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Cubic-Quartic Optical Solitons in Fiber Bragg Gratings with Dispersive Reflectivity Having Parabolic Law of Nonlinear Refractive Index by Lie Symmetry

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Abstract: This work recovers cubic-quartic optical solitons with dispersive reflectivity in fiber Bragg gratings and parabolic law of nonlinearity. The Lie symmetry analysis first reduces the governing partial differential equations to the corresponding ordinary differential equations which are subsequently integrated. This integration is conducted using two approaches which are the modified Kudryashov's approach as well as the generalized Arnous' scheme. These collectively yielded a full spectrum of cubic-quartic optical solitons that have been proposed to control the depletion of the much-needed chromatic dispersion. They range from bright, dark, singular to combo solitons. These solitons are considered with dispersive reflectivity, which maintains the necessary balance between chromatic dispersion and nonlinear refractive index structure for an uninterrupted transmission of solitons along intercontinental distances. Their respective surface and contour plots are also exhibited. A few closing words are included with some prospective future avenues of research to extend this topic further.

Keywords: solitons; Bragg gratings; Lie symmetry; Kudryashov; Arnous

1. Introduction

Optical soliton transmission through fibers and other forms of waveguides is of paramount importance in the modern day telecommunication industry. This technological marvel has become a part of the daily lives of the global population ever since this concept was introduced into the telecommunications industry about half a century ago. Today, there are several models that describe these dynamics in polarization—preserving fibers, dispersion-flattened fibers, optical metamaterials, optical couplers and others [1–6]. One of the innovative engineering marvels that have been proposed to control the depletion of the much-needed chromatic dispersion (CD) is the introduction of gratings structure along the walls of the core [7–11]. This would lead to dispersive reflectivity which maintains the necessary balance between CD and nonlinear refractive index structure for an uninterrupted transmission of solitons along intercontinental distances. Therefore, solitons have been addressed in fiber Bragg gratings [12–17], such as new nonstationary soliton-like solutions are reported for Bragg-resonant wave propagation in a periodic Kerr medium [12]. In addition, slow Bragg solitons have been studied in [13]. Additionally, such solitons come from gap solitons [12–14]. Bragg-grating solitons in a cubic-quartic medium are also considered in [15]. Gap solitons have been retrieved in Bragg gratings with dispersive reflectivity [16]. Dynamics and collisions of moving solitons are studied in Bragg gratings with dispersive reflectivity [17].

The governing model is the nonlinear Schrödinger's equation that is studied with the cubic-quartic form of nonlinear refractive index AKA parabolic law of nonlinearity. The integration of this model is the focus of the paper to retrieve soliton solutions. The preliminary results stem from the reduction of the governing partial differential equation (PDE) to an ordinary differential equation (ODE) by the usage of Lie symmetry analysis. Many methods for obtaining soliton solutions have been proposed [18–31] such as the Jacobi elliptic function method, the modified extended tanh-function method, the inverse scattering approach, Wronskian formulation, the improved F-expansion method, Lie group analysis, power series solution, and the (G/G)-expansion method, but this ODE is subsequently handled by the implementation of two newly introduced algorithms which are due to Kudryashov [32–34] and Arnous [35,36]. Exact solutions of nonlinear differential equations are considered in [32]. Soliton wave solutions, solitary wave solutions, elliptic wave solutions, and periodic wave solutions are determined in [33]. New traveling wave solutions for nonlinear directional couplers with optical metamaterials are obtained in [34]. Pure-cubic optical solitons in a polarization-preserving fiber with Kerr law nonlinearity are reported in [35]. Optical solitons for the cubic quartic Bragg-gratings having an anti-cubic nonlinear form are extracted in [36]. Thus, these integration schemes yielded a complete spectrum of optical solitons that range from bright, dark, singular to combo solitons. The details of the derivation and their respective surface and contour plots are exhibited. To conclude, a few closing words are included with some prospective future avenues of research to extend this topic further.

Governing Model

Equations (1) and (2) represent the model for fiber Bragg gratings with dispersive reflectivity that originates from the CD. The fields $q(x,t)$ and $r(x,t)$ represent the wave profile along the two components of the dual-core fibers. The independent variables are x and t that represent spatial and temporal co-ordinates, respectively. Then, β_j $(j=1,2)$ stem from the detuning parameters, while c_j, η_j and ζ_j arise from cross-phase modulation (XPM). α_j come from inter-modal dispersion, while b_j and ξ_j arise from self-phase modulation (SPM). a_j describe dispersive reflectivity, while $q(x,t)$ depicts the forward wave profile. The first terms come from the temporal evolution in which $i=\sqrt{-1}$. $r(x,t)$ purports the backward wave profile, while t depicts time in the moving frame, and x signifies the propagation variable [7–11]:

$$iq_t + a_1 r_{xx} + \left(b_1|q|^2 + c_1|r|^2\right)q + \left(\xi_1|q|^4 + \eta_1|q|^2|r|^2 + \zeta_1|r|^4\right)q + i\alpha_1 q_x + \beta_1 r = 0, \quad (1)$$

$$ir_t + a_2 q_{xx} + \left(b_2|r|^2 + c_2|q|^2\right)r + \left(\xi_2|r|^4 + \eta_2|q|^2|r|^2 + \zeta_2|r|^4\right)r + i\alpha_2 r_x + \beta_2 q = 0. \quad (2)$$

In fiber Bragg gratings, when CD is replaced by the collective count of third-order dispersion (3OD) and fourth-order dispersions (4OD), the model modifies to [35–43]

$$iq_t + ia_1 r_{xxx} + b_1 r_{xxxx} + \left(c_1|q|^2 + d_1|r|^2\right)q + \left(\xi_1|q|^4 + \eta_1|q|^2|r|^2 + \zeta_1|r|^4\right)q + i\alpha_1 q_x + \beta_1 r = 0, \tag{3}$$

$$ir_t + ia_2 q_{xxx} + b_2 q_{xxxx} + \left(c_2|r|^2 + d_2|q|^2\right)r + \left(\xi_2|r|^4 + \eta_2|q|^2|r|^2 + \zeta_2|r|^4\right)r + i\alpha_2 r_x + \beta_2 q = 0. \tag{4}$$

Here, c_j $(j = 1,2)$ are associated with SPM, while a_j is related to 3OD. d_j stands for XPM, while b_j comes from 4OD.

