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Symmetric Functional Set-Valued Integral Equations and Bihari–LaSalle Inequality

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Abstract: In the paper, we consider functional set-valued integral equations whose representation contains set-valued integrals occurring symmetrically on both sides of the equation. On the coefficients of the equation, we impose certain conditions, more general than the standard Lipschitz condition, which allow the application of the Bihari–LaSalle inequality in the proofs of the obtained theorems. In this way, we obtain a result about the existence and uniqueness of the solution of the equation under consideration and the insensitivity of the solution in the case of minor changes in the parameters of the equation.

Keywords: set-valued integral equations; existence and uniqueness of solution; Bihari–LaSalle inequality

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1. Introduction

This paper is a continuation of the research presented in [1], where certain directions were indicated in which the theory of symmetric functional set-valued integral equations can develop. The current paper presents the achievements in this subject, going in the direction of replacing the Lipschitz condition of the continuity of the coefficients of the considered equation by a weaker condition. Although in the aforementioned paper [1] the facts used to embed our research in the existing mathematical framework are given initially, here we will also mention and repeat the most important ones for the reader's comfort.

The symbol $\mathcal{P}_{cc}(\mathbb{R}^d)$ stands for the family of nonempty compact and convex subsets of $\mathbb{R}^d$. We will work in the metric space $(\mathcal{P}_{cc}(\mathbb{R}^d), \rho_H)$, where ρ_H is the Hausdorff–Pompeiu metric, i.e.,

$$\rho_H(A,B) := \max\left\{ \sup_{a\in A} \inf_{b\in B} \|a-b\|, \sup_{b\in B} \inf_{a\in A} \|a-b\| \right\}, \quad A,B \in \mathcal{P}_{cc}(\mathbb{R}^d),$$

where $\|\cdot\|$ denotes norm in $\mathbb{R}^d$. The space $(\mathcal{P}_{cc}(\mathbb{R}^d), \rho_H)$ is Polish and locally compact. The set $\mathcal{P}_{cc}(\mathbb{R}^d)$ can be supplied with addition and multiplication by a real number

$$A+B := \{a+b \mid a\in A, b\in B\}, \quad k\cdot A := \{k\cdot a \mid a\in A\}$$

$$\text{for} \quad A,B\in\mathcal{P}_{cc}(\mathbb{R}^d) \quad \text{and} \quad k\in\mathbb{R},$$

thus obtaining a semilinear structure. It is worth recalling that the opposite element may not exist, and defining set subtraction can be cumbersome. In the paper, we will use the concept of the Hukuhara difference of two sets, denoting this operation as $A\ominus B$. Such a set $A\ominus B$ is defined by the equality $A = B + (A\ominus B)$. The Hukuhara differences may not exist, but if they exist, they are unique. We also recall some properties of the metric ρ_H that will be useful in our considerations (see [2]). For $A,B,C,D\in\mathcal{P}_{cc}(\mathbb{R}^d)$,

(P1) $\rho_H(A+C, B+C) = \rho_H(A,B)$;
(P2) $\rho_H(A+B, C+D) \le \rho_H(A,C) + \rho_H(B,D)$;

(P3) If there exist $A \ominus B$ and $C \ominus D$ then $\rho_H(A \ominus B, C \ominus D) \leq \rho_H(A, C) + \rho_H(B, D)$.

Since we are interested in functional equations, we also consider the set $C_\theta = C([-\theta, 0], \mathcal{P}_{cc}(\mathbb{R}^d))$ of all ρ_H-continuous set-valued mappings acting from $[-\theta, 0]$ to $\mathcal{P}_{cc}(\mathbb{R}^d)$, where θ is a positive real number. The set C_θ is equipped with the supremum metric ρ^*, i.e.,

$$\rho^*(\chi_1, \chi_2) = \sup_{u \in [-\theta, 0]} \rho_H(\chi_1(u), \chi_2(u)) \quad \text{for} \quad \chi_1, \chi_2 \in C_\theta.$$

The set-valued integrals appearing in this paper are in the sense of Aumann (see [3]), which means that for a set-valued mapping $F\colon [a, b] \to \mathcal{P}_{cc}(\mathbb{R}^d)$

$$\int_a^b F(u)du := \Big\{ \int_a^b f(u)du \mid f \in S(F) \Big\},$$

where $S(F)$ is the set of integrable selections of F and this set is nonempty. Let us recall the following (see [2]):

(P4) $\int_a^b F(u)du \in \mathcal{P}_{cc}(\mathbb{R}^d)$;
(P5) $\int_a^b F(u)du = \int_a^c F(u)du + \int_c^b F(u)du$ if $a \leq c \leq b$;
(P6) If F, G are integrable set-valued mappings then
$\rho_H\Big(\int_a^b F(u)du, \int_a^b G(u)du\Big) \leq \int_a^b \rho_H\big(F(u), G(u)\big)du$.

