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Certain Identities Involving the General Kampé de Fériet Function and Srivastava's General Triple Hypergeometric Series

Mohd Idris Qureshi [1], Junesang Choi [2,*] and Mohd Shaid Baboo [3]

[1] Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University), New Delhi 110025, India
[2] Department of Mathematics, Dongguk University, Gyeongju 38066, Republic of Korea
[3] School of Basic Sciences and Research, Sharda University, Greater Noida 201306, India
* Correspondence: junesang@dongguk.ac.kr; Tel.: +82-010-6525-2262

Abstract: Due to the great success of hypergeometric functions of one variable, a number of hypergeometric functions of two or more variables have been introduced and explored. Among them, the Kampé de Fériet function and its generalizations have been actively researched and applied. The aim of this paper is to provide certain reduction, transformation and summation formulae for the general Kampé de Fériet function and Srivastava's general triple hypergeometric series, where the parameters and the variables are suitably specified. The identities presented in the theorems and additional comparable outcomes are hoped to be supplied by the use of computer-aid programs, for example, Mathematica. Symmetry occurs naturally in ${}_{p+1}F_p$, the Kampé de Fériet function and the Srivastava's function $F^{(3)}[x,y,z]$, which are three of the most important functions discussed in this study.

Keywords: Catalan's constant; hypergeometric and generalized hypergeometric functions; Riemann zeta function; Dirichlet beta function; (general) Kampé de Fériet function; Srivastava's general triple hypergeometric series; integral formulae for inverse trigonometric functions

MSC: 11M06; 11M41; 33B15; 33C20; 33C70

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1. Introduction and Definitions

A number of reduction, transformation and summation formulae for the general Kampé de Fériet's function have been developed (see, e.g., [1] (pp. 26–32), [2–8], and the references cited therein, in particular, [2]). For example, Srivastava and Miller [8] (Equation (6)) presented an interesting reducible Kampé de Fériet's general double hypergeometric series:

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix} 1: & \frac{1}{2}, 1; & \frac{1}{2}, 1; \\ 2: & \frac{3}{2}; & \frac{3}{2}; \end{matrix} \; -1, -1\right] = \pi\, G - \frac{7\,\zeta(3)}{4}. \tag{1}$$

As an intriguing by-product of analysis, researchers [8] (Equation (15)) offered

$$\int_0^{\frac{\pi}{2}} \frac{t^2}{\sin t}\,\mathrm{d}t = 2\pi\, G - \frac{7\,\zeta(3)}{2}. \tag{2}$$

Choi and Srivastava [3] (Equations (1.5) and (3.1)) used a similar technique in [8] to offer the following identities:

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix} 1: \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \\ 2: \quad \frac{3}{2}; \quad \frac{3}{2}; \end{matrix} 1, 1\right] = {}_4F_3\left[\begin{matrix} 1, 1, 1, 1; \\ 2, 2, \frac{3}{2}; \end{matrix} 1\right] \quad = \frac{\pi^2 \log 2}{2} - \frac{7\,\zeta(3)}{4} \tag{3}$$

and

$$\int_0^1 \frac{(\arcsin t)^2}{t}\,\mathrm{d}t = \frac{\pi^2 \log 2}{4} - \frac{7\,\zeta(3)}{8}. \tag{4}$$

For numerous identities of ${}_pF_q$, such as (3), one may refer, for instance, to [9]. Here, G denotes the Catalan's constant given by

$$G := \frac{1}{2}\int_0^1 \mathbf{K}(x)\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.915965594177219015\cdots, \tag{5}$$

where $\mathbf{K}(x)$ is the complete elliptic integral of the first kind (see, e.g., [10]) (p. 43, Equations (16) and (17)). Bradley [11] compiled a comprehensive set of formulas involving Catalan's constant, including single integral, double integral, infinite series and others. The following representation for G may be added to the above-mentioned list in [11]:

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2})_n (\frac{1}{2})_n (1)_n}{(\frac{3}{2})_n (\frac{3}{2})_n \, n!} = {}_3F_2\left[\begin{matrix} 1, \frac{1}{2}, \frac{1}{2}; \\ \frac{3}{2}, \frac{3}{2}; \end{matrix} -1\right], \tag{6}$$

where $(\lambda)_n$ $(\lambda \in \mathbb{C},\ n \in \mathbb{Z}_{\geqslant 0})$ is the Pochhammer symbol (see, e.g., [10]) (p. 2 and p. 5) and ${}_pF_q[\cdot]$ $(p,\, q \in \mathbb{Z}_{\geqslant 0})$ denotes the generalized hypergeometric series (or functions) (see, e.g., [12]):

$$\begin{aligned} {}_pF_q\left[\begin{matrix} \alpha_1, \ldots, \alpha_p\,; \\ \beta_1, \ldots, \beta_q\,; \end{matrix}\ z\right] &= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_n}{\prod_{j=1}^{q} (\beta_j)_n} \frac{z^n}{n!} \\ &= {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z). \end{aligned} \tag{7}$$

Here, and in the following, an empty product is interpreted as 1, and it is assumed that the variable z, the numerator parameters $\alpha_1, \ldots, \alpha_p$ and the denominator parameters $\beta_1, \ldots, \beta_q$ take on complex values, provided that

$$(\beta_j \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0};\ j = 1, \ldots, q). \tag{8}$$

Throughout this paper, let $\mathbb{C}$, $\mathbb{N}$, $\mathbb{Z}_{\geqslant 0}$ and $\mathbb{Z}_{\leqslant 0}$ be the sets of complex numbers, positive integers, nonnegative integers and non-positive integers, respectively.

The $\zeta(s)$ is Riemann zeta function defined by (see, e.g., [10]) (Section 2.3)

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1) \\ \displaystyle\frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} & (\Re(s) > 0,\, s \neq 1). \end{cases} \tag{9}$$

We also recall the Dirichlet beta function $\beta(s)$ defined by

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} \quad (\Re(s) > 0). \tag{10}$$

The tremendous popularity and use of the hypergeometric function ${}_2F_1$ and generalized hypergeometric functions ${}_pF_q$ of one variable have inspired and motivated a huge number of scholars to propose and explore hypergeometric functions of two or more variables. Appell [13] launched serious, substantial and systematic research into the hypergeometric functions of two variables by introducing the so-called Appell functions F_1, F_2, F_3 and F_4, which are extensions of Gauss' hypergeometric function. Humbert explored the confluent forms of the Appell functions [14]. The classic literature, such as [15], has a comprehensive listing of these functions.

