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Lie Symmetry Analysis of a Nonlinear System Characterizing Endemic Malaria

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Abstract: In this paper, the integrability of a nonlinear system developing endemic Malaria was demonstrated using Prelle–Singer techniques. In addition, Lie symmetry techniques were employed to identify additional independent variables that led to the modification of the nonlinear model and the development of analytical solutions.

Keywords: malaria; group theory; Lie symmetry; invariant solutions

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1. Introduction

Malaria is a parasitic disease spread by female Anopheles mosquito bites that is induced by the Plasmodium parasite [1]. It is still one of the most common and deadly human illnesses on the planet. Furthermore, clinical characteristics include the likelihood of infection, severity, and relapse risk. P. falciparum has been identified as the most dangerous of all the species to humans [2]. Malaria-infected areas are home to roughly 40% of the world's population. However, the majority of cases and deaths occur in Sub-Saharan Africa. Every year, 300 to 500 million cases and 1.5 to 2.7 million deaths are estimated to occur over the world. Africa is responsible for 80% of the cases and 90% of the deaths.

In a paper modeling the transmission dynamics of malaria endemic, researchers used rescaling to achieve a cosmetic simplification in order to predict disease propagation [1]. As a result of this scaling, the original five-dimensional system of first-order ordinary differential equations was reduced to the three first-order equations shown below

$$\begin{aligned} \frac{ds_h}{dt} &= 1 - \beta s_h i_h - \alpha s_h \\ \frac{di_h}{dt} &= \beta s_h i_h - (\alpha + \gamma) i_h \\ \epsilon \frac{di_v}{dt} &= \theta(1 - i_v) i_h - \delta i_v \end{aligned} \tag{1}$$

where s_h, i_h, and i_v are rescaled variables that indicate the number of susceptible, infected humans, and infected mosquitoes, respectively, at a given point in time. Nondimensional parameters are described as follows

$$\beta = \frac{\beta_h N_v}{\mu_h}, \ \alpha = \frac{\alpha_h}{\mu_h}, \ \gamma = \frac{\rho_h + \gamma_h}{\mu_h}, \ \epsilon = \frac{\alpha_v}{\mu_h}, \ \delta = \frac{\mu_v}{\mu_h}, \ \theta = \frac{\beta_v N_h}{\mu_v} \tag{2}$$

with β_h, the rate of human contact with mosquitoes; α_h, the rate of human natural death per capita; ρ_h, the human disease-induced death rate per capita; γ_h, the humans' per capita recovery rate; α_v, the natural death rate of mosquitos per capita; β_v, the frequency of mosquito contact with humans; μ_h, the human population's per capita birth rate; μ_v, the mosquitoes' per capita birth rate.

Over the last 40 years, a variety of techniques, including numerical and stability techniques, analytical techniques, approximation techniques, and others, have been utilized to explore and solve nonlinear systems of differential equations. Another major technique, Lie Theory of Symmetry Groups, was utilized to analyze nonlinear differential equations near the end of the twentieth century. Marius Sophus Lie, a talented Norwegian mathematician who lived near the end of the nineteenth century, established the Lie's theory of symmetry groups. Sophus Lie used symmetry groups theory to solve differential equations. He combined all differential equation approaches and deduced that his Lie group theory could account for them all. Lie groups are mathematical objects that represent the properties of groups as defined by group theory. To produce Lie symmetries, the Lie group theory employs appropriate transformations of independent and dependent variables.

The goal of this study, however, is to look at the integrability of a nonlinear system (1). In addition, to employ the modified Prelle–Singer (PS) approach of Chandrasekar et al. [3] to uncover transformations that lead to model linearization. Furthermore, the Lie symmetry technique was used to determine the model's explicit solutions.

This study is organised as follows. Section 2 introduces a heuristic background of the concepts underlying the Prelle–Singer (PS) procedure and Lie symmetry analysis. In Section 3, we used the PS procedure to solve the determining equations of the nonlinear system. In Section 4, we used a Lie symmetry method on the reduced equations to obtain explicit solutions. Section 5 contains the conclusion.

2. Theorems and Fundamental Concepts

This Section provides a comprehensive review of the Prelle–Singer (PS) procedure and Lie symmetry analysis approaches to solving differential equations. The theory includes the tools that will be used in the following sections of the paper. In [4,5], Matadi provided a fundamental definition and theorems that can be found in the literature (see [6,7]).

