symmetry

MDPI

Article

On Ulam Stability of Functional Equations in 2-Normed Spaces—A Survey II

El-sayed El-hady [1] and Janusz Brzdęk [2,*]

[1] Mathematics Department, College of Science, Jouf University, Sakaka P.O. Box 2014, Saudi Arabia; eaelhady@ju.edu.sa
[2] Faculty of Applied Mathematics, AGH University of Science and Technology, Mickiewicza 30, 30-059 Kraków, Poland
* Correspondence: brzdek@agh.edu.pl

Abstract: Ulam stability is motivated by the following issue: how much an approximate solution of an equation differs from the exact solutions to the equation. It is connected to some other areas of investigation, e.g., optimization, approximation theory and shadowing. In this paper, we present and discuss the published results on such stability for functional equations in the classes of function-taking values in 2-normed spaces. In particular, we point to several pitfalls they contain and provide possible simple improvements to some of them. Thus we show that the easily noticeable symmetry between them and the analogous results proven for normed spaces is, in fact, mainly apparent. Our article complements the earlier similar review published in Symmetry (13(11), 2200) because it concerns the outcomes that have not been discussed in this earlier publication.

Keywords: Ulam stability; functional equation; 2-Banach space; 2-norm

MSC: primary 39B62; 39B82; secondary 47H10; 47J20

Citation: El-hady, E.-s.; Brzdęk, J. On Ulam Stability of Functional Equations in 2-Normed Spaces—A Survey II. *Symmetry* **2022**, *14*, 1365. https://doi.org/10.3390/sym14071365

Academic Editors: Calogero Vetro and Juan Luis García Guirao

Received: 3 June 2022
Accepted: 28 June 2022
Published: 2 July 2022

1. Introduction

The theory of stability in the sense of Ulam has become a popular subject of research, which goes in several directions and is somehow connected with issues studied in some other areas of mathematics, e.g., shadowing (see [1]), approximation theory and optimization. It mainly concerns various equations (difference, differential, integral, functional, etc.), and very roughly speaking, the following subject is investigated: how much an approximate solution to an equation differs from the exact solutions of it. It has been motivated by a problem formulated by Ulam in 1940 for the equation of group homomorphism, and the first answer to it was provided by Hyers in [2]. The question and the answer to it inspired many further papers, and we refer to [3–5] for further information on this subject.

It should be mentioned here that shortly after Hyers' publication, a new wider approach in this area was suggested by T. Aoki [6]. The result of Aoki was later complemented in [7–9]. The main outcome that thus arose and is considered to be very representative of the Ulam stability reads as follows (see Theorem 3.5 of [10]).

Theorem 1. *Let W be a normed space, $W_0 := W \setminus \{0\}$, B be a Banach space and $\eta \geq 0$ and $r \neq 1$ be real numbers. Assume that $h\colon W \to B$ satisfies*

$$\|h(x+y) - h(x) - h(y)\| \leq \eta(\|x\|^r + \|y\|^r), \qquad x, y \in W_0\,. \tag{1}$$

Then there is a unique additive mapping $\alpha : W \to B$ with

$$\|h(x) - \alpha(x)\| \leq \frac{\eta\|x\|^r}{|1 - 2^{r-1}|}, \qquad x \in W_0. \tag{2}$$

Let us remember that a mapping $\alpha : W \to B$ is additive if it satisfies the Cauchy equation

$$\alpha(x+y) = \alpha(x) + \alpha(y) \tag{3}$$

for every $x, y \in W$ (here, W is a linear space as in Theorem 1).

Further, an example was provided in [9] showing that for $r = 1$ an analogous result, as in Theorem 1, is not valid. Moreover, estimate (2) is optimal (see [11]) and, in the case $r < 0$, each function $h : W \to B$ fulfilling (1) must be additive, and the completeness of B is not necessary for this situation (see Theorem 3.5 of [10] and [12]). For some examples of related results concerning the stability of modified versions of Equation (3) and their applications, we refer to [13].

The following abstract definition makes the notion of Ulam stability a bit more precise in the case of a general equation in k variables ($\mathbb{R}_+$ denotes the set of nonnegative reals, and A^B means a family of all functions mapping a set $B \neq \emptyset$ to a set $A \neq \emptyset$).

Definition 1. *Let $k \in \mathbb{N}$, (E, ρ) be a metric space, $U \neq \emptyset$ be a set, $\mathcal{D}_0 \subset \mathcal{D} \subset E^U$ and $\mathcal{V} \subset \mathbb{R}_+^{U^k}$ be nonempty, $\mathcal{S} : \mathcal{V} \to \mathbb{R}_+^U$, and $\mathcal{F}_1, \mathcal{F}_2 : \mathcal{D} \to E^{U^k}$. The equation*

$$(\mathcal{F}_1\psi)(t_1, \ldots, t_k) = (\mathcal{F}_2\psi)(t_1, \ldots, t_k) \tag{4}$$

is said to be $\mathcal{S}$-stable in $\mathcal{D}_0$ if, for any $\psi \in \mathcal{D}_0$ and $\delta \in \mathcal{V}$ with

$$\rho((\mathcal{F}_1\psi)(t_1, \ldots, t_k), (\mathcal{F}_2\psi)(t_1, \ldots, t_k)) \leq \delta(t_1, \ldots, t_k), \qquad t_1, \ldots, t_k \in U,$$

there is a mapping $\phi \in \mathcal{D}$ satisfying Equation (4) for all $t_1, \ldots, t_k \in U$ and such that $\rho(\phi(t), \psi(t)) \leq (\mathcal{S}\delta)(t)$ for $t \in U$.

If $(\mathcal{S}\delta)(t) = 0$ for $\delta \in \mathcal{V}$ and $t \in U$, then we say that the equation is hyperstable in $\mathcal{D}_0$.

Note that Equation (4) is the Cauchy functional Equation (3) with $k = 2$, $U = W$, $(\mathcal{F}_1\alpha)(s, t) = \alpha(s+t)$ and $(\mathcal{F}_2\alpha)(s, t) = \alpha(s) + \alpha(t)$ for $\alpha \in \mathcal{D}$ and $s, t \in U = W$.

Clearly, Theorem 1 states that for each real number $r \neq 1$, the Cauchy Equation (3) is $\mathcal{S}$-stable in $\mathcal{D}_0 = \mathcal{D} = B^W$ with $\mathcal{S} : \mathcal{V} \to \mathbb{R}_+^U$ defined by

$$(\mathcal{S}\delta_\eta)(x) := \frac{1}{|2 - 2^r|}\delta_\eta(x, x) = \frac{\eta\|x\|^r}{|1 - 2^{r-1}|}, \qquad \delta_\eta \in \mathcal{V}, x \in W,$$

where

$$\delta_\eta(x, y) = \eta(\|x\|^r + \|y\|^r), \qquad x, y \in W, \eta \in \mathbb{R}_+,$$

and

$$\mathcal{V} = \{\delta_\eta \in \mathbb{R}_+^{W \times W} : \eta \in \mathbb{R}_+\}.$$

However, if $r < 0$, then a stronger property holds, i.e., the already mentioned result in [12] is valid, which states that Equation (3) is hyperstable in $\mathcal{D}_0 = B^W$ (that is every $h\colon W \to B$ satisfying (1) is additive).

Very recently, a more precise outcome (but only for mappings taking values in the set of reals $\mathbb{R}$) has been proven in [14] using the technique of the Banach limit (as in [15]). Namely, the following has been obtained in Theorem 8 of [14] (cf. Remark 7 of [14]).

Theorem 2. *Let W be a normed space, $W_0 := W \setminus \{0\}$, $r, \mu, \xi \in \mathbb{R}$, $r \neq 1$, and $\mu \leq \xi$. Assume that $h\colon W \to \mathbb{R}$ satisfies the inequality*

$$\mu(\|x\|^r + \|y\|^r) \leq h(x+y) - h(x) - h(y) \leq \xi(\|x\|^r + \|y\|^r), \qquad x, y \in W_0.$$

Then there is a unique additive mapping $\alpha\colon W \to \mathbb{R}$ such that, in the case $r < 1$,

$$\frac{\mu}{1-2^{r-1}}\|x\|^r \le \alpha(x) - h(x) \le \frac{\xi}{1-2^{r-1}}\|x\|^r, \qquad x \in W_0, \tag{5}$$

and, in the case $r > 1$,

$$\frac{\mu}{2^{r-1}-1}\|x\|^r \le h(x) - \alpha(x) \le \frac{\xi}{2^{r-1}-1}\|x\|^r, \qquad x \in W_0. \tag{6}$$

Moreover, if h is continuous at some point, then α is continuous.

Certainly, condition (1) can be replaced by various other inequalities of the form

$$\|h(x+y) - h(x) - h(y)\| \le \phi(x,y), \qquad x, y \in W, \tag{7}$$

and we should mention here that, for instance, the inequality

$$\|h(x+y) - h(x) - h(y)\| \le \eta\|x\|^p\|y\|^q, \qquad x, y \in W \setminus \{0\}, \tag{8}$$

with $p, q \in \mathbb{R}$ and $\eta > 0$, was studied in [16,17] (see also [18]). Moreover, the stability of numerous other equations has been investigated in various ways, and we refer to [3–5,19] for more details and examples.

Very roughly, we can say (see Definition 1) that an equation is Ulam stable if, for every mapping fulfilling the equation approximately (in some sense), there is an accurate solution of the equation that is close to this mapping (in some way).

Clearly, the notions of an approximate solution and of the closeness of two mappings can be understood in different ways. Therefore, considering the Ulam stability for various ways of measuring distance makes sense. One non-classical distance-measuring method can be introduced by the concept of 2-norms, which was proposed in 1964 by Gähler (see [20,21]). Let us mention that a natural generalization of this concept is n-normed space (see, e.g., [22,23]), i.e., the 2-normed space is n-normed space with $n = 2$. However, in this article, we limit ourselves only to the case of 2-normed spaces due to the large amount of material we present.

