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Family of Asymptotic Solutions to the Two-Dimensional Kinetic Equation with a Nonlocal Cubic Nonlinearity

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Abstract: We apply the original semiclassical approach to the kinetic ionization equation with the nonlocal cubic nonlinearity in order to construct the family of its asymptotic solutions. The approach proposed relies on an auxiliary dynamical system of moments of the desired solution to the kinetic equation and the associated linear partial differential equation. The family of asymptotic solutions to the kinetic equation is constructed using the symmetry operators acting on functions concentrated in a neighborhood of a point determined by the dynamical system. Based on these solutions, we introduce the nonlinear superposition principle for the nonlinear kinetic equation. Our formalism based on the Maslov germ method is applied to the Cauchy problem for the specific two-dimensional kinetic equation. The evolution of the ion distribution in the kinetically enhanced metal vapor active medium is obtained as the nonlinear superposition using the numerical–analytical calculations.

Keywords: kinetic model; symmetry operators; Maslov germ; nonlinear superposition principle; dense plasma; active media; semiclassical approximation; WKB–Maslov method

MSC: 45K05; 81Q20; 82B40; 82D10



Citation: Shapovalov, A.V.; Kulagin, A.E.; Siniukov, S.A. Family of Asymptotic Solutions to the Two-Dimensional Kinetic Equation with a Nonlocal Cubic Nonlinearity. *Symmetry* **2022**, *14*, 577. <https://doi.org/10.3390/sym14030577>

Academic Editors: Sergei D. Odintsov and Shou-Fu Tian

Received: 24 February 2022

Accepted: 11 March 2022

Published: 15 March 2022

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1. Introduction

Kinetic equations are the theoretical footing for the dynamic phenomena of various nature that occur in physical systems of many interacting elements (particles). Examples of such system are diluted gases, gas discharges, a plasma, processes of coagulation [1], and biological systems such as, e.g., population systems [2–4]. In some systems, nonlocal collective (averaged) interactions of elements substantially contribute to dynamics. Such interactions are modeled by integral terms in kinetic equations that become integro-differential. In spatially heterogeneous kinetic phenomena, the interelement interactions occur along with the diffusion. Then, the model kinetic equation belongs to the class of reaction-diffusion (RD) equations. The study of RD equations with both local and nonlocal terms have formed an independent branch of mathematical physics.

Due to the mathematical complexity of the study of RD equation with nonlocal interactions, methods of computer modeling prevail here. However, the demand for the analytical methods stimulates the development of approximate and asymptotically exact solutions. For a number of RD kinetic equations with nonlocal interactions, one can succeed using the WKB–Maslov theory of semiclassical approximation or the Maslov complex germ method [5–7]. Based on the WKB–Maslov theory, the method of semiclassical asymptotics was developed for a generalized Fisher–Kolmogorov–Petrovskii–Piskunov

equation (Fisher–KPP) with a quadratic nonlocal term in [8,9] and for the nonlocal Gross–Pitaevskii equation in [10,11].

In this work, using the results of [9,12], we construct semiclassical asymptotics for the model kinetic equation with the nonlocal cubic nonlinearity of the form

$$\begin{aligned} \partial_t u(\vec{x}, t) &= D\tilde{D}_a(t)\Delta_x u(\vec{x}, t) + a(\vec{x}, t)u(\vec{x}, t) \\ &- \varkappa u(\vec{x}, t) \int_{\mathbb{R}^n} d\vec{y} \int_{\mathbb{R}^n} d\vec{z} b(\vec{x}, \vec{y}, \vec{z}, t)u(\vec{y}, t)u(\vec{z}, t). \end{aligned} \tag{1}$$

Here, t is a time, $u(\vec{x}, t)$ is a distribution function (e.g., the particle density in a system), $\partial_t = \frac{\partial}{\partial t}$. In a general case, the method under consideration is applicable for n -dimensional space, $\vec{x} = (x_1, x_2, \dots, x_n) = (x_i) \in \mathbb{R}^n$. The nonlinearity parameter \varkappa and the small diffusion parameter D are introduced explicitly for the sake of convenience. The n -dimensional Laplace operator in the Cartesian space $\vec{x} \in \mathbb{R}^n$ is denoted by Δ_x . The coefficients $a(\vec{x}, t)$ and $b(\vec{x}, \vec{y}, \vec{z}, t)$ are smooth functions of their spatial arguments that grow not faster than polynomially at each point t .

In the physical two-dimensional or three-dimensional space, Equation (1) is considered as a model of the optical metal vapor active medium (MVAM) excited by an electrical discharge [12]. The MVAM is a mixture of a buffer inert gas and metal vapors in a gas discharge tube (GDT) (see [13,14] and references therein). In the active medium excited by an electrical discharge, the ionization and recombination processes are mainly caused by the inelastic electron impact. For typical pressures of a buffer gas and metal vapors, preferentially metal atoms are ionized in the mixture. The process of triple recombination of an ion with two electrons is responsible for the deionization (see, e.g., [15]). Such dense plasma formed by metal ions and electrons can be considered as quasineutral. The contracted electrical discharge generates ions and electrons localized in the neighborhood of the GDT center. It means that the concentration of the charges rapidly decreases with the distance from the GDT center. In [12,16], the description of the plasma kinetics under assumptions made was based on the following equation:

$$\partial_t n_i = D_a(t)\Delta_x n_i + q_i n_e n_{neut} - q_{tr} n_i (n_e)^2, \tag{2}$$

where \vec{x} are Cartesian coordinates of a point in \mathbb{R}^2 or \mathbb{R}^3 depending on the problem statement. The quantity $q_i = q_i(\vec{x}, t)$ is the kinetic coefficient for the electron impact ionization of neutral atoms with a concentration $n_{neut} = n_{neut}(\vec{x}, t)$ ($n_e = n_e(\vec{x}, t)$ is an electron concentration). In the same sense, the coefficient $q_{tr} = q_{tr}(\vec{x}, t)$ meets the process of triple recombination of ions with a concentration $n_i = n_i(\vec{x}, t)$. We assume the plasma to be dense so that the triple recombination dominates over the dielectronic recombination. The coefficient $D_a(t)$ is an ambipolar diffusion coefficient. The dependence of q_i, n_{neut}, q_{tr} , and D_a on \vec{x} and t is due to their dependence on electron temperature that substantially depends on \vec{x} and t .

Assuming the quasineutrality of plasma, the concentrations of ions and electrons are the same, i.e.,

$$n_e(\vec{x}, t) = n_i(\vec{x}, t). \tag{3}$$

Then, for given $a(\vec{x}, t) = q_i(\vec{x}, t)n_{neut}(\vec{x}, t)$ and $q_{tr}(\vec{x}, t)$, Equation (2) becomes closed and determines the concentration $n_i(\vec{x}, t)$ for the given initial and boundary conditions.

To apply the method of semiclassical asymptotics borrowed from papers [9,12], we write Equation (2) in the nonlocal form (1). For the space \mathbb{R}^2 or \mathbb{R}^3 , the function $b(\vec{x}, \vec{y}, \vec{z}, t)$ is the probability density of a triple recombination due to the collision of an ion with two electrons. The ambipolar diffusion coefficient $D_a(t)$ in Equation (2) is written as $D\tilde{D}_a(t)$ where D is the asymptotic small parameter.

In this work, following the method of semiclassical asymptotics [8,9], we have constructed approximate solutions of Equation (1) in an explicit analytical form for the special

set of equation coefficients. The obtained expressions are leading terms of semiclassical expansion for the solutions of Equation (1) within the accuracy of $O(D^{3/2})$ in the following class \mathcal{P}_D^t of trajectory concentrated functions (TCF):

$$\mathcal{P}_D^t = \left\{ \Phi : \Phi(\vec{x}, t, D) = \varphi\left(\frac{\Delta\vec{x}}{\sqrt{D}}, t, D\right) \exp\left[\frac{1}{D}S(t, D)\right] \right\}, \quad (4)$$

where $\Phi(\vec{x}, t, D)$ is a generic element of the class \mathcal{P}_D^t ; $\Delta\vec{x} = \vec{x} - \vec{X}(t, D)$; the real function $\varphi(\vec{\eta}, t, D)$ belongs to the Schwartz space \mathbb{S} in variables $\vec{\eta}$, smoothly depends on t , and regularly depends on \sqrt{D} as $D \rightarrow 0$. The real smooth functions $S(t, D)$ and $\vec{X}(t, D)$, characterizing the class \mathcal{P}_D^t , regularly depend on \sqrt{D} as $D \rightarrow 0$ and are to be determined.

Note that the approach proposed can be useful for other models based on nonlinear equations similar to (2). Nonlinear kinetic equations arise in various areas such as cosmology models (see review [17]), superfluidity models [18], etc.

In the next section, we expound the main ideas of the our approach and basic notations, and we introduce the linear equation associated with the original kinetic equation whose solutions include the asymptotic solutions to the original kinetic equation. In Section 3, we obtain the particular solution to the associated linear equation for the special choice of equation coefficients in the two-dimensional case. In Section 4, the main object of Maslov theory, the germ, is obtained. Here, we present the symmetry operators to the associated linear equation and construct the family of its solutions. In Section 5, we apply the algebraic conditions on the solutions to the associated linear equation that allow us to obtain the countable set of new asymptotic solutions to the nonlinear kinetic equation. Moreover, the new method for constructing asymptotic solutions to the Cauchy problem for the kinetic equation based on the nonlinear superposition principle is proposed. In Section 6, the specific physically motivated example of the two-dimensional kinetic equation is considered. We illustrate the general formalism of our semiclassical approach by constructing the evolution of the initial ion distribution in the relaxing kinetically enhanced active medium. In Section 7, we conclude with some remarks.

2. Leading Term of Semiclassical Asymptotics for the Cauchy Problem Solution

In this section, we recapitulate the general scheme for the method of constructing the leading term of semiclassical asymptotics for Equation (1). The detailed description of this method can be found in [12].