2.1. The Prelle–Singer (PS) procedure

In [3], Chandrasekar et al. updated the original Prelle–Singer (PS) technique and used it to solve autonomous and non-autonomous nonlinear systems of ordinary differential equations (ODEs) in the following way:

Given a three-dimensional system of nonlinear first-order ordinary differential equations [3]

$$\begin{aligned} \frac{dx_1}{dt} &= \frac{M_1(t,x_1,x_2,x_3)}{N_1(t,x_1,x_2,x_3)} \\ \frac{dx_2}{dt} &= \frac{M_2(t,x_1,x_2,x_3)}{N_2(t,x_1,x_2,x_3)}, \\ \frac{dx_3}{dt} &= \frac{M_3(t,x_1,x_2,x_3)}{N_3(t,x_1,x_2,x_3)}, \end{aligned} \tag{3}$$

Given x_1, x_2, x_3 with M_i's and N_i's, $i = 1,2,3$ analytic functions. Equation (3) admits a first integral $I(t,x_1,x_2,x_3) = K$, on the solutions, with K constant, resulting in a total differential of

$$dI = I_t dt + I_{x_1} dx_1 + I_{x_2} dx_2 + I_{x_3} dx_3 = 0 \tag{4}$$

Equation (3) can be written as follows:

$$\begin{aligned} \frac{M_1}{N_1} dt - dx_1 &= 0, \\ \frac{M_2}{N_2} dt - dx_2 &= 0, \\ \frac{M_3}{N_3} dt - dx_3 &= 0. \end{aligned} \tag{5}$$

By multiplying the first, second, and third equations in (5) by the functions P, L, and Q we get

$$dI = (P\phi_1 + K\phi_2 + Q\phi_3)dt - Pdx_1 - Ldx_2 - Qdx_3 = 0, \tag{6}$$

where $\phi_i = \frac{M_i}{N_i}, i = 1,2,3$. The following equations are obtained by Comparing Equations (6) and (4)

$$\begin{aligned} I_t &= (P\phi_1 + L\phi_2 + Q\phi_3), \\ I_{x_1} &= -P, \\ I_{x_2} &= -L \\ I_{x_3} &= -Q. \end{aligned} \tag{7}$$

The resulting determining equations for the integrating factors P, L, and Q are derived from the compatibility criteria between Equation (7)

$$\begin{aligned} P_t + \phi_1 P_{x_1} + \phi_2 P_{x_2} + \phi_3 P_{x_3} &= -(P\phi_{1x_1} + L\phi_{2x_1} + Q\phi_{3x_1}), \\ L_t + \phi_1 L_{x_1} + \phi_2 L_{x_2} + \phi_3 L_{x_3} &= -(P\phi_{1x_2} + L\phi_{2x_2} + Q\phi_{3x_2}), \\ Q_t + \phi_1 Q_{x_1} + \phi_2 Q_{x_2} + \phi_3 Q_{x_3} &= -(P\phi_{1x_3} + L\phi_{2x_3} + Q\phi_{3x_3}), \\ P_{x_2} &= L_{x_1}, \\ P_{x_3} &= Q_{x_1}, \\ L_{x_3} &= Q_{x_2}. \end{aligned} \tag{8}$$

with the given condition

$$\begin{aligned} &\phi_{3x_1}\left(\phi_{2t} + \phi_2\phi_{2x_1x_2} + \phi_3\phi_{2x_1x_3} - \phi_{2x_3}\phi_{3x_1} - \phi_{2x_1}\phi_{2x_2}\right) \\ &-\phi_{2x_1}\left(\phi_{3t} + \phi_2\phi_{3x_1x_2} + \phi_3\phi_{3x_1x_3} - \phi_{2x_1}\phi_{3x_2} - \phi_{3x_1}\phi_{3x_3}\right) = 0 \end{aligned} \tag{9}$$

Integrating Equation (7) produces the given integral of motion

$$I = r_1 + r_2 + r_3 - \int \left[Q + \frac{d}{dx_3}(r_1 + r_2 + r_3)\right]dx_3, \tag{10}$$

with

$$\begin{aligned} r_1 &= \int (P\phi_1 + L\phi_2 + Q\phi_3)dt \\ r_2 &= -\int (P + \frac{dr_1}{dx_1})dx_1 \\ r_3 &= -\int (L + \frac{d(r_1 + r_2)}{dx_2})dx_2 \end{aligned} \tag{11}$$

