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Algebra of the Symmetry Operators of the Klein–Gordon–Fock Equation for the Case When Groups of Motions G_3 Act Transitively on Null Subsurfaces of Spacetime

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Abstract: The algebras of the symmetry operators for the Hamilton–Jacobi and Klein–Gordon–Fock equations are found for a charged test particle, moving in an external electromagnetic field in a spacetime manifold on the isotropic (null) hypersurface, of which a three-parameter groups of motions acts transitively. We have found all admissible electromagnetic fields for which such algebras exist. We have proved that an admissible field does not deform the algebra of symmetry operators for the free Hamilton–Jacobi and Klein–Gordon–Fock equations. The results complete the classification of admissible electromagnetic fields, in which the Hamilton–Jacobi and Klein–Gordon–Fock equations admit algebras of motion integrals that are isomorphic to the algebras of operators of the r-parametric groups of motions of spacetime manifolds if $(r \leq 4)$.

Keywords: Klein–Gordon–Fock equation; algebra of symmetry operators; theory of symmetry; separation of variables; linear partial differential equations

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1. Introduction

The Klein–Gordon–Fock equation describes the dynamics of massive spinless test particles interacting with fields of a gauge nature. It is used to study quantum field effects in external electromagnetic and gravitational fields for scalar particles, as well as to build approximate models for fermions. In this case, the problem of finding the exact basic solutions of the Klein–Gordon–Fock equation in the external intensive fields is of great importance. The basic solution is a common eigenfunction of the complete set of symmetry operators. In order to obtain the exact basic solution, it is necessary to find a commutative algebra consisting of three linear differential symmetry operators no more than quadratic in momenta. The problem of constructing such algebras has been sufficiently studied. Suppose this algebra forms the traditional complete set of symmetry operators for the Klein–Gordon–Fock equation. In this case, spacetime admits a complete set of geometric objects, consisting of mutually commuting vector and tensor killing fields, and belongs to the set of Stäckel spaces.

A Stäckel space (V_n) is an n-dimensional Riemannian space of an arbitrary signature, in which the free n-dimensional Hamilton–Jacobi equation for a massive test particle is integrated by the method of the complete separation of variables. The Stäckel space admits the complete set of killing fields. It is proved that the n-dimensional Klein–Gordon–Fock equation can be integrated by the method of complete separation of variables, only if it admits the traditional complete set of symmetry operators.

This is only possible in certain classes of Stäckel spaces. The method for finding basic solutions based on the complete separation of variables is also called the commutative integration method. For information on the method of the complete separation of the variables and results obtained with its help, see [1–16] and the articles cited there.

In the paper [17,18], a method for the integration of linear partial differential equations in n-dimensional Riemannian spaces (and also in the Hamilton–Jacobi equation) of an arbitrary signature admitting noncommutative groups of motions (G_r) was proposed (these spaces are also denoted by V_n) The algebras of the symmetry operators of the Klein–Gordon–Fock equation of rank $r(n-1 \leq r \leq n)$, constructed using the algebras of operators of the noncommutative group of motions of the space V_n, are complemented to a commutative algebra by the operators of differentiation of the first order in n essential parameters. Basic solutions are found using these parameters. By analogy with the method of complete separation of variables, such algebras are called complete sets, and the integration method is called noncommutative integration. The methods are related because they both reduce the problem of finding the basic solution of the test particle equation of motion to the problem of integrating systems of ordinary differential equations.

The noncommutative integration method is based on the complete classification of spacetime manifolds admitting groups of motions, as described in the book [19]. The method made it possible to considerably extend the set of fields in which the construction of a complete system of solutions of the classical and quantum equations of a charged test particle motion is reduced to the integration of compatible systems of first-order differential equations. For the further development of the method and its application in gravitational theory, it is necessary (using the proposed classification) to make a classification of electromagnetic fields in which the classical and quantum equations of motion of a charged test particle (the Hamilton–Jacobi and Klein–Gordon–Fock equations) admits noncommutative algebras of symmetry operators that are linear in momenta. Such electromagnetic fields are called admissible.

For the first time, this problem was formulated and partially solved in [20,21], where the potentials of all admissible electromagnetic fields in spacetime manifolds, admitting the transitive action of four-parameter groups of motions, are given. A similar classification problem was solved for homogeneous spaces with a three-parameter group of motions [22], as well as for spaces with a two-parameter movement group [23]. Moreover, the problem is solved for the case when a four-parameter group of motions, with a three-dimensional hypersurface of transitivity, acts on a spacetime manifold [24]. In the present work, the classification of admissible fields is carried out when the three-parameter group of motions (G_3) acts transitively on the isotropic hypersurfaces of the space (V_4) with a spacetime signature. We have found all relevant admissible electromagnetic fields.

The article is organized as follows.

The second section contains the necessary information and definitions required for the implementation of this classification. The conditions that must be met by admissible electromagnetic fields are obtained and investigated for compatibility.

In the third and fifth sections, the obtained conditions are used to find the potential of the admissible electromagnetic field for resolvable groups of motion. The cases of groups with a singular operator are considered separately.

In the fourth section, unsolvable groups of motion are considered.

In conclusion, possible applications of the obtained results are considered.

2. Admissible Electromagnetic Fields

2.1. Conditions for the Existence of the Symmetry Operators Algebra in the Case of a Charged Test Particle Motion

Consider a spacetime manifold (V_4) on an null hypersurface, on which the three-parameter movement group (G_3) acts transitively. The coordinate indices of the canonical coordinate system $[u^i]$ of the space V_4 are denoted by lower case Latin letters: $i, j, k = 0, 1 \ldots 3$. The coordinate indices of the canonical coordinate system on the isotropic hypersurface V_3^* will be denoted by lower case Greek letters: α, β, γ=1, …3. A non-ignored variable is indexed as 0. The repeated superscripts and subscripts are summed within the limits of the indices change. The papers [20,22] show that, for a charged test particle in an external electromagnetic field, with potential A_i, the Hamilton–Jacobi and Klein–Gordon–Fock equations:

$$H = g^{ij}P_iP_j = m, \quad P_i = p_i + A_i, \quad p_i = \partial_i\varphi \tag{1}$$

$$\hat{H}\varphi = (g^{ij}\hat{P}_i\hat{P}_j)\varphi = m\varphi, \quad \hat{P}_j = -\imath\hat{\nabla}_i + A_i, \tag{2}$$

admit algebras of symmetry operators (in the case of the Hamilton–Jacobi equation, integrals of motion) in the same electromagnetic fields.

Here, $\hat{\nabla}_i$ is the operator of the covariant derivative corresponding to the operator of the partial derivative $\hat{\partial}_i = \imath\hat{p}_i$ along the coordinate u^i; φ is the field of a scalar particle with mass m.