In this paper, we complement the content of [23], where the (less complicated) results from [24–39] have been surveyed. Here, we present and discuss the (more involved) outcomes on Ulam stability in 2-normed spaces provided in [40–58].

Some information on the solutions to functional equations considered in this paper can be found in monographs [59–62].

Let us also add here that in this paper, $\mathbb{R}$ denotes a set of real numbers, $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$, $\mathbb{Q}$ denotes a set of rational numbers, $\mathbb{N}$ stands for the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

2. 2-Normed Spaces

The notion of 2-normed spaces was introduced by Gähler (see, e.g., [21,63]). We present this concept in a somewhat generalized form.

To avoid any ambiguities, let us start with definitions of the notions that we use.

Definition 2. *Given a field $\mathbb{K}$, we say that a mapping $|\cdot| : \mathbb{K} \to \mathbb{R}_+$ is a valuation in $\mathbb{K}$ if, for all $a, b \in \mathbb{K}$,*

(a) $|a| = 0$ *if and only if* $a = 0$;
(b) $|ab| = |a|\,|b|$;
(c) $|a+b| \le |a| + |b|$.

A valuation $|\cdot|$ in field $\mathbb{K}$ is nontrivial if $|a| \notin \{0,1\}$ for some $a \in \mathbb{K}$.
If condition (c) is replaced by the following stronger inequality

(c′) $|a+b| \le \max\{|a|, |b|\}$,

then we say that the valuation is non-Archimedean.

Definition 3. *Let $\mathbb{K}$ be a field with a nontrivial valuation $|\cdot| : \mathbb{K} \to \mathbb{R}_+$ and X be a linear space over $\mathbb{K}$ with a dimension greater than 1.*

We say (cf., e.g., [21,63]) that a mapping $\|\cdot,\cdot\| : X \times X \to \mathbb{R}_+$ is a 2-norm in X if, for every $x_1, x_2, x_3 \in X$ and $\beta \in \mathbb{K}$, the following four conditions are fulfilled:

(1) $\|x_1, x_2\| = 0$ *if and only if* x_1 *and* x_2 *are linearly dependent;*
(2) $\|x_1, x_2\| = \|x_2, x_1\|$;
(3) $\|x_1, x_2 + x_3\| \le \|x_1, x_2\| + \|x_1, x_3\|$;
(4) $\|\beta x_1, x_2\| = |\beta|\|x_1, x_2\|$.

If inequality (3) is replaced by the subsequent stronger condition

(3′) $\|x_1, x_2 + x_3\| \le \max\{\|x_1, x_2\|, \|x_1, x_3\|\}$,

then we say that the 2-norm is non-Archimedean.

Let $\|\cdot,\cdot\| : X \times X \to \mathbb{R}_+$ be a 2-norm in X. Then we say that a pair $(X, \|\cdot,\cdot\|)$ is a 2-normed space. If $\mathbb{K}$ is the field of reals $\mathbb{R}$ and the valuation in $\mathbb{K}$ is the usual absolute value, then we say that $(X, \|\cdot,\cdot\|)$ is a real 2-normed space; if $\mathbb{K}$ is the field of complex numbers $\mathbb{C}$ and the valuation in $\mathbb{K}$ is the usual complex modulus, then we say that $(X, \|\cdot,\cdot\|)$ is a complex 2-normed space.

Definition 4. *A sequence $(x_n)_{n\in\mathbb{N}}$ in a 2-normed space X is a* Cauchy sequence *if there exist two linearly independent vectors $z_1, z_2 \in X$ with*

$$\lim_{n,m\to\infty} \|x_n - x_m, z_i\| = 0, \qquad i = 1, 2.$$

A sequence $(x_n)_{n\in\mathbb{N}}$ in a linear 2-normed space X is convergent if there is a vector $x \in X$ and two linearly independent vectors $z_1, z_2 \in X$, such that

$$\lim_{n\to\infty} \|x_n - x, z_i\| = 0, \qquad i = 1, 2;$$

such vector x is called a limit of $(x_n)_{n\in\mathbb{N}}$ and we denote it by $\lim_{n\to\infty} x_n$.

2-Banach space is a 2-normed space in which every Cauchy sequence is convergent.

Further, it should be remarked that in a 2-normed space, a limit of a sequence is unique. Next, the following property can be easily proven.

Lemma 1. *Let X be a 2-normed space, $x, y, z \in X$, and the vectors y and z be linearly independent. If*

$$\|x, y\| = 0 = \|x, z\|,$$

then $x = 0$.

From the Cauchy–Schwarz inequality, it easily follows that if X is a real linear space with a dimension greater than 1, and $\langle\cdot,\cdot\rangle$ is an inner product in X, then the mapping $\|\cdot,\cdot\| : X^2 \to \mathbb{R}_+$ given by

$$\|x, y\| := \sqrt{\|x\|^2\|y\|^2 - \langle x, y\rangle^2}, \qquad x, y \in X, \tag{9}$$

fulfills conditions (1)–(4), i.e., it is a 2-norm in X. Further (see Proposition 2.3 of [41]), if $(X, \langle\cdot,\cdot\rangle)$ is a real Hilbert space, then X is a 2-Banach space (with the 2-norm defined by (9)).

If an inner product in $\mathbb{R}^2$ is given by: $\langle(x_1, x_2), (y_1, y_2)\rangle = x_1y_1 + x_2y_2$ for (x_1, x_2), $(y_1, y_2) \in \mathbb{R}^2$, then the 2-norm depicted by formula (9) has the following form:

$$\|(x_1, x_2), (y_1, y_2)\| := |x_1y_2 - x_2y_1|, \qquad (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2. \tag{10}$$

Finally, observe that the expressions

$$c\|\cdot,\cdot\|_1 + d\|\cdot,\cdot\|_2, \qquad \min\{c\|\cdot,\cdot\|_1, d\|\cdot,\cdot\|_2\}, \qquad \max\{c\|\cdot,\cdot\|_1, d\|\cdot,\cdot\|_2\}$$

define 2-norms for any two 2-norms $\|\cdot,\cdot\|_1$ and $\|\cdot,\cdot\|_2$ in a real linear space X and every positive reals c, d.

3. Stability in 2-Normed Spaces

An analogue of Definition 1 for 2-normed spaces could be formulated as follows.

Definition 5. *Let $k \in \mathbb{N}$, $(Y, \|\cdot,\cdot\|)$ be a 2-normed space, U be a nonempty set, $\mathcal{B} \subset \mathbb{R}^{U^k \times Y}$ and $\mathcal{D}_0 \subset \mathcal{D} \subset Y^U$ be nonempty, $\mathcal{S} : \mathcal{B} \to \mathbb{R}^{U \times Y}$, and $\mathcal{F}_1, \mathcal{F}_2 : \mathcal{D} \to Y^{U^k}$. Then Equation (4) is said to be $\mathcal{S}$-stable in $\mathcal{D}_0$ if, for any $\psi \in \mathcal{D}_0$ and $\delta \in \mathcal{B}$ such that*

$$\|(\mathcal{F}_1\psi)(t_1, \ldots, t_k) - (\mathcal{F}_2\psi)(t_1, \ldots, t_k), y\| \le \delta(t_1, \ldots, t_k, y), \qquad t_1, \ldots, t_k \in U, y \in Y,$$

there is $\phi \in \mathcal{D}$ satisfying (4) for all $t_1, \ldots, t_k \in U$ with

$$\|\phi(t) - \psi(t), y\| \le (\mathcal{S}\delta)(t, y), \qquad t \in U, y \in Y.$$

If $(\mathcal{S}\delta)(t, y) = 0$ for $\delta \in \mathcal{V}$, $t \in U$ and $y \in Y$, then we say that the equation is hyperstable in $\mathcal{D}_0$.

In this section, we present the Ulam stability results in 2-normed spaces that have been investigated for various interesting equations. In what follows, $(Y, \|\cdot,\cdot\|)$ is always a real 2-Banach space and $(Y_1, \|\cdot,\cdot\|)$ is a real 2-normed space.

We start with a result from [40] concerning the stability of a modification of the Cauchy functional equation, which is called the Pexider equation. This result is not actually covered by Definition 5, but it is easy to reformulate the definition accordingly. The outcome in [40] can be stated as follows (we reformulate it but preserve the assumptions given in [40]).

Theorem 3. *Let X be a normed linear space, $k \in \mathbb{N}$, $k > 1$, and $\varphi : X \times X \times X \to \mathbb{R}_+$ be a function such that, for all $x, y, z \in X$,*

$$\lim_{n\to\infty} \frac{1}{k^n}\varphi(k^n x, k^n y, z) = 0, \tag{11}$$

$$\widetilde{M}_k(x, z) := \sum_{n=0}^{\infty}\sum_{i=1}^{k-1} \frac{M(k^n x, ik^n x, z)}{k^n} < \infty, \tag{12}$$

where $M(x, y, z) := \varphi(x, y, z) + \varphi(0, y, z) + \varphi(x, 0, z)$. Let $f, g, h : X \to Y$ be mappings with

$$f(0) = g(0) = h(0) = 0, \tag{13}$$

$$\|f(x+y) - g(x) - h(y), z\| \le \varphi(x, y, z), \qquad x, y, z \in X. \tag{14}$$

Then there is a unique additive mapping $A_k : X \to Y$ such that

$$\|f(x) - A_k(x), z\| \le \frac{1}{k}\widetilde{M}_k(x, z), \qquad x, z \in X.$$

Let us remind here that the additivity of A_k means that

$$A_k(x+y) = A_k(x) + A_k(y), \qquad x, y \in X.$$

Remark 1. *First, it is clear that if Theorem 3 is to make sense, then either we must have $X = Y$, or (14) should have the form*

$$\|f(x+y) - g(x) - h(y), z\| \le \varphi(x, y, z), \qquad x, y \in X, z \in Y, \tag{15}$$

with $\varphi : X \times X \times Y \to [0, +\infty)$ and (11) and (12) should be assumed for all $x, y \in X$ and $z \in Y$. Further, from the proof given in [40], it follows that the norm in X is not necessary; it is enough to assume that, e.g., X is a real linear space.