According to [12], the following asymptotic estimates hold for functions $u(\vec{x}, t, D)$ from the class \mathcal{P}_D^t :

$$\frac{1}{\|u\|} \|\hat{p}^k \Delta x^l u\| = O(D^{(k+1)/2}), \quad \frac{1}{\|u\|} \|\hat{T}(\vec{X}(t, D), t) u\| = O(D), \quad (5)$$

where $\Delta\vec{x} = \vec{x} - \vec{X}(t, D)$, $\hat{p} = D\nabla$, ∇ is the gradient operator with respect to \vec{x} , $\hat{T}(\vec{X}(t, D), t) = D\partial_t + \langle \dot{\vec{X}}(t, D), \hat{p} \rangle - \dot{S}(t, D)$, $\dot{S}(t, D) = \frac{dS(t, D)}{dt}$, $\|\cdot\|$ is the L_2 -norm, $\langle \cdot, \cdot \rangle$ is the scalar product of vectors, and k is the non-negative integer. In particular, (5) yields $\hat{p} = \hat{O}(\sqrt{D})$, $\Delta\vec{x} = \hat{O}(\sqrt{D})$. Here, $\hat{F} = \hat{O}(D^\mu)$ means that $\frac{\|\hat{F}\varphi\|}{\|\varphi\|} = O(D^\mu)$, $\varphi \in \mathcal{P}_D^t$.

For simplicity, we will omit the parameter D in expressions where it does not cause confusion.

In view of the estimates (5), the asymptotic expansion of the coefficient in Equation (1) in powers of $\Delta\vec{x}$ in a neighborhood of the trajectory $\vec{x} = \vec{X}(t, D)$ allows us to transform Equation (1) to the approximate one with the given accuracy. The residual of the approximate equation has the estimate $O(D^{\frac{1+M}{2}})$ in the class \mathcal{P}_D^t , where $M \geq 2$ is the highest power of Δx_i accounted in the expansion. The leading term of asymptotics of the solution

to (1) is determined by the expansion of up to $O(D^{3/2})$. Following [12], the respective expansions in matrix notations can be written as

$$\begin{aligned} a(\vec{x}, t) &= a(\vec{X}, t) + a_x \Delta \vec{x} + \frac{1}{2} \Delta \vec{x}^\top a_{xx} \Delta \vec{x} + \hat{O}(D^{3/2}), \\ b(\vec{x}, \vec{y}, \vec{z}, t) &= b(\vec{X}, \vec{X}, \vec{X}, t) + b_x \Delta \vec{x} + b_y \Delta \vec{y} + b_z \Delta \vec{z} \\ &\quad + \frac{1}{2} \Delta \vec{x}^\top b_{xx} \Delta \vec{x} + \frac{1}{2} \Delta \vec{y}^\top b_{yy} \Delta \vec{y} + \frac{1}{2} \Delta \vec{z}^\top b_{zz} \Delta \vec{z} \\ &\quad + \Delta \vec{x}^\top b_{xy} \Delta \vec{y} + \Delta \vec{x}^\top b_{xz} \Delta \vec{z} + \Delta \vec{y}^\top b_{yz} \Delta \vec{z} + \hat{O}(D^{3/2}). \end{aligned} \quad (6)$$

Here, $\vec{X} = \vec{X}(t) = \vec{X}(t, D)$, $\Delta \vec{x} = \vec{x} - \vec{X}(t)$, $\Delta \vec{y} = \vec{y} - \vec{X}(t)$, and $\Delta \vec{z} = \vec{z} - \vec{X}(t)$ are column vectors; $(\cdot)^\top$ is a transposed matrix; a_x , b_x , b_y , and b_z are row vectors of the form $\left(a_x = \frac{\partial a}{\partial x_i} \Big|_{\vec{x}=\vec{X}(t)}\right)$, $\left(b_x = \frac{\partial b}{\partial x_i} \Big|_{\vec{x}=\vec{y}=\vec{z}=\vec{X}(t)}\right)$, row vectors b_y , b_z have the analogous form; a_{xx} and b_{xx} are symmetric matrices of the form $a_{xx} = \left(\frac{\partial^2 a}{\partial x_i \partial x_j} \Big|_{\vec{x}=\vec{X}(t)}\right)$, $b_{xx} = \left(\frac{\partial^2 b}{\partial x_i \partial x_j} \Big|_{\vec{x}=\vec{y}=\vec{z}=\vec{X}(t)}\right)$, matrices b_{yy} , b_{zz} , b_{xy} , b_{yx} , b_{zy} , b_{yz} , and b_{xz} have the analogous form.

The key point of the considered approach [12] is that the nonlocal nonlinearity enters into the approximate kinetic equation obtained with the help of an asymptotic expansion of (1) in the form of the moments of the solution $u(\vec{x}, t, D)$, and dynamical equations that determine the evolution of these moments can be solved separately.

In order to construct the leading term of asymptotics, we need corresponding moments of up to the second order that are defined as follows

$$\begin{aligned} \sigma_u(t, D) &= \int_{\mathbb{R}^n} u(\vec{x}, t, D) d\vec{x}, & \vec{x}_u(t, D) &= \frac{1}{\sigma_u(t, D)} \int_{\mathbb{R}^n} \vec{x} u(\vec{x}, t, D) d\vec{x}, \\ (\alpha_{u,ij}^{(2)}(t, D)) &= \frac{1}{\sigma_u(t, D)} \int_{\mathbb{R}^n} \Delta x_i \Delta x_j u(\vec{x}, t, D) d\vec{x}. \end{aligned} \quad (7)$$

Here, $\alpha_u^{(2)}(t, D) = (\alpha_{u,ij}^{(2)}(t, D))$ is the symmetric matrix of the central second-order moments of the function $u(\vec{x}, t, D)$. The first-order moment will determine the functional parameter $\vec{X}(t, D)$ of the class \mathcal{P}_D^t (4) as

$$\vec{x}_u(t, D) = \vec{X}(t, D). \quad (8)$$

Dynamical equations for the moments (7) are obtained by differentiation with respect to t , from the definitions (7), and using $\partial_t u(\vec{x}, t)$ from (1). Taking into consideration expansions (6) and estimates (5) with accuracy of $O(D^{3/2})$, we arrive at the following moment equations:

$$\begin{aligned} \dot{\sigma}_u &= \sigma_u \left(a(\vec{x}_u, t) + \frac{1}{2} \text{Sp} [a_{xx} \alpha_u^{(2)}] \right) \\ &\quad - \varkappa \sigma_u^3 \left(b(\vec{x}_u, \vec{x}_u, \vec{x}_u, t) + \frac{1}{2} \text{Sp} [(b_{xx} + b_{yy} + b_{zz}) \alpha_u^{(2)}] \right), \\ \dot{\vec{x}}_u &= (a_x - \varkappa \sigma_u^2 b_x) \alpha_u^{(2)}, & \dot{\alpha}_u^{(2)} &= 2D \check{D}_a(t) \mathbb{I}_n, \end{aligned} \quad (9)$$

where \mathbb{I}_n is the identity matrix of size n , and the expression $a_{xx} \alpha_u^{(2)}$ implies a product of matrices a_{xx} and $\alpha_u^{(2)}$.

Let us consider Equation (9) as the system where the aggregate of moments

$$g_u(t) = (\sigma_u(t), \vec{x}_u(t), \alpha_u^{(2)}(t)) \quad (10)$$

is substituted for the set of independent variables $(\sigma(t), \vec{x}(t), \alpha^{(2)}(t))$, and $\alpha^{(2)} = (\alpha_{ij}^{(2)}(t))$ that are not related to the function $u(\vec{x}, t)$ in the general case. Thus, the resulting system can be treated as an independent dynamical system. According to [12], this system is termed the Einstein–Ehrenfest (EE) system of the second order for the nonlocal kinetic Equation (1) in the class of TCF (4). The second order of the system implies that we preserve the terms of order not higher than $O(\sqrt{D}^2)$.

Let the general solution of this system be

$$g(t, \mathbf{C}) = (\sigma(t, \mathbf{C}), \vec{x}(t, \mathbf{C}), \alpha^{(2)}(t, \mathbf{C})), \tag{11}$$

where \mathbf{C} is a set of arbitrary integration constants. Then, the substitution of the expansion (6) into Equation (1) with the replacement of moments (10) by the general solution (11) yields the following linear equation:

$$\hat{\mathcal{L}}(\vec{x}, t, \mathbf{C})v(\vec{x}, t) = 0, \tag{12}$$

where the linear operator $\hat{\mathcal{L}}(\vec{x}, t, \mathbf{C})$ is given by

$$\begin{aligned} \hat{\mathcal{L}}(\vec{x}, t, \mathbf{C}) &= -\partial_t + D\tilde{D}_a(t)\Delta_x + L(t, \mathbf{C}) + L_x(t, \mathbf{C})\Delta\vec{x} + \frac{1}{2}\Delta\vec{x}^\top L_{xx}(t, \mathbf{C})\Delta\vec{x}, \\ L(t, \mathbf{C}) &= a(\vec{x}(t, \mathbf{C}), t) - \varkappa\sigma^2(t, \mathbf{C}) \left(b(\vec{x}(t, \mathbf{C}), \vec{x}(t, \mathbf{C}), \vec{x}(t, \mathbf{C}), t) \right. \\ &+ \left. \frac{1}{2} \text{Sp} \left[\left(b_{yy}(\vec{x}(t, \mathbf{C}), \vec{x}(t, \mathbf{C}), \vec{x}(t, \mathbf{C}), t) + b_{zz}(\vec{x}(t, \mathbf{C}), \vec{x}(t, \mathbf{C}), \vec{x}(t, \mathbf{C}), t) \right) \alpha^{(2)}(t, \mathbf{C}) \right] \right), \\ L_x(t, \mathbf{C}) &= a_x(\vec{x}(t, \mathbf{C}), t) - \varkappa\sigma^2(t, \mathbf{C})b_x(\vec{x}(t, \mathbf{C}), \vec{x}(t, \mathbf{C}), \vec{x}(t, \mathbf{C}), t), \\ L_{xx}(t, \mathbf{C}) &= a_{xx}(\vec{x}(t, \mathbf{C}), t) - \varkappa\sigma^2(t, \mathbf{C})b_{xx}(\vec{x}(t, \mathbf{C}), \vec{x}(t, \mathbf{C}), \vec{x}(t, \mathbf{C}), t). \end{aligned} \tag{13}$$

Equations (12) and (13) in [12] are termed the associated linear equation for Equation (1).