2.2. Lie Symmetry Procedure

In accordance with the theory of Lie symmetry, the given three dimensional system of first-order differential equation [5]

$$\begin{aligned} \dot{x_1} &= f_1(t, x_1, x_2, x_3), \\ \dot{x_2} &= f_2(t, x_1, x_2, x_3), \\ \dot{x_3} &= f_3(t, x_1, x_2, x_3), \end{aligned}$$

admits the following Lie group of transformations of one-parameter (a) [5]

$$\begin{aligned}
\tilde{t} &\approx t + aT(t, x_1, x_2, x_3), \\
\tilde{x_1} &\approx x_1 + aX_1(t, x_1, x_2, x_3), \\
\tilde{x_2} &\approx x_2 + aX_2(t, x_1, x_2, x_3), \\
\tilde{x_3} &\approx x_3 + aX_3(t, x_1, x_2, x_3),
\end{aligned}$$

with infinitesimal Lie operators [5]

$$G = T\frac{\partial}{\partial t} + X_1\frac{\partial}{\partial x_1} + X_2\frac{\partial}{\partial x_2} + X_3\frac{\partial}{\partial x_3}. \tag{12}$$

The group transformations $\tilde{t}$, $\tilde{x_1}$, $\tilde{x_2}$ and $\tilde{x_3}$ are obtained by solving the following Lie equations [4,5]

$$\begin{aligned}
\frac{d\tilde{t}}{da} &= T(t, x_1, x_2, x_3), \\
\frac{d\tilde{x_1}}{da} &= X_1(t, x_1, x_2, x_3), \\
\frac{d\tilde{x_2}}{da} &= X_2(t, x_1, x_2, x_3), \\
\frac{d\tilde{x_3}}{da} &= X_3(t, x_1, x_2, x_3),
\end{aligned}$$

with the initial conditions:

$$\tilde{t}\mid_{a=0} = t, \tilde{x_1}\mid_{a=0} = x_1, \tilde{x_2}\mid_{a=0} = x_2, \tilde{x_3}\mid_{a=0} = x_3.$$

The first extension of Lie operators above is defined as follows [5]

$$G^{[1]} = G + X_1^{[t]}\frac{\partial}{\partial \dot{x}_1} + X_2^{[t]}\frac{\partial}{\partial \dot{x}_2} + X_3^{[t]}\frac{\partial}{\partial \dot{x}_3}, \tag{13}$$

where

$$\begin{aligned}
X_1^{[t]} &= D_t(X_1) - \dot{x}_1 D_t(T), \\
X_2^{[t]} &= D_t(X_2) - \dot{x}_2 D_t(T), \\
X_3^{[t]} &= D_t(X_3) - \dot{x}_3 D_t(T),
\end{aligned}$$

with D_t representing the total differential operator describe as follows

$$D_t = \frac{\partial}{\partial t} + \dot{x_1}\frac{\partial}{\partial x_1} + \dot{x_2}\frac{\partial}{\partial x_2} + \dot{x_3}\frac{\partial}{\partial x_3} + \ddot{x_1}\frac{\partial}{\partial \dot{x_1}} + \ddot{x_2}\frac{\partial}{\partial \dot{x_2}} + \ddot{x_3}\frac{\partial}{\partial \dot{x_3}} + \ldots$$

The infinitesimals transformation obtained will be used to solve the following equation [5]

$$\begin{aligned}
Tr_t + X_1 r_{x_1} + X_2 r_{x_2} + +X_3 r_{x_3} &= 0, \\
Tu_t + X_1 u_{x_1} + X_2 u_{x_2} + +X_3 u_{x_3} &= 0, \\
Tv_t + X_1 v_{x_1} + X_2 v_{x_2} + +X_3 v_{x_3} &= 0, \\
Tw_t + X_1 w_{x_1} + X_2 w_{x_2} + +X_3 w_{x_3} &= 1.
\end{aligned} \tag{14}$$

Equation (14) will provide a set of new independent variable, r, and dependent variables, u, v and w, which can be used to transform the nonlinear system (1) to a linear system. The following section explore the existence of integrals to the nonlinear system (1).