Below, we show that under a very weak assumption on φ (that is somewhat complementary to (11)), we obtain a result similar to Theorem 3, but with a better statement. To this end, we need the following hypothesis.

(L) $(X, +)$ is a groupoid (which is not necessarily commutative), Y_0 is a linear subspace of Y_1, $\varphi : X^2 \times Y_0 \to \mathbb{R}_+$ and, for every $x, y \in X$, there exist linearly independent $z_1, z_2 \in Y_0$ and two real sequences $(\xi_n^1)_{n\in\mathbb{N}}, (\xi_n^2)_{n\in\mathbb{N}}$ such that $\xi_n^i \neq 0$ for $i = 1, 2$, $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \frac{1}{\xi_n^i} \varphi(x, y, \xi_n^i z_i) = 0, \qquad i = 1, 2. \tag{16}$$

Now we are in a position to prove the following.

Theorem 4. *Let hypothesis (L) be valid and $f, g, h : X \to Y_1$ fulfill the inequality*

$$\|f(x+y) - g(x) - h(y), z\| \le \varphi(x, y, z), \qquad x, y \in X, z \in Y_0. \tag{17}$$

Then, f, g, h satisfy the Pexider equation

$$f(x+y) = g(x) + h(y), \qquad x, y \in X. \tag{18}$$

Moreover, if $(X, +)$ has a neutral element denoted by 0, then there exist a unique additive mapping $A : X \to Y_1$ and unique $u, v \in Y_1$ such that

$$\begin{aligned} f(x) = A(x) + u + v, \qquad g(x) = A(x) + u, \\ h(x) = A(x) + v, \qquad x \in X. \end{aligned} \tag{19}$$

In the particular case where (13) holds, we have $f = g = h = A$.

Proof. Fix $x, y \in X$. Then, according to hypothesis (L), there exist linearly independent $z_1, z_2 \in Y_0$ and two real sequences $(\xi_n^1)_{n\in\mathbb{N}}, (\xi_n^2)_{n\in\mathbb{N}}$ such that $\xi_n^i \neq 0$ for $i = 1, 2$, $n \in \mathbb{N}$, and condition (16) holds. Hence, by (17),

$$\|f(x+y) - g(x) - h(y), \xi_n^i z_i\| \le \varphi(x, y, \xi_n^i z_i), \qquad n \in \mathbb{N}, i = 1, 2,$$

which yields

$$\|f(x+y) - g(x) - h(y), z_i\| \le \frac{1}{|\xi_n^i|} \varphi(x, y, \xi_n^i z_i), \qquad n \in \mathbb{N}, i = 1, 2. \tag{20}$$

Letting $n \to \infty$ in (20), we get

$$\|f(x+y) - g(x) - h(y), z_i\| = 0, \qquad i = 1, 2.$$

Since z_1 and z_2 are linearly independent, this yields $f(x+y) = g(x) + h(y)$ (see Lemma 1).

Thus we have shown that f, g, h fulfill the Pexider equation

$$f(x+y) = g(x) + h(y). \tag{21}$$

The remaining part of the reasoning is very well known (see, e.g., [59,60]), but for the convenience of readers we present it.

First, putting $y = 0$ and next $x = 0$ in (21), we get $f(x) = g(x) + h(0)$ for $x \in X$ and $f(y) = g(0) + h(y)$ for $y \in X$, which implies that

$$g(x) = f(x) - h(0), \qquad h(x) = f(x) - g(0), \qquad x \in X. \tag{22}$$

Therefore, now (21) takes the form

$$f(x+y) = f(x) - h(0) + f(y) - g(0), \qquad x, y \in X, \tag{23}$$

whence

$$f(x+y) - g(0) - h(0) = f(x) - g(0) - h(0) + f(y) - g(0) - h(0), \qquad x, y \in X. \tag{24}$$

Define $A : X \to Y_1$ by $A(x) = f(x) - g(0) - h(0)$ for $x \in X$. Then, by (24), A is additive and it is easily seen that (19) holds with $u = g(0)$ and $v = h(0)$. It remains to show the uniqueness of u, v and A.

Therefore, suppose that $u_0, v_0 \in X$ and additive $A_0 : X \to Y_1$ are such that

$$\begin{aligned} f(x) = A_0(x) + u_0 + v_0, \qquad g(x) = A_0(x) + u_0, \\ h(x) = A_0(x) + v_0, \qquad x \in X. \end{aligned} \tag{25}$$

Then $A(0) = 0 = A_0(0)$ and $A(x) + u = g(x) = A_0(x) + u_0$ for $x \in X$, whence with $x = 0$ we get $u = u_0$ and consequently $A = A_0$. Analogously we obtain $v = v_0$.

Finally, if (13) holds, then from (19) we get $f = g = h = A$, because $u = h(0)$ and $v = g(0)$. □

In general, condition (16) cannot be derived from (11), but the next remark shows that hypothesis (L) holds for many natural examples of φ.

Remark 2. *It is easy to check that given linearly independent vectors $y_1, y_2 \in Y_1$, we can define a norm $\|\cdot\|_0$ in Y_1 by*

$$\|z\|_0 := \|y_1, z\| + \|y_2, z\|, \qquad z \in Y_1.$$

Thus a 2-norm in Y_1 generates a very large family of norms in Y_1.

Let X and Y_0 be as in hypothesis (L). Define $\varphi : X^2 \times Y_0 \to \mathbb{R}_+$ by

$$\varphi(x, y, z) = \psi_0(x, y)\|z\|_0^r, \qquad x, y \in X, z \in Y_0,$$

where $\|\cdot\|_0$ is a norm in Y_1, $\psi_0 : X^2 \to \mathbb{R}_+$ is an arbitrary given mapping, $r \in \mathbb{R}_+$ and $r \neq 1$. Then φ satisfies hypothesis (L). Moreover, if $X = Y_1$, then hypothesis (L) is also fulfilled by φ given by one of the following two formulas:

$$\varphi(x, y, z) = \psi_1(x, y)\|x, z\|^r + \psi_2(x, y)\|y, z\|^r, \qquad x, y \in X, z \in Y_0,$$

$$\varphi(x, y, z) = \psi_1(x, y)\|x, z\|^p\|y, z\|^q, \qquad x, y \in X, z \in Y_0,$$

where $\psi_1, \psi_2 : X^2 \to \mathbb{R}_+$ are arbitrary given mappings, $p, q, r \in \mathbb{R}_+$, $r \neq 1$ and $p + q \neq 1$.

However, there also exist numerous natural examples of φ that do not satisfy hypothesis (L) (with $Y_1 = Y$) but fulfill the assumptions of Theorem 3 (the corrected versions of them). For instance, let $\|\cdot\|_0$ be a norm in Y and

$$\varphi(x, y, z) = \psi(x, y)\|z\|_0, \qquad x, y \in X, z \in Y, \tag{26}$$

with arbitrary given $\psi : X^2 \to \mathbb{R}_+$. Then clearly, (L) does not hold for such φ with $Y_0 = Y_1 = Y$, and for

$$\psi(x, y) = c_1\|x\|_1^r + c_2\|y\|_2^r, \qquad x, y \in X,$$

or

$$\psi(x,y) = c_1\|x\|_1^p\|y\|_2^q, \qquad x,y \in X,$$

with some $c_1, c_2, p, q, r \in \mathbb{R}_+$, $r < 1$, $p+q < 1$ *and some norms* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *in* $X = Y$, *mapping* φ *given by* (26) *satisfies conditions* (11) *and* (12) *for all* $x,y,z \in X$.

A useful fixed point theorem for 2-Banach spaces has been proven in [41], and one of its direct applications is the Ulam stability result for a very general functional equation with a single variable. To present it, we need the following four hypotheses.

(H1) S is a nonempty set, $Y_0 \subset Y$ contains two linearly independent vectors, $j \in \mathbb{N}$, $f_i\colon S \to S$, $g_i\colon Y_0 \to Y_0$ and $L_i\colon S \times Y_0 \to \mathbb{R}_+$ for $i = 1, \ldots, j$;

(H2) $\Lambda\colon {\mathbb{R}_+}^{S\times Y_0} \to {\mathbb{R}_+}^{S\times Y_0}$ is an operator defined by

$$\Lambda\delta(x,y) := \sum_{i=1}^{j} L_i(x,y)\delta(f_i(x), g_i(y)), \qquad \delta \in {\mathbb{R}_+}^{S\times Y_0}, x \in S, y \in Y_0. \tag{27}$$

(H3) $\Psi : S \times Y^j \to Y$ satisfies the inequality

$$\|\Psi(x, y_1, ..., y_j) - \Psi(x, z_1, ..., z_j), y\| \le \sum_{i=1}^{j} L_i(x,y)\|y_i - z_i, g_i(y)\| \tag{28}$$

for any $x \in S$, $y \in Y_0$ and $(y_1, ..., y_j), (z_1, ..., z_j) \in Y^j$;

(H4) $T : Y^S \to Y^S$ is defined by

$$(T\varphi)(x) := \Psi(x, \varphi(f_1(x)), ..., \varphi(f_j(x))), \qquad \varphi \in Y^S,\ x \in S. \tag{29}$$

Moreover, if Φ is a mapping from a nonempty set A into A, then Φ^n denotes the n-th iterate of Φ for each $n \in \mathbb{N}$, i.e., $\Phi^0(x) = x$ for all $x \in A$ and $\Phi^n(x) = \Phi(\Phi^{n-1}(x))$ for all $x \in A$ and $n \in \mathbb{N}$.

Now, we are in a position to present the subsequent result from Theorem 2 of [41].