Let us pose the Cauchy problem for Equation (1) in the class of TCF (4):

$$u(\vec{x}, t, D)|_{t=0} = \varphi(\vec{x}, D) \in \mathcal{P}_0^D, \quad \mathcal{P}_0^D = \mathcal{P}_t^D|_{t=0}. \tag{14}$$

Next, we impose a restriction on the integration constants \mathbf{C} involved in the general solutions (11). The restriction is given by the following algebraic condition:

$$g(0, \mathbf{C}) = g_\varphi, \tag{15}$$

which yields $\mathbf{C} = \mathbf{C}_\varphi$. Here, g_φ is the aggregate of the moments (10) that is determined by the initial condition $\varphi(\vec{x}, D)$ as

$$\begin{aligned} \sigma_\varphi &= \int_{\mathbb{R}^n} \varphi(\vec{x}, D)d\vec{x}, & \vec{x}_\varphi &= \frac{1}{\sigma_\varphi} \int_{\mathbb{R}^n} \vec{x}\varphi(\vec{x}, D)d\vec{x}, \\ (\alpha_{\varphi, ij}^{(2)}) &= \frac{1}{\sigma_\varphi} \int_{\mathbb{R}^n} (x_i - x_{\varphi, i})(x_j - x_{\varphi, j})\varphi(\vec{x}, D)d\vec{x}. \end{aligned} \tag{16}$$

Let us consider the Cauchy problem for the associated linear Equations (12) and (13) with the initial condition

$$v(\vec{x}, t)|_{t=0} = \varphi(\vec{x}, D). \tag{17}$$

According to [9,12], the solution of the Cauchy problem for Equation (1) with the initial condition (14) and the solutions of the Cauchy problems (12), (17) in the class of TCF are related as follows:

$$u(\vec{x}, t, D) = v(\vec{x}, t, \mathbf{C}_\varphi) + O(D^{3/2}), \tag{18}$$

where \mathbf{C}_φ is given by the algebraic condition (15).

3. Semiclassical Asymptotics in a Two-Dimensional Plane-Parallel Case

In this section, based on the method proposed in the previous section, we will obtain an explicit expression for a family of asymptotic solutions for the special case of Equation (1) analogous to the one considered in [12].

Let us consider the problem in the plane orthogonal to the GDT axis. Let $\vec{x} = (x_1, x_2) \in \mathbb{R}^2$ be Cartesian coordinates in this plane and the coefficients in (1) be given by

$$\begin{aligned} a(\vec{x}, t) &= a(t), & b(\vec{x}, \vec{y}, \vec{z}, t) &= b(t)p(\vec{x} - \vec{y}, \vec{x} - \vec{z}, \mu), \\ p(\vec{r}_1, \vec{r}_2, \mu) &= \exp\left[-\frac{\vec{r}_1^2 + \vec{r}_2^2}{2\mu^2}\right], \end{aligned} \quad (19)$$

where the functions $a(t)$ and $b(t)$ are assumed to be monotone decreasing and increasing, respectively, and the parameter μ characterizes the nonlocality of the nonlinearity kernel $p(\vec{r}_1, \vec{r}_2, \mu)$, $\vec{x}^2 = \langle \vec{x}, \vec{x} \rangle$.

For functions (19), Equations (8) and (9) yield $\dot{\vec{X}}(t) = 0$. The identity $\dot{\vec{X}} = 0$ also holds in a more general case for the problem with the symmetric configuration of a GDT. We choose the origin of coordinates so that $\vec{X}(t)|_{t=0} = 0$. It leads to $\vec{X}(t) = 0$. Ions are usually localized on the GDT axis, which is taken as the origin of coordinates in our case. Then, $\Delta\vec{x} = \vec{x}$.

In the case under consideration, Equation (9) reads

$$\dot{\sigma}(t) = a(t)\sigma(t) - \varkappa\beta(t)\sigma^3(t), \quad \vec{x}(t) = \vec{X}(t) = 0, \quad \alpha^{(2)}(t) = 2Dd(t). \quad (20)$$

Here, we denoted

$$\beta(t) = b(t)\left(1 - \frac{4}{\mu^2}D \cdot \text{Sp}[d(t)]\right), \quad d(t) = \bar{d} + \mathbb{I}_2 \int_0^t \bar{D}_a(\tau)d\tau, \quad d(0) = \bar{d}, \quad (21)$$

where \bar{d} is a diagonal 2×2 -matrix, diagonal elements of the matrix $2D\bar{d}$ characterize the degree of localization (dispersion) of the initial axial ion distribution with respect to x_1 , and x_2 , \mathbb{I}_2 is the identity 2×2 -matrix.

The general solution of Equation (20) is given by

$$\sigma(t) = z^{-2}(t), \quad z(t) = c^{-1}\omega(t, 0) + 2\varkappa \int_0^t \beta(\tau)\omega(t, \tau)d\tau, \quad (22)$$

where $\omega(t, \tau) = \exp\left(-2 \int_\tau^t a(\zeta)d\zeta\right)$ and c is an arbitrary integration constant, $\sigma(0) = c^2$.

Note that $\frac{\partial\omega(t, \tau)}{\partial t} = -2a(t)\omega(t, \tau)$ and $\omega(t, t) = 1$.

Thus, the relation (20) for $\alpha^{(2)}(t) = \alpha^{(2)}(t, \mathbf{C})$ with $\vec{x}(t) = 0$ and the relation (22) for $\sigma(t) = \sigma(t, \mathbf{C})$ yield the general solution of the EE system with arbitrary integration constant \mathbf{C} :

$$g(t, \mathbf{C}) = (\sigma(t, \mathbf{C}), \vec{x}(t, \mathbf{C}) = 0, \alpha^{(2)}(t, \mathbf{C})), \quad (23)$$

where the set of integration constants reads

$$\mathbf{C} = (c^2, 0, 2D\bar{d}). \quad (24)$$

Let us proceed to the construction of the family of particular solutions for the kinetic equation. For the two-dimensional case under consideration, $\vec{x} \in \mathbb{R}^2$, with the coefficients (19), Equation (1) can be written as

$$\begin{aligned} \partial_t u(\vec{x}, t) &= D\tilde{D}_a(t)\Delta_x u(\vec{x}, t) + a(t)u(\vec{x}, t) \\ &- \varkappa b(t)u(\vec{x}, t) \int_{\mathbb{R}^2} d\vec{y} \int_{\mathbb{R}^2} d\vec{z} p(\vec{x} - \vec{y}, \vec{x} - \vec{z})u(\vec{y}, t)u(\vec{z}, t), \end{aligned} \quad (25)$$

where Δ_x is a two-dimensional Laplace operator. In view of (5) and (6), Equation (25) reads

$$\begin{aligned} \partial_t u(\vec{x}, t) &= D\tilde{D}_a(t)\Delta_x u(\vec{x}, t) + a(t)u(\vec{x}, t) - \\ &- \frac{\varkappa}{\mu^2} \sigma_u^2(t) \left[\mu^2 - \text{Sp } \alpha_u^{(2)}(t) - \vec{x}^2 \right] + O(D^{3/2}). \end{aligned} \quad (26)$$

Next, we obtain the associated linear Equation (12) from Equation (26) by the replacement of moments $\sigma_u(t)$, $\alpha_u^{(2)}(t)$ of the required solution $u(\vec{x}, t)$ by the general solutions of the dynamical EE system (23).

In view of (20), the relation (13) reads

$$\begin{aligned} L(t, \mathbf{C}) &= a(t) - \frac{\varkappa}{\mu^2} \sigma^2(t) \left(\mu^2 - 2D \cdot \text{Sp} [d(t)] \right), \\ L_x(t, \mathbf{C}) &= 0, \\ L_{xx}(t, \mathbf{C}) &= 2 \frac{\varkappa}{\mu^2} \sigma^2(t, \mathbf{C}) b(t) \mathbb{I}_2, \end{aligned} \quad (27)$$

which allows us to write the associated linear Equation (12) as

$$\begin{aligned} \hat{\mathcal{L}}(\vec{x}, t, \mathbf{C})v(\vec{x}, t) &= \left(-\partial_t + D\tilde{D}_a(t)\Delta_x + a(t) \right. \\ &\left. - \frac{\varkappa}{\mu^2} b(t) \sigma^2(t, \mathbf{C}) \left(\mu^2 - 2D \cdot \text{Sp} [d(t)] - \vec{x}^2 \right) \right) v(\vec{x}, t) = 0. \end{aligned} \quad (28)$$

We are looking for the particular solutions of Equation (28) in the form of the Gaussian function:

$$v_0(\vec{x}, t, \mathbf{C}) = N_0 \frac{1}{D} \exp \left\{ \frac{1}{D} \left(S(t, D) + \frac{1}{2} \vec{x}^\top Q(t) \vec{x} \right) + \phi(t, D) \right\}, \quad (29)$$

where N_0 is a normalization constant that is related to the initial number of ions. The multiplier $\frac{1}{D}$ is introduced for convenience. Note that this multiplier does not contradict the ansatz (4) since it can be included in the function $S(t, D)$ as the summand $(-D \ln D)$. The term $(-D \ln D) \rightarrow 0$ as $D \rightarrow 0$, so it does not violate the regularity of $S(t, D)$ with respect to \sqrt{D} as $D \rightarrow 0$.