In this paper, we examine functional set-valued equations which have a representation

$$X(t) + \int_{t_0}^t F(s, X_s)ds = \chi_0(0) + \int_{t_0}^t G(s, X_s)ds \quad \text{for} \quad t \in [t_0, t_0 + T] \tag{1}$$

with initial condition

$$X_{t_0} = \chi_0,$$

where t_0 symbolizes initial instant of time, T is a time horizon, $\chi_0 \in C_\theta$ is an initial history, $F, G\colon [t_0, t_0 + T] \times C_\theta \to \mathcal{P}_{cc}(\mathbb{R}^d)$ are the coefficients of the equation and $X_s \in C_\theta$ is understood as $X_s(u) = X(s + u)$ for $u \in [-\theta, 0]$, where s is fixed from $I := [t_0, t_0 + T]$. In the setting of this paper, X is a set-valued mapping that belongs to $C(J, \mathcal{P}_{cc}(\mathbb{R}^d))$, where $J := [t_0 - \theta, t_0 + T]$. Such equations have been called symmetric because of the symmetrical occurrence of integrals on both sides of the equation. Since integrals are sets, it is not possible in general to reduce the form of this equation to one that contains only one integral. At this point, it should be emphasized that equations of the one-sided, asymmetric type are a special case of symmetric equations of type (1). This fact motivates even more to consider symmetric equations. When we talk about asymmetric equations in the context of Equation (1), we mean the equations of the form

$$X(t) = \chi_0(0) + \int_{t_0}^t G(s, X_s)ds \quad \text{for} \quad t \in [t_0, t_0 + T] \tag{2}$$

with initial condition

$$X_{t_0} = \chi_0,$$

and

$$X(t) + \int_{t_0}^t F(s, X_s)ds = \chi_0(0) \quad \text{for} \quad t \in [t_0, t_0 + T] \tag{3}$$

with initial condition

$$X_{t_0} = \chi_0,$$

respectively. It is worth recalling and realizing that equations in integral form (2) can be treated as equivalent to differential equations with the so-called Hukuhara derivative D_H of set-valued mappings, namely

$$D_H X(t) = G(t, X_t) \quad \text{for} \quad t \in [t_0, t_0 + T] \tag{4}$$

with initial condition $X_{t_0} = \chi_0$ and they were the basic form of set-valued functional differential equations, the study of which forms the basis of the theory of such equations (cf. [2,4–8]). Such differential equations, in which the mappings are set valued, should be thought of as mathematical models of processes that change their states dynamically and in which the state of the process cannot be described with a single number, but a set of numbers must be used.

Additionally, integral Equation (3) has its counterpart in the differential equation with the Hukuhara derivative of the second kind D_H^*, i.e.,

$$D_H^* X(t) = (-1) \cdot F(t, X_t) \quad \text{for} \quad t \in [t_0, t_0 + T] \tag{5}$$

with initial condition $X_{t_0} = \chi_0$. Such equations have become interesting because of a certain property that distinguishes them significantly from Equation (4). Namely, every solution of equation of type (5) has the property that its values become more and more precise with the increase in time in the sense that the diameter of the set that is the value of the solution at a given moment t does not increase with the passage of time t (see [9–12]). On the other hand, the values of solutions of Equation (4) have diameters that are not diminished in time, which can be interpreted with non-decreasing uncertainty about the state of the process that is modeled by such a differential equation.

The equations we consider in this paper have the good property that they cover both equations of type (2) and (3) and thus can serve to model real-life processes, whose uncertainty about the exact value of the state can change the nature of monotonicity. Although the current paper mentions the potential of applications of the studied equations, it is a theoretical research that may be the basis for application in practical issues in the future.

The study presented here is a continuation and extension of some achievements collected in [1]. We are now engaged in proving the existence and uniqueness of a solution to Equation (1) under more general conditions than in the paper [1], where we required the Lipschitz condition to be met. Now we use a condition weaker than the Lipschitz one. With this more general condition, described precisely in the next section, we will also justify the stability of the solution in relation to small changes in the initial history or small changes in the coefficients of the equation. All these results are obtained by applying the Bihari–LaSalle inequality.

The theory of set-valued equations began in the 1960s. Since then, they have formed a separate stream of research with their own methods and techniques. An extensive collection of results in this field is contained in the monograph [2]. The importance of these studies was also confirmed by recently published articles, e.g., [9–20], including those that combine this theory with application, for example, in the diagnosis of cancer [21,22]. The applicability of set-valued differential equations confirmed by the last mentioned papers gives a good chance to use the results of the current article in analyses related to mathematical modeling in medicine.

2. Main Results

Since one of the main tools that will allow us to obtain the presented results is the Bihari–LaSalle inequality, we recall it below for the convenience of the reader.

Lemma 1. *(Bihari–LaSalle inequality [23,24]). Let $f, g\colon I \to [0,\infty)$ be continuous, and ξ be a continuous and non-decreasing function such that $\xi(t) > 0$ for $t > 0$. If f satisfies*

$$f(t) \leq \alpha + \int_0^t g(s)\xi(f(s))ds, \quad \textit{for} \quad t \in I,$$

where α is a non-negative constant, then

$$f(t) \leq V^{-1}\left(V(\alpha) + \int_0^t g(s)ds\right)$$

for all $t \in I$ such that $V(\alpha) + \int_0^t g(s)ds \in \mathrm{Dom}(V^{-1})$, where

$$V(r) = \int_1^r \frac{ds}{\xi(s)}, \quad r \geqslant 0,$$

and V^{-1} is the inverse function of V.
Moreover, if $\alpha = 0$ and $\int_{0+} \frac{dt}{\xi(t)} = +\infty$ then $f(t) = 0$ for every $t \in I$.

Remark 1. *If $\xi(t) = t$ in the assumptions of the Bihari–LaSalle inequality, then the inequality in the thesis will be*

$$f(t) \leq \alpha \exp\left\{\int_0^t g(s)\, ds\right\} \quad \textit{for} \quad t \in I$$

and it is well known as the Gronwall–Bellman inequality.