3. Application of (PS) Procedure to Nonlinear System (1)

Considering the three-dimensional Equation (1)

$$\begin{aligned} \frac{ds_h}{dt} &= 1-\beta s_h i_h-\alpha s_h=\phi_1 \\ \frac{di_h}{dt} &= \beta s_h i_h-(\alpha+\gamma)i_h=\phi_2 \\ \epsilon\frac{di_v}{dt} &= \theta(1-i_v)i_h-\delta i_v=\phi_3 \end{aligned} \tag{15}$$

Case 1: $I_{s_h}=0$ and $I_t, I_{i_h}, I_{i_v}\neq 0$

The substitution of Equation (15) into (9) gives

$$-\alpha\delta\beta^2 i_h^2=0. \tag{16}$$

From Equation (16) we have, $\delta=0$ or $\beta=0$ or $\alpha=0$. The determining equation for Q in (9) becomes

$$Q_t-\alpha s_h Q_{s_h}-(\alpha+\gamma)i_h Q_{i_h}+\theta(1-i_v)i_h=\theta i_h, \tag{17}$$

in which we have taken $Q_{s_h}=0$ (since $I_{s_h}=0$). A simple solution for (17) is $Q=-i_v i_h$ with $\gamma=\theta$. Using the restriction $\delta=\beta=\alpha=0$ and $\gamma=\theta$, the solution of the determining equation for P, L is given by $P=i_h, L=0$. Hence, from Equation (11), we obtain

$$\begin{aligned} r_1 &= i_h t-\theta(1-i_v)i_v i_h^2 t \\ r_2 &= -i_h s_h \\ r_3 &= -(i_h+\theta(1-i_v)i_v i_h^2)t \end{aligned} \tag{18}$$

therefore, the integral of motion is given by

$$I=-i_h s_h-\frac{1}{2}i_v^2 i_h \tag{19}$$

Case 2: $I_{i_h}=0$, $I_t, I_{s_h}, I_{i_v}\neq 0$ and $I_{i_v}=0$, $I_t, I_{s_h}, I_{i_h}\neq 0$

According to [3], the determining equations and conditions is obtained by introducing the following transformation

$$P=SQ \text{ and } L=UQ, \tag{20}$$

with

$$U=-\frac{\phi_{3s_h}}{\phi_{2s_h}} \text{ and } S=-\frac{(\phi_3+\phi_2 U)}{\phi_1} \tag{21}$$

4. Lie Symmetry Analysis of the System (1)

4.1. Lie Symmetry of one Dimensional Second-Order Differential Equation

From the first equation of the nonlinear system (1), we obtain

$$i_h=-\frac{\dot{s}_h}{\beta s_h}+\frac{1}{\beta s_h}-\frac{\alpha}{\beta}, \tag{22}$$

differentiating Equation (22) with respect to t, we obtain

$$\dot{i}_h=-\frac{\ddot{s}_h}{\beta s_h}+\frac{\dot{s}_h^2}{\beta s_h^2}-\frac{\dot{s}_h}{\beta s_h^2}, \tag{23}$$

the substitution of Equation (23) into the second equation of the nonlinear system (1) gives

$$\dot{s}_h^2-\beta\dot{s}_h-\ddot{s}_h s_h-\beta s_h^2+\beta\dot{s}_h s_h^2+\alpha\beta s_h^3+(\alpha+\gamma)s_h-(\alpha+\gamma)\dot{s}_h s_h+\alpha(\alpha+\gamma)s_h^2 \tag{24}$$

Lie group analysis to (24) is performed using SYM packages [4–6], yielding the following cases

Case 1: $\beta = 0$ and $\alpha + \gamma = 0$

The determining equations for the classical symmetries of the nonlinear Equation (24) are

$$\begin{aligned}
\frac{\partial \xi}{\partial s_h} + s_h \frac{\partial^2 \xi}{\partial s_h^2} &= 0 \\
\frac{\partial \eta}{\partial t} + s_h \frac{\partial^2 \eta}{\partial t^2} &= 0 \\
\eta + s_h \left(2\frac{\partial \eta}{\partial t} - \frac{\partial \xi}{\partial t} - 2s_h \frac{\partial^2 \xi}{\partial t \partial s_h}\right) &= 0 \\
-\eta + s_h \left(\frac{\partial \eta}{\partial s_h} - 2\frac{\partial \xi}{s_h} - s_h \frac{\partial^2 \eta}{\partial s_h^2} + 2s_h \frac{\partial^2 \xi}{\partial t \partial s_h}\right) &= 0
\end{aligned} \tag{25}$$

The coefficients of the infinitesimal generator are obtained by solving the overdetermining Equation (25)

$$\xi(s_h, t) = c_1 + c_2 t \tag{26}$$

$$\eta(s_h, t) = c_2 s_h, \tag{27}$$

as a result, the two-dimensional Lie algebra is given by

$$G_1 = \partial_t \tag{28}$$

$$G_2 = t\partial_t + s_h \partial_{s_h}, \tag{29}$$

This case reduces Equation (24) to

$$\dot{s}_h^2 - \ddot{s}_h s_h = 0 \tag{30}$$

Equation (38) can be linearized using the transformation

$$S = \frac{1}{s_h} \tag{31}$$

Hence, we obtain

$$\frac{d^2 S}{dt^2} = 0 \tag{32}$$

The solution to Equation is

$$S(t) = ct + d \tag{33}$$

where c, d are constant of integration. Substituting Equation (33) into transformation (31) results in the number of susceptible humans

$$s_h(t) = \frac{1}{ct + d} \tag{34}$$

Equation (34) is substituted into Equation (22) to obtain the number of infected persons, i_h.