Theorem 5. *Let hypotheses (H1)–(H4) be fulfilled,* $\varepsilon : S \times Y_0 \to \mathbb{R}_+$ *satisfy*

$$\varepsilon^*(x,y) := \sum_{m=0}^{\infty} (\Lambda^m \varepsilon)(x,y) < \infty, \qquad x \in S, y \in Y_0, \tag{30}$$

and $\varphi : S \to Y$ *be such that*

$$\|\varphi(x) - \Psi(x, \varphi(f_1(x)), ..., \varphi(f_j(x))), y\| \le \varepsilon(x,y), \qquad x \in S, y \in Y_0. \tag{31}$$

Then, for every $x \in S$ *the limit*

$$\psi(x) = \lim_{m\to\infty} (T^m\varphi)(x) \tag{32}$$

exists and the function $\psi : S \to Y$, *defined in this way, is the unique solution of the functional equation*

$$\Psi(x, \psi(f_1(x)), ..., \psi(f_j(x))) = \psi(x), \qquad x \in S, \tag{33}$$

such that

$$\|\varphi(x) - \psi(x), y\| \le \varepsilon^*(x,y), \qquad x \in S, y \in Y_0. \tag{34}$$

However, arguing analogously as in the first part of the proof of Theorem 4, under an assumption that it is a modified version of hypothesis (L), we can easily obtain the following complementary hyperstability result.

Theorem 6. *Let S be a nonempty set, $Y_0 \subset Y_1$ contains two linearly independent vectors, $j \in \mathbb{N}$, $\Psi : S \times Y_1^j \to Y_1$, $f_1, \ldots, f_j \colon S \to S$, and $\varepsilon : S \times Y_1 \to \mathbb{R}_+$. Assume that for every $x \in X$ there exist linearly independent $z_1, z_2 \in Y_0$ and two real sequences $(\xi_n^1)_{n\in\mathbb{N}}$, $(\xi_n^2)_{n\in\mathbb{N}}$ such that $\xi_n^i \neq 0$ for $i = 1, 2$, $n \in \mathbb{N}$, and*

$$\lim_{n\to\infty} \frac{1}{\xi_n^i} \varepsilon(x, \xi_n^i z_i) = 0, \qquad i = 1, 2. \tag{35}$$

If $\psi : S \to Y_1$ satisfies the inequality

$$\|\psi(x) - \Psi(x, \psi(f_1(x)), \ldots, \psi(f_j(x))), y\| \leq \varepsilon(x, y), \qquad x \in S, y \in Y_0, \tag{36}$$

then (33) holds.

However the main stability results in [41] (motivated by the approach proposed in [64]) concerns the Cauchy equation

$$f(x+y) = f(x) + f(y)$$

and can be rewritten as follows.

Theorem 7. *Assume that $(G, +)$ is a commutative group, $Aut\, G$ denotes the set of all automorphisms of $(G, +)$, $G_0 := G \setminus \{0\} \neq \emptyset$, $Y_0 \subset Y$ contains two linearly independent vectors, $H : G_0^2 \times Y_0 \to \mathbb{R}_+$,*

$$\mathcal{K} := \Big\{ u \in Aut\, G : u' \in Aut\, G \text{ and } \lambda(u') + \lambda(u) < 1 \Big\} \neq \emptyset,$$

and $\mathcal{V} \subset \mathcal{K}$ is nonempty and commutating (i.e., $\xi \circ \eta = \xi \circ \eta$ for every $\xi \circ \eta \in \mathcal{V}$), where, for any $u \in Aut\, G$, $u'(x) = x - u(x)$ for $x \in G$ and

$$\lambda(u) := \inf \Big\{ t \in \mathbb{R}_+ : H(u(x), u(y), z) \leq tH(x, y, z) \text{ for } x, y \in G_0, z \in Y_0 \Big\}.$$

Let $f : G \to Y$ satisfy

$$\|f(x+y) - f(x) - f(y), z\| \leq H(x, y, z), \qquad x, y \in G_0, z \in Y_0. \tag{37}$$

Then there is a unique additive mapping $T : G \to Y$ such that

$$\|f(x) - T(x), z\| \leq H_{\mathcal{V}}(x, z), \qquad x \in G_0, z \in Y_0,$$

where

$$H_{\mathcal{V}}(x, z) := \inf \Big\{ \frac{H(u'(x), u(x), z)}{1 - \lambda(u) - \lambda(u')} : u \in \mathcal{V} \Big\}, \qquad x \in G_0, z \in Y_0.$$

The following hyperstability result, given in Corollary 6.1 of [41], can be easily derived from Theorem 7.

Corollary 1. *Let G, G_0, Y_0 and H be as in Theorem 7. Assume that there is a nonempty and commutating $\mathcal{V} \subset \mathcal{K}$ such that*

$$\inf \{H(u'x, ux, z) : u \in \mathcal{V}\} = 0, \qquad x \in G_0, z \in Y_0, \tag{38}$$

and

$$\sup \{\lambda(u') + \lambda(u) : u \in \mathcal{V}\} < 1. \tag{39}$$

Then each function $f : G \to Y$ fulfilling inequality (37) is additive.

In the same way, as in the first part of the proof of Theorem 4, we can obtain the subsequent outcome that is complementary to Corollary 1.

Corollary 2. *Let hypothesis (L) be valid and $D \subset X^2$ be nonempty. Assume that $F : Y_1^2 \to Y_1$ and $f : X \to Y_1$ satisfy*

$$\|f(x+y) - F(f(x), f(y)), z\| \leq \varphi(x,y,z), \qquad (x,y) \in D, z \in Y_0. \tag{40}$$

Then

$$f(x+y) = F(f(x), f(y)), \qquad (x,y) \in D. \tag{41}$$

A somewhat different and more involved result on stability of the Cauchy equation has been obtained in [42], for functions mapping a nonempty subset X of an abelian semigroup $(S, +)$ into Y, under the assumption that there is $k_0 \in \mathbb{N}$, $k_0 > 1$, with

$$mx, mx+1 \in X, \qquad x \in X, m \in N_0 := \{k_0^n : n \in \mathbb{N}\}, \tag{42}$$

where $1x = x$ and $(n+1)x = nx + x$ for $x \in S, n \in \mathbb{N}$. We present it below (the result in [42] has been formulated under an assumption on X a bit weaker than (42), but actually (42) is necessary there).

To this end, we write $\mathbb{R}y := \{ay : a \in \mathbb{R}\}$ for every $y \in Y$. Let $A_1, A_2 : S \to Y$ be additive mappings (i.e., $A_i(x+z) = A_i(x) + A_i(z)$ for every $x, z \in S$), and $C, D : Y \to Y$ be such that the set $D^{-1}(Y \setminus \mathbb{R}u) \cap C^{-1}(Y \setminus \mathbb{R}v)$ contains two linearly independent vectors for every $u, v \in Y$. Next, let $c, d \in \mathbb{R}_+$, $p, q \in (-\infty, 0)$, and $\psi : X^2 \times Y \to \mathbb{R}$ satisfy the condition

$$\psi^*_{m,i}(x,y) = \sum_{n=0}^{\infty} (\Lambda_m^n \psi_{m,i})(x,y) < \infty, \qquad x \in X, y \in Y, i = 1,2, m \in N_0, m > \kappa, \tag{43}$$

with some $\kappa \in \mathbb{N}$, where $\psi_{m,1}(x,y) := \psi(x, mx, y)$, $\psi_{m,2}(x,y) := \psi(mx, x, y)$, $\Lambda_m : \mathbb{R}_+^{X\times Y} \to \mathbb{R}_+^{X\times Y}$ is given by

$$(\Lambda_m \delta)(x,y) := \delta((m+1)x, y) + \delta(mx, y), \qquad x \in X, y \in Y, \delta \in \mathbb{R}_+^{X\times Y},$$

and $\Lambda_m^0 \delta = \delta$, $\Lambda_m^n = \Lambda_m \circ \Lambda_m^{n-1}$ for $\delta \in \mathbb{R}_+^{X\times Y}$, $n \in \mathbb{N}$.

Define $\Psi : X^2 \times Y \to \mathbb{R}$ by

$$\Psi(x_1, x_2, y) := c\|A_1(x_1), C(y)\|^p + d\|A_2(x_2), D(y)\|^q$$

when

$$\|A_1(x_1), C(y)\| \cdot \|A_2(x_2), D(y)\| \neq 0,$$

and $\Psi(x_1, x_2, y) := \psi(x_1, x_2, y)$ otherwise.

The main result in [42] reads as follows.

Theorem 8. *Assume that $f : X \to Y$ satisfies*

$$\|f(x_1 + x_2) - f(x_1) - f(x_2), y\| \leq \Psi(x_1, x_2, y)$$

for all $y \in Y$ and for all $x_1, x_2 \in X$ with $x_1 + x_2 \in X$. Then there is a unique $h : X \to Y$ such that

$$h(x_1 + x_2) = h(x_1) + h(x_2), \qquad x_1, x_2 \in X, x_1 + x_2 \in X,$$

and

$$\|f(x) - h(x), y\| \leq \min\left\{c\|A_1(x), C(y)\|^p, d\|A_2(x), D(y)\|^q\right\}$$

for every $x \in X$ and every $y \in Y$ with $\|A_1(x), C(y)\| \cdot \|A_2(x), D(y)\| \neq 0$.

Moreover,

$$\|f(x) - h(x), y\| \leq \inf_{m \in N_0} \psi^*_{m,0}(x,y)$$

for every $x \in X$ and for every $y \in Y$ with $\|A_1(x), C(y)\| \cdot \|A_2(x), D(y)\| = 0$, where $\psi^*_{m,0}(x,y) := \min\{\psi^*_{m,1}(x,y), \psi^*_{m,2}(x,y)\}$ and $\psi^*_{m,i}$ is defined by (43).

In [43], the authors applied the fixed point theorem from [41] to investigate, in the real 2-Banach spaces, the stability of a generalized Cauchy functional equation. The main result in [43] is somewhat similar to Theorem 7 ($Aut\, G$ and u' have the same meaning) and can be written as follows.