After recalling the above inequalities, we start by writing what we mean by the solution to Equation (1).

Definition 1. *A set-valued mapping $X\colon J \to \mathcal{P}_{cc}(\mathbb{R}^d)$, which is ρ_H-continuous, is said to be a solution to Equation (1), if $X_{t_0} = \chi_0$ and $X(t)$ meets*

$$X(t) + \int_{t_0}^t F(s, X_s)ds = \chi_0(0) + \int_{t_0}^t G(s, X_s)ds \quad \textit{for every} \quad t \in I = [t_0, t_0 + T].$$

To obtain the results presented in this article, we will use the following conditions:

(A0) $\chi_0 \in C_\theta$;
(A1) $F, G\colon I \times C_\theta \to \mathcal{P}_{cc}(\mathbb{R}^d)$ are jointly continuous;
(A2) There is a positive constant C such that for every $t \in I$

$$\max\{\rho_H(F(t, \mathbf{0}), \{0\}), \rho_H(G(t, \mathbf{0}), \{0\})\} \leq C,$$

where $\mathbf{0}$ is the zero element in the space C_θ;
(A3) There is a continuous, non-decreasing, concave function $\xi\colon [0,\infty) \to [0,\infty)$ satisfying $\xi(0) = 0$, $\xi(t) > 0$ for $t > 0$, $\int_{0^+}^1 \frac{dt}{\xi(t)} = +\infty$ and such that for every $(t, \chi) \in I \times C_\theta$

$$\max\{\rho_H(F(t, \chi_1), F(t, \chi_2)), \rho_H(G(t, \chi_1), G(t, \chi_2))\} \leq \xi(\rho^*(\chi_1, \chi_2));$$

(A4) There is $\tilde{T} \in (0, T]$ such that the sequence $\{X^n\}_{n=0}^\infty$, $X^n\colon \tilde{I} \to \mathcal{P}_{cc}(\mathbb{R}^d)$, $\tilde{I} = [t_0, t_0 + \tilde{T}]$ described as

$$X^0(t) = \begin{cases} \chi_0(t - t_0), & t \in [t_0 - \theta, t_0], \\ \chi_0(0), & t \in \tilde{I}, \end{cases}$$

and for $n \in \{1, 2, \ldots\}$

$$X^n(t) = \begin{cases} \chi_0(t - t_0), & t \in [t_0 - \theta, t_0], \\ \left[\chi_0(0) + \int_{t_0}^t G(s, X_s^{n-1})ds\right] \ominus \int_{t_0}^t F(s, X_s^{n-1})ds, & t \in \tilde{I}, \end{cases}$$

is well defined, i.e., the Hukuhara differences exist.

Remark 2. *If we put $\xi(t) = Lt$ in (A3), where L is a positive constant, then ξ is a continuous, concave, non-negative function satisfying $\xi(0) = 0$, $\xi(t) > 0$ for $t > 0$ and $\int_{0+}^{1} \frac{1}{\xi(t)} dt = +\infty$. Thus, the condition (A3) takes the form of a Lipschitz condition in the case of $\xi(t) = Lt$, and this is the Lipschitz continuity condition found in [1]. For this reason, the more general form of the function ξ in (A3) causes the current results to expand the range of possible coefficients F and G in the Equation (1).*

To signal how the new condition is more effective and better in the sense of being more general and expanding the class of admissible F and G, we will recall a few well-known examples of the functions ξ meeting the conditions listed in (A3). They are, for instance,

$$\xi_1(t) = \begin{cases} t\log(t^{-1}), & 0 \le t \le \varepsilon, \\ \varepsilon\log(\varepsilon^{-1}) + \xi_1'(\varepsilon-)(t-\varepsilon), & t > \varepsilon, \end{cases}$$

$$\xi_2(t) = \begin{cases} t\log(t^{-1})\log\log(t^{-1}), & 0 \le t \le \varepsilon, \\ \varepsilon\log(\varepsilon^{-1})\log\log(\varepsilon^{-1}) + \xi_2'(\varepsilon-)(t-\varepsilon), & t > \varepsilon, \end{cases}$$

where $\varepsilon \in (0,1)$ is sufficiently small and $\xi_k'(\varepsilon-)$ $(k = 1,2)$ stands for left-sided derivative of ξ_k at ε.

Before we proceed to the proper analysis, let us remind that the compositions F with continuous X and G with continuous X in integrals in Equation (1) are the continuous mappings due to assumption (A1). Therefore, the integrals in (1) can be defined. The assumption (A4) in which Hukuhara's differences occur is indelible in general and is a consequence of the symmetric form of the Equation (1).

Below, we present the result indicating the boundedness of the approximation sequence $\{X^n\}$ which will be used to justify the existence of a solution to Equation (1).

Lemma 2. *Let assumptions (A0)–(A4) be satisfied. Then, there is a positive constant M such that for every $n \in \mathbb{N}$*

$$\sup_{t\in\tilde{J}} \rho_H(X^n(t), \{0\}) \le M,$$

where $\tilde{J} := [t_0 - \theta, t_0] \cup \tilde{I} = [t_0 - \theta, t_0 + \tilde{T}]$.

Proof. Firstly notice that $\sup_{t\in[t_0-\theta,t_0]} \rho_H(X^n(t), \{0\}) = \rho^*(\chi_0, \mathbf{0})$.