Kampé de Fériet [16] later generalized the four Appell functions and their confluent forms by introducing more general hypergeometric functions of two variables. Burchnall and Chaudndy [17,18] subsequently shortened the notation that Kampé de Fériet devised and introduced for their double hypergeometric functions of superior order. We remember here the formulation of a generic double hypergeometric function (more generic than that specified by Kampé de Fériet) given by Srivastava and Panda [19] (p. 423, Equation (26)) in a slightly modified notation. The following is a handy generalization of the Kampé de Fériet function:

$$\begin{aligned} &F^{H:A;B}_{G:C;D}\left[\begin{array}{lll} (h_H): & (a_A); & (b_B); \\ (g_G): & (c_C); & (d_D); \end{array} x, y\right] \\ &\qquad = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{((h_H))_{m+n}\,((a_A))_m\,((b_B))_n}{((g_G))_{m+n}\,((c_C))_m\,((d_D))_n}\,\frac{x^m}{m!}\,\frac{y^n}{n!}, \end{aligned} \tag{11}$$

where (h_H) denotes the sequence of parameters $(h_1, h_2, \ldots, h_H)$ and $((h_H))_n$ is defined by the following product of Pochhammer symbols

$$((h_H))_n := (h_1)_n (h_2)_n \cdots (h_H)_n \quad (n \in \mathbb{Z}_{\geqslant 0}),$$

where, when $n = 0$, the product is to be accepted as unity. For more details about the function (11), including its convergence, one may refer, for example, to [1] (pp. 26–32), [20] (pp. 63–64), [21,22].

Srivastava (see, e.g., [1] (pp. 44–45); see also [20] (pp. 69–71)) introduced a general triple hypergeometric series $F^{(3)}[x, y, z]$, which is a unification of Lauricella's fourteen hypergeometric functions $F_1, \ldots, F_{14}$ as well as the Srivastava's functions H_A, H_B and H_C (see [23]), defined by (see also [24–27])

$$\begin{aligned} F^{(3)}[x, y, z] &\equiv F^{(3)}\left[\begin{array}{lllllll} (a):: & (b); & (b'); & (b''): & (c); & (c'); & (c''); \\ (e):: & (g); & (g'); & (g''): & (h); & (h'); & (h''); \end{array} x, y, z\right] \\ &= \sum_{m,n,p=0}^{\infty} \Lambda(m, n, p)\,\frac{x^m}{m!}\,\frac{y^n}{n!}\,\frac{z^p}{p!}, \end{aligned} \tag{12}$$

where

$$\begin{aligned} \Lambda(m, n, p) = &\frac{\prod_{j=1}^{A}(a_j)_{m+n+p} \prod_{j=1}^{B}(b_j)_{m+n} \prod_{j=1}^{B'}(b'_j)_{n+p} \prod_{j=1}^{B''}(b''_j)_{p+m}}{\prod_{j=1}^{E}(e_j)_{m+n+p} \prod_{j=1}^{G}(g_j)_{m+n} \prod_{j=1}^{G'}(g'_j)_{n+p} \prod_{j=1}^{G''}(g''_j)_{p+m}} \\ &\times \frac{\prod_{j=1}^{C}(c_j)_m \prod_{j=1}^{C'}(c'_j)_n \prod_{j=1}^{C''}(c''_j)_p}{\prod_{j=1}^{H}(h_j)_m \prod_{j=1}^{H'}(h'_j)_n \prod_{j=1}^{H''}(h''_j)_p} \end{aligned} \tag{13}$$

and (a) abbreviates the array of A parameters $a_1, \ldots, a_A$, with similar interpretations for (b), (b'), (b'') and so on. The triple hypergeometric series in (12) converges absolutely when

$$\begin{cases} 1 + E + G + G'' + H - A - B - B'' - C \geqslant 0, \\ 1 + E + G + G' + H' - A - B - B' - C' \geqslant 0, \\ 1 + E + G' + G'' + H'' - A - B' - B'' - C'' \geqslant 0, \end{cases} \tag{14}$$

where the equalities hold true for appropriately restricted values of $|x|$, $|y|$ and $|z|$.

Symmetry issues may arise overtly or indirectly in any discipline or aspect of human existence. It is clear that symmetry occurs in the ${}_pF_q$, the generalized Kampé de Fériet function in (11) and the Srivastava's general triple hypergeometric series $F^{(3)}[x,y,z]$ in (12)—three of the most significant functions considered in this paper. For example,

$$ {}_pF_q(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;z) = {}_pF_q(\xi_p,\ldots,\xi_1;\eta_q,\ldots,\eta_1;z), $$

where every reordering of the numerator parameters yields the same function and so does every reordering of the denominator parameters.

In this paper, we aim to establish certain reduction, transformation and summation formulae, such as (1) and (3), which are evaluated in terms of mathematical constants, for the general Kampé de Fériet function (11) and Srivastava's general triple hypergeometric series (12), where the parameters and the variables are appropriately specified.

2. Preliminary Results

This section recalls several known identities that are essential to construct our main findings, which are presented as lemmas below.

Lemma 1. *The following identities hold true:*

$$ \frac{\tan^{-1} z}{z} = {}_2F_1\left[\begin{matrix}1,\ \frac{1}{2};\\ \frac{3}{2};\end{matrix}\ -z^2\right] \quad (|z|<1); \tag{15} $$

$$ \frac{\sin^{-1} z}{z} = {}_2F_1\left[\begin{matrix}\frac{1}{2},\ \frac{1}{2};\\ \frac{3}{2};\end{matrix}\ z^2\right] \quad (|z|<1); \tag{16} $$

$$ \left(\frac{\sin^{-1} z}{z}\right)^2 = {}_3F_2\left[\begin{matrix}1,1,1;\\ 2,\frac{3}{2};\end{matrix}\ z^2\right] \quad (|z|<1); \tag{17} $$

For $|z|<1$,

$$ \left(\frac{\sin^{-1} z}{z}\right)^3 = F^{2:1;3}_{2:0;2}\left[\begin{matrix}\frac{3}{2},\frac{3}{2}:1;1,\frac{1}{2},\frac{1}{2};\\ 2,\frac{5}{2}:-;\ \frac{3}{2},\ \frac{3}{2};\end{matrix}\ z^2,z^2\right]; \tag{18} $$

$$ \left(\frac{\sin^{-1} z}{z}\right)^4 = F^{2:1;3}_{2:0;2}\left[\begin{matrix}2,2:1;1,1,1;\\ 3,\frac{5}{2}:-;\ 2,2;\end{matrix}\ z^2,z^2\right]; \tag{19} $$