$$i_h = \frac{At + B}{\beta(ct + d)} \tag{35}$$

with $A = -\alpha c,\ B = 1 - c - \alpha d$. The substitution of Equations (34) and (35) into the last equation in (1) gives

$$\frac{di_v}{dt} + \left(\frac{Et + F}{Kt + L}\right) i_v = \frac{Ht + G}{Kt + L} \tag{36}$$

with $G = F + L\delta,\ H = E + K\delta,\ E = \theta A,\ F = \theta B,\ L = \beta d,\ K = \beta c$ Hence, the number of infected mosquitoes is given by

$$\begin{aligned} i_v &= c_1 \exp\left[\frac{Et}{K} - \frac{(FK - EL)\ln(L + Kt)}{K^2}\right] \\ &\quad + \frac{1}{K^2}\exp\left[-\frac{F + Et}{K} - \frac{(FK - EL)\ln(L + Kt)}{K^2}\right] \\ &\quad \times (L + Kt)^{\frac{(FK-EL)}{K^2}}\left(-\frac{L + Kt}{K^2}\right)^{\frac{(FK-EL)}{K^2}} \end{aligned} \tag{37}$$

Case 2: $\beta \neq 0$ and $\alpha + \gamma \neq 0$

The determining equations for the classical symmetries of the nonlinear Equation (24) are given by

$$\begin{aligned} \frac{\partial \xi}{\partial s_h} + s_h \frac{\partial^2 \xi}{\partial s_h^2} &= 0 \\ \eta - s_h\left[\frac{\partial \eta}{\partial s_h} + 2(\beta s_h - 1)\frac{\partial \xi}{\partial s_h}\right] - s_h\left[\frac{\partial^2 \eta}{\partial s_h^2} - 2\frac{\partial^2 \xi}{\partial t \partial s_h}\right] &= 0 \\ -\eta + 3s_h^2\Big[(\beta - (\alpha + \gamma)s_h)\Big]\frac{\partial \xi}{\partial s_h} - 2\frac{\partial \eta}{\partial t} + \frac{\partial \xi}{\partial t} - \beta s_h \frac{\partial \xi}{\partial t} &= 0 \\ -\eta(\alpha + \gamma)s_h + s_h\Big[-\beta + (\alpha + \gamma)s_h\Big]\frac{\partial \eta}{\partial s_h} + \frac{\partial \eta}{\partial t} - \beta s_h \frac{\partial \eta}{\partial t} &= 0 \\ \beta s_h \frac{\partial \eta}{\partial t} + 2\beta s_h \frac{\partial \xi}{\partial t} - 2(\alpha + \gamma)s_h^2 \frac{\partial \xi}{\partial t} + s_h \frac{\partial^2 \eta}{\partial t^2} &= 0 \end{aligned} \tag{38}$$

from the above overdetermining equation yields the coefficients of the infinitesimal generator

$$\xi(s_h, t) = \frac{\exp\left(\sqrt{\alpha + \gamma}\right)t}{\alpha + \gamma}c_1 + c_2 \tag{39}$$

$$\eta(s_h, t) = \exp\left(\sqrt{\alpha + \gamma}\right)t c_1 s_h, \tag{40}$$

As a result, we have the two-dimensional Lie algebra shown below

$$G_1 = \left[\exp\left(\sqrt{\alpha + \gamma}\right)t + \exp\left(\sqrt{\alpha + \gamma}\right)t s_h\right]\partial_t \tag{41}$$

$$G_2 = \partial_{s_h}, \tag{42}$$

Solving the nonlinear Equation (24) for case 1, we obtain the number of susceptible humans

$$s_h(t) = \frac{-1 + A\exp(At + B)}{A}, \tag{43}$$

with A and B constants of integration. The number of infected humans, i_h is calculated by substituting Equation (43) into Equation (22)

$$i_h = \frac{A(A - \alpha\beta)\exp(At + B) + A - \alpha\beta}{\beta\left[A\exp(AT + B) - 1\right]} \tag{44}$$

The substitution of (43) and (44) gives the first-order nonlinear first-order ordinary differential equation

$$\epsilon\frac{di_v}{dt} + i_v\left(\frac{Fk(t) + G}{Dk(t) - \beta}\right) = \frac{Lk(t) + N}{Dk(t) - \beta} \tag{45}$$

with

$$\begin{array}{rcl} k(t) & = & \exp(At+B) \\ F & = & L-M \\ G & = & N-P \\ L & = & \theta C \\ M & = & \delta D \\ N & = & \theta B \\ P & = & \delta B \end{array}$$

The solution to Equation (45) is a hypergeometric function, and the numerical solution can be found in [8].