The substitution of (29) into (28) yields

$$\begin{aligned} \dot{S} &= D \left[a(t) - \varkappa b(t) \sigma^2(t) \left(1 - \frac{2D}{\mu^2} \text{Sp} [d(t)] \right) \right], & \phi(t, D) &= \int_0^t \tilde{D}_a(\tau) \text{Sp} Q(\tau) d\tau, \\ \frac{1}{2} \dot{Q} &= \tilde{D}_a(t) Q^2 + D \frac{\varkappa}{\mu^2} b(t) \sigma^2(t) \mathbb{I}_2. \end{aligned} \quad (30)$$

Equation (30) determines the functions $S(t, D)$ and $\phi(t, D)$ through quadratures:

$$S(t, D) = D \int_0^t d\tau \left[a(\tau) - \varkappa b(t)\sigma^2(t) \left(1 - \frac{2D}{\mu^2} \text{Sp} [d(t)] \right) \right], \tag{31}$$

$$\phi(t, D) = \int_0^t d\tau \tilde{D}_a \text{Sp} Q(\tau).$$

The matrix Riccati equation in (30) can be represented as the linear system by the substitution:

$$Q(t) = B(t)C^{-1}(t), \tag{32}$$

where $B(t)$ and $C(t)$ are nondegenerate matrices that satisfy the following matrix linear system of differential equations:

$$\begin{cases} \dot{C}(t) = -2\tilde{D}_a(t)B(t), \\ \dot{B}(t) = 2\frac{\varkappa}{\mu^2}Db(t)\sigma^2(t)C(t). \end{cases} \tag{33}$$

4. Countable Set of Solutions to the Associated Linear Equation

In this section, we describe the approach to constructing the family of solutions of the associated linear Equation (28) based on solutions of the system (33).

Denote $\tilde{b}(t) = \frac{\varkappa}{\mu^2}Db(t)\sigma^2(t)$. Since the coefficients in the system (33) are scalar, the matrices $B(t)$ and $C(t)$ can be sought in the diagonal form:

$$B(t) = \text{diag} \{W_1(t), W_2(t)\}, \quad C(t) = \text{diag} \{Z_1(t), Z_2(t)\}, \tag{34}$$

where $W_i(t)$ and $Z_i(t)$, $i, j = 1, 2$, are scalar functions.

Note that the matrix $Q(t)$ is also diagonal in this case:

$$Q(t) = \text{diag} \left\{ \frac{W_1(t)}{Z_1(t)}, \frac{W_2(t)}{Z_2(t)} \right\} \tag{35}$$

For the functions in (34), the system (33) reads

$$\dot{Z}_i(t) = -2\tilde{D}_a(t)W_i(t), \quad \dot{W}_i(t) = 2\tilde{b}(t)Z_i(t). \tag{36}$$

The equations for functions $(W_1(t), Z_1(t))$ and $(W_2(t), Z_2(t))$ are identical. Hence, these functions can differ only due to their different initial condition. Let us consider the system of equations for the functions $W(t)$ and $Z(t)$,

$$\dot{Z}(t) = -2\tilde{D}_a(t)W(t), \quad \dot{W}(t) = 2\tilde{b}(t)Z(t), \tag{37}$$

whose solutions for different initial conditions determine the matrices $B(t)$ and $C(t)$ and, correspondingly, the matrix $Q(t)$. The system (37) was coined “the variational system” in [9].

The system (37) has two linearly independent solutions. Let us denote them by the following formulae:

$$\left(W^{(\pm)}(t), Z^{(\pm)}(t) \right). \tag{38}$$

These solutions are determined by the following initial conditions:

$$W^{(+)}(0) = \beta > 0, \quad W^{(-)}(0) = -\beta, \quad Z^{(\pm)}(0) = 1. \tag{39}$$

For the functions $(W_i^\pm(t), Z_i^\pm(t))$, $i, j = 1, 2$, the numbers β_i that determine the initial conditions by

$$W_i^{(+)}(0) = \beta_i > 0, \quad W_i^{(-)}(0) = -\beta_i, \quad Z_i^{(\pm)}(0) = 1 \tag{40}$$

can be different in a general case.

Now, we can construct the family of solutions to Equation (28) based on the particular solution (29). For this purpose, we use the well-known quantum mechanics method widely used in various problems [19–21].

Denote the two-dimensional symplectic identity matrix as $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Introduce the two-dimensional column vectors

$$a^{(\pm)}(t) = (W^{(\pm)}(t), Z^{(\pm)}(t))^T \tag{41}$$

and the skew scalar product

$$\{a^{(-)}(t), a^{(+)}(t)\} = (a^{(-)}(t))^T J a^{(+)}(t) = Z^{(-)}(t)W^{(+)}(t) - Z^{(+)}(t)W^{(-)}(t), \tag{42}$$

which is a conserved quantity in view of (37):

$$\begin{aligned} \frac{d}{dt} \{a^{(-)}(t), a^{(+)}(t)\} &= \dot{Z}^{(-)}(t)W^{(+)}(t) + Z^{(-)}(t)\dot{W}^{(+)}(t) - \dot{Z}^{(+)}(t)W^{(-)}(t) \\ &\quad - Z^{(+)}(t)\dot{W}^{(-)}(t) = -2\tilde{D}_a(t)W^{(-)}(t)W^{(+)}(t) + Z^{(-)}(t)2\tilde{b}(t)Z^{(+)}(t) \\ &\quad + 2\tilde{D}_a(t)W^{(+)}(t)W^{(-)}(t) - Z^{(+)}(t)2\tilde{b}(t)Z^{(+)}(t) = 0. \end{aligned} \tag{43}$$

Therefore, for $t = 0$, we have

$$\{a^{(-)}(0), a^{(+)}(0)\} = Z^{(-)}(0)W^{(+)}(0) - Z^{(+)}(0)W^{(-)}(0) = 2\beta, \tag{44}$$

i.e.,

$$\{a^{(-)}(t), a^{(+)}(t)\} = 2\beta. \tag{45}$$

Following the general Maslov complex germ method, the linear *two*-dimensional space with the basis vectors $a_1^{(-)}(t)$ and $a_2^{(-)}(t)$ is called the germ r_t^2 . In our case, the germ is chosen to be real, since we seek the real solutions to (25). The pair (Λ_t^0, r_t^2) , where Λ_t^0 is the trajectory (the time-dependent 0-dimensional manifold) $\vec{x} = \vec{X}(t)$, determines the set of asymptotic solutions to Equation (25). In order to construct such solutions, we present the symmetry operators associated with the vectors $a_1^{(-)}(t), a_2^{(-)}(t)$ and with the vectors $a_1^{(+)}(t), a_2^{(+)}(t)$.

Define the operators

$$\begin{aligned} \hat{a}^{(-)}(t) &= -N_a \left(Z^{(-)}(t)D\partial_x - W^{(-)}(t)x \right), \\ \hat{a}^{(+)}(t) &= N_a \left(Z^{(+)}(t)D\partial_x - W^{(+)}(t)x \right). \end{aligned} \tag{46}$$

These operators satisfy the following commutation relations:

$$\begin{aligned} [\hat{a}^{(-)}(t), \hat{a}^{(-)}(t)] &= [\hat{a}^{(+)}(t), \hat{a}^{(+)}(t)] = 0, \\ [\hat{a}^{(-)}(t), \hat{a}^{(+)}(t)] &= N_a^2 \left[-Z^{(-)}(t)D\partial_x + W^{(-)}(t)x, Z^{(+)}(t)D\partial_x - W^{(+)}(t)x \right] \\ &= N_a^2 D \left(Z^{(-)}(t)W^{(+)}(t) - Z^{(+)}(t)W^{(-)}(t) \right) = N_a^2 D \cdot 2\beta. \end{aligned} \tag{47}$$

We define the normalization multiplier as $N_a = \frac{1}{\sqrt{2\beta D}}$. Then, we have

$$\begin{aligned} \hat{a}^{(-)}(t) &= -\frac{1}{\sqrt{2\beta D}} \left(Z^{(-)}(t) D \partial_x - W^{(-)}(t)x \right), \\ \hat{a}^{(+)}(t) &= -\frac{1}{\sqrt{2\beta D}} \left(Z^{(+)}(t) D \partial_x - W^{(+)}(t)x \right), \end{aligned} \tag{48}$$

and

$$[\hat{a}^{(-)}(t), \hat{a}^{(+)}(t)] = 1. \tag{49}$$

For the diagonal matrix $Q(t)$ (35), the solution (29) can be written as

$$v_0(\vec{x}, t, \mathbf{C}) = N_0 \frac{1}{D} \exp \left\{ \frac{1}{D} \left[S(t, D) + \frac{1}{2} (Q_1(t)x_1^2 + Q_2(t)x_2^2) \right] + \phi(t, D) \right\}. \tag{50}$$

Here, $Q_i(t) = \frac{W_i^{(-)}(t)}{Z_i^{(-)}(t)}$, and the functions $(W_i^{(\pm)}, Z_i^{(\pm)})$ are solutions of the system (37) with initial conditions (40).

For the two-dimensional case, the operators (48) read

$$\begin{aligned} \hat{a}_i^{(-)}(t) &= -\frac{1}{\sqrt{2\beta_i D}} \left(Z_i^{(-)}(t) D \partial_{x_i} - W_i^{(-)}(t)x_i \right), \\ \hat{a}_i^{(+)}(t) &= -\frac{1}{\sqrt{2\beta_i D}} \left(Z_i^{(+)}(t) D \partial_{x_i} - W_i^{(+)}(t)x_i \right), \end{aligned} \tag{51}$$

and the commutators are as follows:

$$[\hat{a}_i^{(-)}(t), \hat{a}_j^{(+)}(t)] = \delta_{ij}. \tag{52}$$

It can be shown that the operators $\hat{a}_i^{(-)}(t)$ nullify the function $v_0(\vec{x}, t, \mathbf{C})$ of the form (50)

$$\hat{a}_i^{(-)}(t)v_0(\vec{x}, t, \mathbf{C}) = 0, \tag{53}$$

and that operators $\hat{a}^{(\pm)}(t)$ commute with the operator $\hat{\mathcal{L}}(\vec{x}, t, \mathbf{C})$ of Equation (28):

$$[\hat{\mathcal{L}}(\vec{x}, t, \mathbf{C}), \hat{a}_i^{(\pm)}(t)] = 0. \tag{54}$$

It means that the operators $\hat{a}_i^{(\pm)}(t)$ are the symmetry operators for Equation (28). Hence, the action of the operators $\hat{a}_i^{(+)}(t)$ on $v_0(\vec{x}, t, \mathbf{C})$ generates the family of new solutions to Equation (28). Let us define this family of solutions by

$$v_n(\vec{x}, t, \mathbf{C}) = \frac{N_n}{N_0} \frac{1}{n!} (\hat{a}_1^{(+)}(t))^{n_1} (\hat{a}_2^{(+)}(t))^{n_2} v_0(\vec{x}, t, \mathbf{C}). \tag{55}$$

Here, $n = (n_1, n_2)$ is the two-dimensional multi-index, $n_i \in \{0, 1, 2, \dots\}$, $|n| = n_1 + n_2$, $n! = n_1!n_2!$, N_n are normalization coefficients.