For $|z|<1$,

$$ \left(\frac{\tan^{-1} z}{z}\right)^2 = F^{1:2;1}_{1:1;0}\left[\begin{matrix}1:\ \frac{1}{2},1;1;\\ 2:\ \ \frac{3}{2};-;\end{matrix}\ -z^2,-z^2\right], \tag{20} $$

where $F^{p:q;k}_{l:m;n}[\cdot]$ *is the general Kampé de Fériet's function in* (11)*;*

$$ \begin{aligned} \left(\frac{\tan^{-1} z}{z}\right)^3 &= F^{(3)}\left[\begin{matrix}\frac{3}{2}::-;1;-:1;1;\frac{1}{2},1;\\ \frac{5}{2}::-;2;-:-;-;\ \frac{3}{2};\end{matrix}\ -z^2,-z^2,-z^2\right] \\ &= {}_2F_1\left[\begin{matrix}1,\frac{1}{2};\\ \frac{3}{2};\end{matrix}\ -z^2\right] F^{1:2;1}_{1:1;0}\left[\begin{matrix}1:\frac{1}{2},1;1;\\ 2:\ \frac{3}{2};-;\end{matrix}\ -z^2,-z^2\right], \end{aligned} \tag{21} $$

where $F^{(3)}$ *is Srivastava's general triple hypergeometric series in* (12).

Proof. The (15) and (16) are well-known representations via ${}_2F_1$ (see, e.g., [10] (p. 67)). For (17)–(19), one may consult [28]. For (20) and (21), one may refer to [29]. □

Li and Chu [30] presented several elegant recursive improper integral formulas involving powers of inverse trigonometric and hyperbolic functions, some of whose particular instances are recalled in the subsequent lemmas (see [30] (Tables 2 and 3); see also [31]).

Lemma 2. *The following integral formulas hold true:*

$$\int_0^1 (\arctan u)^2\,\mathrm{d}u = \tfrac{\pi^2}{16} + \tfrac{\pi\log 2}{4} - G; \tag{22}$$

$$\int_0^1 u(\arctan u)^2\,\mathrm{d}u = \tfrac{\pi^2}{16} - \tfrac{\pi}{4} + \tfrac{\log 2}{2}; \tag{23}$$

$$\int_0^1 u^2(\arctan u)^2\,\mathrm{d}u = \tfrac{1}{3} - \tfrac{\pi}{6} - \tfrac{\pi\log 2}{12} + \tfrac{\pi^2}{48} + \tfrac{G}{3}; \tag{24}$$

$$\int_0^1 u^3(\arctan u)^2\,\mathrm{d}u = \tfrac{1}{12} + \tfrac{\pi}{12} - \tfrac{\log 2}{3}; \tag{25}$$

$$\int_0^1 \frac{(\arctan u)^2}{u}\,\mathrm{d}u = \tfrac{\pi G}{2} - \tfrac{7\zeta(3)}{8}; \tag{26}$$

$$\int_0^1 \frac{(\arctan u)^2}{u^2}\,\mathrm{d}u = \tfrac{\pi\log 2}{4} - \tfrac{\pi^2}{16} + G. \tag{27}$$

Lemma 3. *The following integral formulas hold true:*

$$\int_0^1 (\arctan u)^3\,\mathrm{d}u = \tfrac{\pi^3}{64} + \tfrac{3\pi^2\log 2}{32} - \tfrac{3\pi G}{4} + \tfrac{63\,\zeta(3)}{64}; \tag{28}$$

$$\int_0^1 u(\arctan u)^3\,\mathrm{d}u = \tfrac{\pi^3}{64} - \tfrac{3\pi^2}{32} - \tfrac{3\,\pi\log 2}{8} + \tfrac{3G}{2}; \tag{29}$$

$$\int_0^1 u^2(\arctan u)^3\,\mathrm{d}u = \tfrac{\pi}{4} - \tfrac{\log 2}{2} - \tfrac{\pi^2}{16} - \tfrac{\pi^2\log 2}{32} + \tfrac{\pi^3}{192} + \tfrac{\pi G}{4} - \tfrac{21\zeta(3)}{64}; \tag{30}$$

$$\int_0^1 u^3(\arctan u)^3\,\mathrm{d}u = \tfrac{\pi}{8} - \tfrac{1}{4} + \tfrac{\pi\,\log 2}{4} + \tfrac{\pi^2}{32} - G; \tag{31}$$

$$\int_0^1 \frac{(\arctan u)^3}{u}\,\mathrm{d}u = \tfrac{3\pi^2 G}{16} - \tfrac{3\beta(4)}{2}; \tag{32}$$

$$\int_0^1 \frac{(\arctan u)^3}{u^2}\,\mathrm{d}u = \tfrac{3\pi^2\log 2}{32} - \tfrac{\pi^3}{64} + \tfrac{3\pi G}{4} - \tfrac{105\zeta(3)}{64}; \tag{33}$$

$$\int_0^1 \frac{(\arctan u)^3}{u^3}\,\mathrm{d}u = \tfrac{3\pi\log 2}{8} - \tfrac{\pi^3}{64} - \tfrac{3\pi^2}{32} + \tfrac{3G}{2}. \tag{34}$$

Lemma 4. *The following integral formulas hold true:*

$$\int_0^1 (\arcsin u)^2\,\mathrm{d}u = \tfrac{\pi^2}{4} - 2; \tag{35}$$

$$\int_0^1 u(\arcsin u)^2\,\mathrm{d}u = \tfrac{\pi^2}{16} - \tfrac{1}{4}; \tag{36}$$

$$\int_0^1 u^2(\arcsin u)^2\,\mathrm{d}u = \tfrac{\pi^2}{12} - \tfrac{14}{27}; \tag{37}$$

$$\int_0^1 u^3(\arcsin u)^2\,\mathrm{d}u = \tfrac{5\pi^2}{128} - \tfrac{1}{8}; \tag{38}$$

$$\int_0^1 u^4(\arcsin u)^2\,\mathrm{d}u = \tfrac{\pi^2}{20} - \tfrac{298}{1125}; \tag{39}$$

$$\int_0^1 \frac{(\arcsin u)^2}{u}\,\mathrm{d}u = \tfrac{\pi^2\log 2}{4} - \tfrac{7\zeta(3)}{8}; \tag{40}$$

$$\int_0^1 \frac{(\arcsin u)^2}{u^2}\,\mathrm{d}u = 4G - \tfrac{\pi^2}{4}. \tag{41}$$