4.2. Lie Symmetry of Three Dimensional System of First-Order Differential Equation

Equations (12) and (13) are applied to the Non-dimensional model Equation (1), yielding the following:

$$\begin{array}{rcl} G\Big(1-\beta s_h i_h - \alpha s_h\Big) & = & -\beta S I_1 - \alpha S, \\ G\Big(\beta s_h i_h - (\alpha+\gamma) i_h\Big) & = & \beta S I_1 - (\alpha+\gamma) I_1, \\ G\Big(\theta(1-i_v) i_h - \delta i_v\Big) & = & -\theta I_1 I_2 - \delta I_2. \end{array} \tag{46}$$

The extended infinitesimal transformation is obtained by using Equation (13)

$$\begin{aligned} -\beta S I_1 - \alpha S &= S^{[t]} + s_h' S^{[s_h]} + i_h' S^{[i_h]} + i_v' S^{[i_v]} \\ &\quad - s_h'(\mathcal{T}^{[t]} + s_h' \mathcal{T}^{[s_h]} + i_h' \mathcal{T}^{[i_h]} + i_v' \mathcal{T}^{[i_v]}), \\ \beta S I_1 - (\alpha+\gamma) I_1 &= I_1^{[t]} + s_h' I_1^{[s_h]} + i_h' I_1^{[i_h]} + i_v' I_1^{[i_v]} \\ &\quad - i_h'(\mathcal{T}^{[t]} + s_h' \mathcal{T}^{[s_h]} + i_h' \mathcal{T}^{[i_h]} + i_v' \mathcal{T}^{[i_v]}), \\ -\theta I_1 I_2 - \delta I_2 &= I_2^{[t]} + u_1' I_2^{[u_1]} + u_2' I_2^{[i_h]} + i_v' I_2^{[i_v]} \\ &\quad - i_v'(\mathcal{T}^{[t]} + s_h' \mathcal{T}^{[s_h]} + i_h' \mathcal{T}^{[i_h]} + i_v' \mathcal{T}^{[i_v]}). \end{aligned} \tag{47}$$

with

$$s_h' = \frac{ds_h}{dt}; i_h' = \frac{du_2}{dt}; i_v' = \frac{du_3}{dt}.$$

Substituting Equation (1) into (47) yields

$$\begin{aligned}
-\beta S I_1 - \alpha S &= S^{[\tau]} + \Big(1 - \beta s_h i_h - \alpha s_h\Big)\Big(S^{[s_h]} - \mathcal{T}^{[t]}\Big) \\
&+ \Big(\beta s_h i_h - (\alpha+\gamma) i_h\Big) S^{[i_h]} \\
&+ \Big(\theta(1 - i_v) i_h - \delta i_v\Big) S^{[i_v]} \\
&- \Big(1 - \beta s_h i_h - \alpha s_h\Big)^2 \mathcal{T}^{[s_h]} \\
&- \Big(1 - \beta s_h i_h - \alpha s_h\Big)\Big(\beta s_h i_h - (\alpha+\gamma) i_h\Big) \mathcal{T}^{[i_h]} \\
&- \Big(1 - \beta s_h i_h - \alpha s_h\Big)\Big(\theta(1 - i_v) i_h - \delta i_v\Big) \mathcal{T}^{[i_v]}, \\
\beta S I_1 - (\alpha+\gamma) I_1 &= T_1^{[t]} + \Big(1 - \beta s_h i_h - \alpha s_h\Big) I_1^{[s_h]} \\
&+ \Big(\beta s_h i_h - (\alpha+\gamma) i_h\Big)\Big(I_1^{[i_h]} - \mathcal{T}^{[t]}\Big) \\
&+ \Big(\theta(1 - i_v) i_h - \delta i_v\Big) I_1^{[i_v]} \\
&- \Big(1 - \beta s_h i_h - \alpha s_h\Big)\Big(\beta s_h i_h - (\alpha+\gamma) i_h\Big) \mathcal{T}^{[s_h]} \\
&- \Big(\beta s_h i_h - (\alpha+\gamma) i_h\Big)^2 \mathcal{T}^{[i_v]} \\
&- \Big(\beta s_h i_h - (\alpha+\gamma) i_h\Big)\Big(\theta(1 - i_v) i_h - \delta i_v\Big) \mathcal{T}^{[i_v]}, \\
-\theta I_1 I_2 - \delta I_2 &= U_3^{[t]} + \Big(1 - \beta s_h i_h - \alpha s_h\Big) I_2^{[s_h]} \\
&+ \Big(\beta s_h i_h - (\alpha+\gamma) i_h\Big) I_2^{[i_h]} \\
&+ \Big(\theta(1 - i_v) i_h - \delta i_v\Big)\Big(I_2^{[i_v]} - \mathcal{T}^{[t]}\Big) \\
&- \Big(1 - \beta s_h i_h - \alpha s_h\Big)\Big(\theta(1 - i_v) i_h - \delta i_v\Big) \mathcal{T}^{[s_h]} \\
&- \Big(\beta s_h i_h - (\alpha+\gamma) i_h\Big)\Big(\theta(1 - i_v) i_h - \delta i_v\Big) \mathcal{T}^{[i_h]} \\
&+ \Big(\theta(1 - i_v) i_h - \delta i_v\Big)^2 \mathcal{T}^{[i_v]}.
\end{aligned} \tag{48}$$