Therefore, for the implementation of the admissible electromagnetic fields classification, the Hamilton–Jacobi equation would suffice. The integrals of motion of the free Hamilton–Jacobi equation have the form:

$$Y_\alpha = \xi^i_\alpha p_i, \tag{3}$$

where ξ^j_α are the killing vector fields, satisfying the equations:

$$g^{ik}\xi^j_{\alpha,k} + g^{jk}\xi^i_{\alpha,k} - g^{ij}_{,k}\xi^k_\alpha = 0. \tag{4}$$

ξ^j_α defines the movement groups (G_3) of the space V_4. It can be shown that if the Equation (1) has r independent integrals of motion of the first order, then these operators have the form Equation (2). The Hamilton–Jacobi equation Equation (1) admits a motion integral of the form if H and Y_α commute, with respect to the Poisson brackets:

$$[H, Y_\alpha]_P = \frac{\partial H}{\partial p_i}\frac{\partial Y_\alpha}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial Y_\alpha}{\partial p_i} = \tag{5}$$

$$(g^{ik}\zeta^j_{\alpha,k} + g^{jk}\zeta^i_{\alpha,k} - g^{ij}_{,k}\zeta^k_\alpha)P_iP_j + 2g^{ik}(\zeta^j_\alpha F_{ji} + (\zeta^\beta_\alpha A_\beta)_{,i})P_k = 0.$$

It is possible if and only if the potential of the electromagnetic field satisfies the system of equations:

$$(\xi^j_\alpha A_j)_{,i} = \xi^j_\alpha F_{ij}, \quad F_{ji} = A_{i,j} - A_{j,i}. \tag{6}$$

Unlike the free Hamilton–Jacobi equation, the Equation (1), in a space with a group of motions, in the general case, has no integrals of motion. The system of Equation (5) defines the set of admissible electromagnetic fields, in which Equation (1) has the first-order r integrals of motion, given by the algebra of the group G_3. It can be shown [24] that, since the vector fields (ξ^j_α) define the movement group of the space (V_4), the set Equation (5) can be represented in the form:

$$\mathbf{A}_{\alpha|\beta} = C^\gamma_{\beta\alpha}\mathbf{A}_{\text{fl}}, \tag{7}$$

$$A_{0|\alpha} = -\xi^i_{\alpha,0}A_i, \tag{8}$$

where $\mathbf{A}_\alpha = \xi^i_\alpha A_i$, $A_\alpha = \lambda^\beta_\alpha\mathbf{A}_\beta$, $X_{|\alpha} = \xi^i_\alpha X_{,i}$, $\lambda^\beta_\alpha\xi^\gamma_\beta = \delta^\gamma_\beta$, $C^\gamma_{\alpha\beta}$—structural constants of the group G_r. For arbitrary r and n, the following statement is true [24]:

If the group of motions (G_r) of the space (V_n) acts transitively on the subspace (V_r,), Equations (6) and (7) form a completely integrable system. This system specifies the necessary and sufficient conditions for the existence of symmetry operators that are linear in momenta.

2.2. Notations and Necessary Information from Petrov Group Classification

Petrov classification of spacetime manifolds (V_4,), according to the groups of motions (G_r), is based on the works of Petrov, Fubini, and Kruchkovich (see [19]). The method of constructing the classification consists of using the group structural constants to find the killing vector fields components in the simplest (canonical) holonomic coordinate system.

Then, the integration of the killing equations allows for determining the components of the metric tensor. Structural constant groups (G_3) are known, due to the classification of real groups of motions by real non-isomorphic structures for two- and three-parameter Bianchi groups [25].

According to Bianchi classification, there are nine nonisomorphic structures for three-parameter movement groups (G_3).

Seven classes consist of solvable groups (containing a two-parameter subgroup (G_2)).

$$\begin{cases} G_3(I): & C^{\gamma}_{\alpha\beta} = 0; \\ G_3(II): & C^{\alpha}_{12} = 0, \quad C^{\alpha}_{13} = 0 \quad C^{\alpha}_{23} = \delta^{\alpha}_1; \\ G_3(III): & C^{\alpha}_{12} = 0, \quad C^{\alpha}_{13} = \delta^{\alpha}_1 \quad C^{\alpha}_{23} = 0; \\ G_3(IV): & C^{\alpha}_{12} = 0, \quad C^{\alpha}_{13} = \delta^{\alpha}_1 \quad C^{\alpha}_{23} = \delta^{\alpha}_1 + \delta^{\alpha}_2; \\ G_3(V): & C^{\alpha}_{12} = 0, \quad C^{\alpha}_{13} = \delta^{\alpha}_1 \quad C^{\alpha}_{23} = \delta^{\alpha}_2; \\ G_3(VI): & C^{\alpha}_{12} = 0, \quad C^{\alpha}_{13} = \delta^{\alpha}_1, \quad C^{\alpha}_{23} = q\delta^{\alpha}_2. \quad (q \neq 0, 1); \\ G_3(VII): & C^{\alpha}_{12} = 0, \quad C^{\alpha}_{13} = \delta^{\alpha}_1 \quad C^{\alpha}_{23} = 2\delta^{\alpha}_2 \cos\alpha, \quad \alpha = const. \end{cases} \tag{9}$$

Two classes consist of unsolvable groups:

$$\begin{cases} G_3(VIII): & C^{\alpha}_{12} = \delta^{\alpha}_1, \quad C^{\alpha}_{13} = 2\delta^{\alpha}_2 \quad C^{\alpha}_{23} = -\delta^{\alpha}_3. \\ G_3(IX): & C^{\alpha}_{12} = \delta^{\alpha}_3, \quad C^{\alpha}_{13} = -\delta^{\alpha}_2 \quad C^{\alpha}_{23} = \delta^{\alpha}_1. \end{cases} \tag{10}$$

Crucial step in constructing the Petrov classification is to find the canonical coordinate system. Since, in our case, G_3 acts transitively on the isotropic (null) hypersurface (V^*_3), the canonical coordinate system can be chosen as semi-geodesic. In this case, the hypersurface itself will be given by the equation:

$$u^0 = const.$$

Groups ($G_3(N)$), except $G_3(IX)$, have a two-parameter subgroup (G_2). Thus, we can first construct the operators of the subgroup, and then define the canonical coordinate system. If the subgroup (G_2) is abelian and contains a singular operator, it acts on the null subspace (V^*_2) of the hypersurface (V^*_3). In this case, the canonical coordinate system can be chosen, so that the operators of the group X_1, X_2 have the form:

$$X_1 = p_1, \quad X_2 = p_2.$$

In the case when G_2 does not contain a special operator, the operators of the group X_1, X_2, in the canonical coordinate system, can be reduced to one of the following forms:

$$\begin{cases} A: & X_1 = p_2, \quad X_2 = p_3; \\ B: & X_1 = p_2, \quad X_2 = p_1 + u^3 p_2; \\ C: & X_1 = p_2, \quad X_2 = p_2 + u^0 p_2. \end{cases} \tag{11}$$

Subgroups (G_2), in this case, are denoted as $G_2(K)$, where K can take the value A, B, C. Among the unsolvable groups, only the group $G_2(VIII)$ contains an (abelian) subgroup G_2. By choosing the canonical coordinate system, the operators X_1, X_2 can be reduced to the form:

$$X_1 = p_2, \quad X_2 = p_1 + u^2 p_2.$$

The operator X_3 is found from the equations of the structure, whereupon the killing equations are integrated.

The group $G_3(IX)$ has no subgroup (G_2) or special operator. In this case, the operator X_1 can be reduced to the form: $X_1 = p_1$. The remaining operators of the group and canonical coordinate system follow from the equations of the structure.

Note that, in all cases, the metric tensor components contain specific functions of the variables of the local coordinate system $[u^\alpha]$ of the hypersurface (V^*_3) (we call them ignored) and arbitrary functions of the variable u^0 (we call this variable non-ignored). As before, we will stick to the notations accepted in the work of A.Z. Petrov [19], with minor exceptions. For example, a non-ignored variable (x^4) would be denoted by u^0, etc. The letters $a, b, \alpha, \beta, \gamma$ (with and without indices) denote functions that depend only on the variable u^0).

3. Solvable G_3 Groups. Killing Vector Fields Do Not Depend on a Non-Ignored Variable

When a three-parameter group of motions acts transitively on a null hypersurface, the components of the vector ξ^i_α may depend on the nonignored variable u^0. In this section, we consider groups in which $(\xi^\alpha_\beta)_{,0} = 0$. According to Equation (7), this implies: $A_0 = A_0(u^0)$, which is equivalent to the condition:

$$A_0 = 0.$$

This is used in Sections Equations (3) and (4). Each subsection is devoted to integration of Equations (6) and (7) for specific groups. The metrics and group operators are given in the canonical coordinate system (taken from [19])).