Lemma 5. *The following integral formulas hold true:*

$$\int_0^1 (\arcsin u)^3\, \mathrm{d}u = 6 - 3\pi + \tfrac{\pi^3}{8}; \tag{42}$$

$$\int_0^1 u(\arcsin u)^3\, \mathrm{d}u = \tfrac{\pi^3}{32} - \tfrac{3\pi}{16}; \tag{43}$$

$$\int_0^1 u^2(\arcsin u)^3\, \mathrm{d}u = \tfrac{40}{27} - \tfrac{7\pi}{9} + \tfrac{\pi^3}{24}; \tag{44}$$

$$\int_0^1 u^3(\arcsin u)^3\, \mathrm{d}u = \tfrac{5\pi^3}{256} - \tfrac{51\pi}{512}; \tag{45}$$

$$\int_0^1 u^4(\arcsin u)^3\, \mathrm{d}u = \tfrac{4144}{5625} - \tfrac{149\pi}{375} + \tfrac{\pi^3}{40}; \tag{46}$$

$$\int_0^1 \frac{(\arcsin u)^3}{u}\, \mathrm{d}u = \tfrac{\pi^3 \log 2}{8} - \tfrac{9\pi\zeta(3)}{16}; \tag{47}$$

$$\int_0^1 \frac{(\arcsin u)^3}{u^2}\, \mathrm{d}u = 6\pi G - \tfrac{21\zeta(3)}{2} - \tfrac{\pi^3}{8}; \tag{48}$$

$$\int_0^1 \frac{(\arcsin u)^3}{u^3}\, \mathrm{d}u = \tfrac{3\pi \log 2}{2} - \tfrac{\pi^3}{16}. \tag{49}$$

Lemma 6. *The following integral formulas hold true:*

$$\int_0^1 (\arcsin u)^4\, \mathrm{d}u = 24 - 3\pi^2 + \tfrac{\pi^4}{16}; \tag{50}$$

$$\int_0^1 u(\arcsin u)^4\, \mathrm{d}u = \tfrac{3}{4} - \tfrac{3\pi^2}{16} + \tfrac{\pi^4}{64}; \tag{51}$$

$$\int_0^1 u^2(\arcsin u)^4\, \mathrm{d}u = \tfrac{488}{81} - \tfrac{7\pi^2}{9} + \tfrac{\pi^4}{48}; \tag{52}$$

$$\int_0^1 u^3(\arcsin u)^4\, \mathrm{d}u = \tfrac{3}{8} - \tfrac{51\pi^2}{512} + \tfrac{5\pi^4}{512}; \tag{53}$$

$$\int_0^1 u^4(\arcsin u)^4\, \mathrm{d}u = \tfrac{254{,}728}{84{,}375} - \tfrac{149\pi^2}{375} + \tfrac{\pi^4}{80}; \tag{54}$$

$$\int_0^1 \frac{(\arcsin u)^4}{u}\, \mathrm{d}u = \tfrac{\pi^4 \log 2}{16} - \tfrac{9\pi^2\zeta(3)}{16} + \tfrac{93\zeta(5)}{32}; \tag{55}$$

$$\int_0^1 \frac{(\arcsin u)^4}{u^2}\, \mathrm{d}u = 6\pi^2 G - 48\beta(4) - \tfrac{\pi^4}{16}; \tag{56}$$

$$\int_0^1 \frac{(\arcsin u)^4}{u^3}\, \mathrm{d}u = \tfrac{3\pi^2 \log 2}{2} - \tfrac{\pi^4}{32} - \tfrac{21\zeta(3)}{4}; \tag{57}$$

$$\int_0^1 \frac{(\arcsin u)^4}{u^4}\, \mathrm{d}u = 8G + \pi^2 G - 8\beta(4) - \tfrac{\pi^4}{48} - \tfrac{\pi^2}{2}. \tag{58}$$

3. Formulas for the Kampé de Fériet's Functions

This section offers formulas for transformation, reduction and summation for the general Kampé de Fériet functions.

Theorem 1. *The following identities hold:*

$$
\begin{aligned}
F^{1:2;2}_{1:1;1}\left[\begin{matrix}\frac{3}{2}:1,\frac{1}{2};1,\frac{1}{2};\\ \frac{5}{2}:\ \ \frac{3}{2};\ \ \frac{3}{2};\end{matrix}-1,-1\right] &= F^{2:1;2}_{2:0;1}\left[\begin{matrix}1,\frac{3}{2}:1;1,\frac{1}{2};\\ 2,\frac{5}{2}:-;\ \ \frac{3}{2};\end{matrix}-1,-1\right] \\
&= \frac{3\pi^2}{16}+\frac{3\pi\log 2}{4}-3G;
\end{aligned} \tag{59}
$$

$$
\begin{aligned}
F^{1:2;2}_{1:1;1}\left[\begin{matrix}2:1,\frac{1}{2};1,\frac{1}{2};\\ 3:\ \ \frac{3}{2};\ \ \frac{3}{2};\end{matrix}-1,-1\right] &= F^{1:2;1}_{1:1;0}\left[\begin{matrix}1:\ 1,\frac{1}{2};\ 1;\\ 3:\ \ \frac{3}{2};-;\end{matrix}-1,-1\right] \\
&= \frac{\pi^2}{4}-\pi+2\log 2;
\end{aligned} \tag{60}
$$

$$
\begin{aligned}
F^{1:2;2}_{1:1;1}\left[\begin{matrix}\frac{5}{2}:1,\frac{1}{2};1,\frac{1}{2};\\ \frac{7}{2}:\ \ \frac{3}{2};\ \ \frac{3}{2};\end{matrix}-1,-1\right] &= F^{2:1;2}_{2:0;1}\left[\begin{matrix}1,\frac{5}{2}:1;1,\frac{1}{2};\\ 2,\frac{7}{2}:-;\ \ \frac{3}{2};\end{matrix}-1,-1\right] \\
&= \frac{5}{3}-\frac{5\pi}{6}-\frac{5\pi\log 2}{12}+\frac{5\pi^2}{48}+\frac{5G}{3};
\end{aligned} \tag{61}
$$