Theorem 9. *Let $m, l \in \mathbb{N}$, $(G, +)$ be a commutative group such that $G_0 := G \setminus \{0\} \neq \emptyset$ and $mx \neq 0$, $lx \neq 0$ for $x \in G_0$, Y_0 be a subset of Y containing two linearly independent vectors, and $h : G_0 \times G_0 \times Y_0 \to \mathbb{R}_+$ be such that*

$$M(G) := \{u \in Aut\, G : u' \in Aut\, G \text{ and } ms(u') + ls(u) < 1\} \neq \emptyset$$

where

$$s(u) := \inf\{t \in \mathbb{R}_+ : h(u(x), u(y), z) \leq t\, h(x,y,z) \text{ for all } x, y \in G_0, z \in Y_0\}$$

for $u \in Aut\, G$. Suppose that $f : G \to Y$ fulfills the inequality

$$\|f(mx + ly) - mf(x) - lf(y), z\| \leq h(mx, ly, z)$$

for all $x, y \in G_0$ and $z \in Y_0$. Then, for any nonempty $\mathcal{V} \subset M(G)$ with $u \circ v = v \circ u$ for $u, v \in \mathcal{V}$, there is a unique $T : G \to Y$ such that

$$f(mx + ly) = mf(x) + lf(y), \qquad x, y \in G_0, mx + ly \in G_0, \tag{44}$$

$$\|f(x) - T(x), z\| \leq K(x,z), \qquad x \in G_0, z \in Y_0,$$

where

$$K(x,z) := \inf\left\{\frac{h(x - lu(x), u(x), z)}{1 - ms(u') - ls(u)} : u \in \mathcal{V}\right\}. \tag{45}$$

Actually, Theorem 9 has been formulated in Theorem 3.1 of [43] in a way suggesting that f satisfies

$$f(mx + ly) = mf(x) + lf(y)$$

for all $x, y \in G$, or at least for all $x, y \in G_0$; but from the proof of Theorem 3.1 of [43] it follows that only (44) has been shown there. Moreover, we have corrected a small mistake in the denominator of the fraction in (45).

Clearly, under hypothesis (L), we can derive from Corollary 2 a result that complements Theorem 9.

In [44], the author also used the fixed point theorem from [41] to investigate the Ulam stability of the following radical functional equation

$$f\left(\sqrt[4]{x^4 + y^4}\right) = f(x) + f(y), \qquad x, y \in \mathbb{R}. \tag{46}$$

For information on solutions to equations of such type we refer to [65].

The stability result in Theorem 4.1 of [44] is following (we preserve its form as in [44]).

Theorem 10. *Assume that $h_1, h_2 : \mathbb{R}^2 \to \mathbb{R}_+$ are such that*

$$\mathcal{M}_0 := \{n \in \mathbb{N} : a_n := s_1(n^4)s_2(n^4) + s_1(1+n^4)s_2(1+n^4) < 1\} \neq \emptyset,$$

where, for $i = 1, 2$ and $n \in \mathbb{N}$,

$$s_i(n) := \inf\{t \in \mathbb{R}_+ : h_i(nx^4, z) \le t h_i(x^4, z) \text{ for } x, z \in \mathbb{R}\}.$$

Suppose that $f : \mathbb{R} \to Y$ satisfies the inequality

$$\left\| f\left(\sqrt[4]{x^4+y^4}\right) - f(x) - f(y), z \right\| \le h_1(x^4, z)h_2(y^4, z), \qquad x, y, z \in \mathbb{R}. \tag{47}$$

Then there exists a unique additive function $T : \mathbb{R} \to Y$ such that

$$\|f(x) - T(x^4), z\| \le s_0 h_1(x^4, z)h_2(y^4, z), \qquad x, z \in \mathbb{R},$$

where

$$s_0 := \inf\left\{\frac{s_2(n^4)}{1-a_n} : n \in \mathcal{M}_0\right\}.$$

It is easily seen that there are several small mistakes in the theorem. Namely, the domains of h_1, h_2 should be $\mathbb{R} \times Y$, and z should belong to Y everywhere and not to $\mathbb{R}$.

The other stability result in Theorem 4.2 of [44] is analogous (with the same mistakes, which also occur in Corollaries 5.1–5.3 in [44]), but with $h_1(x^4, z)h_2(y^4, z)$ in (47) replaced by $h(x^4, z) + h(y^4, z)$ with some $h : \mathbb{R}^2 \to \mathbb{R}$ satisfying similar assumptions as h_i in Theorem 10.

Moreover, in this case, under hypothesis (L), we can derive from Corollary 2 a result that complements Theorem 10 and Theorem 4.2 of [44] (i.e., the corrected versions of them).

In [45], the authors investigated the stability of the functional equations

$$h\left(\sqrt{x^2+y^2}\right) = h(x) + h(y), \tag{48}$$

$$h\left(\sqrt{x^2+y^2}\right) + h\left(\sqrt{|x^2-y^2|}\right) = 2h(x) + 2h(y), \tag{49}$$

for function $h : \mathbb{R} \to Y$. They have proposed several interesting outcomes. We present below only one example of them (Theorem 3.2 of [45]) in a bit modified form to increase its readability. The other results in [45] are of similar type.

Theorem 11. *Let $\ell \in \{-1, 1\}$, $\phi : \mathbb{R}^3 \to \mathbb{R}_+$ be such that*

$$\begin{aligned}\widehat{\varphi}(x,z) := \sum_{i=\frac{1}{2}(1-\ell)}^{\infty} \sqrt{2}^{\,-3\ell i}\Big(&\phi(\sqrt{2}^{\,li}x, \sqrt{2}^{\,li}x, \sqrt{2}^{\,li}z) + \phi(\sqrt{2}^{\,li+1}x, 0, \sqrt{2}^{\,li}z) \\ &+ \frac{1+\ell}{2}\phi(0, 0, \sqrt{2}^{\,li}z)\Big) < \infty\end{aligned} \tag{50}$$

for $x, z \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{2^{\ell n}}\phi(\sqrt{2}^{\,ln}x, \sqrt{2}^{\,ln}y, z) = 0, \qquad x, y, z \in \mathbb{R}, \tag{51}$$

and $g : \mathbb{R} \to Y$ satisfy the condition

$$g(\alpha x) = \alpha g(x), \qquad \alpha \in \mathbb{R}. \tag{52}$$

Assume that $f : \mathbb{R} \to Y$ *satisfies*

$$\left\| f\left(\sqrt{x^2+y^2}\right) - f(x) - f(y), g(z) \right\| \le \phi(x,y,z), \qquad x,y,z \in \mathbb{R}. \tag{53}$$

Then the limit

$$h(x) := \lim_{n\to\infty} \frac{1}{2^{\ell n}} f\left(\sqrt{2}^{\,\ell n} x\right)$$

exists for every $x \in \mathbb{R}$ *and* $h : \mathbb{R} \to Y$ *is a unique mapping satisfying Equation* (48) *for all* $x, y \in \mathbb{R}$ *and the inequality*

$$\| f(x) - h(x), g(z) \| \le \frac{1}{2} \widehat{\phi}(x,z), \qquad x,z \in \mathbb{R}. \tag{54}$$

Unfortunately, the assumptions on g are not sufficient in this theorem. Namely, it is necessary for the set $g(\mathbb{R})$ to include at least two linearly independent vectors because otherwise, the statement is not true. In fact, if $g(\mathbb{R}) \subset \mathbb{R}v$ with some $v \in Y$, then for every mapping $f : \mathbb{R} \to Y$ with $f(\mathbb{R}) \subset \mathbb{R}v$ condition (53) holds with $\phi(x,y,u) \equiv 0$, while f does not necessarily satisfy (48) for all $x, y \in \mathbb{R}$ (in this case $\widehat{\phi}(x,z) \equiv 0$, whence (54) means that $f = h$).

Further, assumption (52) with a fixed $x \in \mathbb{R} \setminus \{0\}$ (it is not explained in [45] what is x in this assumption) simply means that $g(\alpha) = \alpha \frac{1}{x} g(x)$ for all $\alpha \in \mathbb{R}$, that is $g(X) \subset \mathbb{R}v$ with $v := \frac{1}{x} g(x) \in Y$.

We should add here that the main reasoning in the proof of Theorem 3.2 of [45] can be modified in such a way that assumption (52) is superfluous; and then the final outcome is similar to Theorem 11, but with some formulas modified. We will publish this modified version (and some complementary results) in a separate article.

Moreover, note that under hypothesis (L), we can derive from Corollary 2 a result that complements Theorem 11 (in this modified form).

The other main stability outcomes in [45] for 2-normed spaces contain similar imperfections.

The other functional equations, for which stability has been investigated in [45] in 2-normed spaces (with similar doubts), are

$$h\left(\sqrt{xx^* + yy^*}\right) = h(x) + h(y), \tag{55}$$

$$h\left(\sqrt{xx^* + yy^*}\right) + h\left(\sqrt{xx^* - yy^*}\right) = 2h(x) + 2h(y), \tag{56}$$

for mappings h from a C^*-algebra $\mathcal{A}$ into Y, where $\sqrt{z}$ for $z \in \mathcal{A}$ denotes the unique positive element $u \in \mathcal{A}$ with $u^2 = z$ (Equation (56) is considered for such $x, y \in \mathcal{A}$ that $xx^* - yy^*$ is a positive element).

In [46], the authors have investigated the stability of the functional equation

$$F\left(\sqrt[p]{\sum_{i=1}^{k} x_i^p}\right) = \sum_{i=1}^{k} F(x_i), \qquad x_1, \ldots, x_k \in \mathbb{R}, \tag{57}$$

for functions $F : \mathbb{R} \to Y$, where $p, k \in \mathbb{N} \setminus \{1\}$ are fixed. The first main stability result in Theorem 3.1 of [46] reads as follows (some typos made in [46] have been corrected).