In addition, the explicit form of equations Equation (6), and its solutions are given, as well as holonomic components of the vector potential A_i, which are calculated in accordance with the given relations to Equations (6) and (7). Note another fact that distinguishes the variant considered in this paper from the case with homogeneous spaces. For all G_3 groups (except $G_3(IX)$) acting on isotropic (null) hypersurfaces, there are several nonequivalent sets of killing vectors, depending on which $G_2(K)$ subgroup they contain (see (11)). Therefore, for each such set, there are several nonequivalent solutions of the killing equations. Groups $G_3(N)$, with a subgroup $G_2(K)$, will be denoted as $G_3(N[K])$. In the following the results of the sets of Equations (7) and (8) integration integration in the following order. First, using information from work [19], the metrics of the spaces on which the considered groups, group operators, and structure constants act are presented. Then, the matrix $\hat{\lambda}$, as well as the nonholonomic and holonomic components of the vector potential of the electromagnetic field ($\mathbf{A}_\alpha$ and A_α), are given (with explanations of the integration procedure, if necessary).

3.1. *Groups $G_3(II) - G_3(VI)$ with the Singular Operators*

If the groups $G_3(II) - G_3(VI)$ acts on the hypersurface V^*_3, the subgroup G_2 may contain a singular operator. In this case, the subgroup G_2 acts on the null subspace V^*_2 of the hypersurface V^*_3. The metrics of appropriate spaces and group operators can be represented as:

$$ds^2 = 2\exp{(-ku^3)}du^0(du^1 - \varepsilon u^3 du^2) + a_1 \exp{(-2lu^3)}du^{2^2} + 2a_2 \exp{(-lu^3)}du^2 du^3 + a_3 du^{2^2}.$$

The group operators can be presented as follows:

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = (ku^1 + \varepsilon u^2)p_1 + lu^2 p_2 + p_3$$

From here the structural constants follow:

$$C^\gamma_{12} = 0, \quad C^\alpha_{13} = k\delta^\alpha_1, \quad C^\gamma_{23} = \varepsilon\delta^\alpha_1 + l\delta^\alpha_2.$$

Matrix $\hat{\lambda}$, has the form:

$$||\lambda^\alpha_\beta|| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -(ku^1 + \varepsilon u^2) & -lu^2 & 1 \end{pmatrix}.$$

Set Equation (6) can be represented in the form:

$$\mathbf{A}_{1|\beta} = \delta_{3\beta} k \mathbf{A_1} \rightarrow \mathbf{A_1} = \alpha_1(u^0) \exp(-ku^3);$$

$$\mathbf{A}_{2|\beta} = -\delta_{3\beta}(\varepsilon \mathbf{A_1} + l\mathbf{A_2});$$

$$\mathbf{A}_{3|\beta} = k\delta_{1\beta}\mathbf{A_1} + \delta_{2\beta}(\varepsilon \mathbf{A_1} + l\mathbf{A_2})$$

One can find next solutions of the set:

$(\mathbf{A})$ $l \neq k.$

$$\mathbf{A_1} = \alpha_1 \exp(-ku^3);$$

$$\mathbf{A_2} = (\frac{\varepsilon}{k-l})\alpha_1 \exp(-ku^3) + \alpha_2 \exp(-lu^3);$$

$$\mathbf{A_3} = \alpha_3 + k(u^1 + \frac{\varepsilon u^2}{k-l})\alpha_1 \exp(-ku^3) + lu^2\alpha_2 \exp(-lu^3).$$

The holonomic components of the electromagnetic potential are as follows:

$$A_\alpha = \mathbf{A}_\beta \lambda^\beta_\alpha \tag{12}$$

and have the form:

$$A_1 = \alpha_1 \exp(-ku^3), \quad A_2 = (\frac{\varepsilon}{k-l})\alpha_1 \exp(-ku^3) + \alpha_2 \exp(-lu^3), \quad A_3 = \alpha_3.$$

$(\mathbf{B})$ $k = l.$

The non-holonomic components of the electromagnetic potential are as follows:

$$\mathbf{A_1} = \alpha_1 \exp(-ku^3);$$

$$\mathbf{A_2} = (\alpha_2 - \varepsilon\alpha_1 u^3) \exp(-ku^3);$$

$$\mathbf{A_3} = \alpha_3 + (k\alpha_1 u^1 + u^2(\varepsilon\alpha_1 + k(\alpha_2 - \varepsilon\alpha_1 u^3)) \exp(-ku^3).$$

The holonomic components of the electromagnetic potential are as follows:

$$A_1 = \alpha_1 \exp(-ku^3), \quad A_2 = (\alpha_2 - \varepsilon\alpha_1 u^3) \exp(-ku^3), \quad A_3 = \alpha_3.$$

This exhausts the classification of admissible electromagnetic fields for movement groups of spacetime with a special operator. Groups without a special operator are considered below.

3.2. *Groups* $G_3(III)$

The metrics of the spaces and the group operators can be represented as:

$$ds^2 = 2du^0du^1 + 2du^0(\frac{du^2 b_0 + du^3(b_1 - a_0u^1)}{u^3}) +$$

$$2du^2du^3(\frac{a_3 - a_1u^1}{u^{3^2}}) + du^{2^2}(\frac{a_1}{u^{3^2}}) + du^{3^2}(\frac{a_1 u^{1^2} - 2a_3u^1 + a_2}{u^{3^2}}).$$

The group operators can be presented as follows:

$$X_1 = p_1 + u^3 p_2, \quad X_2 = p_2, \quad X_3 = u^2 p_2 + u^3 p_3$$

From here the structural constants follow:

$$C^{\alpha}_{12} = C^{\alpha}_{31} = 0, \quad C^{\alpha}_{23} = \delta^{\alpha}_{2}.$$

$\hat{\lambda}$ has the form:

$$||\lambda^{\alpha}_{\beta}|| = \begin{pmatrix} 1 & -u^3 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{u^2}{u^3} & \frac{1}{u^3} \end{pmatrix}.$$

Set Equation (6) can be represented in the form:

$$\mathbf{A}_{1,\beta} = 0 \quad \mathbf{A}_{2|\beta} = -\delta_{3\beta}\mathbf{A}_2, \quad \mathbf{A}_{3|\beta} = \delta_{2\beta}\mathbf{A}_2 \rightarrow$$

$$\mathbf{A}_1 = \alpha_1, \quad \mathbf{A}_2 = \frac{\alpha_2}{u^3}; \quad \mathbf{A}_3 = \alpha_3 + \frac{\alpha_2 u^2}{u^3}.$$

The holonomic components of the electromagnetic potential are as follows:

$$A_1 = (\alpha_1 - \alpha_2), \quad A_2 = \frac{\alpha_2}{u^3}, \quad A_3 = \frac{\alpha_3}{u^3}.$$

3.3. *Groups* $G_3(IV[A])$

The metrics of the spaces and the group operators can be represented as:

$$ds^2 = 2du^0du^1 + 2du^0(b_0du^2 + (b_1 - b_0u^1)du^3))\exp(-u^1) + (a_1du^{2^2} +$$

$$2(a_3 - a_1u^1)du^2du^3 + (a_1u^{1^2} - 2a_2u^1 + a_3)du^{3^2})\exp(-2u^1).$$

The group operators can be presented as follows:

$$X_1 = p_1 + (u^2 + u^3)p_2 + u^3p_3, \quad X_2 = p_2, \quad X_3 = p_3$$

From here the structural constants follow:

$$C^{\gamma}_{12} = \delta^{\alpha}_{2}, \quad C^{\gamma}_{31} = \delta^{\alpha}_{2} + \delta^{\alpha}_{2}, \quad C^{\alpha}_{23} = 0.$$