$$
\begin{aligned}
F^{1:2;2}_{1:1;1}\left[\begin{matrix}3:1,\frac{1}{2};1,\frac{1}{2};\\ 4:\ \ \frac{3}{2};\ \ \frac{3}{2};\end{matrix}-1,-1\right] &= F^{2:1;2}_{2:0;1}\left[\begin{matrix}1,3:1;1,\frac{1}{2};\\ 2,4:-;\ \ \frac{3}{2};\end{matrix}-1,-1\right] \\
&= \frac{1}{2}+\frac{\pi}{2}-2\log 2;
\end{aligned} \tag{62}
$$

$$
\begin{aligned}
F^{1:2;2}_{1:1;1}\left[\begin{matrix}1:1,\frac{1}{2};1,\frac{1}{2};\\ 2:\ \ \frac{3}{2};\ \ \frac{3}{2};\end{matrix}-1,-1\right] &= F^{2:1;2}_{2:0;1}\left[\begin{matrix}1,1:1;1,\frac{1}{2};\\ 2,2:-;\ \ \frac{3}{2};\end{matrix}-1,-1\right] \\
&= \pi G-\frac{7\zeta(3)}{4};
\end{aligned} \tag{63}
$$

$$
\begin{aligned}
F^{1:2;2}_{1:1;1}\left[\begin{matrix}\frac{1}{2}:1,\frac{1}{2};1,\frac{1}{2};\\ \frac{3}{2}:\ \ \frac{3}{2};\ \ \frac{3}{2};\end{matrix}-1,-1\right] &= F^{2:1;2}_{2:0;1}\left[\begin{matrix}1,\frac{1}{2}:1;1,\frac{1}{2};\\ 2,\frac{3}{2}:-;\ \ \frac{3}{2};\end{matrix}-1,-1\right] \\
&= \frac{\pi\log 2}{4}-\frac{\pi^2}{16}+G.
\end{aligned} \tag{64}
$$

Proof. Using (15) gives

$$
\int_0^1 (\arctan u)^2\,\mathrm{d}u = \sum_{m,n=0}^{\infty} \frac{(1)_m(\frac{1}{2})_m\,(1)_n(\frac{1}{2})_n}{(\frac{3}{2})_m\,(\frac{3}{2})_n}\,\frac{(-1)^{m+n}}{m!\,n!}\,\frac{1}{2(m+n)+3}. \tag{65}
$$

Note that

$$
\frac{1}{2(m+n)+3} = \frac{(2(m+n)+2)!}{(2(m+n)+3)!} = \frac{(1)_{2(m+n)+2}}{(1)_{2(m+n)+3}}.
$$

Employing

$$
(\lambda)_{2n} = 2^{2n}\left(\frac{\lambda}{2}\right)_n\left(\frac{\lambda+1}{2}\right)_n \quad (\lambda\in\mathbb{C},\ n\in\mathbb{Z}_{\geqslant 0}),
$$

one may find

$$
\frac{1}{2(m+n)+3} = \frac{1}{3}\,\frac{\left(\frac{3}{2}\right)_{m+n}}{\left(\frac{5}{2}\right)_{m+n}}. \tag{66}
$$

Substituting the right member of (66) for $\frac{1}{2(m+n)+3}$ in the right member of (65), in terms of (11), one may find

$$
F^{1:2;2}_{1:1;1}\left[\begin{matrix}\frac{3}{2}:1,\frac{1}{2};1,\frac{1}{2};\\ \frac{5}{2}:\ \ \frac{3}{2};\ \ \frac{3}{2};\end{matrix}-1,-1\right] = 3\int_0^1 (\arctan u)^2\,\mathrm{d}u.
$$

Finally, using the identity (22), one may obtain the formula involving the first expression and the third evaluation in (59). Similarly, using (20) in (22) may provide the formula involving the second expression and the third evaluation in (59).

Proofs of the other identities would parallel those of (59) by using the Formulas (23) through (27), one after the other. The specifics are omitted. □

Theorem 2. *The following identities hold:*

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix}\frac{3}{2}: \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \\ \frac{5}{2}: \quad \frac{3}{2}; \quad \frac{3}{2};\end{matrix} 1, 1\right] = \frac{3\pi^2}{4} - 6; \tag{67}$$

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix}2: \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \\ 3: \quad \frac{3}{2}; \quad \frac{3}{2};\end{matrix} 1, 1\right] = \frac{\pi^2}{4} - 1; \tag{68}$$

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix}\frac{5}{2}: \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \\ \frac{7}{2}: \quad \frac{3}{2}; \quad \frac{3}{2};\end{matrix} 1, 1\right] = \frac{5\pi^2}{12} - \frac{70}{27}; \tag{69}$$

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix}3: \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \\ 4: \quad \frac{3}{2}; \quad \frac{3}{2};\end{matrix} 1, 1\right] = \frac{15\pi^2}{64} - \frac{3}{4}; \tag{70}$$

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix}\frac{7}{2}: \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \\ \frac{9}{2}: \quad \frac{3}{2}; \quad \frac{3}{2};\end{matrix} 1, 1\right] = \frac{7\pi^2}{20} - \frac{2086}{1125}; \tag{71}$$

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix}1: \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \\ 2: \quad \frac{3}{2}; \quad \frac{3}{2};\end{matrix} 1, 1\right] = \frac{\pi^2 \log 2}{2} - \frac{7\zeta(3)}{4}; \tag{72}$$

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix}\frac{1}{2}: \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \\ \frac{3}{2}: \quad \frac{3}{2}; \quad \frac{3}{2};\end{matrix} 1, 1\right] = 4G - \frac{\pi^2}{4}. \tag{73}$$

Proof. With (16) and the identities in Lemma 4, the similar technique as in the proof of Theorem 1 may prove the results here. The details are omitted. □

Employing the reduction formula [1] (p. 29, Equation (36)) to the identities in Theorem 2, one may obtain a set of particular instances for ${}_3F_2(1)$ and ${}_4F_3(1)$ in the following corollary.