Theorem 12. *Let* $h_1, \ldots, h_k : \mathbb{R}^2 \to \mathbb{R}_+$ *be such that*

$$U := \left\{ n \in \mathbb{N} : \alpha_n := \prod_{i=1}^{k} \lambda_i\left((k-1)n^p + 1\right) + (k-1) \prod_{i=1}^{k} \lambda_i(n^p) < 1 \right\} \neq \emptyset$$

where

$$\lambda_i(n) := \inf\Big\{t \in \mathbb{R}_+ : h_i(nx^p, z) \le t\, h_i(x^p, z),\ x, z \in \mathbb{R}\Big\}, \qquad n \in \mathbb{N}.$$

Assume that $g : \mathbb{R} \to Y$ is surjective and $f : \mathbb{R} \to Y$ satisfies

$$\left\| f\left(\sqrt[p]{\sum_{i=1}^{k} x_i^p}\right) - \sum_{i=1}^{k} f(x_i), g(z)\right\| \le \prod_{i=1}^{k} h_i(x_i^p, z), \qquad x_1, \ldots, x_k, z \in \mathbb{R}. \tag{58}$$

Then there exists a unique solution $F : \mathbb{R} \to Y$ of Equation (57) such that

$$\|f(x) - F(x), g(z)\| \le \lambda_0 \prod_{i=1}^{k} h_i(x^p, z), \qquad x, z \in \mathbb{R},$$

where

$$\lambda_0 := \inf_{n\in U}\left\{\frac{\prod_{i=1}^{k-1}\lambda_i(n^p)}{1 - \prod_{i=1}^{k}\lambda_i((k-1)n^p + 1) - (k-1)\prod_{i=1}^{k}\lambda_i(n^p)}\right\}.$$

Below we show that for $k > 2$, with a short reasoning, we can obtain an improved version of Theorem 12. Namely, we have the following.

Theorem 13. *Let $k > 2$ and $h_1, \ldots, h_k : \mathbb{R}^2 \to \mathbb{R}_+$ be such that there is $l \in \mathbb{N}$ with $1 < l \le k$ and*

$$U_l := \left\{n \in \mathbb{N} : \alpha_n := \prod_{i=1}^{l}\lambda_i\Big((l-1)n^p + 1\Big) + (k-1)\prod_{i=1}^{l}\lambda_i(n^p) < 1\right\} \ne \emptyset,$$

where $\lambda_i(n)$ is defined as in Theorem 12. Assume that $g : \mathbb{R} \to Y$ is surjective and $f : \mathbb{R} \to Y$ fulfills inequality (58). Then f is a solution to Equation (57).

Proof. Since $U_l \ne \emptyset$, there is $n \in \mathbb{N}$ with $\alpha_n < 1$, which means that $\lambda_i(n^p) < 1$ for some $i \in \{1, \ldots, l\}$.

Suppose that $h_i(0, z) > 0$ for some $z \in \mathbb{R}$. Then, taking $x = 0$ in the definition of $\lambda_i(n)$, we see that we must have $\lambda_i(n) \ge 1$, which is a contradiction. Thus we have shown that $h_i(0, z) = 0$ for $z \in \mathbb{R}$. Without a loss of generality, we can assume that $i = 1$.

Now taking $x_1 = \ldots = x_{k-2} = 0$ in (58) we get

$$\left\| f\left(\sqrt[p]{\sum_{i=k-1}^{k} x_i^p}\right) - \sum_{i=k-1}^{k} f(x_i), g(z)\right\| \le 0, \qquad x_2, z \in \mathbb{R}.$$

As g is surjective, this means that

$$\left\| f\left(\sqrt[p]{\sum_{i=k-1}^{k} x_i^p}\right) - \sum_{i=k-1}^{k} f(x_i), u\right\| = 0, \qquad x_{k-1}, x_k \in \mathbb{R}, u \in Y, \tag{59}$$

and consequently (see Lemma 1)

$$f\left(\sqrt[p]{x_1^p + x_2^p}\right) = f(x_1) + f(x_2), \qquad x_1, x_2 \in \mathbb{R}. \tag{60}$$

Since Equations (57) and (60) have the same solutions in the class of functions $f : \mathbb{R} \to Y$ (see Theorem 2.1 of [44]), this completes the proof. $\square$

In Theorem 3.2 of [46], the authors obtained an outcome that is similar to Theorem 12 and reads as follows.

Theorem 14. *Let $h : \mathbb{R}^2 \to \mathbb{R}_+$ be such that*

$$U := \Big\{ n \in \mathbb{N} : \beta_n := \lambda\big((k-1)n^p + 1\big) + (k-1)\lambda(n^p) < 1 \Big\} \neq \emptyset,$$

where

$$\lambda(n) := \inf \Big\{ t \in \mathbb{R}_+ : h(nx^p, z) \le t\, h(x^p, z) \text{ for } x, z \in \mathbb{R} \Big\}, \qquad n \in \mathbb{N}. \tag{61}$$

Assume that $g : \mathbb{R} \to Y$ is surjective and $f : \mathbb{R} \to Y$ satisfies

$$\left\| f\left(\sqrt[p]{\sum_{i=1}^{k} x_i^p} \right) - \sum_{i=1}^{k} f(x_i), g(z) \right\| \le \sum_{i=1}^{k} h(x_i^p, z), \qquad x_1, \ldots, x_k, z \in \mathbb{R}. \tag{62}$$

Then there exists a unique solution $F : \mathbb{R} \to Y$ of Equation (57) such that

$$\|f(x) - F(x), g(z)\| \le \eta h(x^p, z), \qquad x, z \in \mathbb{R},$$

where

$$\eta := \inf_{n \in U} \left\{ \frac{1 + (k-1)\lambda(n^p)}{1 - \lambda((k-1)n^p + 1) - (k-1)\lambda(n^p)} \right\}.$$

Below we show that for $k > 2$, by additional reasoning, we can obtain the following improved version of Theorem 14.

Theorem 15. *Let $h : \mathbb{R}^2 \to \mathbb{R}_+$ be such that there is $l \in \mathbb{N}$ with $1 < l \le k$ and*

$$U_l := \Big\{ n \in \mathbb{N} : \beta_n := \lambda\big((l-1)n^p + 1\big) + (l-1)\lambda(n^p) < 1 \Big\} \neq \emptyset,$$

where λ is given by (61). Assume that $g : \mathbb{R} \to Y$ is surjective and $f : \mathbb{R} \to Y$ satisfies (62). Then there exists a unique solution $F : \mathbb{R} \to Y$ of Equation (57) such that

$$\|f(x) - F(x), g(z)\| \le \eta_l\, h(x^p, z), \qquad x, z \in \mathbb{R}, \tag{63}$$

where

$$\eta_l := \inf_{n \in U} \left\{ \frac{1 + (l-1)\lambda(n^p)}{1 - \lambda((l-1)n^p + 1) - (l-1)\lambda(n^p)} \right\}.$$

Proof. The case $l = k$ is just Theorem 14. Therefore, assume that $l < k$.

First, we prove that $h(0, z) = 0$ for $z \in \mathbb{R}$. For the proof by contradiction, suppose that there is $z \in \mathbb{R}$ with $h(0, z) > 0$. Then taking $x = 0$ in the definition of $\lambda(n)$, we see that we must have $\lambda(n) \ge 1$ for $n \in \mathbb{N}$, which means that $U_l = \emptyset$ and, therefore, contradicts the assumptions.

Therefore, we have proven that $h(0, z) = 0$ for $z \in \mathbb{R}$. Now taking $x_{l+1} = \ldots = x_k = 0$ in (58) we get

$$\left\| f\left(\sqrt[p]{\sum_{i=1}^{l} x_i^p} \right) - \sum_{i=1}^{l} f(x_i), g(z) \right\| \le \sum_{i=1}^{l} h(x_i^p, z), \qquad x_1, \ldots, x_l, z \in \mathbb{R}.$$

Now, by Theorem 14 with $k = l$, there exists a unique solution $F : \mathbb{R} \to Y$ of the equation

$$F\left(\sqrt[p]{\sum_{i=1}^{l} x_i^p} \right) = \sum_{i=1}^{l} F(x_i), \qquad x_1, \ldots, x_l \in \mathbb{R}, \tag{64}$$

such that (63) holds. As Equations (57) and (60) have the same solutions in the class of mappings from $\mathbb{R}$ into Y (see Theorem 2.1 of [44]), this completes the proof. □

Please note that the form of Theorem 14 is quite simple for $l = 2$. Moreover, if $L = \{l \in \{2, \ldots, k\} : U_l \neq \emptyset\}$, then in (63) we can replace η_l by

$$\eta_0 := \inf_{l \in L} \eta_l.$$

In [47], the authors used the fixed point theorem from [41] to investigate the stability of the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{65}$$

The first main result in Theorem 2.1 of [47] reads as follows.

Theorem 16. *Let X be a normed space, $X_0 := X \setminus \{0\}$, and mappings $h_1, h_2 : X_0 \times X_0 \to \mathbb{R}_+$ be such that*

$$\mathcal{V} = \Big\{ n \in \mathbb{N} : \alpha_n = \frac{1}{2}\lambda_1(1+n)\lambda_2(1+n) + \frac{1}{2}\lambda_1(1-n)\lambda_2(1-n) + \lambda_1(n)\lambda_2(n) < 1 \Big\} \neq \emptyset, \tag{66}$$

with

$$\lambda_i(n) := \inf\{t \in \mathbb{R}_+ : h_i(nx, z) \leq t\, h_i(x, z) \text{ for } x, z \in X_0\}, \qquad n \in \mathbb{N}, i = 1, 2.$$

Assume that $g : X_0 \to Y$ is a surjective mapping and $f : X \to Y$ satisfies

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y), g(z)\| \leq h_1(x, z) h_2(y, z) \tag{67}$$

for all $x, y, z \in X_0$, such that $x + y \neq 0$ and $x - y \neq 0$. Then there exists a unique quadratic mapping $F : X \to Y$ such that

$$\|f(x) - F(x), g(z)\| \leq \eta h_1(x, z) h_2(x, z), \qquad x, z \in X_0, \tag{68}$$

where

$$\eta = \inf_{n \in \mathcal{V}} \left\{ \frac{\lambda_2(n)}{2 - \lambda_1(1+n)\lambda_2(1+n) - \lambda_1(1-n)\lambda_2(1-n) - 2\lambda_1(n)\lambda_2(n)} \right\}.$$

It is not sure what is meant in [47] by the statement that F is *a quadratic mapping*. It seems that it means the following condition

$$F(x+y) + F(x-y) = 2F(x) + 2F(y), \qquad x, y \in X_0, x+y, x-y \in X_0, \tag{69}$$

because in the proof of Theorem 2.1 in [47], the authors have only shown condition (69) for F. Moreover, it is not necessary to assume the existence of a norm in X (in Theorem 2.1 of [47], i.e., in our Theorem 16).