2. Lie Symmetry Analysis

Let us characterize $q(x,t)$ and $r(x,t)$ as follows:

$$q(x,t) = u_1(x,t)e^{i(-kx+wt+\theta)}, \tag{5}$$

$$r(x,t) = u_2(x,t)e^{i(-kx+wt+\theta)}, \tag{6}$$

where $u_1(x,t)$ and $u_2(x,t)$ are the soliton amplitudes, k is the soliton frequency, w is the soliton wave number and θ is the phase constant. Putting (5) and (6) into (3) and (4) exposes the governing equations

$$\begin{aligned}(k\alpha_l - w)u_l + (b_l k^4 - a_l k^3 + \beta_l)u_{\tilde{l}} + (3a_l k - 6b_l k^2)\left(\frac{\partial^2 u_{\tilde{l}}}{\partial x^2}\right)\\ + b_l\left(\frac{\partial^4 u_{\tilde{l}}}{\partial x^4}\right) + (c_l u_l^2 + d_l u_{\tilde{l}}^2)u_l + (\xi_l u_l^4 + \eta_l u_l^2 u_{\tilde{l}}^2 + \zeta_l u_{\tilde{l}}^4)u_l = 0,\end{aligned} \tag{7}$$

$$\left(\frac{\partial u_l}{\partial t}\right) - (3a_l k^2 - 4b_l k^3)\left(\frac{\partial u_{\tilde{l}}}{\partial x}\right) + (a_l - 4b_l k)\left(\frac{\partial^3 u_{\tilde{l}}}{\partial x^3}\right) + \alpha_l\left(\frac{\partial u_l}{\partial x}\right) = 0, \tag{8}$$

where $l = 1,2$ and $\tilde{l} = 3 - l$. The Lie group symmetries for the system (7) and (8) are enumerated as [44–47]

$$\begin{aligned}x^* &= x + \epsilon\,\xi(x,t,u_1,u_2) + O(\epsilon^2),\\ t^* &= t + \epsilon\,\tau(x,t,u_1,u_2) + O(\epsilon^2),\\ u_1^* &= u_1 + \epsilon\,\phi_1(x,t,u_1,u_2) + O(\epsilon^2),\\ u_2^* &= u_2 + \epsilon\,\phi_2(x,t,u_1,u_2) + O(\epsilon^2),\end{aligned} \tag{9}$$

where ϕ_2, ϕ_1, τ and ξ are infinitesimals. The vector field for (7) and (8) is therefore extracted as

$$V = \xi(x,t,u_1,u_2)\frac{\partial}{\partial x} + \tau(x,t,u_1,u_2)\frac{\partial}{\partial t} + \phi_1(x,t,u_1,u_2)\frac{\partial}{\partial u_1} + \phi_2(x,t,u_1,u_2)\frac{\partial}{\partial u_2}. \tag{10}$$

The fourth prolongation formula for (7) and (8) is also presented as below [44,46]

$$\begin{aligned}pr^{(4)}V = V + \phi_1^t\frac{\partial}{\partial u_{1t}} + \phi_2^t\frac{\partial}{\partial u_{2t}} + \phi_1^x\frac{\partial}{\partial u_{1x}} + \phi_2^x\frac{\partial}{\partial u_{2x}} + \phi_1^{xx}\frac{\partial}{\partial u_{1xx}} + \phi_2^{xx}\frac{\partial}{\partial u_{2xx}}\\ + \phi_1^{xxx}\frac{\partial}{\partial u_{1xxx}} + \phi_2^{xxx}\frac{\partial}{\partial u_{2xxx}} + \phi_1^{xxxx}\frac{\partial}{\partial u_{1xxxx}} + \phi_2^{xxxx}\frac{\partial}{\partial u_{2xxxx}},\end{aligned} \tag{11}$$

where ϕ_2^{xxxx}, ϕ_1^{xxxx}, ϕ_2^{xxx}, ϕ_1^{xxx}, ϕ_2^{xx}, ϕ_1^{xx}, ϕ_2^{x}, ϕ_1^{x}, ϕ_2^{t} and ϕ_1^{t} denote the extended infinitesimals. With the usage of $pr^{(4)}V(\Delta) = 0$, whenever $\Delta = 0$ in (7) and (8), the invariance condition enables us the fundamental equations

$$\phi_l\Big(\alpha_l k - w + 3c_l u_l^2 + d_l u_{\bar{l}}^2 + 5\xi_l u_l^4 + 3\eta_l u_l^2 u_{\bar{l}}^2 + \zeta_l u_{\bar{l}}^4\Big)$$

$$+\phi_{\bar{l}}\Big(b_l k^4 - a_l k^3 + \beta_l + 2d_l u_l u_{\bar{l}} + 2\eta_l u_l^3 u_{\bar{l}} + 4\zeta_l u_l u_{\bar{l}}^3\Big) + (3a_l k - 6b_l k^2)\phi_{\bar{l}}^{xx} + b_l\phi_{\bar{l}}^{xxxx} = 0, \tag{12}$$

$$\phi_l^t + \alpha_l\phi_l^x - (3a_l k^2 - 4b_l k^3)\phi_{\bar{l}}^x + (a_l - 4b_l k)\phi_{\bar{l}}^{xxx} = 0. \tag{13}$$

From (12) and (13), the infinitesimals are enlisted as

$$\xi = C_1, \qquad \tau = C_2, \qquad \phi_1 = 0, \qquad \phi_2 = 0, \tag{14}$$

where C_1 and C_2 are constants. As a result, the infinitesimal generators for (7) and (8) are structured below

$$V_1 = \tfrac{\partial}{\partial t}, \qquad V_2 = \tfrac{\partial}{\partial x}. \tag{15}$$

The vector field

$$V_1 + \nu V_2 = \frac{\partial}{\partial t} + \nu\frac{\partial}{\partial x}, \tag{16}$$

where ν is non-zero real number, leaves us with the similarity variables:

$$\sigma = x - \nu t,$$
$$q(x,t) = P_1(\sigma)e^{i(-kx+wt+\theta)},$$
$$r(x,t) = P_2(\sigma)e^{i(-kx+wt+\theta)}, \tag{17}$$

where P_1 and P_2 are new dependent variables. Inserting (17) into (3) and (4) enables us the strategic equations

$$b_l P_{\bar{l}}'''' + (3a_l k - 6b_l k^2)P_{\bar{l}}'' + (c_l P_l^2 + d_l P_{\bar{l}}^2)P_l + (\xi_l P_l^4 + \eta_l P_l^2 P_{\bar{l}}^2 + \zeta_l P_{\bar{l}}^4)P_l$$

$$+ (k\alpha_l - w)P_l + (b_l k^4 - a_l k^3 + \beta_l)P_{\bar{l}} = 0, \tag{18}$$

$$(a_l - 4b_l k)P_{\bar{l}}''' + (\alpha_l - \nu)P_l' - (3a_l k^2 - 4b_l k^3)P_{\bar{l}}' = 0. \tag{19}$$