The solutions to (55) can be written in the explicit form. Define

$$\xi_i = \sqrt{\frac{\beta_i}{DZ_i^{(-)}(t)Z_i^{(+)}(t)}} x_i. \tag{56}$$

In view of the formula for the Hermitian polynomials [22]

$$H_{n_i}(\xi_i) = (-1)^{n_i} \exp(\xi_i^2) (\partial_{\xi_i})^{n_i} \exp(-\xi_i^2), \tag{57}$$

the relation (55) reads

$$v_n(\vec{x}, t, \mathbf{C}) = \frac{N_n (-1)^{|n|}}{N_0 \sqrt{2^{|n|} n!}} \left(\frac{Z_1^{(+)}(t)}{Z_1^{(-)}(t)} \right)^{\frac{n_1}{2}} \left(\frac{Z_2^{(+)}(t)}{Z_2^{(-)}(t)} \right)^{\frac{n_2}{2}} H_{n_1}(\xi_1) H_{n_2}(\xi_2) v_0(\vec{x}, t, \mathbf{C}). \tag{58}$$

The set (58) is the parametric family of solutions to the associated linear Equation (28). Our next task is to find a countable set of asymptotic solutions to the kinetic Equation (25) among these solutions with the free parameter \mathbf{C} .

5. Algebraic Conditions and Solutions of the Kinetic Equation

In this section, we construct solutions $u_n(\vec{x}, t)$ to Equation (26) with help of the solutions $v_n(\vec{x}, t, \mathbf{C})$ (58) to the associated linear equation (28). For this, we impose the algebraic conditions (15) and (16) on arbitrary integration constants $\mathbf{C} = (c^2, 0, 2D\vec{d})$ that are included in the solution of the EE system. Then, we have

$$u_n(\vec{x}, t) = v_n(\vec{x}, t, \mathbf{C}_n), \tag{59}$$

where

$$\mathbf{C}_n = (c_n^2, 0, 2D\vec{d}_n), \quad n = (n_1, n_2). \tag{60}$$

The functions $u_n(\vec{x}, t)$ are the leading terms of asymptotics for solutions to the original nonlocal kinetic Equation (25) with accuracy of $O(D^{3/2})$.

Next, we obtain the constants (60) in an explicit form. Let us write the solutions to (58) in the form

$$v_n(\vec{x}, t, \mathbf{C}) = Y_n(t) H_{n_1}(\xi_1) H_{n_2}(\xi_2) \frac{1}{D} \exp \left\{ \frac{1}{2D} (Q_1(t)x_1^2 + Q_2(t)x_2^2) \right\},$$

$$Y_n(t) = \frac{(-1)^{|n|}}{\sqrt{2^{|n|} n!}} \left(\frac{Z_1^{(+)}(t)}{Z_1^{(-)}(t)} \right)^{\frac{n_1}{2}} \left(\frac{Z_2^{(+)}(t)}{Z_2^{(-)}(t)} \right)^{\frac{n_2}{2}} N_n \exp \left\{ \frac{1}{D} S(t, D) + \phi(t, D) \right\}, \tag{61}$$

$$Q_i(t) = \frac{W_i^{(-)}}{Z_i^{(-)}(t)}, \quad \xi_i = \sqrt{\frac{\beta_i}{D Z_i^{(-)}(t) Z_i^{(+)}(t)}} x_i.$$

In order to obtain the moments for the whole set of solutions $v_n(\vec{x}, t, \mathbf{C})$, we use the representation of the Hermitian polynomial through the generating function [22]:

$$\exp[2xt - t^2] = \sum_{k=0}^{\infty} H_k(x) \frac{t^k}{k!}. \tag{62}$$

Then, the generating function $V(\vec{x}, t, t_1, t_2, \mathbf{C})$ for solutions (61) is given as follows,

$$V(\vec{x}, t, t_1, t_2, \mathbf{C}) = \exp \left\{ 2 \sqrt{\frac{\beta_1}{D Z_1^{(-)}(t) Z_1^{(+)}(t)}} t_1 x_1 + 2 \sqrt{\frac{\beta_2}{D Z_2^{(-)}(t) Z_2^{(+)}(t)}} t_2 x_2 \right\}$$

$$\times \frac{1}{D} \exp \left\{ \frac{1}{2D} \left(\frac{W_1^{(-)}(t)}{Z_1^{(-)}(t)} x_1^2 + \frac{W_2^{(-)}(t)}{Z_2^{(-)}(t)} x_2^2 \right) \right\}, \tag{63}$$

$$V(\vec{x}, t, t_1, t_2, \mathbf{C}) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} Y_n^{-1}(t) v_n(\vec{x}, t, \mathbf{C}) \frac{t_1^{n_1} t_2^{n_2}}{n!}.$$

From (7), one can see that $\sigma_u(t)$ and $\alpha_u^{(2)}(t) \cdot \sigma_u(t)$ are linear functionals with respect to $u(\vec{x}, t)$. This property allows us to obtain moments $\sigma_n(t)$ and $\alpha_n^{(2)}(t) \cdot \sigma_n(t)$ of the functions

$v_n(\vec{x}, t, \mathbf{C})$ (61) with the help of respective generating functions derived from (63). Introduce the following functions:

$$\begin{aligned} \Omega(t, t_1, t_2) &= \int_{\mathbb{R}^2} V(\vec{x}, t, t_1, t_2, \mathbf{C}) d\vec{x}, \\ \Omega(t, t_1, t_2) &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} Y_n^{(-1)}(t) \sigma_n(t) \frac{t_1^{n_1} t_2^{n_2}}{n!}, \\ A_{ii}(t, t_1, t_2) &= \int_{\mathbb{R}^2} x_i^2 V(\vec{x}, t, t_1, t_2, \mathbf{C}) d\vec{x}, \quad i = 1, 2, \\ A_{ii}(t, t_1, t_2) &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} Y_n^{-1}(t) \alpha_{ii,n}^{(2)}(t) \sigma_n(t) \frac{t_1^{n_1} t_2^{n_2}}{n!}. \end{aligned} \tag{64}$$

Here, the subscripts ii indicate the number of a matrix element.

Straightforward calculations of integral in the formulae (64) yield

$$\begin{aligned} \Omega(t, t_1, t_2) &= \frac{2\pi \sqrt{Z_1^{(-)}(t) Z_2^{(-)}(t)}}{\sqrt{W_1^{(-)}(t) W_2^{(-)}(t)}} \exp \left[-2 \left(\frac{\beta_1 t_1^2}{Z_1^{(+)}(t) W_1^{(-)}(t)} + \frac{\beta_2 t_2^2}{Z_2^{(+)}(t) W_2^{(-)}(t)} \right) - t_1^2 - t_2^2 \right], \\ A_{11}(t, t_1, t_2) &= \frac{2\pi D \sqrt{(Z_1^{(-)}(t))^5 Z_2^{(-)}(t)}}{\sqrt{(W_1^{(-)}(t))^5 W_2^{(-)}(t)}} \left(-\frac{W_1^{(-)}(t)}{Z_1^{(-)}(t)} + 4t_1^2 \frac{\beta_1}{Z_1^{(-)}(t) Z_1^{(+)}(t)} \right) \\ &\times \exp \left[-2 \left(\frac{\beta_1 t_1^2}{Z_1^{(+)}(t) W_1^{(-)}(t)} + \frac{\beta_2 t_2^2}{Z_2^{(+)}(t) W_2^{(-)}(t)} \right) - t_1^2 - t_2^2 \right]. \end{aligned} \tag{65}$$

The formula for $A_{22}(t, t_1, t_2)$ can be obtained from one for $A_{11}(t, t_1, t_2)$ (65) by the formal interchanging $1 \leftrightarrow 2$ in all subscripts. In view of (64), the expansion of functions (65) in powers of t_1, t_2 yields the following expression for moments:

$$\begin{aligned} \sigma_{(2n_1, 2n_2)}(t) &= Y_{(2n_1, 2n_2)}(t) \frac{2\pi \sqrt{Z_1^{(-)}(t) Z_2^{(-)}(t)}}{\sqrt{W_1^{(-)}(t) W_2^{(-)}(t)}} \frac{(2n_1)!(2n_2)!}{n_1!n_2!} \\ &\times \left[-\frac{2\beta_1}{Z_1^{(+)}(t) W_1^{(-)}(t)} - 1 \right]^{n_1} \left[-\frac{2\beta_2}{Z_2^{(+)}(t) W_2^{(-)}(t)} - 1 \right]^{n_2}, \\ \sigma_{(2n_1+1, 2n_2)}(t) &= \sigma_{(2n_1, 2n_2+1)}(t) = \sigma_{(2n_1+1, 2n_2+1)}(t) = 0, \end{aligned} \tag{66}$$

$$\begin{aligned} \alpha_{11, (2n_1, 2n_2)}^{(2)}(t) &= -D \frac{Z_1^{(-)}(t)}{W_1^{(-)}(t)} \left\{ 1 + \left[\frac{2\beta_1}{Z_1^{(+)}(t) W_1^{(-)}(t)} + 1 \right]^{-1} \frac{4\beta_1}{W_1^{(-)}(t) Z_1^{(+)}(t)} n_1 \right\}, \\ \alpha_{22, (2n_1, 2n_2)}^{(2)}(t) &= -D \frac{Z_2^{(-)}(t)}{W_2^{(-)}(t)} \left\{ 1 + \left[\frac{2\beta_2}{Z_2^{(+)}(t) W_2^{(-)}(t)} + 1 \right]^{-1} \frac{4\beta_1}{W_2^{(-)}(t) Z_2^{(+)}(t)} n_2 \right\}. \end{aligned}$$

Thus, among the obtained solutions $v_{(n_1, n_2)}(\vec{x}, t, \mathbf{C})$ to the associated linear Equation (28), only those ones with even indices n_1, n_2 generate the asymptotic solutions to the kinetic Equation (25).