Let us denote $f_n(t) = \sup_{u\in[t_0-\theta,t]} \rho_H(X^n(t), \{0\})$ for $n \in \mathbb{N}$ and $t \in \tilde{I}$. Then one can write

$$f_n(t) \le \max\Big\{ \sup_{t\in[t_0-\theta,t_0]} \rho_H(X^n(t), \{0\}), \sup_{t\in[t_0,t]} \rho_H(X^n(t), \{0\}) \Big\}.$$

Now, we will deal with the second component of the maximum in the above inequality

$$\sup_{u\in[t_0,t]} \rho_H(X^n(u), \{0\}) = \sup_{u\in[t_0,t]} \rho_H\Big(\big[\chi_0(0) + \int_{t_0}^{u} G(s, X_s^{n-1}) ds\big] \ominus \int_{t_0}^{u} F(s, X_s^{n-1}) ds, \{0\} \Big).$$

Due to properties (P3), (P2) and (P6), we obtain

$$\begin{aligned}
\sup_{u\in[t_0,t]} \rho_H(X^n(u),\{0\}) &\leq \sup_{u\in[t_0,t]} \rho_H\Big(\chi_0(0)+\int_{t_0}^{u} G(s,X_s^{n-1})ds,\{0\}\Big) \\
&\quad + \sup_{u\in[t_0,t]} \rho_H\Big(\int_{t_0}^{u} F(s,X_s^{n-1})ds,\{0\}\Big) \\
&\leq \rho^*(\chi_0,\mathbf{0}) + \sup_{u\in[t_0,t]} \rho_H\Big(\int_{t_0}^{u} G(s,X_s^{n-1})ds,\{0\}\Big) \\
&\quad + \sup_{u\in[t_0,t]} \rho_H\Big(\int_{t_0}^{u} F(s,X_s^{n-1})ds,\{0\}\Big) \\
&\leq \rho^*(\chi_0,\mathbf{0}) + \int_{t_0}^{t} \rho_H\Big(G(s,X_s^{n-1}),\{0\}\Big)ds \\
&\quad + \int_{t_0}^{t} \rho_H\Big(F(s,X_s^{n-1}),\{0\}\Big)ds.
\end{aligned}$$

The use of a triangle inequality and assumptions (A2) and (A3) leads us to

$$\begin{aligned}
\sup_{u\in[t_0,t]} \rho_H(X^n(u),\{0\}) &\leq \rho^*(\chi_0,\mathbf{0}) + \int_{t_0}^{t} \rho_H\Big(G(s,X_s^{n-1}),G(s,\mathbf{0})\Big)ds \\
&\quad + \int_{t_0}^{t} \rho_H\Big(G(s,\mathbf{0}),\{0\})\Big)ds \\
&\quad + \int_{t_0}^{t} \rho_H\Big(F(s,X_s^{n-1}),F(s,\mathbf{0})\Big)ds \\
&\quad + \int_{t_0}^{t} \rho_H\Big(F(s,\mathbf{0}),\{0\})\Big)ds \\
&\leq \rho^*(\chi_0,\mathbf{0}) + 2C\tilde{T} + 2\int_{t_0}^{t} \xi\big(\rho^*(X_s^{n-1},\mathbf{0})\big)ds \\
&= \rho^*(\chi_0,\mathbf{0}) + 2C\tilde{T} + 2\int_{t_0}^{t} \xi\big(\sup_{r\in[-\theta,0]} \rho_H(X^{n-1}(s+r),\mathbf{0})\big)ds \\
&\leq \rho^*(\chi_0,\mathbf{0}) + 2C\tilde{T} + 2\int_{t_0}^{t} \xi\big(\sup_{u\in[t_0-\theta,s]} \rho_H(X^{n-1}(u),\mathbf{0})\big)ds.
\end{aligned}$$

Hence

$$f_n(t) \leq \rho^*(\chi_0,\mathbf{0}) + 2C\tilde{T} + 2\int_{t_0}^{t} \xi\big(f_{n-1}(s)\big)ds.$$

Since function ξ is concave, we have that $\xi(u) \leq au+b$ for $u \geq 0$, where a,b are positive constants. Thus

$$f_n(t) \leq E_1 + E_2\int_{t_0}^{t} f_{n-1}(s)ds,$$

where $E_1 = \rho^*(\chi_0,\mathbf{0}) + 2(C+b)\tilde{T}$ and $E_2 = 2a$. Therefore, we can also write

$$\max_{1\leq n\leq k} f_n(t) \leq E_1 + E_2\int_{t_0}^{t} \max_{1\leq n\leq k} f_{n-1}(s)ds$$

for $k \in \mathbb{N}$. Since $\max_{1\leq n\leq k} f_{n-1}(s) \leq \rho^*(\chi_0,\mathbf{0}) + \max_{1\leq n\leq k} f_n(s)$, we get

$$\max_{1\leq n\leq k} f_n(t) \leq E_1 + E_2\tilde{T}\rho^*(\chi_0,\mathbf{0}) + E_2\int_{t_0}^{t} \max_{1\leq n\leq k} f_n(s)ds.$$

By the Gronwall–Bellman inequality, we arrive at

$$\max_{1\le n\le k} f_n(t) \le \Big(E_1 + E_2\tilde{T}\rho^*(\chi_0,\mathbf{0})\Big)\Big) \exp\{E_2(t-t_0)\} \quad \text{for } t\in\tilde{I} \quad \text{and } k\in\mathbb{N}.$$

Thus

$$\sup_{t\in\tilde{J}} \rho_H(X^n(t),\{0\}) \le M,$$

where $M = \Big(E_1 + E_2\tilde{T}\rho^*(\chi_0,\mathbf{0})\Big)\Big) \exp\{E_2\tilde{T}\}$. □