With the criterion,

$$P_2 = \omega P_1, \tag{20}$$

Equations (18) and (19) evolve as

$$\omega b_1 P_1'''' + \omega(3a_1 k - 6b_1 k^2)P_1'' + (\xi_1 + \omega^2\eta_1 + \omega^4\zeta_1)P_1^5$$

$$+ (c_1 + \omega^2 d_1)P_1^3 + (k\alpha_1 - w + \omega(b_1 k^4 - a_1 k^3 + \beta_1))P_1 = 0, \tag{21}$$

$$\omega(a_1 - 4b_1 k)P_1''' + (\alpha_1 - \nu - \omega(3a_1 k^2 - 4b_1 k^3))P_1' = 0, \tag{22}$$

with the usage of the constraints

$$\omega b_1 = b_2,$$
$$c_1 + \omega^2 d_1 = \omega(d_2 + \omega^2 c_2),$$
$$(a_1 - 2kb_1)\omega = a_2 - 2b_2 k,$$
$$\omega^4\zeta_1 + \omega^2\eta_1 + \xi_1 = \omega(\omega^4\xi_2 + \omega^2\eta_2 + \zeta_2),$$
$$b_1\omega k^4 - a_1\omega k^3 + \alpha_1 k + \beta_1\omega - w = b_2 k^4 - a_2 k^3 + \alpha_2\omega k + \beta_2 - \omega w. \tag{23}$$

Equation (22) provides us with the velocity

$$\nu = \alpha_1 - \omega k^2(3a_1 - 4b_1 k), \tag{24}$$

and the certain restrictions

$$\omega(4\omega b_1 k^3 - 3\omega a_1 k^2 + \alpha_1) = 4b_2 k^3 - 3a_2 k^2 + \omega\alpha_2, \tag{25}$$

$$\omega(a_1 - 4b_1 k) = a_2 - 4b_2 k = 0, \tag{26}$$

while Equation (21) comes out as

$$\omega P_1'''' + A_1 P_1'' + A_2 P_1 + A_3 P_1^3 + A_4 P_1^5 = 0, \tag{27}$$

where

$$A_1 = \frac{\omega(3a_1 k - 6b_1 k^2)}{b_1}, A_2 = \frac{(k\alpha_1 - w + \omega(b_1 k^4 - a_1 k^3 + \beta_1))}{b_1},$$
$$A_3 = \frac{(c_1 + \omega^2 d_1)}{b_1}, A_4 = \frac{(\xi_1 + \omega^2 \eta_1 + \omega^4 \zeta_1)}{b_1}. \tag{28}$$

3. Modified Kudryashov's Method

Equation (27) holds the solution form [32–34]

$$P_1(\sigma) = B_0 + B_1 R(\sigma), \tag{29}$$

with the aid of the ancillary equation

$$R'(\sigma) = (R^2(\sigma) - R(\sigma))\ln(a),\ B_1 \neq 0, \tag{30}$$

and the analytical solution

$$R(\sigma) = \frac{1}{1 + d a^{\sigma}}, \tag{31}$$

where B_0 and B_1 are constants. Plugging (29) together with (30) into (27) enables us with the simplest equations

$$\begin{gathered}
A_2 B_0 + A_3 B_0^3 + A_4 B_0^5, = 0,\\
\omega\, B_1 \ln(a)^4 + A_1 B_1 \ln(a)^2 + A_2 B_1 + 3\, A_3 B_0^2 B_1 + 5\, A_4 B_0^4 B_1 = 0,\\
-15\,\omega\, B_1 \ln(a)^4 - 3\, A_1 B_1 \ln(a)^2 + 3\, A_3 B_0 B_1^2 + 10\, A_4 B_0^3 B_1^2 = 0,\\
50\,\omega\, B_1 \ln(a)^4 + 2\, A_1 B_1 \ln(a)^2 + A_3 B_1^3 + 10\, A_4 B_0^2 B_1^3 = 0,\\
-60\,\omega\, B_1 \ln(a)^4 + 5\, A_4 B_0 B_1^4 = 0,\\
24\,\omega\, B_1 \ln(a)^4 + A_4 B_1^5 = 0.
\end{gathered} \tag{32}$$

Thus, Equation (32) satisfies the results:

$$B_0 = \pm\frac{1}{2}\sqrt{\frac{3A_1\ln(a)^2 - 10A_2}{A_3}}, B_1 = \mp\sqrt{\frac{3A_1\ln(a)^2 - 10A_2}{A_3}}, \omega = \frac{A_1\ln(a)^2 - 2\,A_2}{2\ln(a)^4},$$
$$A_4 = \frac{12(2A_2 - A_1\ln(a)^2)A_3^2}{100A_2^2 - 60A_2A_1\ln(a)^2 + 9A_1^2\ln(a)^4}, A_3\left(3A_1\ln(a)^2 - 10A_2\right) > 0. \tag{33}$$

As a result, the analytical solutions read as

$$q(x,t) = \pm\left\{\sqrt{\frac{3A_1\ln(a)^2 - 10A_2}{A_3}}\left(\frac{(1 - d a^{x-\nu t})}{2(1 + d a^{x-\nu t})}\right)\right\}e^{i(-kx + wt + \theta)}, \tag{34}$$

$$r(x,t) = \pm\left\{\sqrt{\frac{3A_1 \ln(a)^2 - 10A_2}{A_3}}\left(\frac{\left(1 - da^{x-\nu t}\right)\left(A_1 \ln(a)^2 - 2\,A_2\right)}{4\left(1 + da^{x-\nu t}\right)\ln(a)^4}\right)\right\}\mathrm{e}^{i\,(-kx+wt+\theta)}. \tag{35}$$

Setting $d = \pm 1$ and $a = e$ paves the way to the dark solitons

$$q(x,t) = \pm\left\{\sqrt{\frac{3A_1 - 10A_2}{A_3}}\left(\frac{1}{2}\tanh\left(\frac{x - \nu t}{2}\right)\right)\right\}\mathrm{e}^{i\,(-kx+wt+\theta)}, \tag{36}$$