Matrix $\hat{\lambda}$ has the form:

$$||\lambda^{\alpha}_{\beta}|| = \begin{pmatrix} 1 & -(u^2 + u^3) & -u^3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Set Equation (6) can be represented in the form:

$$\mathbf{A_{1,1}} + \mathbf{A_2}(u^2 + 2u^3) + u^3\mathbf{A_3} = 0, \quad \mathbf{A_{1,2}} = \mathbf{A_2}, \quad \mathbf{A_{1,3}} = \mathbf{A_2} + \mathbf{A_3};$$

$$\mathbf{A_{2,1}} = -\mathbf{A_1}, \quad \mathbf{A_{2,2}} = \mathbf{A_{2,3}} = 0;$$

$$\mathbf{A_{3,1}} = -(\mathbf{A_2} + \mathbf{A_3}), \quad \mathbf{A_{3,2}} = \mathbf{A_{3,3}} = 0;$$

$$\rightarrow \quad \mathbf{A_1} = \alpha_1 + (u^2 + u^3)\mathbf{A_2} + u^3\mathbf{A_3}, \quad \mathbf{A_2} = \alpha_2\exp(-u^1);$$

$$\mathbf{A_3} = \alpha_3\exp(-u^1) - u^1\mathbf{A_2}.$$

The holonomic components of the electromagnetic potential are as follows:

$$A_1 = \alpha_1, \quad A_2 = \alpha_2 \exp -u^1, \quad A_3 = (\alpha_3 - \alpha_2u^1)\exp -u^1.$$

3.4. *Groups* $G_3(IV[B])$

The metrics of the spaces and the group operators can be represented as:

$$ds^2 = 2du^0 du^1 \exp u^3 + 2du^0 du^3 a_0 +$$

$$2du^2 du^3 (a_3 \exp u^3 - a_2 u^1 \exp 2u^3) + du^{2^2} a_1 \exp 2u^3 + du^{3^2} (u^{1^2} a_1 \exp 2u^3 - 2a_3 u^1 \exp u^3 + a_2).$$

The group operators can be presented as follows:

$$X_1 = p_2 + u^3 p_2, \quad X_2 = p_1, +u^3 p_2 \quad X_3 = u^1 p_1 + u^2 p_2 - p_3$$

From here the structural constants follow:

$$C^\alpha_{12} = 0, \quad C^\alpha_{31} = \delta^\alpha_1, \quad C^\alpha_{23} = \delta^\alpha_1 + \delta^\alpha_2.$$

$\hat{\lambda}$ has the form:

$$||\lambda^\alpha_\beta|| = \begin{pmatrix} -u^3 & 1 & 0 \\ 1 & 0 & 0 \\ (u^2 - u^1 u^3) & u^1 & -1. \end{pmatrix}.$$

Set Equation (6) can be represented in the form:

$$\mathbf{A_{1,2}} = \mathbf{A_{1,1}} = 0, \quad \mathbf{A_{1,3}} = \mathbf{A_1}, \quad \mathbf{A_{2,2}} = \mathbf{A_{2,1}} = 0, \quad \mathbf{A_{2,3}} = \mathbf{A_1} + \mathbf{A_2},$$

$$\mathbf{A_{3,2}} = \mathbf{A_1}, \quad \mathbf{A_{3,1}} + u^3 \mathbf{A_{3,2}} = \mathbf{A_1} + \mathbf{A_2}, \quad u^1 \mathbf{A_{3,1}} + u^2 \mathbf{A_{3,2}} - \mathbf{A_{3,3}} = 0.$$

One can find next solutions of the set:

$$\mathbf{A_1} = \alpha_2 \exp u^3, \quad \mathbf{A_2} = (\alpha_1 u^3 + \alpha_2) \exp u^3, \quad \mathbf{A_3} = -\alpha_3 + (\alpha_1 (u^2 + u^1) + \alpha_2 u^1) \exp u^3.$$

The holonomic components of the electromagnetic potential are as follows:

$$A_1 = \alpha_2 \exp u^3, \quad A_2 = \alpha_1 \exp u^3, \quad A_3 = \alpha_3 - \alpha_1 u^1 \exp u^3.$$

3.5. Groups $G_3(V[A]) - G_3(VI[A])$

The metrics of the spaces and the group operators can be represented as:

$$ds^2 = 2du^0 du^1 + 2du^0 (b_0 du^2 + (b_1 - b_0 u^1) du^3)) \exp(-u^1) + (a_1 du^{2^2} +$$
$$2(a_3 - a_1 u^1) du^2 du^3 + (a_1 u^{1^2} - 2a_2 u^1 + a_3) du^{3^2}) \exp(-2u^1).$$

$$ds^2 = 2du^0 du^1 + 2du^0 (b_0 du^2 \exp(-u^1) + b_1 du^3 \exp(-q u^1)) + a_1 du^{2^2} \exp(-2u^1) +$$
$$2a_3 du^2 du^3 \exp(-(q+1) u^1) + a_3 du^{3^2} \exp(2q u^1).$$

The group operators can be presented as follows:

$$X_1 = p_1 + u^2 p_2 + q u^3 p_3, \quad X_2 = p_2, \quad X_3 = p_3$$

From here the structural constants follow:

$$C^\gamma_{12} = \delta^\alpha_2, \quad C^\gamma_{31} = q\delta^\alpha_3, \quad C^\alpha_{23} = 0.$$

If $q = 1$, the group of motions is of type $G_3(V)$. In opposite case it has type $G_3(VI)$.

Matrix $\hat{\lambda}$ has the form:

$$||\lambda^{\alpha}_{\beta}|| = \begin{pmatrix} 1 & -u^2 & -qu^3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Set Equation (6) can be represented in the form:

$$\mathbf{A_{1,1}} + u^2\mathbf{A_{1,3}} + q^2u^3\mathbf{A_{1,3}} = 0, \quad \mathbf{A_{1,2}} = \mathbf{A_2}, \quad \mathbf{A_{1,3}} = q\mathbf{A_3};$$

$$\mathbf{A_{2,1}} = -\mathbf{A_2}, \quad \mathbf{A_{3,1}} = -q\mathbf{A_3}, \quad \mathbf{A_{3,2}} = \mathbf{A_{3,3}} = \mathbf{A_{2,2}} = \mathbf{A_{2,3}} = 0;$$

$$\rightarrow \mathbf{A_1} = \alpha_1 + qu^3\alpha_3 \exp -qu^1 + u^2\alpha_2 \exp -u^1, \quad \mathbf{A_2} = \alpha_2 \exp -u^1, \quad \mathbf{A_3} = \alpha_3 \exp -qu^1.$$

The holonomic components of the electromagnetic potential are as follows:

$$A_1 = \alpha_1, \quad A_2 = \alpha_2 \exp -u^1, \quad A_3 = \alpha_3 \exp -qu^1.$$