As a consequence of Theorem 2.1 of [47], the authors obtained several hyperstability outcomes, stating that under some additional assumptions on h_1 and h_2, every mapping $f : X \to Y$ satisfying (67) must fulfill (69) (with $F = f$).

Clearly, arguing analogously as in the first part of the proof of Theorem 4, we can obtain, for instance, the following theorem corresponding to Theorem 2.3 of [47].

Theorem 17. *Let $k > 1$ be an integer, $\alpha \in \mathbb{R}$, $(X, +)$ be a group (not necessarily commutative), $g : X \to Y_1$, $g(kx) = \alpha g(x)$ for $x \in X$, $D \subset X^2$, and $\varphi : X^3 \to \mathbb{R}_+$ be such that for every $(x, y) \in D$, there exist $z_1, z_2 \in X$ such that $g(z_1)$ and $g(z_2)$ are linearly independent and*

$$\lim_{n\to\infty} \alpha^{-n}\varphi(x, y, k^n z_i) = 0, \qquad i = 1, 2. \tag{70}$$

Assume that $f : X \to Y_1$ satisfies

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y), g(z)\| \leq \varphi(x, y, z), \qquad (x, y) \in D, z \in X. \tag{71}$$

Then

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \qquad (x, y) \in D. \tag{72}$$

Proof. Fix $(x, y) \in D$ and $i \in \{1, 2\}$. Then there exist $z_1, z_2 \in X$ such that $g(z_1)$ and $g(z_2)$ are linearly independent and (70) holds and, by (71), for $i = 1, 2$,

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y), g(k^n z_i)\| \leq \varphi(x, y, k^n z_i) \tag{73}$$

whence

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y), g(z_i)\| \leq \alpha^{-n}\varphi(x, y, k^n z_i). \tag{74}$$

Letting $n \to \infty$ in (74), by (70) we get

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y), g(z_i)\| = 0, \qquad i = 1, 2.$$

As vectors z_1 and z_2 are linearly independent, this implies that $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ (see Lemma 1), which completes the proof. □

In view of Remark 2, it is easy to find numerous examples of functions φ and g satisfying the assumptions of Theorem 17.

The other main result in [47] (Theorem 2.2) is analogous with Theorem 16, but with inequality (67) replaced by

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y), g(z)\| \leq h(x, z) + h(y, z), \tag{75}$$

where $h : X_0^2 \to \mathbb{R}$ satisfies analogous assumptions as h_1 and h_2 in Theorem 16. The ambiguities regarding Theorem 2.2 of [47] are similar as for Theorem 2.1 of [47].

The authors in [48] obtained stability results for the functional equations

$$f(x + y, z + w) + f(x - y, z - w) = 2f(x, z) + 2f(y, w), \tag{76}$$

$$f(x + y, z - w) + f(x - y, z + w) = 2f(x, z) + 2f(y, w), \tag{77}$$

$$f(x + y, z - w) + f(x - y, z + w) = 2f(x, z) - 2f(y, w). \tag{78}$$

The main outcome in [48] for Equation (76) (i.e., Theorem 2.1 of [48]) can be written as follows.

Theorem 18. *Let X be a normed space, $p \in (0, 2)$, $\epsilon, \delta, \eta \in \mathbb{R}_+$, and let $f : X \times X \to Y$ be a surjective mapping such that*

$$\begin{aligned} \|f(x + y, z + w) &+ f(x - y, z - w) - 2f(x, z) - 2f(y, w), f(u, v)\| \\ &\leq \epsilon + \delta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) + \eta(\|u\| + \|v\|) \end{aligned} \tag{79}$$

for all $x, y, z, w, u, v \in X$. Then there exists a unique mapping $F : X^2 \to Y$ satisfying (76) for all $x, y, z, w, u, v \in X$ such that

$$\|f(x,y) - F(x,y), f(u,v)\| \leq \frac{\epsilon}{3} + \frac{2\delta}{4-2^p}(\|x\|^p + \|y\|^p) + \frac{\eta}{3}(\|u\| + \|v\|)$$

for all $x, y, u, v \in X$.

Actually, it has been assumed in Theorem 2.1 of [48] that $\epsilon > 0$, but it is easily seen that the theorem is also true for $\epsilon = 0$; this can be deduced from Theorem 2.1 of [48] (i.e., from our Theorem 18) with $\epsilon \to 0$, or from the proof of it.

Similar results have been obtained in [48] for Equations (77) and (78).

In [49], the author applied the fixed point theorem in [41] to study the stability of the subsequent functional equation (of the p-Wright affine functions)

$$g(px_1 + (1-p)x_2) + g((1-p)x_1 + px_2) = g(x_1) + g(x_2), \tag{80}$$

with a fixed $p \in \mathbb{R}$ and a mapping g from a nonempty set $E \subset Y$ into Y. The main stability result in [49] can be written as follows.

Theorem 19. *Let $E \subset Y$ be nonempty, $Y_0 \subset Y$ contain two linearly independent vectors, $p \in \mathbb{R}$, $A, k \in (0, \infty)$,*

$$|p|^k + |1-p|^k < 1,$$

$$px_1 + (1-p)x_2 \in E, \qquad x_1, x_2 \in E,$$

and $g : E \to Y$ satisfy

$$\begin{aligned} \|g(px_1 + (1-p)x_2) + g((1-p)x_1 + px_2) - g(x_1) - g(x_2), y\| \\ \leq A(\|x_1, y\|^k + \|x_2, y\|^k), \qquad x_1, x_2 \in E, y \in Y_0. \end{aligned} \tag{81}$$

Then there exists a unique mapping $G : E \to Y$ such that

$$G(px_1 + (1-p)x_2) + G((1-p)x_1 + px_2) = G(x_1) + G(x_2), \qquad x, y \in E, \tag{82}$$

$$\|g(x) - G(x), y\| \leq \frac{A\|x, y\|}{1 - |p|^k - |1-p|^k}, \qquad x \in E, y \in Y_0.$$

Moreover, G is the unique mapping fulfilling (82) such that there exists a constant $M \in (0, \infty)$ with

$$\|g(x) - G(x), y\| \leq M\|x, y\|^k, \qquad x \in E, y \in Y_0.$$

However, under an additional assumption of Y_0, in the case $k \neq 1$, we can obtain the following better result.

Theorem 20. *Let E be a nonempty subset of a real linear space X, $D \subset E^2$ be nonempty, $p \in \mathbb{R}$, $A_1, A_2, k \in \mathbb{R}$, and $Y_0 \subset Y_1$ be such that there exist two linearly independent vectors $z_1, z_2 \in Y_1$ and $l_1, l_2 \in \mathbb{R} \setminus \{0\}$ such that $|l_i|^{k-1} > 1$ and $l_i^n z_i \in Y_0$ for $i = 1, 2$ and $n \in \mathbb{N}$. Assume that $g : E \to Y_1$ satisfies the inequality*

$$\begin{aligned} \|g(px_1 + (1-p)x_2) + g((1-p)x_1 + px_2) - g(x_1) - g(x_2), y\| \\ \leq A_1\|x_1, y\|^k + A_2\|x_2, y\|^k \end{aligned} \tag{83}$$

for every $y \in Y_0$ *and every* $(x_1, x_2) \in D$ *such that* $px_1 + (1-p)x_2 \in E$. *Then*

$$g(px_1 + (1-p)x_2) + g((1-p)x_1 + px_2) = g(x_1) + g(x_2) \tag{84}$$

for every $(x_1, x_2) \in D$ *such that* $px_1 + (1-p)x_2 \in E$.

Proof. Fix $(x_1, x_2) \in D$ such that $px_1 + (1-p)x_2 \in E$. Then, for each $n \in \mathbb{N}$ and $i = 1, 2$, we have $l_i^n z_i \in Y_0$ and, by (83),

$$\begin{aligned} \|g(px_1 + (1-p)x_2) + g((1-p)x_1 + px_2) - g(x_1) - g(x_2), l_i^n z_i\| \\ \leq A_1 \|x_1, l_i^n z_i\|^k + A_2 \|x_2, l_i^n z_i\|^k, \end{aligned} \tag{85}$$

whence

$$\begin{aligned} \|g(px_1 + (1-p)x_2) + g((1-p)x_1 + px_2) - g(x_1) - g(x_2), z_i\| \\ \leq l_i^{(k-1)n} \big(A_1 \|x_1, z_i\|^k + A_2 \|x_2, z_i\|^k\big). \end{aligned} \tag{86}$$

Letting $n \to \infty$ in (86), we get

$$\|g(px_1 + (1-p)x_2) + g((1-p)x_1 + px_2) - g(x_1) - g(x_2), z_i\| = 0$$

for $i = 1, 2$. Since z_1 and z_2 are linearly independent vectors, this means that $g(px_1 + (1-p)x_2) + g((1-p)x_1 + px_2) - g(x_1) - g(x_2) = 0$ (see Lemma 1), which completes the proof. □

The first main stability result in [50] can be rewritten as follows.