Corollary 1. *The following identities hold:*

$${}_3F_2\left[\begin{matrix}1, 1, 1; \\ 2, \frac{5}{2};\end{matrix} 1\right] = \frac{3\pi^2}{4} - 6; \tag{74}$$

$${}_3F_2\left[\begin{matrix}1, 1, 1; \\ 3, \frac{3}{2};\end{matrix} 1\right] = \frac{\pi^2}{4} - 1; \tag{75}$$

$${}_4F_3\left[\begin{matrix}\frac{5}{2}, 1, 1, 1; \\ \frac{7}{2}, \frac{3}{2}, 2;\end{matrix} 1\right] = \frac{5\pi^2}{12} - \frac{70}{27}; \tag{76}$$

$${}_4F_3\left[\begin{matrix}3, 1, 1, 1; \\ 4, \frac{3}{2}, 2;\end{matrix} 1\right] = \frac{15\pi^2}{64} - \frac{3}{4}; \tag{77}$$

$${}_4F_3\left[\begin{matrix}\frac{7}{2}, 1, 1, 1; \\ \frac{9}{2}, \frac{3}{2}, 2;\end{matrix} 1\right] = \frac{7\pi^2}{20} - \frac{2086}{1125}; \tag{78}$$

$${}_4F_3\left[\begin{matrix}1, 1, 1, 1; \\ 2, 2, \frac{3}{2};\end{matrix} 1\right] = \frac{\pi^2 \log 2}{2} - \frac{7\zeta(3)}{4}; \tag{79}$$

$${}_4F_3\left[\begin{matrix}\frac{1}{2}, 1, 1, 1; \\ \frac{3}{2}, \frac{3}{2}, 2;\end{matrix} 1\right] = 4G - \frac{\pi^2}{4}. \tag{80}$$

Theorem 3. *The following identities hold:*

$$F^{2:1;3}_{2:0;2}\left[\begin{matrix}2,2:1;1,1,1;\\3,\frac{7}{2}:-;\ 2,2;\end{matrix}1,1\right]=120-15\pi^2+\frac{5\pi^4}{16};\tag{81}$$

$$F^{2:1;3}_{2:0;2}\left[\begin{matrix}2,2:1;1,1,1;\\4,\frac{5}{2}:-;\ 2,2;\end{matrix}1,1\right]=\frac{9}{2}-\frac{9\pi^2}{8}+\frac{3\pi^4}{32};\tag{82}$$

$$F^{3:1;3}_{3:0;2}\left[\begin{matrix}2,2,\frac{7}{2}:1;1,1,1;\\3,\frac{5}{2},\frac{9}{2}:-;\ 2,2;\end{matrix}1,1\right]=\frac{3416}{81}-\frac{49\pi^2}{9}+\frac{7\pi^4}{48};\tag{83}$$

$$F^{3:1;3}_{3:0;2}\left[\begin{matrix}2,2,4:1;1,1,1;\\3,5,\frac{5}{2}:-;\ 2,2;\end{matrix}1,1\right]=3-\frac{51\pi^2}{64}+\frac{5\pi^4}{64};\tag{84}$$

$$F^{3:1;3}_{3:0;2}\left[\begin{matrix}2,2,\frac{9}{2}:\ 1;1,1,1;\\3,\frac{5}{2},\frac{11}{2}:-;\ 2,2;\end{matrix}1,1\right]=\frac{254{,}728}{9375}-\frac{447\pi^2}{125}+\frac{9\pi^4}{80};\tag{85}$$

$$F^{3:1;3}_{3:0;2}\left[\begin{matrix}2,2,2:\ 1;1,1,1;\\3,3,\frac{5}{2}:-;\ 2,2;\end{matrix}1,1\right]=\frac{\pi^4\log 2}{4}-\frac{9\pi^2\zeta(3)}{4}+\frac{93\zeta(5)}{8};\tag{86}$$

$$F^{3:1;3}_{3:0;2}\left[\begin{matrix}2,2,\frac{3}{2}:\ 1;1,1,1;\\3,\frac{5}{2},\frac{5}{2}:-;\ 2,2;\end{matrix}1,1\right]=18\pi^2G-144\beta(4)-\frac{3\pi^4}{16};\tag{87}$$

$$F^{2:1;3}_{2:0;2}\left[\begin{matrix}1,2:\ 1;1,1,1;\\3,\frac{5}{2}:-;\ 2,2;\end{matrix}1,1\right]=3\pi^2\log 2-\frac{\pi^4}{16}-\frac{21\zeta(3)}{2};\tag{88}$$

$$F^{3:1;3}_{3:0;2}\left[\begin{matrix}2,2,\frac{1}{2}:\ 1;1,1,1;\\3,\frac{3}{2},\frac{5}{2}:-;\ 2,2;\end{matrix}1,1\right]=8G+\pi^2G-8\beta(4)-\frac{\pi^4}{48}-\frac{\pi^2}{2}.\tag{89}$$

Proof. With (19) and the identities in Lemma 6, one may use the same technique as in the proof of Theorem 1 to find the results here. The details are omitted. □

4. Formulas for the Srivastava's General Triple Hypergeometric Series

This section establishes certain transformation, reduction and summation formulae for the Srivastava's general triple hypergeometric series (12).

Theorem 4. *The following identities hold:*

$$\begin{aligned}&F^{(3)}\left[\begin{matrix}2::-;-;-:1,\frac{1}{2};1,\frac{1}{2};1,\frac{1}{2};\\3::-;-;-:\ \frac{3}{2};\ \ \frac{3}{2};\ \ \frac{3}{2};\end{matrix}-1,-1,-1\right]\\&=F^{(3)}\left[\begin{matrix}\frac{3}{2},2::-;1;-\ :\ 1;1;\frac{1}{2},1;\\\frac{5}{2},3::-;2;-\ :-;-;\ \ \frac{3}{2};\end{matrix}-1,-1,-1\right]\\&=\frac{\pi^3}{16}+\frac{3\pi^2\log 2}{8}-3\pi G+\frac{63\,\zeta(3)}{16};\end{aligned}\tag{90}$$

$$\begin{aligned}&F^{(3)}\left[\begin{matrix}\frac{5}{2}::-;-;-:1,\frac{1}{2};1,\frac{1}{2};1,\frac{1}{2};\\\frac{7}{2}::-;-;-:\ \frac{3}{2};\ \ \frac{3}{2};\ \ \frac{3}{2};\end{matrix}-1,-1,-1\right]\\&=F^{(3)}\left[\begin{matrix}\frac{3}{2}::-;1;-\ :\ 1;1;\frac{1}{2},1;\\\frac{7}{2}::-;2;-\ :-;-;\ \ \frac{3}{2};\end{matrix}-1,-1,-1\right]\\&=\frac{5\pi^3}{64}-\frac{15\pi^2}{32}-\frac{15\,\pi\log 2}{8}+\frac{15G}{2};\end{aligned}\tag{91}$$