3.6. Group $G_3(VII[A])$

The metrics of the spaces and the group operators can be represented as:

$$ds^2 = 2du^0du^3 + 2du^0du^1(a_4\cos(u^3\sin c) + a_0\sin(u^3\sin c))\exp(-u^3\cos c) +$$
$$2du^0du^2((a_0\sin c - a_4\cos c)\cos(u^3\sin c) - (a_0\cos c + a_4\sin c)\sin(u^3\sin c))\exp(-u^3\cos c) +$$
$$2du^1du^2(a_1\cos c + 2a_2\cos 2(u^3\sin c) + 2a_3\sin 2(u^3\sin c))\exp(-2u^3\cos c) +$$
$$du^{1^2}(2a_1 + 2(a_3\sin c + a_2\cos c)\cos 2(u^3\sin c) + 2(a_3\cos c - a_2\sin c)\sin 2(u^3\sin c))\exp(-2u^3\cos c) -$$
$$du^{2^2}(2a_1 - 2(a_3\sin c - a_2\cos c)\cos 2(u^3\sin c) + 2(a_3\cos c + a_2\sin c)\sin 2(u^3\sin c))\exp(-2u^3\cos c).$$

The group operators can be presented as follows:

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = p_3 + (2u^2\cos c + u^1)p_2 - u^2p_1$$

From here the structural constants follow:

$$C^{\gamma}_{12} = 0, \quad C^{\alpha}_{13} = \delta^{\alpha}_2, \quad C^{\alpha}_{23} = -\delta^{\alpha}_1 + 2\delta^{\alpha}_2\cos c.$$

where $c = const$. Matrix $\hat{\lambda}$ has the form:

$$||\lambda^{\alpha}_{\beta}|| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u^2 & -(u^1 + 2u^2\cos c) & 1 \end{pmatrix}.$$

Set of Equation (7) has the form:

$$\mathbf{A_{1,1}} = \mathbf{A_{1,2}} = 0, \quad \mathbf{A_{1,3}} = \mathbf{A_2}, \quad \mathbf{A_{2,1}} = \mathbf{A_{2,2}} = 0, \quad \mathbf{A_{2,3}} = 2\mathbf{A_2}\cos c - \mathbf{A_1} \tag{13}$$

$$\mathbf{A}_{3|\beta} = 0 \rightarrow \mathbf{A_3} = \alpha_3.$$

From Equation (12) it follows:

$$\mathbf{A_1} = D_1(u^3, u^0), \quad \mathbf{A_2} = D_2(u^3, u^0).$$

Let us denote:

$$D_2 = B(u^3, u^0)\exp(u^3\cos c).$$

Then, set Equation (12) can be present in the form (the dot denotes the derivative, with respect to u^3):

$$\dot{D}_1 = B\exp(u^3\cos c), \quad \ddot{B} + B\sin^2 c = 0 \to B = \alpha_1 \sin(\alpha_2 + u^3\sin c).$$

Set Equation (6) has the form:

$$\mathbf{A_1} = \alpha_1 \exp(u^3\cos c)\sin(\alpha_2 + u^3\sin c), \quad \mathbf{A_2} = \alpha_1\exp(u^3\cos c)\sin(\sin c + \alpha_2 + u^3\sin c),$$

The holonomic components of the electromagnetic potential are as follows:

$$A_1 = \mathbf{A_1}, \quad A_2 = \mathbf{A_2},$$

$$A_3 = \alpha_3 - \alpha_1\exp(u^3\cos c)(u^1\sin(\sin c + \alpha_2 + u^3\sin c) + u^2\sin(2\sin c + \alpha_2 + u^3\sin c)).$$

3.7. Group $G_3(VI[B])$

The metrics of the spaces and the group operators can be represented as:

$$ds^2 = 2du^0du^1u^{3(1+\omega)} + 2du^0du^3\frac{a_0}{u^3} +$$

$$2du^2du^3(a_3u^{3(\omega-1)} - a_1u^1u^{32\omega}) + a_1u^{32\omega}du^{22} + du^{32}(a_1u^{12}u^{32\omega} - 2a_3u^1u^{3(\omega-1)} + \frac{a_2}{u^{32}}),$$

where it is denoted as: $\omega = \frac{1}{q-1}, \quad q = const.$

The group operators can be presented as follows:

$$X_1 = p_2, \quad X_2 = p_1 + u^3p_2 \quad X_3 = qu^1p_1 + u^2p_2 + (1-q)p_3$$

From here the structural constants follow:

$$C^\alpha_{12} = 0, \quad C^\alpha_{31} = \delta^\alpha_1, \quad C^\alpha_{23} = \delta^\alpha_1 + \delta^\alpha_2.$$

Matrix $\hat{\lambda}$ has the form:

$$||\lambda^\alpha_\beta|| = \begin{pmatrix} -u^3 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{(u^2 - qu^1u^3)}{u^3(q-1)} & \frac{qu^1}{u^3(q-1)} & -\frac{1}{u^3(q-1)}. \end{pmatrix}.$$

Set Equation (6):

$$\mathbf{A}_{\alpha|\beta} = C^\gamma_{\beta\alpha}\mathbf{A}_\gamma$$

has the form:

$$\mathbf{A_{1,2}} = \mathbf{A_{1,1}} = 0, \quad (q-1)u^3\mathbf{A_{1,3}} = \mathbf{A_1};$$

$$\mathbf{A_{2,2}} = \mathbf{A_{2,1}} = 0, \quad (q-1)u^3\mathbf{A_{2,3}} = q\mathbf{A_2};$$

$$\mathbf{A_{3,2}} = \mathbf{A_1}, \quad \mathbf{A_{3,1}} + u^3\mathbf{A_{3,2}} = q\mathbf{A_2} \quad qu^1\mathbf{A_{3,1}} + u^2\mathbf{A_{3,2}} + (1-q)u^3\mathbf{A_{3,3}} = 0.$$

One can find next solutions of the set:

$$\mathbf{A_1} = \alpha_1 u^{3^\omega}, \quad \mathbf{A_2} = \alpha_2 u^{3^{(\omega+1)}}, \quad \mathbf{A_3} = -\alpha_3 + \alpha_1 u^2 u^{3^\omega} + (q\alpha_2 - \alpha_1)u^1u^{3^{\omega+1}}.$$

The non-holonomic components of the electromagnetic potential are as follows:

$$A_\alpha = \mathbf{A}_\beta\lambda^\beta_\alpha, \to A_1 = (\alpha_2 - \alpha_1)u^{3^{(\omega+1)}}, \quad A_2 = \alpha_1 u^{3^\omega}, \quad A_3 = \frac{\alpha_3}{u^3} - \alpha_1 u^1 u^{3^\omega}.$$

3.8. Group $G_3(VII[B])$

The metrics of the spaces and the group operators can be represented as:

$$ds^2 = 2du^0du^1r_3S + 2du^0du^2(a_0 - u^3){r_3}^{-1}S + 2du^0du^3(u^3 - a_0)u^1{r_3}^{-1}S$$

$$2du^2du^3(a_3{r_3}^{-3}S - a_1u^1a_1{r_3}^{-1}S^2) + {du^2}^2a_1{r_3}^{-2}S^2 + {du^3}^2(a_1{u^1}^2{r_3}^{-2}S^2 - 2a_3{r_3}^{-3}S + a_2{r_3}^{-4}).$$

The group operators can be presented as follows:

$$X_1 = p_2, \quad X_2 = p_1 + u^3p_2 \quad X_3 = (u^2 + u^1(u^3 - 2\cos c))p_1 + u^2u^3p_2 + {r_3}^2p_3$$

From here the structural constants follow:

$$C^\alpha_{12} = 0, \quad C^\alpha_{31} = \delta^\alpha_2, \quad C^\alpha_{23} = q\delta^\alpha_2 - \delta^\alpha_1,$$

where $r_3 = ({u^3}^2 - 2u^3\cos c + 1), S = \exp(-2ctgcarctg\frac{u^3-\cos c}{\sin c})$ $a_i = a_i(u^0), c = const.$ Matrix $\hat{\lambda}$ has the form:

$$||\lambda^\alpha_\beta|| = \begin{pmatrix} -u^3 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{u^1u^3(2\cos c - u^3)}{r_3} & -\frac{u^2 + u^1(2\cos c - u^3)}{r_3} & -\frac{1}{r_3}. \end{pmatrix}.$$

