t)}{\partial \delta_p}\right) - a_k x_k(\delta,t) + \sum_{d=1}^{m} b_{kd} f_d(x_d(\delta,t)) \\ & + \sum_{d=1}^{m} c_{kd} f_d(x_d(\delta, t-\tau_d(t))) + \sum_{d=1}^{m} \tilde{c}_{kd} \int_{-\infty}^{t} K_d(t-s) f_d(x_d(\delta,s))ds \\ & + B_k s_k(\delta,t) + I_k(t) \\ LTM : \frac{\partial s_k(\delta,t)}{\partial t} & = \sum_{p=1}^{P} \frac{\partial}{\partial \delta_p}\left(d^*_{kp}\frac{\partial s_k(\delta,t)}{\partial \delta_p}\right) - \alpha_k s_k(\delta,t) + \beta_k f_k(x_k(\delta,t)), \end{cases} \tag{16}$$

where $K_d(\cdot)$ is the delay kernel function which satisfies the following assumption:
(H_2) (i) $K_d(\cdot) : [0,\infty) \to [0,\infty)$ $(d = 1,2,\cdots,m)$ is continuous;
(ii) $\int_0^\infty K_d(s)ds = 1, \quad \int_0^\infty sK_d(s)ds < \infty$;
(iii) There exists a positive number μ such that $\int_0^\infty se^{\mu s}K_d(s)ds < \infty$. Let system (3.1) have the following initial conditions:

$$\begin{cases} \frac{\partial x_k}{\partial n} = (\frac{\partial x_k}{\partial \delta_1}, \frac{\partial x_k}{\partial \delta_2}, \cdots \frac{\partial x_k}{\partial \delta_P})^T, \ k = 1,2,\cdots,m, \\ \frac{\partial s_k}{\partial n} = (\frac{\partial s_k}{\partial \delta_1}, \frac{\partial s_k}{\partial \delta_2}, \cdots \frac{\partial s_k}{\partial \delta_P})^T, \ k = 1,2,\cdots,m, \end{cases} \tag{17}$$

and

$$\begin{cases} x_k(\delta,s) = \phi_{xk}(\delta,s), \ s \in (-\infty,0], \ k = 1,2,\cdots,m, \\ s_k(\delta,s) = \phi_{sk}(\delta,s), \ s \in (-\infty,0], \ k = 1,2,\cdots,m. \end{cases} \tag{18}$$

Theorem 2. *Suppose that assumptions (H_1) and (H_2) hold. Then, the equilibrium point $Y^* = (x^*, s^*)^T$ of system (16) is globally asymptotically stable under the initial conditions (17) and (18), provided that*

$$-2a_k + |B_k| + 2\sum_{d=1}^{m}\Big(|b_{kd}|F_d + |c_{kd}|F_d\Big) + 2\sum_{d=1}^{m} |\tilde{c}_{kd}|^{2\xi_{kd}} F_d^{2\eta_d} + |\beta_k|F_k < 0 \tag{19}$$

and

$$-2\alpha_k + |\beta_k|F_k < 0, \tag{20}$$

where $k = 1,2,\cdots,m$, $\xi_{kd} + \eta_d = 1$ with $\xi_{kd}, \eta_d \geq 0$.

Proof. Assume that $(x_1(\delta,t), x_2(\delta,t), \cdots, x_m(\delta,t), s_1(\delta,t), s_2(\delta,t), \cdots, s_m(\delta,t))^T$ is any solution of the system (16). We rewrite system (16) as follows:

$$\begin{aligned} \frac{\partial(x_k - x_k^*)}{\partial t} &= \sum_{p=1}^{P} \frac{\partial}{\partial \delta_p}\left(d_{kp}\frac{\partial(x_k - x_k^*)}{\partial \delta_p}\right) - a_k(x_k - x_k^*) \\ &+ \sum_{d=1}^{m} b_{kd}[f_d(x_d) - f_d(x_d^*)] + \sum_{d=1}^{m} c_{kd}[f_d(x_d(\delta, t-\tau_d(t))) - f_d(x_d^*)] \\ &+ \sum_{d=1}^{m} \tilde{c}_{kd} \int_{-\infty}^{t} K_d(t-s)[f_d(x_d(\delta,s)) - f_d(x_d^*)]ds + B_k(s_k - s_k^*) \end{aligned} \tag{21}$$

and

$$\begin{aligned} \frac{\partial(s_k - s_k^*)}{\partial t} &= \sum_{p=1}^{P} \frac{\partial}{\partial \delta_p}\left(d^*_{kp}\frac{(s_k - s_k^*)}{\partial \delta_p}\right) \\ &- \alpha_k(s_k - s_k^*) + \beta_k[f_k(x_k(\delta,t)) - f_k(x_k^*)]. \end{aligned} \tag{22}$$