Theorem 1. *Under assumptions (A0)–(A4), Equation (1) has a unique solution.*

Proof. Let $n,m\in\mathbb{N}$ and $t\in\tilde{I}$. Using (P3) and (P1)

$$\begin{aligned}
&\sup_{u\in[t_0-\theta,t]} \rho_H(X^m(u),X^n(u)) \\
&\quad= \sup_{u\in[t_0,t]} \rho_H(X^m(u),X^n(u)) \\
&\quad= \sup_{u\in[t_0,t]} \rho_H\Big(\Big[\chi_0(0)+\int_{t_0}^u G(s,X_s^{m-1})ds\Big] \ominus \int_{t_0}^u F(s,X_s^{m-1})ds, \\
&\qquad \Big[\chi_0(0)+\int_{t_0}^u G(s,X_s^{n-1})ds\Big] \ominus \int_{t_0}^u F(s,X_s^{n-1})ds\Big) \\
&\quad\le \sup_{u\in[t_0,t]} \rho_H\Big(\int_{t_0}^u G(s,X_s^{m-1})ds, \int_{t_0}^u G(s,X_s^{n-1})ds\Big) \\
&\qquad + \sup_{u\in[t_0,t]} \rho_H\Big(\int_{t_0}^u F(s,X_s^{m-1})ds, \int_{t_0}^u F(s,X_s^{n-1})ds\Big).
\end{aligned}$$

By property (P6), we obtain

$$\begin{aligned}
\sup_{u\in[t_0-\theta,t]} \rho_H(X^m(u),X^n(u)) &\le \int_{t_0}^t \rho_H\big(G(s,X_s^{m-1}),G(s,X_s^{n-1})\big)ds \\
&\quad + \int_{t_0}^t \rho_H\big(F(s,X_s^{m-1}),F(s,X_s^{n-1})\big)ds
\end{aligned}$$

and due to assumption (A2)

$$\sup_{u\in[t_0-\theta,t]} \rho_H(X^m(u),X^n(u)) \le 2\int_{t_0}^t \xi\big(\rho^*(X_s^{m-1},X_s^{n-1})\big)ds.$$

Hence,

$$\sup_{u\in[t_0-\theta,t]} \rho_H(X^m(u),X^n(u)) \le 2\int_{t_0}^t \xi\big(\sup_{u\in[t_0-\theta,s]} \rho_H(X^{m-1}(u),X^{n-1}(u))\big)ds.$$

By the integration of both sides, with $\kappa\in[t_0,t_0+\tilde{T}]$, and using Jensen's inequality, we arrive at

$$\begin{aligned}
&\int_{t_0}^{\kappa} \sup_{u\in[t_0-\theta,t]} \rho_H(X^m(u),X^n(u))dt \\
&\quad\le 2\int_{t_0}^{\kappa}\int_{t_0}^t \xi\big(\sup_{u\in[t_0-\theta,s]} \rho_H(X^{m-1}(u),X^{n-1}(u))\big)dsdt \\
&\quad\le 2(\kappa-t_0)\int_{t_0}^{\kappa} \xi\Big(\frac{1}{t-t_0}\int_{t_0}^t \sup_{u\in[t_0-\theta,s]} \rho_H(X^{m-1}(u),X^{n-1}(u))\Big)ds.
\end{aligned}$$

Let us denote

$$h_{m,n}(t) = \frac{1}{t-t_0}\int_{t_0}^{t} \sup_{u\in[t_0-\theta,s]} \rho_H(X^m(u), X^n(u)))ds.$$

Then

$$h_{m,n}(\kappa) \le 2\int_{t_0}^{\kappa} \xi(h_{m-1,n-1}(t))dt.$$

Owing to Lemma 2, we can state that $\sup_{t\in\tilde{I}}\sup_{m,n\in\mathbb{N}} h_{m,n}(t)$ is finite. Denoting

$$h(t) = \limsup_{m,n\to\infty} h_{m,n}(t)$$

and applying Fatou's lemma we obtain

$$h(\kappa) \le 2\int_{t_0}^{\kappa} \xi(h(t))dt.$$

The Bihari–LaSalle inequality (Lemma 1) allows us to conclude that $h(\kappa) = 0$ for every $\kappa \in \tilde{I}$ which means that

$$\limsup_{m,n\to\infty} \frac{1}{t-t_0}\int_{t_0}^{t} \sup_{u\in[t_0-\theta,s]} \rho_H(X^{m-1}(u), X^{n-1}(u)))ds = 0 \quad \text{for every} \quad t\in\tilde{I}.$$

From this, it is easy to conclude that

$$\lim_{m,n\to\infty} \sup_{u\in[t_0-\theta,t_0+\tilde{T}]} \rho_H(X^m(u), X^n(u)) = 0$$

and this means that the sequence $\{X^n\}$ of ρ_H-continuous mappings from a complete space $C([t_0-\theta, t_0+\tilde{T}], \mathcal{P}_{cc}(\mathbb{R}^d))$ with the supremum metric converges to a certain ρ_H-continuous element X of this space. Of course $X(t) = \chi_0(t-t_0)$ for $t \in [t_0-\theta, t_0]$, because the same equality occurs for every X^n, $n \in \mathbb{N}\cup\{0\}$.