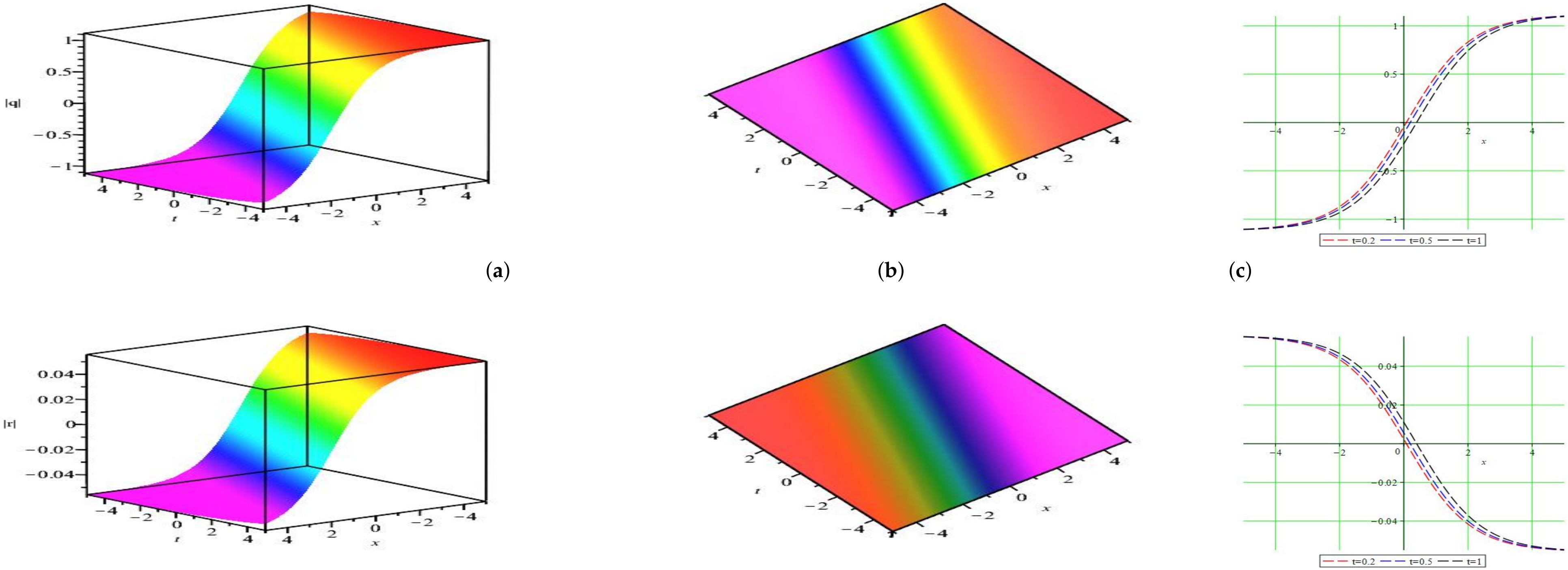
$$r(x,t) = \pm\left\{\sqrt{\frac{3A_1 - 10A_2}{A_3}}\left(\frac{(A_1 - 2\,A_2)}{4}\tanh\left(\frac{x - \nu t}{2}\right)\right)\right\}\mathrm{e}^{i\,(-kx+wt+\theta)}, \tag{37}$$

and the singular solitons

$$q(x,t) = \pm\left\{\sqrt{\frac{3A_1 - 10A_2}{A_3}}\left(\frac{1}{2}\coth\left(\frac{x - \nu t}{2}\right)\right)\right\}\mathrm{e}^{i\,(-kx+wt+\theta)}, \tag{38}$$

$$r(x,t) = \pm\left\{\sqrt{\frac{3A_1 - 10A_2}{A_3}}\left(\frac{(A_1 - 2\,A_2)}{4}\coth\left(\frac{x - \nu t}{2}\right)\right)\right\}\mathrm{e}^{i\,(-kx+wt+\theta)}. \tag{39}$$

The surface plots of solitons (36) and (37) are depicted in Figure 1.



(**a**) (**b**) (**c**)

(**d**) (**e**) (**f**)

Figure 1. Kink behavior of $|q(x,t)|$ shown as (**a**–**c**) and anti-kink behavior of $|r(x,t)|$ shown as (**d**–**f**), respectively, given by (36) and (37) with $A_1 = 0.4$, $A_2 = 0.3$, $A_3 = -0.3, \nu = 0.4$.

4. Generalized Arnous Method

Equation (27) satisfies the explicit solution [35,36]

$$P(\sigma) = \alpha_0 + \frac{\alpha_1 + \beta_1 \Phi'(\sigma)}{\Phi(\sigma)}, \tag{40}$$

by the aid of the auxiliary equation

$$\left[\Phi'(\sigma)\right]^2 = \left[\Phi(\sigma)^2 - \rho\right] \ln(a)^2, \tag{41}$$

$$\Phi^{(n)}(\sigma) = \begin{cases} \Phi(\sigma)\ln(a)^n, & n \text{ is even}, \\ \Phi'(\sigma)\ln(a)^{n-1}, & n \text{ is odd}, \end{cases}, n \geq 2, a > 0, a \neq 1, \tag{42}$$

and the analytical solution

$$\Phi(\sigma) = \kappa \ln(a) a^{\sigma} + \frac{\rho}{4\kappa \ln(a) a^{\sigma}}, \tag{43}$$

where κ, β_1, α_1, ρ, α_0 are constants.