Similar to the proof of Theorem 1, by (21) and (22), we have

$$\begin{aligned}
\frac{d||x_k - x_k^*||_2^2}{dt} &\le -2a_k||x_k - x_k^*||_2^2 + \sum_{d=1}^{m} |b_{kd}|F_d||x_d - x_d^*||_2^2 + \sum_{d=1}^{m} |b_{kd}|F_d||x_k - x_k^*||_2^2 \\
&+ \sum_{d=1}^{m} |c_{kd}|F_d||x_d - x_d^*||_2^2 + \sum_{d=1}^{m} |c_{kd}|F_d||x_k - x_k^*||_2^2 \\
&+ 2\sum_{d=1}^{m} |\tilde{c}_{kd}| \int_{-\infty}^{t} K_d(t-s)F_d||x_d - x_d^*||_2||x_k - x_k^*||_2 ds \\
&+ |B_k|||x_k - x_k^*||_2^2 + |B_k|||s_k - s_k^*||_2^2 \\
&= \Bigg(-2a_k + |B_k| + \sum_{d=1}^{m} |b_{kd}|F_d + \sum_{d=1}^{m} |c_{kd}|F_d \Bigg) ||x_k - x_k^*||_2^2 \\
&+ \sum_{d=1}^{m} \Bigg(|b_{kd}|F_d + |c_{kd}|F_d \Bigg) ||x_d - x_d^*||_2^2 + |B_k|||s_k - s_k^*||_2^2 \\
&+ 2\sum_{d=1}^{m} |\tilde{c}_{kd}| \int_{-\infty}^{t} K_d(t-s)F_d||x_d - x_d^*||_2||x_k - x_k^*||_2 ds
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
\frac{d||s_k - s_k^*||_2^2}{dt} &\le -2\alpha_k||s_k - s_k^*||_2^2 + |\beta_k|F_k|s_k - s_k^*||_2^2 + |\beta_k|F_k|x_k - x_k^*||_2^2 \\
&= \Bigg(-2\alpha_k + |\beta_k|F_k \Bigg) ||s_k - s_k^*||_2^2 + |\beta_k|F_k|x_k - x_k^*||_2^2.
\end{aligned} \tag{24}$$

Construct the following Lyapunov functional:

$$\mathbb{V}(t) = \sum_{k=1}^{m} \Bigg[||x_k - x_k^*||_2^2 + ||s_k - s_k^*||_2^2 + \sum_{d=1}^{m} |\tilde{c}_{kd}|^{2\xi_{kd}} F_d^{2\eta_d} \int_0^\infty K_d(s) \Bigg(\int_{t-s}^{t} ||x_d(\delta,\tau) - x_d^*||_2^2 d\tau \Bigg) ds \Bigg]. \tag{25}$$

Calculating the upper right Dini derivative $D^+\mathbb{V}(t)$ of $\mathbb{V}(t)$ along the solutions of system (16), it follows from (23), (24), (19), (20), and assumption (H_2) that

$$\begin{aligned}
D^+\mathbb{V}(t) &\le \sum_{k=1}^{m} \Bigg\{ -2a_k + |B_k| + \sum_{d=1}^{m} |b_{kd}|F_d + \sum_{d=1}^{m} |c_{kd}|F_d \\
&+ \sum_{d=1}^{m} \big(|b_{kd}|F_d + |c_{kd}|F_d \big) + |\beta_k|F_k \Bigg\} ||x_k - x_k^*||_2^2 \\
&+ \sum_{k=1}^{m} \Bigg(-2\alpha_k + |\beta_k|F_k + |B_k| \Bigg) ||s_k - s_k^*||_2^2 \\
&+ \sum_{k=1}^{m} \Bigg\{ 2\sum_{d=1}^{m} |\tilde{c}_{kd}| \int_0^\infty K_d(s)F_d||x_d - x_d^*||_2||x_k - x_k^*||_2 ds \Bigg\} \\
&+ \sum_{k=1}^{m} \Bigg\{ \sum_{d=1}^{m} |\tilde{c}_{kd}|^{2\xi_{kd}} F_d^{2\eta_d} \int_0^\infty K_d(s) \Bigg(||x_d(\delta,t) - x_d^*||_2^2 - ||x_d(\delta,t-s) - x_d^*||_2^2 \Bigg) ds \Bigg\} \\
&\le \sum_{k=1}^{m} \Bigg\{ -2a_k + |B_k| + \sum_{d=1}^{m} |b_{kd}|F_d + \sum_{d=1}^{m} |c_{kd}|F_d \\
&+ \sum_{d=1}^{m} \big(|b_{kd}|F_d + |c_{kd}|F_d \big) + |\beta_k|F_k + 2\sum_{d=1}^{m} |\tilde{c}_{kd}|^{2\xi_{kd}} F_d^{2\eta_d} \Bigg\} ||x_k - x_k^*||_2^2 \\
&+ \sum_{k=1}^{m} \Bigg(-2\alpha_k + |\beta_k|F_k + |B_k| \Bigg) ||s_k - s_k^*||_2^2 \\
&\le 0.
\end{aligned}$$