Set Equation (6):

$$\mathbf{A}_{\alpha|\beta} = C^\gamma_{\beta\alpha}\mathbf{A}_{\text{fl}}$$

has the form:

$$\mathbf{A_{1,2}} = \mathbf{A_{1,1}} = 0, \quad r_3\mathbf{A_{1,3}} + \mathbf{A_2} = 0; \tag{14}$$

$$\mathbf{A_{2,2}} = \mathbf{A_{2,1}} = 0, \quad r_3\mathbf{A_{2,3}} - \mathbf{A_1} + q\mathbf{A_2} = 0; \tag{15}$$

$$\mathbf{A_{3,2}} = \mathbf{A_2}, \quad \mathbf{A_{3,1}} + u^3\mathbf{A_{3,2}} + \mathbf{A_1} - 2\cos c\mathbf{A_2} = 0; \tag{16}$$

$$(u^2 + u^1(2\cos c - u^3))\mathbf{A_{3,1}} + u^2u^3\mathbf{A_{3,2}} + r_3\mathbf{A_{3,3}} = 0.$$

From Equations (13) and (14) it follows:

$$\mathbf{A_1} = B(u^0, u^3), \quad \mathbf{A_2} = -r_3B_{,3}.$$

The function B satisfies the equation:

$$r_3(r_3B_{,3})_3 + 2(r_3B_{,3})\cos c + B = 0. \tag{17}$$

Equation (16) has the form:

$$B = \sin(\alpha_2 + arctg(\frac{u^3 - \cos c}{\sin c}))\exp(\alpha_1 - (ctgc)arctg(\frac{u^3 - \cos c}{\sin c})).$$

First two equations of Equations (18) have a solution:

$$\mathbf{A_3} = u^1((u^3 - 2\cos c)r_3B_{,3} - B) - u^2r_3B_{,3} + b_3(u^0, u^3).$$

From the last equation of the set it follows: $b_3 = \alpha_3$. As a result, we obtain the nonholonomic components of the potential of the electromagnetic field:

$$\mathbf{A_1} = \sin(\alpha_1 + arctg(\frac{u^3 - \cos c}{\sin c}))\exp(\alpha_2 - (ctgc)arctg(\frac{u^3 - \cos c}{\sin c})),$$

$$\mathbf{A_2} = -\sin(\alpha_1 - c + arctg(\frac{u^3 - \cos c}{\sin c}))\exp(\alpha_2 - (ctgc)arctg(\frac{u^3 - \cos c}{\sin c}))$$

$$\mathbf{A_3} = \alpha_3 - ((u^2 + u^1(2\cos c))\sin(\alpha_1 - c + arctg(\frac{u^3 - \cos c}{\sin c})) +$$
$$u^1 \sin(\alpha_1 + arctg(\frac{u^3 - \cos c}{\sin c}))) \exp(\alpha_2 - (tgc) arctg(\frac{u^3 - \cos c}{\sin c})).$$

The holonomic components of the electromagnetic potential are as follows:

$$A_1 = -\exp(\alpha_1 - ctgcarctg\frac{(u^3 - \cos c)}{\sin c})(\sin(\alpha_1 - c + arctg\frac{(u^3 - \cos c)}{\sin c}) +$$
$$u^3 \sin(\alpha_1 + c + arctg\frac{(u^3 - \cos c)}{\sin c}),$$

$$A_2 = \exp(\alpha_1 - (ctgc)arctg\frac{(u^3 - \cos c)}{\sin c})\sin(\alpha_0 - c + arctg\frac{(u^3 - \cos c)}{\sin c}),$$

$$A_3 = \frac{\alpha_3}{(u^3 - \cos c)^2 + \sin c^2} - u^1 \exp(\alpha_1 - (ctgc)arctg\frac{(u^3 - \cos c)}{\sin c})\sin(\alpha_1 + arctg\frac{(u^3 - \cos c)}{\sin c}).$$

4. Insolvable Groups $G_3(N)$

4.1. Groups $G_3(VIII)$

Let us represent operators of the group:

$$X_1 = p_2, \quad X_2 = p_3 + u^2 p_2 \quad X_3 = \exp u^3 p_1 + (u^{2^2} + \varepsilon \exp u^{3^2}) p_2 + 2u^2 p_3,$$

where $\varepsilon = 0, -1, +1$

From here the structural constants follow:

$$C^{\alpha}_{12} = 0, \quad C^{\alpha}_{13} = 2\delta^{\alpha}_2, \quad C^{\alpha}_{23} = \delta^{\alpha}_2 - \delta^{\alpha}_1.$$

Matrix $\hat{\lambda}$ has the form:

$$||\lambda^{\alpha}_{\beta}|| = \begin{pmatrix} (u^{2^2} - \epsilon \exp 2u^3)\exp -u^3 & -2u^2 \exp -u^3 & \exp -u^3 \\ 1 & 0 & 0 \\ -u^2 & -1 & 0. \end{pmatrix}.$$

Set Equation (6) has the form:

$$\mathbf{A_{1,2}} = 0, \quad \mathbf{A_{1,3}} = -\mathbf{A_1}, \quad \mathbf{A_{1,1}} = 2(u^2\mathbf{A_1} - \mathbf{A_2})\exp -u^3;$$
$$\mathbf{A_{2,2}} = \mathbf{A_1}, \quad \mathbf{A_{2,3}} = -u^2\mathbf{A_1}, \quad \mathbf{A_{2,1}} \exp u^3 + (u^{2^2} - \varepsilon \exp 2u^3)\mathbf{A_1} + \mathbf{A_3} = 0;$$
$$\mathbf{A_{3,2}} = 2\mathbf{A_2}, \quad \mathbf{A_{3,3}} = \mathbf{A_3} - 2u^2\mathbf{A_2} \quad \mathbf{A_{3,1}} \exp u^3 + 2(u^{2^2} + \varepsilon \exp 2u^3)\mathbf{A_2} + 2u^2\mathbf{A_{3,3}} = 0.$$

This implies:

$$\mathbf{A_1} = 2B(u^1, u^0)\exp -u^3, \quad \mathbf{A_2} = 2Bu^2 \exp(-u^3) - \dot{B},$$

$$\mathbf{A_3} = (\ddot{B} - 2\varepsilon B)\exp(-u^3) - 2u^2\dot{B} + 2Bu^{2^2}\exp u^3),$$

where B is a function of (u^0, u^1), satisfying the equation:

$$\dot{B}_{,11} = 4\varepsilon \dot{B} \quad \rightarrow \quad \dot{B}_{,1} = \alpha + 4\varepsilon B. \tag{18}$$

Dots denote the derivatives, with respect to u^1. The holonomic components of the electromagnetic potential are as follows:

$$A_1 = \alpha, \quad A_2 = 2B\exp(-u^3), \quad A_3 = -\dot{B}.$$

Depending on the value of ε, the function B and metric of the space admitting this group have the form:

1. for $\varepsilon = 0$:

$$B = c{u^1}^2 + \beta u^1 + \gamma;$$

$$ds^2 = 2du^0du^1 + 2du^0du^2(a_0 - 2u^1a_4 - {u^1}^2)\exp -u^3 + 2du^0du^3(a_4 - u^1)+$$

$$2du^2du^3(2a_1u^1 + a_2)\exp -u^3 + {du^2}^2(4a_1{u^1}^2 + 4a_2u^1 + a_3)\exp -2u^3 + {du^3}^2a_1.$$