Remark 1. *This is worth mentioning that, in paper [35], there are some typing mistakes in Equations (5) and (7). In Equation (5), summation j should vary from* 1 *to n. Equation (7) has to be replaced by* (42) *of this manuscript.*

Putting (40), together with (41) into (27), gives the ancillary equations

$$\begin{gathered}
A_4\beta_1^5\rho^2 \ln(a)^4 - 10\, A_4\alpha_1^2\beta_1^3\rho \ln(a)^2 + 5\, A_4\alpha_1^4\beta_1 + 24\,\omega \ln(a)^4\beta_1\rho^2 = 0, \\
-20\, A_4\alpha_0\alpha_1\beta_1^3\rho \ln(a)^2 + 20\, A_4\alpha_0\alpha_1^3\beta_1 = 0, \\
-2\, A_4\beta_1^5\rho \ln(a)^4 + \left(10\, A_4\alpha_1^2\beta_1^3 - \left(A_3\beta_1^3 + 10\, A_4\alpha_0^2\beta_1^3 + 8\,\omega \ln(a)^2\beta_1\right)\rho\right) \ln(a)^2 \\
-2\, A_1 \ln(a)^2\beta_1\rho + 30\, A_4\alpha_0^2\alpha_1^2\beta_1 + 3\, A_3\alpha_1^2\beta_1 = 0, \\
20\, A_4\alpha_0\alpha_1\beta_1^3 \ln(a)^2 + 20\, A_4\alpha_0^3\alpha_1\beta_1 + 6\, A_3\alpha_0\alpha_1\beta_1 = 0, \\
A_4\beta_1^5 \ln(a)^4 + \left(A_3\beta_1^3 + 10\, A_4\alpha_0^2\beta_1^3 + 8\,\omega \ln(a)^2\beta_1\right) \ln(a)^2 \\
+A_2\beta_1 - 8\,\omega \ln(a)^4\beta_1 + 3\, A_3\alpha_0^2\beta_1 + 5\, A_4\alpha_0^4\beta_1 = 0, \\
5\, A_4\alpha_1\beta_1^4\rho^2 \ln(a)^4 - 10\, A_4\alpha_1^3\beta_1^2\rho \ln(a)^2 + 24\,\omega \ln(a)^4\alpha_1\rho^2 + A_4\alpha_1^5 = 0, \\
5\, A_4\alpha_0\beta_1^4\rho^2 \ln(a)^4 - 30\, A_4\alpha_0\alpha_1^2\beta_1^2\rho \ln(a)^2 + 5\, A_4\alpha_0\alpha_1^4 = 0, \\
-10\, A_4\alpha_1\beta_1^4\rho \ln(a)^4 + \left(10\, A_4\alpha_1^3\beta_1^2 - \left(3\, A_3\alpha_1\beta_1^2 + 30\, A_4\alpha_0^2\alpha_1\beta_1^2 + 8\,\omega \ln(a)^2\alpha_1\right)\rho\right) \ln(a)^2 \\
+10\, A_4\alpha_0^2\alpha_1^3 - 12\,\omega \ln(a)^4\alpha_1\rho + A_3\alpha_1^3 - 2\, A_1 \ln(a)^2\alpha_1\rho = 0, \\
-10\, A_4\alpha_0\beta_1^4\rho \ln(a)^4 + 3\, A_3\alpha_0\alpha_1^2 \\
+\left(30\, A_4\alpha_0\alpha_1^2\beta_1^2 - \left(10\, A_4\alpha_0^3\beta_1^2 + 3\, A_3\alpha_0\beta_1^2\right)\rho\right) \ln(a)^2 + 10\, A_4\alpha_0^3\alpha_1^2 = 0, \\
A_2\alpha_1 + 5\, A_4\alpha_1\beta_1^4 \ln(a)^4 + \left(3\, A_3\alpha_1\beta_1^2 + 30\, A_4\alpha_0^2\alpha_1\beta_1^2 + 8\,\omega \ln(a)^2\alpha_1\right) \ln(a)^2 \\
+5\, A_4\alpha_0^4\alpha_1 + 3\, A_3\alpha_0^2\alpha_1 + A_1 \ln(a)^2\alpha_1 - 7\,\omega \ln(a)^4\alpha_1 = 0, \\
5\, A_4\alpha_0\beta_1^4 \ln(a)^4 + \left(10\, A_4\alpha_0^3\beta_1^2 + 3\, A_3\alpha_0\beta_1^2\right) \ln(a)^2 + A_2\alpha_0 + A_3\alpha_0^3 + A_4\alpha_0^5 = 0.
\end{gathered} \tag{44}$$

As a result, Equation (44) admits the results:

Result 1.

$$\alpha_0 = \alpha_1 = 0, \beta_1 = \pm \frac{1}{\ln(a)} \sqrt{\frac{6A_1 \ln(a)^2 - 5A_2}{2A_3}}, \omega = \frac{2A_1 \ln(a)^2 - A_2}{16 \ln(a)^4},$$

$$A_4 = \frac{6\left(A_2 - 2A_1 \ln(a)^2\right) A_3^2}{25A_2^2 - 60A_2A_1 \ln(a)^2 + 36A_1^2 \ln(a)^4}, A_3\left(12A_1 \ln(a)^2 - 10A_2\right) > 0. \tag{45}$$

Thus, the exact solutions stick out as

$$q(x,t) = \pm\left\{ \sqrt{\frac{12A_1 \ln(a)^2 - 10A_2}{A_3}} \left(\frac{4\kappa^2 \ln(a)^2 a^{2(x-\nu t)} - \rho}{2\left(4\kappa^2 \ln(a)^2 a^{2(x-\nu t)} + \rho\right)} \right) \right\} e^{i\,(-kx+wt+\theta)}, \tag{46}$$

$$r(x,t) = \pm\left\{ \sqrt{\frac{12A_1 \ln(a)^2 - 10A_2}{A_3}} \left(\frac{\left(4\kappa^2 \ln(a)^2 a^{2(x-\nu t)} - \rho\right)\left(2A_1 \ln(a)^2 - A_2\right)}{32\left(4\kappa^2 \ln(a)^2 a^{2(x-\nu t)} + \rho\right) \ln(a)^4} \right) \right\} e^{i\,(-kx+wt+\theta)}. \tag{47}$$

Taking $\rho = \pm 4\kappa^2$ *and* $a = e$ *provides us with the dark solitons*

$$q(x,t) = \pm\left\{ \sqrt{\frac{12A_1 - 10A_2}{A_3}} \left(\frac{\tanh(x-\nu t)}{2} \right) \right\} e^{i\,(-kx+wt+\theta)}, \tag{48}$$

$$r(x,t) = \pm\left\{ \sqrt{\frac{12A_1 - 10A_2}{A_3}} \left(\frac{(2A_1 - A_2)\tanh(x-\nu t)}{32} \right) \right\} e^{i\,(-kx+wt+\theta)}, \tag{49}$$

and the singular solitons

$$q(x,t) = \pm\left\{ \sqrt{\frac{12A_1 - 10A_2}{A_3}} \left(\frac{\coth(x-\nu t)}{2} \right) \right\} e^{i\,(-kx+wt+\theta)}, \tag{50}$$

$$r(x,t) = \pm\left\{ \sqrt{\frac{12A_1 - 10A_2}{A_3}} \left(\frac{(2A_1 - A_2)\coth(x-\nu t)}{32} \right) \right\} e^{i\,(-kx+wt+\theta)}. \tag{51}$$

The surface plots of solitons (48) and (49) are depicted in Figure 2.