Note that functions (66) are particular solutions to the EE system (20). Hence, the solutions to the EE system (20) are determined by the solutions to the variational system (36). It is a corollary of the fact shown in [12] that the leading term of asymptotics for the function $u(\vec{x}, t)$ uniquely defines the functions $\sigma(t), \alpha^{(2)}(t)$ within accuracy of $O(D^{3/2})$.

Substituting $t = 0$ into the formulae (66) and taking into account (40) and (31), we obtain the following initial conditions for moments included in the integration constants C_n :

$$\begin{aligned} \sigma_{(2n_1, 2n_2)}(0) &= \frac{N_{(2n_1, 2n_2)}}{2^{n_1+n_2-1}} \frac{\pi}{\sqrt{\beta_1 \beta_2}} \frac{\sqrt{(2n_1)!(2n_2)!}}{n_1!n_2!}, \\ \alpha_{11, (2n_1, 2n_2)}^{(2)}(0) &= \frac{D}{\beta_1}(1 + 4n_1), \quad \alpha_{22, (2n_1, 2n_2)}^{(2)}(0) = \frac{D}{\beta_2}(1 + 4n_2), \end{aligned} \tag{67}$$

while the integration constants (60) read

$$c_{(2n_1, 2n_2)}^2 = \sigma_{(2n_1, 2n_2)}(0), \quad \bar{d}_{(2n_1, 2n_2)} = \text{diag} \left(\frac{1}{2\beta_1}(1 + 4n_1), \frac{1}{2\beta_2}(1 + 4n_2) \right). \tag{68}$$

Thus, the functions (59) with the constants $C_{(2n_1, 2n_2)} = (c_{(2n_1, 2n_2)}^2, 0, 2D\bar{d}_{(2n_1, 2n_2)})$ of the form (67) and (68) determine a countable set of solutions to Equation (26) that are the leading terms of semiclassical asymptotics for the kinetic Equation (25).

Note that the solution in Equation (58) for $n \neq 0$ changes its sign in the space $\vec{x} \in \mathbb{R}^2$ due to the properties of Hermitian polynomials. For the main physical applications, the function $u(\vec{x}, t)$ is positive definite (e.g., it corresponds to the ion concentration in the model of the MVAM kinetics). Therefore, the functions $v_n(\vec{x}, t, \mathbf{C})$ are rather of interest not by themselves but as a basis for the expansion of positive definite functions. In such interpretation, the function $v_n(\vec{x}, t, \mathbf{C})$ is a “mode” of the physical state involving $v_n(\vec{x}, t, \mathbf{C})$ in its expansion. In order to clarify the meaning of this statements, let us describe the nonlinear superposition principle for semiclassical solutions of Equation (25).

The functions $v_n(\vec{x}, t)$ (61) can be written as

$$\begin{aligned} v_n(\vec{x}, t) &= Y_n(t)\Psi_n(\vec{x}, t), \quad n = (n_1, n_2), \\ \Psi_n(\vec{x}, t) &= H_{n_1}(\zeta_1)H_{n_2}(\zeta_2) \frac{1}{D} \exp \left\{ \frac{1}{2D} \left(\frac{W_1^{(-)}(t)}{Z_1^{(-)}(t)} x_1^2 + \frac{W_2^{(-)}(t)}{Z_2^{(-)}(t)} x_2^2 \right) \right\}, \\ \zeta_i &= \sqrt{\frac{\beta_i}{DZ_i^{(-)}(t)Z_i^{(+)}(t)}} x_i. \end{aligned} \tag{69}$$

The substitution of $t = 0$ into (69) yields

$$\psi_n(\vec{x}) = \Psi_n(\vec{x}, 0) = H_{n_1} \left(x_1 \sqrt{\frac{\beta_1}{D}} \right) H_{n_2} \left(x_2 \sqrt{\frac{\beta_2}{D}} \right) \frac{1}{D} \exp \left\{ - \frac{1}{2D} (\beta_1 x_1^2 + \beta_2 x_2^2) \right\}. \tag{70}$$

It can be seen that the following orthogonality condition holds for the functions $\psi_n(\vec{x})$:

$$\int_{\mathbb{R}^2} \psi_n(\vec{x}) \psi_m(\vec{x}) d\vec{x} = \frac{2^{|n|} n! \pi}{D \sqrt{\beta_1 \beta_2}} \delta_{n_1 m_1} \delta_{n_2 m_2}, \quad m = (m_1, m_2). \tag{71}$$

Let us expand the initial condition (17) in functions $\psi_n(\vec{x})$:

$$\varphi(\vec{x}) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} k_n \psi_n(\vec{x}). \tag{72}$$

Then, the initial condition $\varphi(\vec{x})$ corresponds to the following solution to the associated linear Equation (28):

$$v(\vec{x}, t, \mathbf{C}) = \exp \left\{ \frac{1}{D} S(t) + \phi(t) \right\} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} k_n \left(\frac{Z_1^{(+)}(t)}{Z_1^{(-)}(t)} \right)^{\frac{n_1}{2}} \left(\frac{Z_2^{(+)}(t)}{Z_2^{(-)}(t)} \right)^{\frac{n_2}{2}} \Psi_n(\vec{x}, t), \tag{73}$$

$$v(\vec{x}, t, \mathbf{C}) \Big|_{t=0} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} k_n \psi_n(\vec{x}), \quad \Psi_n(\vec{x}, 0) = \psi_n(\vec{x}).$$

In order to obtain the asymptotic solution of the original kinetic Equation (25), we must impose the algebraic condition $\mathbf{C} = \mathbf{C}_\varphi$. The integration constants \mathbf{C}_φ for the function $\varphi(\vec{x})$ (72) are given by

$$\sigma(0) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} k_{(2n_1, 2n_2)} \frac{2\pi}{\sqrt{\beta_1 \beta_2}} \left(-\frac{1}{\sqrt{2}} \right)^{n_1+n_2} \frac{\sqrt{(2n_1)!(2n_2)!}}{\sqrt{n_1!n_2!}}, \quad \sigma(0) \neq 0, \tag{74}$$

$$\alpha_{ii}^{(2)}(0) = \frac{1}{\sigma(0)} \frac{D}{\beta_i} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} k_{(2n_1, 2n_2)} \frac{2\pi}{\sqrt{\beta_1 \beta_2}} \left(-\frac{1}{\sqrt{2}} \right)^{n_1+n_2} \frac{\sqrt{(2n_1)!(2n_2)!}}{\sqrt{n_1!n_2!}} (1 + 4n_i),$$

where $i = 1, 2$.

Thus, a set of the integration constants \mathbf{C}_φ is determined by a set of expansion coefficients of an initial condition $\varphi(\vec{x})$ with respect to functions $\psi_n(\vec{x}, t)$, i.e., expansion coefficients nonlinearly determine the asymptotic solution $u(\vec{x}, t) = v(\vec{x}, t, \mathbf{C}) + O(D^{3/2})$ to the nonlinear kinetic Equation (25) with the initial condition $u(\vec{x}, t) \Big|_{t=0} = \varphi(\vec{x})$. The nonlinearity of this superposition principle is caused by the nonlinear dependence of the functions included in (73) (in particular, the solutions of the variational system) on the integration constants in (74).

Although the solutions of the associated linear equation $v_{(2n_1, 2n_2+1)}(\vec{x}, t)$, $v_{(2n_1+1, 2n_2)}(\vec{x}, t)$, $v_{(2n_1+1, 2n_2+1)}(\vec{x}, t)$ do not correspond to any asymptotic solutions of the kinetic equation in themselves, the functions $\Psi_{(2n_1, 2n_2+1)}(\vec{x}, t)$, $\Psi_{(2n_1+1, 2n_2)}(\vec{x}, t)$, and $\Psi_{(2n_1+1, 2n_2+1)}(\vec{x}, t)$ given by (69) enter into the expansion (73) subject to $\sigma(0) \neq 0$. Thus, we can construct asymmetrical solutions with respect to spatial variables (not odd and not even) to the nonlinear kinetic equation with the help of the nonlinear superposition principle.

Note that our functions $v_n(\vec{x}, t, \mathbf{C})$ contain two free positive parameters, β_1 and β_2 . The change of these parameters yields us a new family of the asymptotic solutions (59) or a new basis (69) for the nonlinear superposition principle. For example, the solution $v_0(\vec{x}, t, \mathbf{C}_0)$ is invariant under the interchanging $x_1 \leftrightarrow x_2$ for $\beta_1 = \beta_2$ and is not invariant for $\beta_1 \neq \beta_2$. The parameters β_1, β_2 determine the localization area of the functions $v_n(\vec{x}, t, \mathbf{C})$ and $\Psi_n(\vec{x}, t, \mathbf{C})$ with respect to x_1, x_2 . The nonlinear superposition principle can be applied to the given initial condition under any positive β_1, β_2 . However, the more the localization area of the initial condition differs from the localization area of the functions $\Psi_n(\vec{x}, t, \mathbf{C})$ determined by β_1, β_2 , the slower the series (72) converges. The slow convergence of the series (72) leads to the slow convergence of the series (73) for $t > 0$. Yet, it is not clear how to compare the localization area of two multipeak functions in a general case. It can be completed in the particular case though, which is illustrated in the next section. We can draw an analogy for the parameters β_1, β_2 with the scaling factor (dilations) of the wavelets [23], since both the functions (69) and their Fourier transform with respect to \vec{x} are localized functions at each fixed t with the localization area determined by β_1, β_2 . The proposed rule of thumb for the choice of β_1, β_2 corresponds to the value of dilations for the peak of the wavelet image.