In the next stage of the proof, we will show that X is a solution to Equation (1). For this purpose, it is enough to show that

$$\rho_H\Big(X(t) + \int_{t_0}^{t} F(s,X_s)ds, \chi_0(0) + \int_{t_0}^{t} G(s,X_s)ds\Big) = 0 \quad \text{for every} \quad t\in\tilde{I}.$$

Therefore, we present further estimates, where $U(t)$ denotes the left-hand side of the above equality and $t \in \tilde{I}$

$$\begin{aligned} U(t) &\le \rho_H\Big(X(t) + \int_{t_0}^{t} F(s,X_s)ds, X^n(t) + \int_{t_0}^{t} F(s,X_s^{n-1})ds\Big) \\ &\quad + \rho_H\Big(\int_{t_0}^{t} G(s,X_s)ds, \int_{t_0}^{t} G(s,X_s^{n-1})ds\Big) \\ &\le \rho_H(X(t), X^n(t)) + \int_{t_0}^{t} \rho_H\big(F(s,X_s), F(s,X_s^{n-1})\big)ds \\ &\quad + \int_{t_0}^{t} \rho_H\big(G(s,X_s), G(s,X_s^{n-1})\big)ds \\ &\le \rho_H(X(t), X^n(t)) + 2\int_{t_0}^{t} \xi\big(\rho^*(X_s, X_s^{n-1})\big)ds. \end{aligned}$$

Since $\xi(0) = 0$, ξ is continuous, and the sequence $\{\rho^*(X_s, X_s^{n-1})\}$ is bounded, by the Lebesgue dominated convergence theorem, we obtain that $\lim_{n\to\infty}\int_{t_0}^{t} \xi\big(\rho^*(X_s, X_s^{n-1})\big)ds = 0$. We previously justified that $\lim_{n\to\infty} \rho_H(X(t), X^n(t)) = 0$. Hence, indeed $U(t) = 0$ for $t \in \tilde{I}$.

If Z is the second solution to Equation (1) on the interval $\tilde{J} = [t_0 - \theta, t_0 + \tilde{T}]$, then notice that in accordance with the definition of the solution, it would have to be $Z(t) = \chi_0(t - t_0)$ for $t \in [t_0 - \theta, t_0]$ and this means $Z(t) = X(t)$ for $t \in [t_0 - \theta, t_0]$. Our next goal is to show that Z coincides with X also on the interval $[t_0, t_0 + \tilde{T}]$. To this end, let us observe that

$$\begin{aligned}
\sup_{u\in[t_0,t]} \rho_H\big(Z(u), X(u)\big) &= \sup_{u\in[t_0,t]} \rho_H\Big(\Big[\chi_0(0) + \int_{t_0}^{u} G(s, Z_s)ds\Big] \ominus \int_{t_0}^{u} F(s, Z_s)ds, \\
&\quad \Big[\chi_0(0) + \int_{t_0}^{u} G(s, X_s)ds\Big] \ominus \int_{t_0}^{u} F(s, X_s)ds\Big) \\
&\leq \sup_{u\in[t_0,t]} \rho_H\Big(\int_{t_0}^{u} G(s, Z_s)ds, \int_{t_0}^{u} G(s, X_s)ds\Big) \\
&\quad + \sup_{u\in[t_0,t]} \rho_H\Big(\int_{t_0}^{u} F(s, Z_s)ds, \int_{t_0}^{u} F(s, X_s)ds\Big) \\
&\leq 2\int_{t_0}^{t} \xi(\rho^*(Z_s, X_s))ds \\
&\leq 2\int_{t_0}^{t} \xi(\sup_{u\in[s-\theta,s]} \rho_H(Z(u), X(u)))ds \\
&= 2\int_{t_0}^{t} \xi(\sup_{u\in[t_0,s]} \rho_H(Z(u), X(u)))ds.
\end{aligned}$$

Invoking the Bihari–LaSalle inequality, we receive

$$\sup_{u\in[t_0,t]} \rho_H\big(Z(u), X(u)\big) = 0 \quad \text{for every} \quad t \in \tilde{I}$$

from which the equality $Z(t) = X(t)$ also follows for $t \in [t_0, t_0 + \tilde{T}]$. Hence X is the unique solution on the interval $\tilde{J} = [t_0 - \theta, t_0] \cup [t_0, t_0 + \tilde{T}]$. □

Having the result of the existence of a solution for Equation (1) in the next stage, one can think about some desired properties of the solutions. The first one we will analyze is the slight sensitivity of the solution to the equation with a different initial history $\chi_0^{\varepsilon} \in C_\theta$, which is only slightly different from the original history χ_0. Therefore, we will now consider two equations: Equation (1) and the following equation,

$$Z(t) + \int_{t_0}^{t} F(s, Z_s)ds = \chi_0^{\varepsilon}(0) + \int_{t_0}^{t} G(s, Z_s)ds \quad \text{for} \quad t \in [t_0, t_0 + T] \tag{6}$$

with initial condition

$$Z_{t_0} = \chi_0^{\varepsilon}.$$

The following statement confirms the occurrence of the property just discussed.

Theorem 2. *Suppose that the assumptions (A0), (A1) and (A3) are satisfied. Let X be a solution to Equation (1) on interval $\tilde{J} = [t_0 - \theta, t_0 + \tilde{T}]$, where $\tilde{T} \in (0, T]$. Let Z be a solution to Equation (6) also on interval $\tilde{J}$. Then*

$$\sup_{t\in\tilde{J}} \rho_H\big(Z(t), X(t)\big) \leq V^{-1}\Big(V\big(2\rho^*(\chi_0^{\varepsilon}, \chi_0)\big) + 2\tilde{T}\Big),$$

where V and V^{-1} are connected with ξ from (A3) in a way described in Lemma 2.