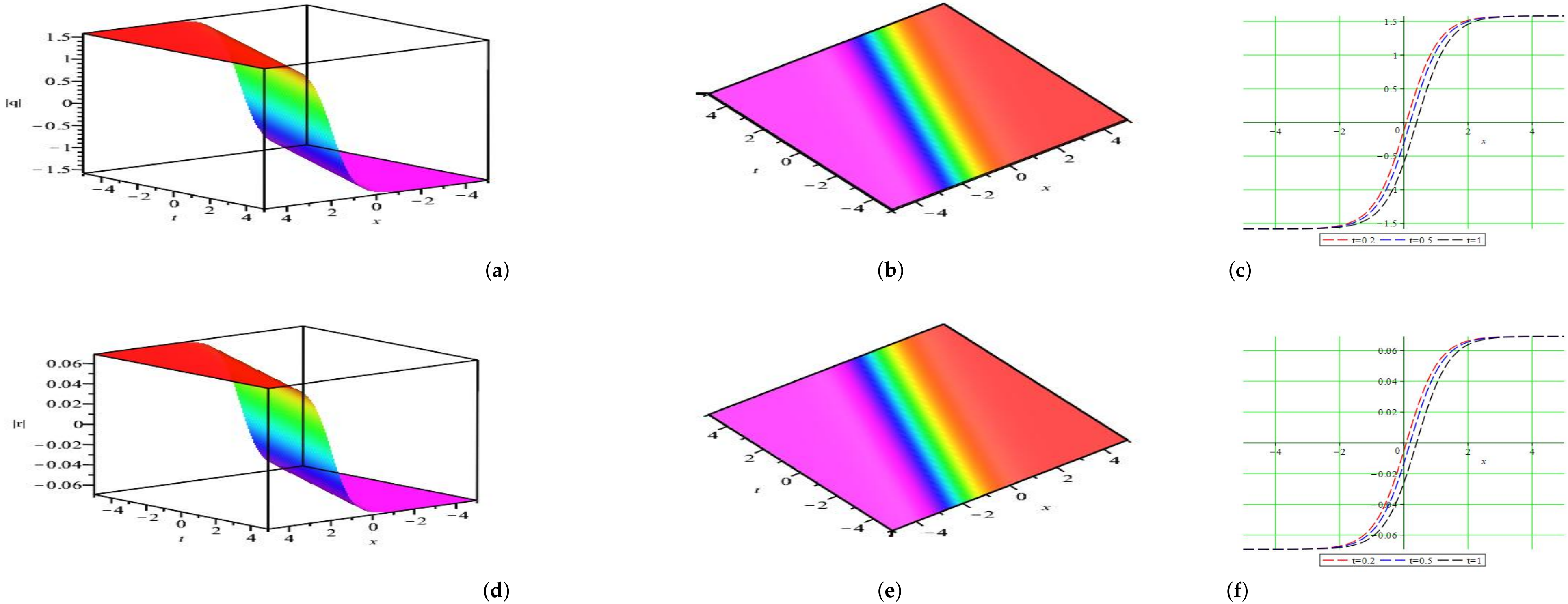


Figure 2. Kink wave representation for $|q(x,t)|$ shown as (**a**–**c**) and $|r(x,t)|$ shown as (**d**–**f**), respectively, given by (48) and (49) with $A_1 = 0.5, A_2 = 0.3, A_3 = 0.3, \nu = 0.4$.

Result 2.

$$\alpha_0 = \beta_1 = 0, \alpha_1 = \pm\sqrt{-\frac{18A_1 \ln(a)^2\rho + 20A_2\rho}{A_3}}, \omega = -\frac{A_2 + A_1 \ln(a)^2}{\ln(a)^4},$$

$$A_4 = \frac{6\left(A_2 + A_1 \ln(a)^2\right)A_3^2}{100A_2^2 + 180A_2A_1 \ln(a)^2 + 81A_1^2 \ln(a)^4}, \rho A_3\left(18A_1 \ln(a)^2 + 20A_2\right) < 0. \tag{52}$$

Hence, the analytical solutions stand as

$$q(x,t) = \pm\left\{\sqrt{-\frac{18A_1 \ln(a)^2\rho + 20A_2\rho}{A_3}}\left(\frac{4\kappa \ln(a)a^{x-\nu t}}{4\kappa^2 \ln(a)^2 a^{2(x-\nu t)} + \rho}\right)\right\}e^{i(-kx+wt+\theta)}, \tag{53}$$

$$r(x,t) = \mp\left\{\sqrt{-\frac{18A_1 \ln(a)^2\rho + 20A_2\rho}{A_3}}\left(\frac{4\kappa \ln(a)a^{x-\nu t}\left(A_2 + A_1 \ln(a)^2\right)}{\left(4\kappa^2 \ln(a)^2 a^{2(x-\nu t)} + \rho\right)\ln(a)^4}\right)\right\}e^{i(-kx+wt+\theta)}. \tag{54}$$

Setting $\rho = \pm 4\kappa^2$and $a = e$ leaves us with the bright solitons

$$q(x,t) = \pm\left\{\sqrt{-\frac{18A_1 + 20A_2}{A_3}}(sech(x-\nu t))\right\}e^{i(-kx+wt+\theta)}, \tag{55}$$

$$r(x,t) = \mp\left\{\sqrt{-\frac{18A_1 + 20A_2}{A_3}}((A_2 + A_1)sech(x-\nu t))\right\}e^{i(-kx+wt+\theta)}, \tag{56}$$

and the singular solitons

$$q(x,t) = \pm\left\{\sqrt{\frac{18A_1 + 20A_2}{A_3}}(csch(x-\nu t))\right\}e^{i(-kx+wt+\theta)}, \tag{57}$$

$$r(x,t) = \mp\left\{\sqrt{\frac{18A_1 + 20A_2}{A_3}}((A_2 + A_1)csch(x-\nu t))\right\}e^{i(-kx+wt+\theta)}. \tag{58}$$

The surface plots of solitons (55)–(58) are depicted in Figures 3 and 4.