2. for $\varepsilon = -1$:

$$B = \alpha_1\sin 2(u^1 + \alpha_2) + \frac{\alpha}{4};$$

$$ds^2 = 2du^0du^1 + 2du^0du^2(a_4\cos 2u^1 + a_0\sin 2u^1 - \frac{1}{2})\exp -u^3 + 2du^0du^3$$

$$(a_0\cos 2u^1 - a_4\sin 2u^1) + 2du^2du^3(a_3\cos 4u^1 - a_2\sin 4u^1)\exp -u^3 + {du^2}^2(a_2\cos 4u^1 +$$

$$a_3\sin 4u^1 - \frac{a_1}{2})\exp -2u^3 - {du^3}^2(a_2\cos 4u^1 + a_3\sin 4u^1 + \frac{a_1}{2}).$$

3. for $\varepsilon = 1$:

$$B = \alpha_1 sh2u^1 + \alpha_2 sh2u^1 - \frac{\alpha}{4}.$$

$$ds^2 = 2du^0du^1 + 2du^0du^2(a_4\exp -2u^1 + a_0\exp 2u^1 + \frac{1}{2})\exp -u^3 + 2du^0du^3(a_4\exp -2u^1$$

$$-a_0\exp 2u^1) + 2du^2du^3(a_2\exp -4u^1 + a_3\sin 4u^1)\exp -u^3 + {du^2}^2(a_2\exp -4u^1 +$$

$$a_3\exp 4u^1 + \frac{a_1}{2})\exp -2u^3 + {du^3}^2(a_2\exp -4u^1 + a_3\exp 4u^1 - \frac{a_1}{2}).$$

4.2. Groups $G_3(IX)$

The metrics of the spaces and the group operators can be represented as:

$$ds^2 = 2du^0du^1 + {du^2}^2(a_1\sin 2u^1 - a_2\cos 2u^1 + a_3)\cos^2 u^3 -$$

$$2du^2du^3(a_1\cos 2u^1 + a_2\sin 2u^1)\cos u^3 + {du^3}^2(a_2\cos 2u^1 - a_1\sin 2u^1 + a_3)+$$

$$2du^0du^2(\sin u^3 + (a_0\cos u^1 + a_4\sin u^1)\cos u^3) + 2du^0du^3(a_0\sin u^1 - a_4\cos u^1).$$

The group operators can be represented as follows:

$$X_1 = p_2, \quad X_2 = \frac{\cos u^2}{\sin u^3}p_1 - tgu^3\cos u^2 p_2 + \sin u^2 p_3$$

$$X_3 = \partial_2(X_2) = -\frac{\sin u^2}{\sin u^3}p_1 + tgu^3\sin u^2 p_2 + \cos u^2 p_3$$

From here the structural constants follow:

$$C^\alpha_{12} = \delta^\alpha_3, \quad C^\alpha_{13} = -\delta^\alpha_2, \quad C^\alpha_{23} = \delta^\alpha_1.$$

Matrix $\hat{\lambda}$ has the form:

$$||\lambda^\alpha_\beta|| = \begin{pmatrix} \frac{{\sin u^3}^2}{\cos u^3} & \cos u^2\sin u^3 & -\sin u^2\sin u^3 \\ 1 & 0 & 0 \\ 0 & \sin u^2 & \cos u^2. \end{pmatrix}.$$

Set Equation (6) has the form:

$$\mathbf{A_{1,2}} = 0, \quad \mathbf{A_{1,1}} \frac{\cos u^2}{\sin u^3} + \mathbf{A_{1,3}} \sin u^2 + \mathbf{A_3} = 0;$$

$$\mathbf{A_{1,1}} \frac{\sin u^2}{\sin u^3} - \mathbf{A_{1,3}} \cos u^2 + \mathbf{A_2} = 0;$$

$$\mathbf{A_{2,2}} = \mathbf{A_3}, \quad \mathbf{A_{2,1}} \frac{\cos u^2}{\sin u^3} - \mathbf{A_{2,2}} tgu^3 \cos u^2 + \mathbf{A_{2,3}} \sin u^2 = 0;$$

$$-\mathbf{A_{2,1}} \frac{\sin u^2}{\sin u^3} + \mathbf{A_{2,2}} tgu^3 \sin u^2 + \cos u^2 \mathbf{A_{2,3}} + \mathbf{A_1} = 0;$$

$$\mathbf{A_{3,2}} = -\mathbf{A_2}, \quad \mathbf{A_{3,1}} \frac{\cos u^2}{\sin u^3} - \mathbf{A_{3,2}} tgu^3 \cos u^2 + \mathbf{A_{3,3}} \sin u^2 - \mathbf{A_1} = 0;$$

$$-\mathbf{A_{3,1}} \frac{\sin u^2}{\sin u^3} + \mathbf{A_{3,2}} tgu^3 \sin u^2 + \cos u^2 \mathbf{A_{3,3}} = 0.$$

This implies:

$$\mathbf{A_1} = \alpha_1 \sin u^3, \quad \mathbf{A_2} = \alpha_1 \cos u^2 \cos u^3, \quad \mathbf{A_3} = -\alpha_1 \cos u^3 \sin u^2.$$

The holonomic components of the electromagnetic potential are as follows:

$$A_1 = \alpha_1 tgu^3, \quad A_2 = \alpha_1 \cos u^2 \cos u^3, \quad A_3 = 0.$$

5. Killing Vector Fields Depend on the Non-Ignored Variable U^0

5.1. Group $G_3(II[C])$

The metrics of the spaces and group operators can be represented as:

$$ds^2 = 2du^0(a_0 du^1 + \varepsilon(2a_2 u^1 + a_3)du^2) + (a_1 + \varepsilon(2a_2 {u^1}^2 + 3a_3 u^1 + a_4)u^1)du^3) +$$
$$\varepsilon u^1 (a_2 {u^1}^3 + 2a_3 {u^1}^2 + a_4 u^1 + 2a_1){du^0}^2 + 4(a_3 + 2a_2 u^1)du^3 du^2 + 4a_2 {du^2}^2 + (a_4 + 4a_3 u^1 + 4a_2 {u^1}^2){du^3}^2.$$

The group operators can be presented as follows:

$$X_1 = p_2, \quad X_2 = p_3, \quad X_3 = -p_1 + u^3 p_2 + \varepsilon u^0 p_3, \quad \varepsilon = 0, 1$$

From here the structural constants follow:

$$C^\gamma_{12} = C^\gamma_{13} = 0, \quad C^\alpha_{23} = \delta^\alpha_1.$$

Matrix $\hat{\lambda}$ has the form:

$$||\lambda^\alpha_\beta|| = \begin{pmatrix} u^3 & \varepsilon u^0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

From Equation (6):

$$\mathbf{A_{ff|fi}} = C^\gamma_{\beta\alpha} \mathbf{A_{fl}}$$

it follows:

$$\mathbf{A_{1|fi}} = 0 \quad \to \mathbf{A_{1,fi}} = 0 \to \mathbf{A_1} = 2\alpha_1;$$

$$\mathbf{A_{2|fi}} = -\delta_{3\beta} \mathbf{A_1}, \to \mathbf{A_2} = 2\alpha_1 u^1 + \alpha_2;$$

$$\mathbf{A_{3|fi}} = \delta_{2\beta} \mathbf{A_1}, \to \mathbf{A_{3,2}} = 0, \quad \mathbf{A_{3,3}} = \mathbf{A_1}, \quad \mathbf{A_{3,1}} = \varepsilon u^0 \mathbf{A_1} \to \mathbf{A_3} = 2\alpha_1(u^3 + \varepsilon u^0 u^1) - \alpha_3.$$