Proof. At the beginning, let us note that

$$\sup_{u\in[t_0-\theta,t_0]} \rho_H(Z(u), X(u)) = \rho^*(\chi_0^{\varepsilon}, \chi_0).$$

For t in the remaining part of the interval $\tilde{J}$, i.e., for $t \in [t_0, t_0 + \tilde{T}]$ we have

$$\begin{aligned}\sup_{u\in[t_0,t]} \rho_H\big(Z(u), X(u)\big) &= \sup_{u\in[t_0,t]} \rho_H\Big(\big[\chi_0^\varepsilon(0) + \int_{t_0}^u F(s, Z_s)ds\big] \ominus \int_{t_0}^u G(s, Z_s)ds,\\ &\quad \big[\chi_0(0) + \int_{t_0}^u F(s, X_s)ds\big] \ominus \int_{t_0}^u G(s, X_s)ds\Big)\\ &\leq \rho_H\big(\chi_0^\varepsilon(0), \chi_0(0)\big) + \int_{t_0}^t \rho_H\big(F(s, Z_s), F(s, X_s)\big)ds\\ &\quad + \int_{t_0}^t \rho_H\big(G(s, Z_s), G(s, X_s)\big)ds.\end{aligned}$$

Due to condition (A3), we obtain

$$\sup_{u\in[t_0,t]} \rho_H\big(Z(u), X(u)\big) \leq \rho_H\big(\chi_0^\varepsilon(0), \chi_0(0)\big) + 2\int_{t_0}^t \xi\big(\rho^*(Z_s, X_s)\big)ds.$$

Hence

$$\begin{aligned}\sup_{u\in[t_0,t]} \rho_H\big(Z(u), X(u)\big) &\leq \rho^*(\chi_0^\varepsilon, \chi_0) + 2\int_{t_0}^t \xi\big(\rho^*(Z_s, X_s)\big)ds\\ &\leq \rho^*(\chi_0^\varepsilon, \chi_0) + 2\int_{t_0}^t \xi\big(\sup_{u\in[t_0-\theta,s]} \rho_H(Z(u), X(u))\big)ds.\end{aligned}$$

Therefore, in the considered case of $t \in [t_0, t_0 + \tilde{T}]$

$$\begin{aligned}\sup_{u\in[t_0-\theta,t]} \rho_H\big(Z(t), X(t)\big) &\leq \sup_{u\in[t_0-\theta,t_0]} \rho_H\big(Z(t), X(t)\big) + \sup_{u\in[t_0,t]} \rho_H\big(Z(t), X(t)\big)\\ &\leq 2\rho^*(\chi_0^\varepsilon, \chi_0) + 2\int_{t_0}^t \xi\big(\sup_{u\in[t_0-\theta,s]} \rho_H(Z(u), X(u))\big)ds.\end{aligned}$$

Now, by the Bihari–LaSalle inequality, we can infer that for every $t \in [t_0, t_0 + \tilde{T}]$

$$\sup_{u\in[t_0-\theta,t]} \rho_H\big(Z(t), X(t)\big) \leq V^{-1}\Big(V\big(2\rho^*(\chi_0^\varepsilon, \chi_0)\big) + 2(t - t_0)\Big).$$

From this inequality, the assertion follows immediately. □

With the above result, it is easy to determine the property of continuous dependence of the solution to Equation (1) on the initial history χ_0. Indeed, if $\rho^*(\chi_0^\varepsilon, \chi_0)$ converges to zero, then $V(2\rho^*(\chi_0^\varepsilon, \chi_0))$ goes to $(-\infty)$. However, $V^{-1}(t) \to 0$ as $t \to -\infty$, and this yields continuous dependence.

In addition to the low sensitivity of the solution to changes in the initial history, the second similar desirable property would be a small change of the solution in the case of small changes in the coefficients of the equation. This feature results directly from the fact written below.

Let us consider Equation (1) and the equation with slightly different coefficients $\tilde{F}, \tilde{G}$, i.e.,

$$Z(t) + \int_{t_0}^t \tilde{F}(s, Z_s)ds = \chi_0(0) + \int_{t_0}^t \tilde{G}(s, Z_s)ds \quad \text{for} \quad t \in [t_0, t_0 + T] \tag{7}$$

with initial condition

$$Z_{t_0} = \chi_0.$$

The fact that the coefficients $\tilde{F}$ and $\tilde{G}$ are slightly different from F and G is understood in the sense of satisfying the condition

$$\int_{t_0}^{t_0+\tilde{T}} [\rho_H(\tilde{F}(t, X_t), F(t, X_t)) + \rho_H(\tilde{G}(t, X_t), G(t, X_t))]dt \le \varepsilon, \tag{8}$$

where $\varepsilon > 0$ is small and X is the solution to Equation (1).