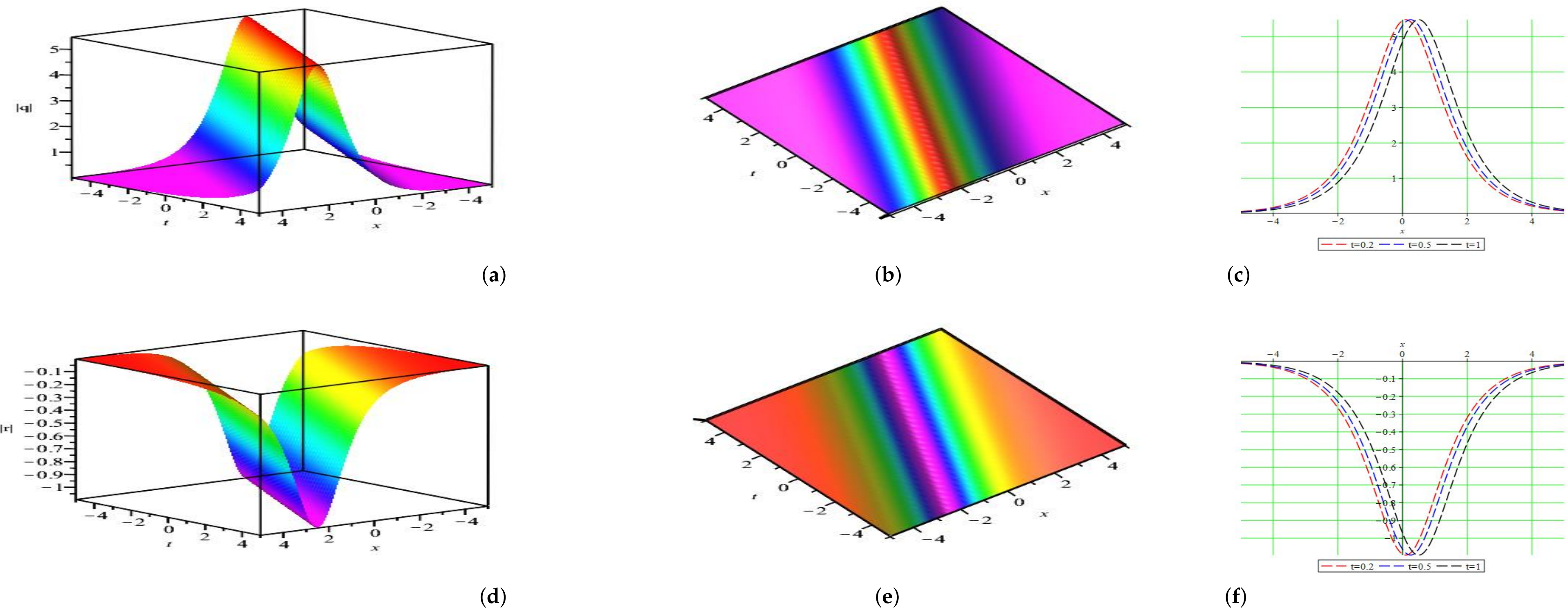


Figure 3. Representation of bright soliton for $|q(x,t)|$ shown as (**a**–**c**) and dark soliton $|r(x,t)|$ shown as (**d**–**f**), respectively, given by (55) and (56) with $A_1 = -0.5$, $A_2 = 0.3$, $A_3 = 0.1$, $\nu = 0.5$.

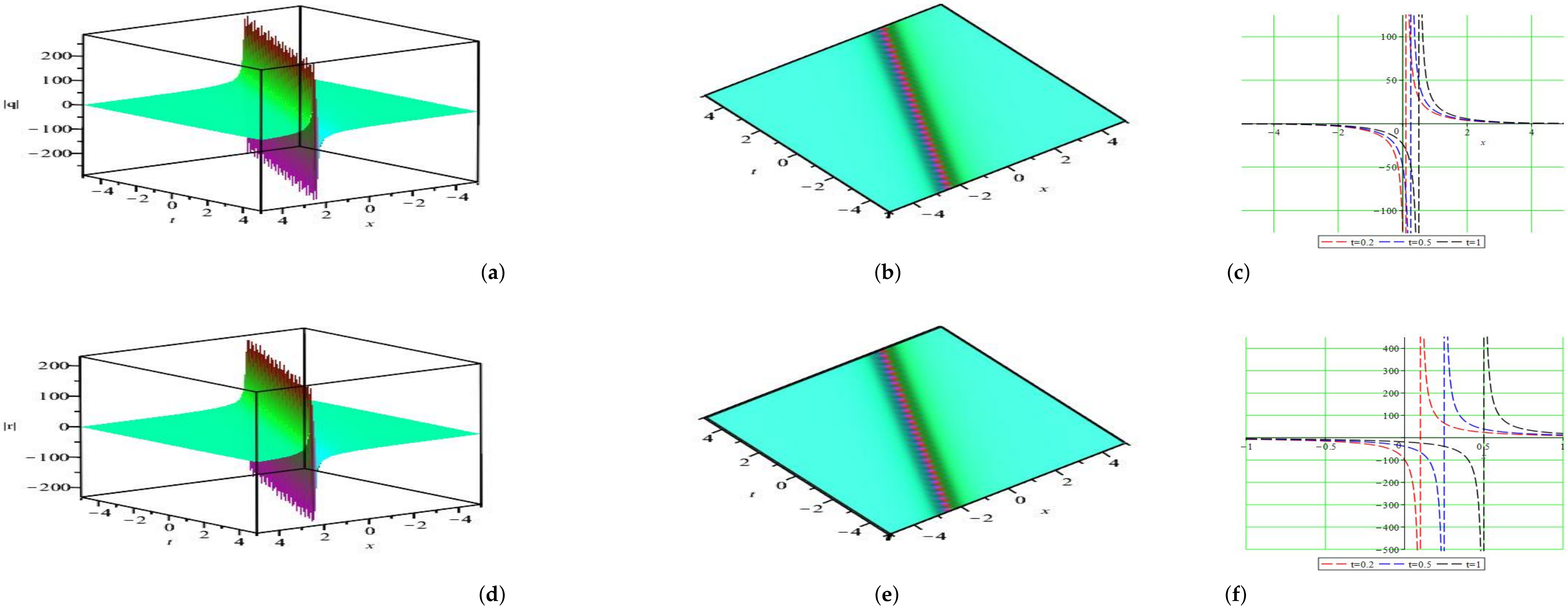


Figure 4. Graphical representation for $|q(x,t)|$ shown as (**a**–**c**) and $|r(x,t)|$ shown as (**d**–**f**), respectively, given by (57) and (58) with $A_1 = 0.5, A_2 = 0.3, A_3 = 0.1, \nu = 0.5$.