In general, solving nonlinear system (48) is challenging. As a result, it is required to use special solutions [5]. In the case of $\mathcal{T} = \mathcal{T}(t)$, $S = S(s_h)$, $I_1 = I_1(i_h)$, $I_2 = I_2(i_v)$, the non-linear Equations (48) are simplified as

$$\Big(1 - \beta s_h i_h - \alpha s_h\Big)\Big(S^{[s_h]} - \mathcal{T}^{[t]}\Big) = -\beta S I_1 - \alpha S, \tag{49}$$

$$\Big(\beta s_h i_h - (\alpha+\gamma) i_h\Big)\Big(I_1^{[i_h]} - \mathcal{T}^{[t]}\Big) = \beta S I_1 - (\alpha+\gamma) I_1, \tag{50}$$

$$\Big(\theta(1 - i_v) i_h - \delta i_v\Big)\Big(T_2^{[i_v]} - \mathcal{T}^{[t]}\Big) = -\theta I_1 I_2 - \delta I_2. \tag{51}$$

The following second-order partial differential equation is obtained by considering the partial derivative of Equation (51) with regard to τ [5]

$$\mathcal{T}^{[tt]} = 0,$$

solving Equation (52), we obtain

$$\mathcal{T}(t) = lt + m, \tag{52}$$

with l and m being the integration constants. Substituting Equation (52) into (49)–(51) yields

$$\begin{aligned} \left(1-\beta s_h i_h-\alpha s_h\right)\left(S^{[s_h]}-l\right) &= -\beta S I_1-\alpha S, & (53)\\ \left(\beta s_h i_h-(\alpha+\gamma) i_h\right)\left(I_1^{[i_h]}-l\right) &= \beta S I_1-(\alpha+\gamma) I_1, & (54)\\ \left(\theta(1-i_v) i_h-\delta i_v\right)\left(I_2^{[i_v]}-l\right) &= -\theta I_1 I_2-\delta I_2. & (55) \end{aligned}$$

After twice partially differentiating Equation (54) with respect to i_h, we obtain

$$I_1^{[i_h i_h]}=0.$$

Hence,

$$I_1(i_h)=p i_h+q. \tag{56}$$

Substituting Equation (56) into (53), we obtain

$$\left(1-\beta s_h i_h-\alpha s_h\right)\left(S^{[s_h]}-l\right)=-\beta S(p i_h+q)-\alpha S. \tag{57}$$

As Equation (57) is dependent on the values of s_h and i_h, we obtain

$$\begin{aligned} s_h &: -(\beta i_h+\alpha)\frac{\partial S}{\partial s_h}=-\beta l i_h-\alpha l,\\ i_h &: s_h\left(\frac{\partial S}{\partial s_h}-l\right)=Sp,\\ - &: \frac{\partial S}{\partial s_h}-l=\beta S q-\alpha S. \end{aligned}$$

Hence,

$$S=c_1\exp\left[(\beta q-\alpha)s_h\right]+\frac{l}{-\beta q+\alpha}. \tag{58}$$