The holonomic components of the electromagnetic potential are as follows:

$$A_\alpha = \mathbf{A}_\beta \lambda^\beta_\alpha, \rightarrow A_1 = \alpha_3 + \alpha_2 \varepsilon u^0, \quad A_2 = 2\alpha_1, \quad A_3 = \alpha_2 + 2\alpha_1 u^1.$$

The holonomic component A_0 can be found from the equation:

$$A_{0|\alpha} = -\xi^\beta_{\alpha,0} A_\beta, \rightarrow A_0 = \varepsilon(\alpha_1 u^1 + \alpha_2) u^1 + \alpha_0.$$

5.2. Group $G_3(III[C])$

The metrics of the spaces and group operators can be represented as:

$$ds^2 = (a_0 + \varepsilon a_2 {u^1}^2) {du^0}^2 + 2(a_1 du^1 - \varepsilon u^1 (a_3 \exp(-u^1) du^3 + a_2 du^2)) du^0 + a_2 {du^2}^2 +$$

$$2a_3 \exp(-u^1) du^2 du^3 + a_4 \exp(-2u^1) {du^3}^2.$$

The group operators can be presented as follows:

$$X_1 = p_1 + \varepsilon u^0 p_2 + u^3 p_3, \quad X_2 = p_2, \quad X_3 = p_3, \quad \varepsilon = 0, 1$$

From here the structural constants follow:

$$C^\gamma_{12} = 0, \quad C^\alpha_{13} = -\delta^\alpha_3, \quad C^\gamma_{23} = 0.$$

Matrix $\hat{\lambda}$ has the form:

$$||\lambda^\alpha_\beta|| = \begin{pmatrix} 1 & -\varepsilon u^0 & -u^3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Set Equation (6) has the form:

$$\mathbf{A_{1|fi}} = \delta_{3\beta} \mathbf{A_3} \rightarrow \mathbf{A_{1,3}} = \mathbf{A_3};$$

$$\mathbf{A_{2|fi}} = 0, \rightarrow \mathbf{A_2} = \alpha_2(u^0);$$

$$\mathbf{A_{3|fi}} = -\delta_{1\beta} \mathbf{A_3}, \rightarrow \mathbf{A_{3,1}} = -\mathbf{A_3} \rightarrow \mathbf{A_3} = \alpha_3(u^0) \exp(-u^1) \rightarrow \mathbf{A_1} = \alpha_1 + u^3 \alpha_3 \exp(-u^1).$$

The holonomic components of the electromagnetic potential are as follows:

$$A_1 = \alpha_1 - \alpha_2 \varepsilon u^0, \quad A_2 = \alpha_2, \quad A_3 = -\varepsilon \alpha_2 u^1.$$

The holonomic component A_0 can be found from the equation:

$$A_{0|\alpha} = -\xi^\beta_{\alpha,0} A_\beta, \rightarrow A_0 = \varepsilon(u^0 \alpha_2 - \alpha_1) u^1.$$

5.3. Groups $G_3(V[C])$ *with the Singular Operators*

The metrics of the spaces and the group operators can be represented as:

$$ds^2 = {du^0}^2 a_1 {u^1}^2 \exp 2u^3 + 2du^0 [du^1 \exp u^3 - du^2 a_1 u^1 \exp 2u^3 +$$

$$(a_0 - a_2 u^1 \exp u^3) du^3] + {du^2}^2 a_1 \exp 2u^3 + 2du^2 du^3 a_2 \exp u^3.$$

The group operators can be represented as follows:

$$X_1 = p_2, \quad X_2 = p_1 + u^0 p_2, \quad X_3 = u^1 p_1 + u^2 p_2 - p_3,$$

where $a_i = a_i(u^0)$,

From here the structural constants follow:

$$C^{\gamma}_{12} = 0, \quad C^{\alpha}_{13} = \delta^{\alpha}_{1}, \quad C^{\gamma}_{23} = \delta^{\alpha}_{2}.$$

Matrix $\hat{\lambda}$ has the form:

$$||\lambda^{\alpha}_{\beta}|| = \begin{pmatrix} -u^0 & 1 & 0 \\ 1 & 0 & 0 \\ u^2 - u^0 u^1 & u^1 & -1 \end{pmatrix}.$$

From the set Equation (6):

$$\mathbf{A}_{\mathbf{ff}|\mathbf{fi}} = C^{\gamma}_{\beta\alpha} \mathbf{A}_{\mathbf{fl}},$$

it follows:

$$\mathbf{A}_{\mathbf{1}|\mathbf{fi}} = -\delta_{3\beta} \mathbf{A}_{\mathbf{1}} \to \mathbf{A}_{\mathbf{1,3}} = \alpha_1(u^0) \exp u^3;$$

$$\mathbf{A}_{\mathbf{2}|\mathbf{fi}} = -\delta_{3\beta} \mathbf{A}_{\mathbf{2}}, \to \mathbf{A}_{\mathbf{2}} = \alpha_2(u^0) \exp u^3;$$

$$\mathbf{A}_{\mathbf{3}|\mathbf{fi}} = \delta_{1\beta} \mathbf{A}_{\mathbf{1}} + \delta_{2\beta} \mathbf{A}_{\mathbf{2}}, \to \mathbf{A}_{\mathbf{3}} = -\alpha_3(u^0) + (\alpha_1 u^2 + (\alpha_2 - \alpha_1 u^0) u^1) \exp u^3.$$

The holonomic components of the electromagnetic potential are as follows:

$$A_{\alpha} = \mathbf{A}_{\beta} \lambda^{\beta}_{\alpha}, \to A_1 = (\alpha_2 - \alpha_1 u^0) \exp u^3, \quad A_2 = \alpha_1 \exp u^3, \quad A_3 = \alpha_3.$$

The holonomic component A_0 can be found from the equation:

$$A_{0|\alpha} = -\xi^{\beta}_{\alpha,0} A_{\beta} \to A_0 = \alpha_1 \exp u^3.$$

6. Conclutions

All admissible electromagnetic fields of greatest interest to gravitational theory have been found. The metric tensor for these admissible fields contains arbitrary functions of nonignorable variables, so that considerable arbitrariness is preserved for them. This arbitrariness can be used, for example, in the search for self-consistent solutions of the gravitational field equations in the general theory of relativity, Brans–Dicke scalar-tensor theory (see, e.g., [16]), or other alternative theories of gravity. The non-ignored variable is either temporary (for homogeneous spaces) or (as in this article) isotropic (null). This is important when considering cosmological problems and obtaining and studying models of spaces with gravitational waves. Let us mention other directions for further research in the framework of the obtained classification.

First, it is possible to consider a similar problem of admissible electromagnetic fields classification for the Dirac–Fock equation, since the method of noncommutative integration is also applicable to this equation (see, e.g., [26]). At the same time, from the physical point of view, the construction of this classification is most justified in the framework of the already obtained classification of admissible electromagnetic fields for the Klein–Gordon–Fock equation.

Second, a complement to the classification carried out in this work will be the classification of generalized privileged coordinate systems, in which the basic solutions of the Klein–Gordon–Fock equation can be found by the method of noncommutative integration.

Third, the resulting classification can be used to find the basic solutions of the Klein–Gordon–Fock equation and other quantum-mechanical equations of motion by the method of noncommutative integration. Note that this problem attracts the attention of many researchers (see, e.g., [27,28]).

Note that group approaches remain the most effective methods for constructing and studying realistic quantum mechanical models in linear and nonlinear physics [29,30].

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