Theorem 3. *Suppose that the assumptions (A0), (A1) and (A3) are satisfied for the data χ_0, F, G of Equation (1) and for the data $\chi_0, \tilde{F}, \tilde{G}$ of Equation (7). Let X be a solution to Equation (1) on interval $\tilde{J} = [t_0 - \theta, t_0 + \tilde{T}]$, where $\tilde{T} \in (0, T]$. Let Z be a solution to Equation (7) also on interval $\tilde{J}$. Assume that (8) is satisfied. Then*

$$\sup_{t\in\tilde{J}} \rho_H(Z(t), X(t)) \le V^{-1}\Big(V(A) + 2\tilde{T}\Big),$$

where V and V^{-1} are connected with ξ from (A3) in a way described in Lemma 2 and $A = \int_{t_0}^{t_0+\tilde{T}} [\rho_H(\tilde{F}(t, X_t), F(t, X_t)) + \rho_H(\tilde{G}(t, X_t), G(t, X_t))]dt$.

Proof. Notice first that for $t \in [t_0 - \theta, t_0]$ we have $Z(t) = \chi_0(t - t_0) = X(t)$. Hence

$$\sup_{t\in[t_0-\theta,t_0]} \rho_H(Z(t), X(t)) = 0.$$

Further, we analyze what we will get for $t \in [t_0, t_0 + \tilde{T}]$ and so

$$\begin{aligned}
&\sup_{u\in[t_0,t]} \rho_H(Z(u), X(u)) \\
&\quad= \sup_{u\in[t_0,t]} \rho_H\Big([\chi_0(0) + \int_{t_0}^{u} \tilde{F}(s, Z_s)ds] \ominus \int_{t_0}^{u} \tilde{G}(s, Z_s)ds, \\
&\qquad [\chi_0(0) + \int_{t_0}^{u} F(s, X_s)ds] \ominus \int_{t_0}^{u} G(s, X_s)ds\Big) \\
&\quad\le \int_{t_0}^{t} \Big[\rho_H(G(s, X_s), \tilde{G}(s, Z_s)) + \rho_H(F(s, X_s), \tilde{F}(s, Z_s))\Big]ds \\
&\quad\le \int_{t_0}^{t} \Big[\rho_H(G(s, X_s), \tilde{G}(s, X_s)) + \rho_H(\tilde{G}(s, X_s), \tilde{G}(s, Z_s))\Big]ds \\
&\qquad + \int_{t_0}^{t} \Big[\rho_H(F(s, X_s), \tilde{F}(s, X_s)) + \rho_H(\tilde{F}(s, X_s), \tilde{F}(s, Z_s))\Big]ds \\
&\quad\le \int_{t_0}^{t} \Big[\rho_H(G(s, X_s), \tilde{G}(s, X_s)) + \rho_H(F(s, X_s), \tilde{F}(s, X_s))\Big]ds \\
&\qquad + \int_{t_0}^{t} \Big[\rho_H(\tilde{G}(s, X_s), \tilde{G}(s, Z_s)) + \rho_H(\tilde{F}(s, X_s), \tilde{F}(s, Z_s))\Big]ds \\
&\quad\le A + \int_{t_0}^{t} \Big[\rho_H(\tilde{G}(s, X_s), \tilde{G}(s, Z_s)) + \rho_H(\tilde{F}(s, X_s), \tilde{F}(s, Z_s))\Big]ds
\end{aligned}$$

Invoking assumption (A3), we obtain

$$\sup_{u\in[t_0,t]} \rho_H(Z(u), X(u)) \quad\le\quad A + 2\int_{t_0}^{t} \xi\Big(\sup_{u\in[t_0,s]} \rho_H(Z(u), X(u))\Big)ds.$$

Applying the Bihari–LaSalle inequality, we have

$$\sup_{u\in[t_0,t]} \rho_H(Z(t), X(t)) \le V^{-1}(V(A) + 2(t - t_0)) \ \text{ for } \ t \in [t_0, t_0 + T],$$

from which the thesis follows. □

This assertion allows us to attribute to the solution of Equation (1) the property of a continuous dependence on the coefficients of the equation.

3. Conclusions

In this paper, the subject of research is symmetric functional set-valued integral equation

$$X(t) + \int_{t_0}^{t} F(s, X_s)ds = \chi_0(0) + \int_{t_0}^{t} G(s, X_s)ds \quad \text{for} \quad t \in [t_0, t_0 + T]$$

with initial condition

$$X_{t_0} = \chi_0,$$

where χ_0 is an initial set-valued history, and F, G are the set-valued coefficients. Solutions of such symmetric equations have the feature of unnecessary monotonicity of the diameter of the solution value in contrast to asymmetric equations, that is, with one integral on only one side of the equation. The research presented in this paper is theoretical and concerns the properties of such symmetric equations. Among the fundamental properties, we confirm the existence of a unique solution, which is the basis of applicability in practice to describe real processes with states described in the form of sets. The assumptions under which we conduct our study are more general than the conditions used in our previous paper [1]. In particular, we use a more general condition of continuity of equation coefficients compared to the Lipschitz continuity used in [1]. With this more general condition, we also justify the stability of the solution in relation to small changes in the initial history and small changes in the coefficients of the equation. All these results are obtained by applying the Bihari–LaSalle inequality. One of the conditions we used assumes the existence of some Hukuhara differences, which at first glance may be puzzling. However, this is an intrinsic feature of equations of the symmetric type and the fact that the space of sets does not have a linear structure.

We hope that the theoretical results established in this paper will be used in modeling real-world processes. It seems particularly interesting to apply our equations to mathematical modeling in medical issues related to edge detection and determining cancer regions in images as was done in [21]. In addition to future research directions involving applications in practical issues, it is worth mentioning that the current article certainly does not completely exhaust future theoretical research. One can think about weakening the Lipschitz-type condition again or consider coefficients that will not necessarily be continuous. Conditions ensuring the periodicity of the solution would also be of interest.

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