From (55), we obtain

$$I_h : \frac{\partial I_2}{\partial i_v}=l.$$

Hence,

$$I_2=l i_v+c_2. \tag{59}$$

As a result, the infinitesimal transformations are as follows:

$$\begin{aligned} S(s_h) &= c_1\exp\left[(\beta q-\alpha)s_h\right]+\frac{l}{-\beta q+\alpha},\\ I_1(i_h) &= p i_h+q,\\ I_2(i_v) &= l i_v+c_2. \end{aligned} \tag{60}$$

It is worth noting that these infinitesimal transformations are not unique. There is, however, an infinite number of infinitesimal transformations [5]. As a result, Equation (12) becomes

$$G=(lt+m)\frac{\partial}{\partial t}+\left(c_1\exp\left[(\beta q-\alpha)s_h\right]+\frac{l}{-\beta q+\alpha}\right)\frac{\partial}{\partial s_h}+(p i_h+q)\frac{\partial}{\partial i_h}+(l i_v+c_2)\frac{\partial}{\partial i_v}.$$

Hence, the following Lie generators are found

$$
\begin{aligned}
G_1 &= t\frac{\partial}{\partial t} + \frac{1}{\alpha}\frac{\partial}{\partial s_h} + i_v\frac{\partial}{\partial i_v},\\
G_2 &= \frac{\partial}{\partial t},\\
G_3 &= i_h\frac{\partial}{\partial i_h},\\
G_4 &= \frac{1}{(\alpha-\beta)}\frac{\partial}{\partial s_h},\\
G_5 &= \exp[-\alpha s_h]\frac{\partial}{\partial s_h},\\
G_6 &= \frac{\partial}{\partial i_v}.
\end{aligned}
$$

By setting the constant of integration to $l = 1$, $m = 1$, $p = 1$, $q = 1$, $c_1 = 1$, $c_2 = 1$. Equation (60) becomes

$$
\begin{aligned}
S &= \frac{1}{\alpha-\beta} + \exp[\beta-\alpha],\\
I_1 &= 1 + i_h,\\
I_2 &= 1 + i_v,\\
\mathcal{T} &= 1 + t.
\end{aligned}
$$

Hence, Equation (14) becomes

$$
\begin{aligned}
(1+t)r_t + \big(\frac{1}{\alpha-\beta} + \exp[\beta-\alpha]\big)r_{s_h} + (1+i_h)r_{i_h} + (1+i_v)r_{i_v} &= 0,\\
(1+t)s_h^{[t]} + \big(\frac{1}{\alpha-\beta} + \exp[\beta-\alpha]\big)S^{[s_h]} + (1+i_h)S^{[i_h]} + (1+i_v)S^{[i_v]} &= 0,\\
(1+t)i_h^{[t]} + \big(\frac{1}{\alpha-\beta} + \exp[\beta-\alpha]\big)I_1^{[s_h]} + (1+i_h)I_1^{[i_h]} + (1+i_v)I_1^{[i_v]} &= 0,\\
(1+t)i_v^{[t]} + \big(\frac{1}{\alpha-\beta} + \exp[\beta-\alpha]\big)I_2^{[s_h]} + (1+i_h)I_2^{[i_v]} + (1+i_v)I_2^{[i_v]} &= 1.
\end{aligned} \tag{61}
$$

The solution of Equation (61) is given by

$$
\begin{aligned}
r &= \frac{t}{g(s_h i_h i_v)},\\
s_h &= \ln t + \frac{t}{g(s_h i_h i_v)},\\
i_h &= \frac{t}{g(s_h i_h i_v)},\\
i_v &= \frac{t}{g(s_h i_h i_v)}.
\end{aligned}
$$

The special case is given by

$$
\begin{aligned}
r &= \frac{t}{(s_h i_h i_v)}, \\
s_h &= \ln t + \frac{t}{(s_h i_h i_v)}, \\
i_h &= \frac{t}{(s_h i_h i_v)}, \\
i_v &= \frac{t}{(s_h i_h i_v)}.
\end{aligned}
$$

5. Conclusions

Understanding physical models requires the analysis of nonlinear differential equations. According to Ove [9], finding a closed form solution of a nonlinear differential requires a thorough comprehension of the phenomena being described. The Prelle–Singer (PS) and Lie symmetry techniques are utilized in this research to show the linear integrability of a mathematical model of endemic malaria and to identify explicit solutions. The results showed that for parameter values $\beta \neq \alpha + \gamma$ and $\alpha + \gamma \neq 0$, the reduced second-order differential equation allows for system linearization and provides the explicit solutions.

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