6. Example of the Semiclassical Two-Dimensional Distribution

In this section, we construct the semiclassical solutions to Equation (25) with the initial condition

$$\varphi(\vec{x}) = \frac{N}{D} \left(\exp \left[-\frac{\vec{x}^2}{2\gamma_1 D} \right] - \varepsilon \cdot \exp \left[-\frac{\vec{x}^2}{2\gamma_2 D} \right] \right), \tag{75}$$

where $\gamma_1 > \gamma_2 > 0, 0 \leq \varepsilon \leq 1$. For such constraints for the parameters $\gamma_1, \gamma_2, \varepsilon$, and $N > 0$, the function $\varphi(\vec{x})$ does not take on negative values. If $\varepsilon > \frac{\gamma_2}{\gamma_1}$, then the initial distribution (75) has a minimum at the center $\vec{x} = 0$. The minimum of the ion distribution described by Equation (25) is physically realizable by the addition of hydrogen into the metal vapor active medium (by creating the so-called kinetically enhanced active medium) [24–26].

Let us apply the nonlinear superposition principle to the initial condition (75). Note that

$$\begin{aligned} h(\vec{x}, t_1, t_2) &= \exp \left\{ 2\sqrt{\frac{\beta_1}{D}} t_1 x_1 + 2\sqrt{\frac{\beta_2}{D}} t_2 x_2 - t_1^2 - t_2^2 \right\} \frac{1}{D} \\ &\times \exp \left\{ -\frac{1}{2D} (\beta_1 x_1^2 + \beta_2 x_2^2) \right\} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \psi_{(n_1, n_2)}(\vec{x}) \frac{t_1^{n_1} t_2^{n_2}}{n_1! n_2!}. \end{aligned} \tag{76}$$

Since the function $\varphi(\vec{x})$ (75) has the symmetry $x_1 \leftrightarrow x_2$, we put

$$\beta_1 = \beta_2 = \beta. \tag{77}$$

In view of (72) and (77), the coefficient k_n in (72) is given by

$$\int_{\mathbb{R}^2} \varphi(\vec{x}) h(\vec{x}, t_1, t_2) d\vec{x} = \frac{\pi}{D\beta} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} k_n 2^{|n|} t_1^{n_1} t_2^{n_2}. \tag{78}$$

The integral in (78) yields

$$\begin{aligned} \int_{\mathbb{R}^2} \varphi(\vec{x}) h(\vec{x}, t_1, t_2) d\vec{x} &= \frac{N}{D} \left(\frac{2\pi\gamma_1}{1 + \beta\gamma_1} \exp \left[\frac{\beta\gamma_1 - 1}{1 + \beta\gamma_1} (t_1^2 + t_2^2) \right] \right. \\ &\quad \left. - \frac{2\pi\varepsilon\gamma_2}{1 + \beta\gamma_2} \exp \left[\frac{\beta\gamma_2 - 1}{1 + \beta\gamma_2} (t_1^2 + t_2^2) \right] \right), \end{aligned} \tag{79}$$

$$k_{2n} = \frac{N\beta}{2^{|2n|} n!} \left(\frac{2\gamma_1}{1 + \beta\gamma_1} \left(\frac{\beta\gamma_1 - 1}{1 + \beta\gamma_1} \right)^{|n|} - \frac{2\varepsilon\gamma_2}{1 + \beta\gamma_2} \left(\frac{\beta\gamma_2 - 1}{1 + \beta\gamma_2} \right)^{|n|} \right), \quad 2n = (2n_1, 2n_2).$$

Let the value of the parameter β be considered optimal when the coefficients k_{2n} converge to zero, as $|n| \rightarrow \infty$ is the most rapid. It corresponds to the minimum value of the following function:

$$f(\beta) = \max \left(\left| \frac{\beta\gamma_1 - 1}{1 + \beta\gamma_1} \right|; \left| \frac{\beta\gamma_2 - 1}{1 + \beta\gamma_2} \right| \right). \tag{80}$$

The function $f(\beta)$ reaches its minimum at

$$\beta = \frac{1}{\sqrt{\gamma_1\gamma_2}}. \tag{81}$$

We use this value of β hereinafter. Note that the search of the value of β_i corresponding to the the most rapid convergence can be more complicated in the general case. However, it is not crucial to obtain the exact value of β_i corresponding to the least $|k_n|$ for large $|n|$. Hence, it can be done using some approximations for k_n (e.g., [27]).

The initial conditions for the EE system (20) can be obtained either by the substitution of the coefficients (79) into the formulae (74) or by the substitution of the initial condition (75) into the relations (16). Both ways yield

$$\begin{aligned}\sigma(0) &= 2N\pi(\gamma_1 - \varepsilon\gamma_2), \\ \alpha_{11}^{(2)}(0) &= \alpha_{22}^{(2)}(0) = D \frac{\gamma_1^2 - \varepsilon\gamma_2^2}{\gamma_1 - \varepsilon\gamma_2}.\end{aligned}\quad (82)$$

The EE system (20) is integrable for the coefficients in Equation (25) of the form

$$a(t) = A_1 e^{-t/\tau_a}, \quad \tilde{D}_a(t) = d_1 e^{-t/\tau_d}, \quad b(t) = B_2 + (B_1 - B_2)e^{-t/\tau_b}. \quad (83)$$

The general solutions to the EE system for the coefficients in (83) were obtained in [12]. This case is treated as the model of the plasma relaxation. Let us illustrate the solutions corresponding to the initial condition (75) for this case. Note that the variational system (37) is not integrable for such coefficients as far as we know. It is a remarkable fact, since the analytical solutions of the EE system, which are quite cumbersome though, can be expressed in terms of the solutions of the variational system by analogy with (66). For our example, we use the solution to the EE system from [12] and construct the solutions to the variational system (37) numerically. Note that the variational system (37) is the system of linear differential equations with constant-sign coefficients. The subsequent calculations are presented for $\varkappa = 1$, $A_1 = 1$, $\tau_a = 1$, $d_1 = 0.5$, $\tau_d = 1$, $B_2 = 0.4$, $B_1 = 0.2$, $\tau_b = 1$, $\mu = 0.5$, $\gamma_1 = 1.5$, $\gamma_2 = 1$, $D = 0.01$, $N = 1$. Figure 1 shows the evolution of $\sigma(t)$ for two values of ε . The physical meaning of this function is the total number of ions in the active medium. Figure 2 shows the solutions of the variational system. We provide the plot of these solutions just for one value of ε , since the difference between these plots for $\varepsilon = 0.85$ and $\varepsilon = 1$ is barely perceptible due to the little difference in $\sigma(t)$. The sign of $W^{(-)}(t)$ was reverted for compactness of the figure. The values of the expansion coefficients are presented in Table 1. Note that they tend to zero rapidly as $|n|$ increases due to the optimal choice of β and they equal zero for odd n_1 or n_2 . Figure 3 shows the asymptotic evolution $u(\vec{x}, t)$ of the initial ion distribution (75) constructed as the proposed nonlinear superposition in the weak diffusion approximation with an accuracy of $O(D^{3/2})$.

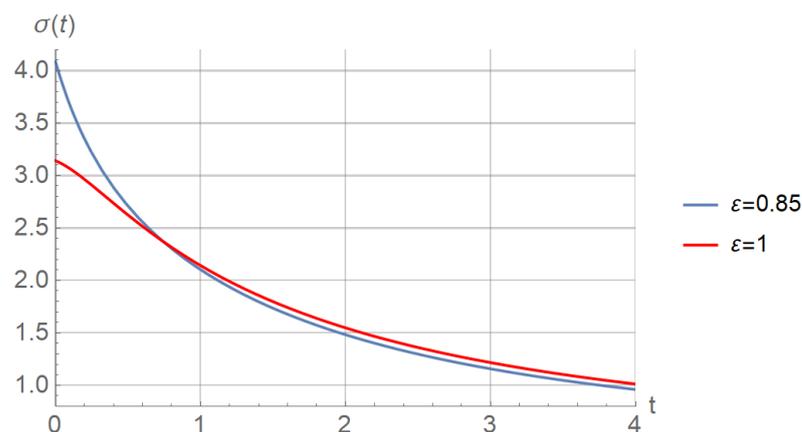


Figure 1. The plot of the function $\sigma(t)$ for various ε . It illustrates the relaxation of the total number of ions for two initial distributions according to the analytic formula derived in [12] and (82).

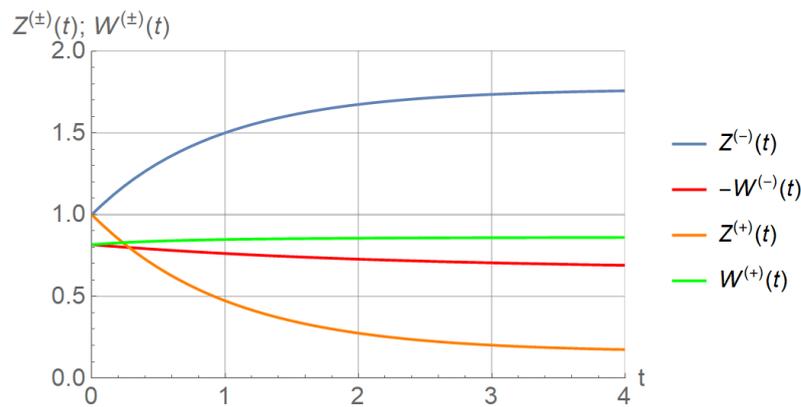


Figure 2. The plot of the solutions to (37) and (39) for $\epsilon = 0.85$. The solutions are obtained numerically.