Hence, $\mathbb{V}(t) \leq \mathbb{V}(0)$ for $t \geq 0$. From (25), we have

$$\mathbb{V}(t) \geq \sum_{k=1}^{m} \left[||x_k - x_k^*||_2^2 + ||s_k - s_k^*||_2^2 \right]$$

and

$$\begin{aligned}\mathbb{V}(0) &= \sum_{k=1}^{m} \left[||x_k(\delta,0) - x_k^*||_2^2 + ||s_k(\delta,0) - s_k^*||_2^2 + \sum_{d=1}^{m} |\tilde{c}_{kd}|^{2\xi_{kd}} F_d^{2\eta_d} \int_0^{\infty} K_d(s) \left(\int_{-s}^{0} ||x_d(\delta,\tau) - x_d^*||_2^2 d\tau \right) ds \right] \\ &\leq \left[1 + \sum_{d=1}^{m} |\tilde{c}_{kd}|^{2\xi_{kd}} F_d^{2\eta_d} \int_0^{\infty} sK_d(s)ds \right] \sup_{s\in(-\infty,0]} \sum_{k=1}^{m} ||x_k(\delta,s) - x_k^*||_2 \\ &+ \sup_{s\in(-\infty,0]} \sum_{k=1}^{m} ||s_k(\delta,s) - s_k^*||_2.\end{aligned}$$

Let

$$M = \max_{k=1,2,\cdots,m} \left(1 + \sum_{d=1}^{m} |\tilde{c}_{kd}|^{2\xi_{kd}} F_d^{2\eta_d} \int_0^{\infty} sK_d(s)ds \right).$$

Then, $M \geq 1$ and

$$\begin{aligned}\sum_{k=1}^{m} \left(||x_k - x_k^*||_2 + ||s_k - s_k^*||_2 \right) &\leq \sqrt{M} \sup_{s\in[-\tau,0]} \sum_{k=1}^{m} \left(||x_k(\delta,s) - x_k^*||_2 + ||s_k(\delta,s) - s_k^*||_2 \right) \\ &= \sqrt{M} ||\phi_x - x^*||_2 + ||\phi_s - s^*||_2.\end{aligned}$$

This implies that the equilibrium point of system (16) is globally asymptotically stable. The proof is completed. □

Corollary 1. *Suppose that assumptions (H_1) and (H_2) hold. Then, the equilibrium point $Y^* = (x^*, s^*)^T$ of system (16) is globally asymptotically stable under the initial conditions (17) and (18), provided that*

$$-2a_k + |B_k| + 2\sum_{d=1}^{m} \left(|b_{kd}|F_d + |c_{kd}|F_d \right) + 2\sum_{d=1}^{m} |\tilde{c}_{kd}|^2 F_d^2 + |\beta_k|F_k < 0 \tag{26}$$

and

$$-2\alpha_k + |\beta_k|F_k < 0, \tag{27}$$

where $k = 1, 2, \cdots, m$.

Remark 1. *In general, constructing a Lyapunov functional is a main research method for studying stability problems of neural networks, see [23–27]. However, constructing a proper Lyapunov functional is very difficult for obtaining the stability criteria of a complicated system. In this paper, a simple Lyapunov functional is constructed. Using this Lyapunov functional, we can easily study the dynamic behavior of a competitive network system.*

Remark 2. *Since system (1) contains reaction-diffusion terms, we develop new ways (see Equations (11) and (12)) to deal with these terms so that we can obtain the stability conclusions of the solution smoothly.*

Remark 3. *In this paper, we only obtain the global asymptotic stability for competitive neural networks with reaction-diffusion terms and mixed delays. However, we cannot obtain the global exponential stability; the main reason is that system (3) contains reaction-diffusion terms and mixed delays and this makes it difficult to construct a suitable Lyapunov function. The global exponential stability of system (3) is a problem that we need to solve in the future.*