$$
\begin{aligned}
&F^{(3)}\left[\begin{matrix} 3 :: -;-;- : 1,\frac{1}{2};1,\frac{1}{2};1,\frac{1}{2}; \\ 4 :: -;-;- : \frac{3}{2}; \frac{3}{2}; \frac{3}{2}; \end{matrix} -1,-1,-1\right] \\
&= F^{(3)}\left[\begin{matrix} \frac{3}{2},3 :: -;1;- : 1;1;\frac{1}{2},1; \\ \frac{5}{2},4 :: -;2;- : -;-; \frac{3}{2}; \end{matrix} -1,-1,-1\right] \\
&= \frac{3\pi}{2} - 3\log 2 - \frac{3\pi^2}{8} - \frac{3\pi^2\log 2}{16} + \frac{\pi^3}{32} + \frac{3\pi G}{2} - \frac{63\zeta(3)}{32};
\end{aligned} \tag{92}
$$

$$
\begin{aligned}
&F^{(3)}\left[\begin{matrix} \frac{7}{2} :: -;-;- : 1,\frac{1}{2};1,\frac{1}{2};1,\frac{1}{2}; \\ \frac{9}{2} :: -;-;- : \frac{3}{2}; \frac{3}{2}; \frac{3}{2}; \end{matrix} -1,-1,-1\right] \\
&= F^{(3)}\left[\begin{matrix} \frac{3}{2},\frac{7}{2} :: -;1;- : 1;1;\frac{1}{2},1; \\ \frac{5}{2},\frac{9}{2} :: -;2;- : -;-; \frac{3}{2}; \end{matrix} -1,-1,-1\right] \\
&= \frac{7\pi}{8} - \frac{7}{4} + \frac{7\pi\log 2}{4} + \frac{7\pi^2}{32} - 7G;
\end{aligned} \tag{93}
$$

$$
\begin{aligned}
&F^{(3)}\left[\begin{matrix} \frac{3}{2} :: -;-;- : 1,\frac{1}{2};1,\frac{1}{2};1,\frac{1}{2}; \\ \frac{5}{2} :: -;-;- : \frac{3}{2}; \frac{3}{2}; \frac{3}{2}; \end{matrix} -1,-1,-1\right] \\
&= F^{(3)}\left[\begin{matrix} \frac{3}{2},\frac{3}{2} :: -;1;- : 1;1;\frac{1}{2},1; \\ \frac{5}{2},\frac{5}{2} :: -;2;- : -;-; \frac{3}{2}; \end{matrix} -1,-1,-1\right] \\
&= \frac{9\pi^2 G}{16} - \frac{9\beta(4)}{2};
\end{aligned} \tag{94}
$$

$$
\begin{aligned}
&F^{(3)}\left[\begin{matrix} 1 :: -;-;- : 1,\frac{1}{2};1,\frac{1}{2};1,\frac{1}{2}; \\ 2 :: -;-;- : \frac{3}{2}; \frac{3}{2}; \frac{3}{2}; \end{matrix} -1,-1,-1\right] \\
&= F^{(3)}\left[\begin{matrix} \frac{3}{2},1 :: -;1;- : 1;1;\frac{1}{2},1; \\ \frac{5}{2},2 :: -;2;- : -;-; \frac{3}{2}; \end{matrix} -1,-1,-1\right] \\
&= \frac{3\pi^2\log 2}{16} - \frac{\pi^3}{32} + \frac{3\pi G}{2} - \frac{105\zeta(3)}{32};
\end{aligned} \tag{95}
$$

$$
\begin{aligned}
&F^{(3)}\left[\begin{matrix} \frac{1}{2} :: -;-;- : 1,\frac{1}{2};1,\frac{1}{2};1,\frac{1}{2}; \\ \frac{3}{2} :: -;-;- : \frac{3}{2}; \frac{3}{2}; \frac{3}{2}; \end{matrix} -1,-1,-1\right] \\
&= F^{(3)}\left[\begin{matrix} \frac{3}{2},\frac{1}{2} :: -;1;- : 1;1;\frac{1}{2},1; \\ \frac{5}{2},\frac{3}{2} :: -;2;- : -;-; \frac{3}{2}; \end{matrix} -1,-1,-1\right] \\
&= \frac{3\pi\log 2}{8} - \frac{\pi^3}{64} - \frac{3\pi^2}{32} + \frac{3G}{2}.
\end{aligned} \tag{96}
$$

Proof. Using (21) and the formulas in Lemma 3, one may employ the technique in the proof of Theorem 1 to obtain the results here. The specifics are omitted. □

Theorem 5. *The following identities hold:*

$$
\begin{aligned}
&F^{(3)}\left[\begin{matrix} 2 :: -;-;- : \frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}; \\ 3 :: -;-;- : \frac{3}{2}; \frac{3}{2}; \frac{3}{2}; \end{matrix} 1,1,1\right] \\
&= F^{2:1;3}_{2:0;2}\left[\begin{matrix} \frac{3}{2},\frac{3}{2} : 1;1,\frac{1}{2},\frac{1}{2}; \\ 3,\frac{5}{2} : -; \frac{3}{2},\frac{3}{2}; \end{matrix} 1,1\right] \\
&= 24 - 12\pi + \frac{\pi^3}{2};
\end{aligned} \tag{97}
$$

$$
\begin{aligned}
&F^{(3)}\left[\begin{matrix} \frac{5}{2} :: -;-;- : \frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}; \\ \frac{7}{2} :: -;-;- : \frac{3}{2}; \frac{3}{2}; \frac{3}{2}; \end{matrix} 1,1,1\right] \\
&= F^{2:1;3}_{2:0;2}\left[\begin{matrix} \frac{3}{2},\frac{3}{2} : 1;1,\frac{1}{2},\frac{1}{2}; \\ 2,\frac{7}{2} : -; \frac{3}{2},\frac{3}{2}; \end{matrix} 1,1\right] \\
&= \frac{5\pi^3}{32} - \frac{15\pi}{16};
\end{aligned} \tag{98}
$$

$$
\begin{aligned}
&F^{(3)}\left[\begin{matrix}3::-;-;-:\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2};\\ 4::-;-;-:\quad\frac{3}{2};\quad\frac{3}{2};\quad\frac{3}{2};\end{matrix}1,1,1\right]\\
&=F^{3:1;3}_{3:0;2}\left[\begin{matrix}3,\frac{3}{2},\frac{3}{2}:1;1,\frac{1}{2},\frac{1}{2};\\ 2,4,\frac{5}{2}:-;\ \frac{3}{2},\ \frac{3}{2};\end{matrix}1,1\right]\\
&=\frac{80}{9}-\frac{42\pi}{9}+\frac{\pi^3}{4};
\end{aligned}\tag{99}
$$