Table 1. The values of the coefficient $k_{(n_1, n_2)}$ for $n_1, n_2 = 0, 1, 2, 3, 4$, and two values of ϵ .

		$\epsilon = 0.85$					$\epsilon = 1$				
		$n_2 = 0$	$n_2 = 1$	$n_2 = 2$	$n_2 = 3$	$n_2 = 4$	$n_2 = 0$	$n_2 = 1$	$n_2 = 2$	$n_2 = 3$	$n_2 = 4$
n_1	0	3.37×10^{-1}	0	4.71×10^{-2}	0	1.07×10^{-4}	2.02×10^{-1}	0	5.05×10^{-2}	0	6.44×10^{-5}
	1	0	0	0	0	0	0	0	0	0	0
	2	4.71×10^{-2}	0	2.15×10^{-4}	0	1.50×10^{-5}	5.05×10^{-2}	0	1.29×10^{-4}	0	1.61×10^{-5}
	3	0	0	0	0	0	0	0	0	0	0
	4	1.07×10^{-4}	0	1.50×10^{-5}	0	3.43×10^{-8}	6.44×10^{-5}	0	1.61×10^{-5}	0	2.05×10^{-8}

Note that $Z^{(+)}(t)$ does not tend to nonzero constant. In a general case, this constant can be negative, which leads to the zero value of $Z^{(+)}(t)$ at some point $t = t_0$. At this point, the condition of nondegeneracy of the matrix $C(t)$ (34) is violated. It is worthy of discussion how it affects the solutions $v_n(\vec{x}, t, C_n)$. It can be shown that the functions (66) have the removable discontinuity at this point. The same is true for the solutions in (58). Hence, the asymptotic solutions $u(\vec{x}, t) = v_n(\vec{x}, t, C_n)$ regularly depend on t in a neighborhood of the point $t = t_0$. It means that if we construct the germ r_t^2 from the point $t = 0$ to a point left of the point $t = t_0$ and then we construct the germ \tilde{r}_t^2 after the point $t = t_0$, the resulting asymptotic solutions $u(\vec{x}, t), t < t_0$, and $\tilde{u}(\vec{x}, t), t > t_0$, generated by the germ r_t^2 and \tilde{r}_t^2 , respectively, determine the continuous asymptotic evolution of the initial state $u(\vec{x}, t)|_{t=0}$ for both $t < t_0$ and $t > t_0$. Therefore, the formal constraint of the nondegeneracy of the matrix $C(t)$ (34) can be bypassed. Note that the error of the semiclassical solutions usually grows over time. It can result in an adverse effect when the semiclassical solution jumps from the asymptotics for one exact solution to another at a sufficiently large time (see the work [28]). Therefore, the estimate of the period of time where the asymptotics are valid is the subject to study separately and is to be obtained in every particular case.

Figure 1 illustrates that the ion distribution with the less pronounced minimum at the GDT center relaxes faster. However, Figure 3 shows that both distributions tend to the Gauss-like profile over time. It means that their evolution will be similar on large times. Hence, the difference is significant only at the initial stage of the relaxation.

Note that our method is applicable for the linear case ($\varkappa = 0$). For $\varkappa = 0$ and $a(\vec{x}, t) = a(t)$, the method proposed yields the exact solution to the kinetic equation.

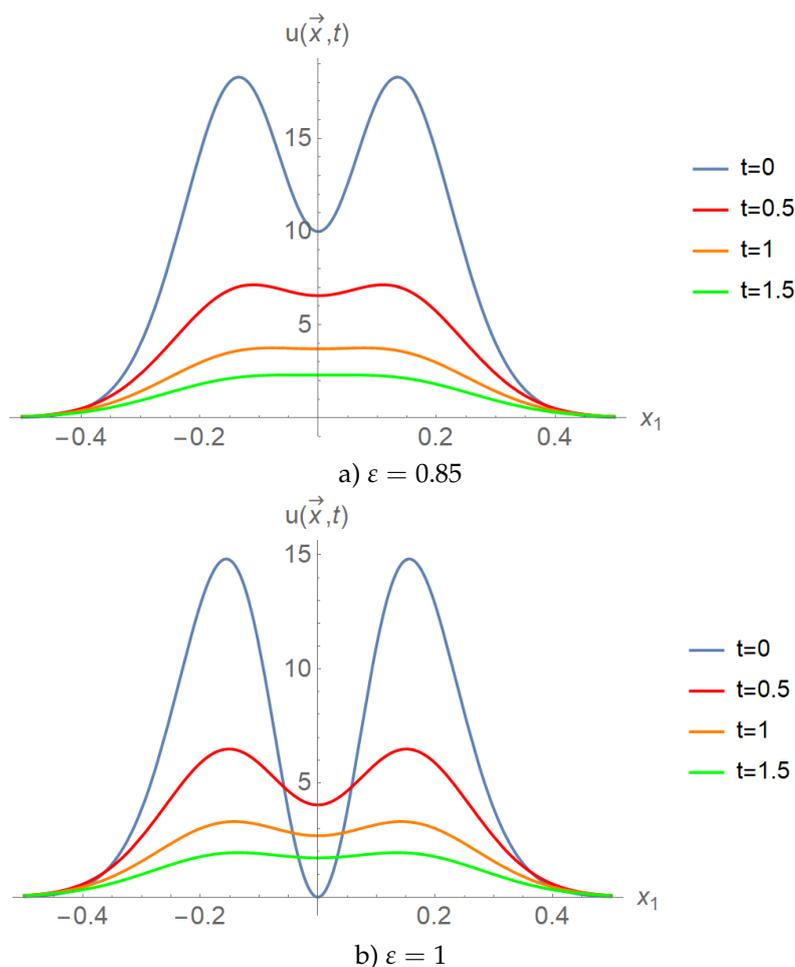


Figure 3. The plot of the function $u(\vec{x}, t)$ in the section $x_2 = 0$. It illustrates the evolution of the ion distribution according to the formula in (73).

7. Conclusions

Within the framework of the approach presented in [12], we have constructed a countable family of asymptotic solutions to the two-dimensional kinetic Equation (25) with the nonlocal cubic nonlinearity. The approach proposed can be generalized for n -dimensional space. We have considered the two-dimensional problem for the sake of simplicity in order to demonstrate our method and to give its physical interpretation. The constructed asymptotic solutions correspond to the weak diffusion approximation. The approach is based on the solutions to the Cauchy problem to the nonlinear dynamical system (the Einstein–Ehrenfest system) (9) and (20). These solutions generate the linear parabolic Equations (13) and (28) associated with the nonlinear kinetic equation. Solutions of this associated linear equation subject to the algebraic condition (15) yield the asymptotic solutions (58) and (59) to the original nonlinear kinetic equation. The family of the asymptotic solutions is constructed based on the set of skew-orthogonal, linearly independent solutions to the system of linear ODEs (37) (the variational system). Such asymptotic solutions form the orthogonal basis for the nonlinear superposition principle that allows one to construct the asymptotic evolution of arbitrary initial distribution $\varphi(\vec{x})$ (17) expanded into the series with respect to the basis formed by the functions (58). The nonlinearity of the superposition principle is caused by the necessity of solving the nonlinear dynamical system (9) and (20) with initial conditions (16) determined by the initial distribution. Note that the functions $v_n(\vec{x}, t, \mathbf{C})$ with \mathbf{C} determined from (74) form an orthogonal basis for the nonlinear superposition principle, while the functions $v_n(\vec{x}, t, \mathbf{C}_n)$ with \mathbf{C}_n given by (67) and (68), which

determine independent asymptotic solutions to the nonlinear kinetic equation, are not orthogonal in the general case.

This work extends the results of our work [12] where the asymptotic evolution operator of the kinetic Equation (25) was constructed via the Green function of the associated linear equation. Here, we construct the asymptotic solutions with the help of the symmetry operators (51) for the associated linear equation based on the nonlinear superposition principle. It allows us to study the properties of the asymptotic solutions through the properties of the variational system (37) that generates the Maslov germ r_i^2 on the zero-dimensional manifold Λ_i^0 and the symmetry operators. Moreover, we have found the relation (66) between the solutions to the Einstein–Ehrenfest system and the variational system. It means that our approach can yield analytical solutions expressed in terms of the solutions to the system of the linear ODEs (37) only. However, in the considered specific case, the nonlinear dynamical system admits the analytical solutions, while the variational system (37) is solved numerically. Note that the asymptotic evolution operator in [12] and the nonlinear superposition principle yield the exact solutions to the associated linear Equation (28), i.e., they yield the same asymptotic solutions for the given initial condition. However, the approach proposed here broadens the possibilities of analysis for these solutions. In addition, the nonlinear superposition principle leads to the expressions with free parameters that can be chosen so that the functional series (73) converges rapidly. Hence, in some cases, the study of the solutions (73) can be practically reduced to the study of the first few terms of the series. It makes the analysis of such solutions even easier, especially when the solutions are numerical–analytical.

The future prospects for this work are related to the study of a more complex two-component kinetic model. We plan to generalize the approach [29] for the latter problem.

Author Contributions: Conceptualization, A.V.S. and A.E.K.; methodology, A.V.S. and A.E.K.; validation, A.V.S. and A.E.K.; formal analysis, A.V.S. and A.E.K.; investigation, A.V.S. and A.E.K. and S.A.S.; writing—original draft preparation, A.V.S. and A.E.K.; writing—review and editing, A.V.S. and A.E.K.; visualization, A.E.K. and S.A.S.; software, A.E.K. and S.A.S.; supervision, A.V.S.; funding acquisition, A.V.S. and A.E.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Russian Foundation for Basic Research (RFBR) and Tomsk region grant number 19-41-700004.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The work is supported by Tomsk Polytechnic University under the International Competitiveness Improvement Program; by IAO SB RAS, Russia, project no. 121040200025-7.

Conflicts of Interest: The authors declare no conflict of interest.

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