Result 3.

$$\alpha_0 = 0, \alpha_1 = \pm\sqrt{\frac{10A_2\rho - 3A_1\ln(a)^2\rho}{4A_3}}, \omega = \frac{A_1\ln(a)^2 - 2A_2}{2\ln(a)^4}, \beta_1 = \pm\frac{1}{\ln(a)}\sqrt{-\frac{10A_2 - 3A_1\ln(a)^2}{4A_3}},$$

$$A_4 = \frac{12\left(2A_2 - A_1\ln(a)^2\right)A_3^2}{100A_2^2 - 60A_2A_1\ln(a)^2 + 9A_1^2\ln(a)^4}, A_3\left(3A_1\ln(a)^2 - 10A_2\right) > 0. \tag{59}$$

Therefore, the explicit solutions shape up as

$$q(x,t) = \pm\left\{\sqrt{\frac{3A_1\ln(a)^2 - 10A_2}{A_3}}\left(\frac{\left(4\sqrt{-\rho}\ln(a)\kappa a^{x-\nu t} + 4\kappa^2\ln(a)^2a^{2(x-\nu t)} - \rho\right)}{2\left(4\kappa^2\ln(a)^2a^{2(x-\nu t)} + \rho\right)}\right)\right\}e^{i\,(-kx+wt+\theta)}, \tag{60}$$

$$r(x,t) = \pm\left\{\sqrt{\frac{3A_1\ln(a)^2 - 10A_2}{A_3}}\left(\frac{(A_1\ln(a)^2 - 2A_2)\left(4\sqrt{-\rho}\ln(a)\kappa a^{x-\nu t} + 4\kappa^2\ln(a)^2a^{2(x-\nu t)} - \rho\right)}{4\ln(a)^4\left(4\kappa^2\ln(a)^2a^{2(x-\nu t)} + \rho\right)}\right)\right\}e^{i\,(-kx+wt+\theta)}. \tag{61}$$

Taking $\rho = -4\kappa^2$ and $a = e$ paves the way to the bright-singular combo solitons

$$q(x,t) = \pm\left\{\sqrt{\frac{3A_1 - 10A_2}{A_3}}\left(\frac{1}{2}(sech(x-\nu t) + \coth(x-\nu t))\right)\right\}e^{i\,(-kx+wt+\theta)}, \tag{62}$$

$$r(x,t) = \pm\left\{\sqrt{\frac{3A_1 - 10A_2}{A_3}}\left(\frac{(A_1 - 2A_2)}{4}(sech(x-\nu t) + \coth(x-\nu t))\right)\right\}e^{i\,(-kx+wt+\theta)}. \tag{63}$$

Remark 2. *The special cases as suggested: $\alpha_1 = -\alpha_2$ and $\beta_1 = \beta_2$ reduce the model (3) and (4) to*

$$iq_t + ia_1r_{xxx} + b_1r_{xxxx} + \left(c_1|q|^2 + d_1|r|^2\right)q + \left(\xi_1|q|^4 + \eta_1|q|^2|r|^2 + \zeta_1|r|^4\right)q - i\alpha_2q_x + \beta_2r = 0, \tag{64}$$

$$ir_t + ia_2q_{xxx} + b_2q_{xxxx} + \left(c_2|r|^2 + d_2|q|^2\right)r + \left(\xi_2|r|^4 + \eta_2|q|^2|r|^2 + \zeta_2|r|^4\right)r + i\alpha_2r_x + \beta_2q = 0. \tag{65}$$

These special cases indeed conform to the fiber Bragg-grating system. The final solutions of this special form also admit all the soliton solutions reported in the current paper by setting the special cases $\alpha_1 = -\alpha_2$ and $\beta_1 = \beta_2$ in (28).

5. Conclusions

A complete spectrum of CQ optical solitons is revealed for fiber Bragg gratings with dispersive reflectivity and parabolic law of SPM. It is not exactly known if the model Equations (1) and (2) are derivable from a particular Lagrangian. However, the models are already studied from a numerical perspective in the past [48–51]. Thus, in order to make the current study novel, the CD from past papers is being replaced by a combination of 3OD and 4OD. This newly structured model is therefore investigated by the aid of Lie symmetry for the first time. Such CQ optical solitons have been proposed to control the depletion of the much-needed CD. These solitons are also considered with dispersive reflectivity, which maintains the necessary balance between chromatic dispersion and nonlinear refractive index structure for an uninterrupted transmission of solitons along intercontinental distances. The Lie symmetry analysis coupled with Kudryashov and Arnous approaches collectively revealed a variety of soliton solutions that are bright, dark, singular and combo solitons and are reported in the current paper for the first time by using these newly proposed integration methods. Dark-soliton solutions are meaningful only if the background which supports the dark solitons is modulationally stable. Bright solitons are formed when the group–velocity dispersion is anomalous. Bright and dark solitons are considered from the perspective of modulation instability. Dark solitons are observed when the carrier wave is modulationally stable, while bright solitons are observed when the carrier wave is unstable with respect to long-wave modulations [12–17,52].

The stability criterion for dark solitons can be expressed in terms of the renormalized soliton momentum as [53]

$$\frac{dM_r}{dv} > 0. \tag{66}$$

This stability criterion was provided with the help of the Lyapunov function [54]. This criterion was considered to address the instability of dark solitons with the variational principle [55,56]. The stability of dark solitons can also be formulated using the following alternative definition of the renormalized momentum [57]

$$M_r = \frac{i}{2}\int\int[(u-1)\nabla_T u^* - (u^*-1)\nabla_T u]dr, \tag{67}$$

where ∇_T denotes the transverse part of the Laplacian operator in the multidimensional case. This definition was used to analyze the transverse instability of dark solitons [58]. This list of solitons therefore opens up future avenues of research. One of the prime prospects would be to address the conservation laws that would subsequently lead to the quasi-monochromatic dynamics of soliton parameters which are of paramount importance when weak perturbation terms to the model sneak in. The numerical studies are also to be conducted later on with the application of Adomian decomposition scheme or Laplace–Adomian decomposition. These would reveal an insightful visual perspective to the model. These studies would give way to results that will be disseminated elsewhere.

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