$$
\begin{aligned}
&F^{(3)}\left[\begin{matrix}\frac{7}{2}::-;-;-:\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2};\\ \frac{9}{2}::-;-;-:\quad\frac{3}{2};\quad\frac{3}{2};\quad\frac{3}{2};\end{matrix}1,1,1\right]\\
&=F^{3:1;3}_{3:0;2}\left[\begin{matrix}\frac{3}{2},\frac{3}{2},\frac{7}{2}:1;1,\frac{1}{2},\frac{1}{2};\\ 2,\frac{5}{2},\frac{9}{2}:-;\ \frac{3}{2},\ \frac{3}{2};\end{matrix}1,1\right]\\
&=\frac{35\pi^3}{256}-\frac{357\pi}{512};
\end{aligned}\tag{100}
$$

$$
\begin{aligned}
&F^{(3)}\left[\begin{matrix}4::-;-;-:\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2};\\ 5::-;-;-:\quad\frac{3}{2};\quad\frac{3}{2};\quad\frac{3}{2};\end{matrix}1,1,1\right]\\
&=F^{3:1;3}_{3:0;2}\left[\begin{matrix}4,\frac{3}{2},\frac{3}{2}:1;1,\frac{1}{2},\frac{1}{2};\\ 2,5,\frac{5}{2}:-;\ \frac{3}{2},\ \frac{3}{2};\end{matrix}1,1\right]\\
&=\frac{33152}{5625}-\frac{1192\pi}{375}+\frac{\pi^3}{5};
\end{aligned}\tag{101}
$$

$$
\begin{aligned}
&F^{(3)}\left[\begin{matrix}\frac{3}{2}::-;-;-:\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2};\\ \frac{5}{2}::-;-;-:\quad\frac{3}{2};\quad\frac{3}{2};\quad\frac{3}{2};\end{matrix}1,1,1\right]\\
&=F^{3:1;3}_{3:0;2}\left[\begin{matrix}\frac{3}{2},\frac{3}{2},\frac{3}{2}:1;1,\frac{1}{2},\frac{1}{2};\\ 2,\frac{5}{2},\frac{5}{2}:-;\ \frac{3}{2},\ \frac{3}{2};\end{matrix}1,1\right]\\
&=\frac{3\pi^3\log 2}{8}-\frac{27\pi\zeta(3)}{16};
\end{aligned}\tag{102}
$$

$$
\begin{aligned}
&F^{(3)}\left[\begin{matrix}1::-;-;-:\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2};\\ 2::-;-;-:\quad\frac{3}{2};\quad\frac{3}{2};\quad\frac{3}{2};\end{matrix}1,1,1\right]\\
&=F^{3:1;3}_{3:0;2}\left[\begin{matrix}1,\frac{3}{2},\frac{3}{2}:1;1,\frac{1}{2},\frac{1}{2};\\ 2,2,\frac{5}{2}:-;\ \frac{3}{2},\ \frac{3}{2};\end{matrix}1,1\right]\\
&=12\pi G-21\zeta(3)-\frac{\pi^3}{4};
\end{aligned}\tag{103}
$$

$$
\begin{aligned}
&F^{(3)}\left[\begin{matrix}\frac{1}{2}::-;-;-:\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2};\\ \frac{3}{2}::-;-;-:\quad\frac{3}{2};\quad\frac{3}{2};\quad\frac{3}{2};\end{matrix}1,1,1\right]\\
&=F^{2:1;3}_{2:0;2}\left[\begin{matrix}\frac{1}{2},\frac{3}{2}:1;1,\frac{1}{2},\frac{1}{2};\\ 2,\frac{5}{2}:-;\ \frac{3}{2},\ \frac{3}{2};\end{matrix}1,1\right]\\
&=\frac{3\pi\log 2}{2}-\frac{\pi^3}{16}.
\end{aligned}\tag{104}
$$

Proof. With (16), (18) and the identities in Lemma 5, one may use the technique in the proof of Theorem 1 to derive the results here. The specifics are omitted. □

5. Concluding Remarks and a Question

The vast popularity and immense usefulness of the hypergeometric function and the generalized hypergeometric functions of one variable have inspired and stimulated a large number of researchers to introduce and investigate hypergeometric functions of two or more variables. Among a variety of proposed two-variable hypergeometric functions, many researchers have presented a number of intriguing and useful identities and properties for the general Kampé de Fériet function in (11).

In this paper, in connection with the known integral formulas for arcsin and arctan functions, we provided a number of explicit formulas for the general Kampé de Fériet function with specific parameters and variables, including some related transformation formulas. In the same line, we also explored the general Srivastava's triple hypergeometric function.

Applying the known relations for the inverse trigonometric functions to the recursive improper integral formulas involving powers of $\sin^{-1} u$ and $\tan^{-1} u$ in [30], for instance,

$$\sin^{-1} u + \cos^{-1} u = \pi/2 \quad \text{and} \quad \tan^{-1} u + \cot^{-1} u = \pi/2,$$

the corresponding recursive improper integral formulas involving powers of $\cos^{-1} u$ and $\cot^{-1} u$ may be obtained and utilized to provide certain similar identities offered in this article.

The formulas in Corollary 1 may also be derived by using (17) and the identities in Lemma 4. The known identities (1) and (3) are listed, respectively, in (63) and (72), for the sake of completeness.

It may be interesting to compare (75) with the following known formula (see [9] (p. 544, 7.4.4–210)):

$$ {}_3F_2\left[\begin{matrix}1,\, 1,\, 1; \\ 2,\, \frac{3}{2};\end{matrix}\, 1\right] = \frac{\pi^2}{4}. \tag{105}$$

Other formulas for ${}_4F_3(1)$ in Corollary 1 do not appear in [9] (pp. 558–560). In addition, it may be intriguing to compare (79) with the following easily-derivable identity:

$$ {}_4F_3\left[\begin{matrix}1,\, 1,\, 1,\, 1; \\ 2,\, 2,\, 2;\end{matrix}\, 1\right] = \zeta(3), \tag{106}$$

which can also be obtained by taking the limits on both sides of the known formula [9] (p. 554, 7.5.3–9) as $a \to 1$ and then $b \to 1$.

Question: Can the results presented in Theorems 1–5 and more similar results be computed by the use of some computer-aid programs, for instance, Mathematica? Or, is there an AI (artificial intelligence) capable of handling the issues discussed in this article?

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