4. An Example

Example 1. *Consider the following system:*

$$\begin{cases} STM: \frac{\partial x_k(\delta,t)}{\partial t} & = \frac{\partial}{\partial \delta_p}\left(d_{kp}\frac{\partial x_k(\delta,t)}{\partial \delta_p}\right) - a_k x_k(\delta,t) + \sum_{d=1}^{m} b_{kd} f_d(x_d(\delta,t)) \\ & + \sum_{d=1}^{m} c_{kd} f_d(x_d(\delta, t-\tau_d(t))) + \sum_{d=1}^{m} \tilde{c}_{kd} \int_{-\infty}^{t} K_d(t-s) f_d(x_d(\delta,s))ds \\ & + B_k s_k(\delta,t) \\ LTM: \frac{\partial s_k(\delta,t)}{\partial t} & = \frac{\partial}{\partial \delta_p}\left(d^*_{kp}\frac{\partial s_k(\delta,t)}{\partial \delta_p}\right) - \alpha_k s_k(\delta,t) + \beta_k f_k(x_k(\delta,t)), \end{cases} \tag{28}$$

where $p = 1$, $k = d = 1,2$, $K_d(t) = te^{-t}$, $d_{kp} = d^*_{kp} = 1$, $\tau_d(t) = |\sin t|$, $f_d(\xi) = |\xi+1| - |\xi-1|$. *Obviously,* $|f_d(\xi_1) - f_d(\xi_2)| \le 2|\xi_1 - \xi_2|$ *and* $F_d = 2$. *Let*

$$a_1 = 10.5,\ a_2 = 12,\ b_{11} = \frac{5}{4},\ b_{12} = -\frac{2}{3}, b_{21} = \frac{2}{3},\ b_{22} = \frac{7}{10},$$

$$c_{11} = -\frac{3}{4},\ c_{12} = -\frac{1}{2}, c_{21} = \frac{5}{6},\ c_{22} = \frac{2}{5},\ B_1 = B_2 = \frac{3}{5},$$

$$\tilde{c}_{11} = \frac{3}{4},\ \tilde{c}_{12} = \frac{1}{5}, \tilde{c}_{21} = \frac{1}{3},\ \tilde{c}_{22} = \frac{4}{3},\ \alpha_1 = \alpha_2 = 3.1,\ \beta_1 = \beta_2 = 1.$$

It is easy to check that

$$-2a_1 + |B_1| + 2\sum_{d=1}^{2}\left(|b_{1d}|F_d + |c_{1d}|F_d + |\tilde{c}_{1d}|F_d\tau\right) + |\beta_1|F_1 \approx -0.81 < 0,$$

$$-2a_2 + |B_2| + 2\sum_{d=1}^{2}\left(|b_{2d}|F_d + |c_{2d}|F_d + |c_{1d}|F_d + |\tilde{c}_{2d}|F_d\tau\right) + |\beta_2|F_2 \approx -3.63 < 0,$$

$$-2\alpha_1 + |\beta_1|F_1 = -4.2 < 0,$$

$$-2\alpha_2 + |\beta_1|F_2 = -4.2 < 0.$$

All the hypotheses of Theorem 1 are satisfied. Since $f_1(0) = f_2(0)$, *then* $(x^*, s^*)^T = (0,0,0,0)^T$ *is a constant solution of system (28) which is globally asymptotically stable.*

5. Conclusions and Discussion

This paper is devoted to studying the global asymptotic stability for competitive neural networks with reaction-diffusion terms and mixed delays by using the mathematical analysis technique and Lyapunov functional method. For achieving the global asymptotic stability of the competitive neural networks, we use some inequality analysis techniques. We construct a suitable Lyapunov functional for the considered system and obtain some new criteria for guaranteeing the global asymptotic stability of competitive neural networks with reaction-diffusion terms and mixed delays. It should be pointed out that we first study the global asymptotic stability of competitive neural networks with reaction-diffusion terms and mixed delays. Finally, a numerical simulation has been shown to verify the correctness of our theoretical results. However, we only obtain the global asymptotic stability in this paper, we cannot obtain the global exponential stability which will be our research focus in the future.

Since the CNNs in the present paper contain reaction-diffusion terms, they belong to a partial differential equation and the traditional methods of dealing with an ordinary differential system are no longer applicable. By using theories of delayed partial differential equation and Lyapunov stability, we construct a suitable Lyapunov function and obtain global asymptotic stability. We believe that the method in the present paper can be used for other types of systems, such as impulsive partial differential equations, stochastic partial differential equations, and so on.

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