Theorem 21. *Let* a *and* b *be nonzero rational numbers,* $h_1, h_2 : \mathbb{R}^2 \to \mathbb{R}_+$ *be two functions such that*

$$\begin{aligned} \mathcal{U} := \Big\{ n \in \mathbb{N} : \alpha_n = 2a^2 \lambda_1\Big(\frac{n+1}{a}\Big)\, \lambda_2\Big(\frac{n+1}{a}\Big) + 2b^2 \lambda_1\Big(\frac{-n}{b}\Big)\lambda_2\Big(\frac{-n}{b}\Big) \\ + \lambda_1(2n+1)\lambda_2(2n+1) < 1 \Big\} \neq \emptyset, \end{aligned} \tag{87}$$

where

$$\lambda_i(\rho) := \inf\{t \in \mathbb{R}_+ : h_i(\rho x^3, z) \leq t\, h_i(x^3, z) \text{ for } x, z \in \mathbb{R}\}$$

for every $\rho \in \mathbb{R}$ *and* $i = 1, 2$. *Assume that* $g : \mathbb{R} \to Y$ *is a surjective mapping with* $g(0) = 0$, $f : \mathbb{R} \to Y$ *satisfies*

$$\Big\| f\Big(\sqrt[3]{ax^3 + by^3}\Big) + f\Big(\sqrt[3]{ax^3 - by^3}\Big) - 2a^2 f(x) - 2b^2 f(y), g(z) \Big\| \leq h_1(x^3, z) h_2(y^3, z) \tag{88}$$

for all $x, y, z \in \mathbb{R}_0 := \mathbb{R} \setminus \{0\}$, *and* $f(0) = (a^2 + b^2) f(0)$. *Then there exists a unique function* $T : \mathbb{R}_0 \to Y$ *satisfying*

$$T\Big(\sqrt[3]{ax^3 + by^3}\Big) + T\Big(\sqrt[3]{ax^3 - by^3}\Big) = a^2 T(x) + 2b^2 T(y), \qquad x, y \in \mathbb{R}_0, \tag{89}$$

and such that

$$\|f(x) - T(x), g(z)\| \leq \beta h_1(x^3, z) h_2(x^3, z), \qquad x, z \in \mathbb{R}_0,$$

where

$$\beta = \inf \Big\{ \frac{1}{1 - \alpha_n} \lambda_1\Big(\frac{n+1}{a}\Big)\lambda_2\Big(\frac{-n}{b}\Big) : n \in \mathcal{U} \Big\}.$$

It is not clear why in the above result a and b are assumed in [50] to be rational numbers and not just reals because it is not necessary for the proof.

Note that also for Equation (89), we can easily prove a result analogous to Theorem 17.

The other main stability result in [50] is analogous to Theorem 21, but with $h_1(x^3,z)h_2(y^3,z)$ in (88) replaced by $h(x^3,z)+h(y^3,z)$, where $h:\mathbb{R}^2\to\mathbb{R}_+$ fulfills an analogous assumption as h_1 and h_2. General remarks on solutions to equations similar to (89) can be found in [66].

In [51], the authors used the fixed point approach to investigate the stability of the functional equation

$$f\Big(\frac{x+y}{2}+z\Big)+f\Big(\frac{x-y}{2}+z\Big)=f(x)+2f(z). \tag{90}$$

The main stability result in [51] can be rewritten as follows.

Theorem 22. *Let $Y_0:=Y\setminus\{0\}$, E be a real linear space, $E_0:=E\setminus\{0\}$ and $h_1,h_2,h_3:\mathbb{R}^2\to\mathbb{R}_+$ be such that the set*

$$\mathcal{U}:=\{n\in\mathbb{N}:\lambda_1(1+n)\lambda_2(1+n)\lambda_3(1+n)+\lambda_1(n)\lambda_2(n)\lambda_3(n)<1\}$$

is nonempty, where

$$\lambda_i(n)=\inf\{t\in\mathbb{R}_+:h_i(nx,w)\le t\,h_i(x,w)\ \text{for}\ x\in E_0, w\in Y_0\}$$

for all $n\in\mathbb{N}$ and $i=1,2,3$. Assume that $f:E\to Y$ satisfies

$$\Big\|f\Big(\frac{x+y}{2}+z\Big)+f\Big(\frac{x-y}{2}+z\Big)-2f(z)-f(x),w\Big\|\le h_1(x,w)h_2(y,w)h_3(z,w)$$

for all $w\in Y_0$ and all $x,y,z\in E_0$ such that $x+y\neq-2z$ and $x-y\neq-2z$. Then there exists a unique $F:E_0\to Y$ such that

$$F\Big(\frac{x+y}{2}+z\Big)+F\Big(\frac{x-y}{2}+z\Big)=F(x)+2F(z)$$

for all $x,y,z\in E_0$ with $x+y\neq-2z$, $x-y\neq-2z$ and

$$\|f(x)-F(x),w\|\le\eta h_1(x,w)h_2(x,w)h_3(x,w),\qquad x\in E_0, w\in Y_0,$$

where

$$\eta=\inf\Big\{\frac{\lambda_1(n)\lambda_2(n)}{1-\lambda_1(1+n)\lambda_2(1+n)\lambda_3(1+n)-\lambda_1(n)\lambda_2(n)\lambda_3(n)}:n\in\mathcal{U}\Big\}. \tag{91}$$

Using ideas already applied earlier, we can obtain, e.g., the improved version of Theorem 22 given below. To this end, we need the following hypothesis, similar to hypothesis (L) (used in Theorem 4).

(L′) Y_0 is a linear subspace of Y_1, E is a real linear space, $D\subset E^3$ is nonempty, $\varphi:E^3\times Y_0\to\mathbb{R}_+$ and, for every $(x,y,z)\in D$, there exist linearly independent $w_1,w_2\in Y_0$ and two real sequences $(\xi_n^1)_{n\in\mathbb{N}}$, $(\xi_n^2)_{n\in\mathbb{N}}$ such that $\xi_n^i\neq0$ for $i=1,2$, $n\in\mathbb{N}$, and

$$\lim_{n\to\infty}\frac{1}{\xi_n^i}\varphi(x,y,z,\xi_n^i w_i)=0,\qquad i=1,2. \tag{92}$$

Theorem 23. *Let (L′) be valid. Assume that $f:E\to Y$ satisfies*

$$\Big\|f\Big(\frac{x+y}{2}+z\Big)+f\Big(\frac{x-y}{2}+z\Big)-2f(z)-f(x),w\Big\|\le\varphi(x,y,z,w) \tag{93}$$

for all $w\in Y_0$ and all $(x,y,z)\in D$. Then (90) holds for all $(x,y,z)\in D$.

Proof. Fix $(x, y, z) \in D$. Then, according to hypothesis (L'), there exist linearly independent $z_1, z_2 \in Y_0$ and two real sequences $(\xi_n^1)_{n\in\mathbb{N}}, (\xi_n^2)_{n\in\mathbb{N}}$ such that $\xi_n^i \neq 0$ for $i = 1, 2$, $n \in \mathbb{N}$, and condition (92) holds. Hence, by (93), for all $n \in \mathbb{N}$ and $i = 1, 2$

$$\left\| f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x-y}{2} + z\right) - 2f(z) - f(x), \xi_n^i w_i \right\| \le \varphi(x, y, z, \xi_n^i w),$$

which yields

$$\left\| f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x-y}{2} + z\right) - 2f(z) - f(x), w_i \right\| \le \frac{1}{|\xi_n^i|} \varphi(x, y, z, \xi_n^i w_i). \quad (94)$$

Letting $n \to \infty$ in the inequality (94), on account of (92) we get

$$\left\| f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x-y}{2} + z\right) - 2f(z) - f(x), w_i \right\| = 0, \qquad i = 1, 2.$$

This implies (90) because z_1 and z_2 are linearly independent (see Lemma 1). □

In the next section, we present the stability results obtained for non-Archimedean 2-normed spaces.

4. Stability in Non-Archimedean 2-Normed Spaces

In [52], the authors have investigated the stability of the following functional inequality

$$\|f(x) + f(y) + f(az), w\| \le \|f(x + y) - f(az), w\|,$$

for mapping f from a non-Archimedean 2-normed space into a non-Archimedean 2-Banach space, where a is a fixed non-zero integer. The first main stability result given in Theorem 2.2 of [52] can be written as follows.

Theorem 24. *Let X be a non-Archimedean 2-normed space and Y be a non-Archimedean 2-Banach space. Assume that $\phi : X^3 \to \mathbb{R}_+$ is such that*

$$\lim_{n\to\infty} \frac{\phi((-2)^n x, (-2)^n y, (-2)^n z)}{|2|^n} = 0, \qquad x, y, z \in X, \quad (95)$$

and the limit

$$\lim_{n\to\infty} \max\left\{ \frac{\phi((-2)^k x, (-2)^k x, (-2)^{k+1} \frac{x}{a})}{|2|^{k-1}} : 0 \le k \le n-1 \right\} \quad (96)$$

exists for all $x \in X$. Let $f : X \to Y$ be such that $f(0) = 0$ and

$$\|f(x) + f(y) + f(az), w\| \le \|f(x + y) - f(-az), w\| + \phi(x, y, z), \quad x, y, z \in X, w \in Y. \quad (97)$$

Then there exists an additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x), w\| \le \lim_{n\to\infty} \max\left\{ \frac{\phi((-2)^k x, (-2)^k x, (-2)^{k+1} \frac{x}{a})}{|2|^{k-1}} : 0 \le k \le n-1 \right\} \quad (98)$$

for all $x \in X$ and $w \in Y$. Moreover, if for every $x \in X$

$$\lim_{k\to\infty} \lim_{n\to\infty} \max\left\{ \frac{\phi((-2)^i x, (-2)^i x, (-2)^{i+1} \frac{x}{a})}{|2|^i} : k \le i \le k+n-1 \right\} = 0,$$

then A is a unique additive mapping satisfying (98).

It seems that the authors assume in [52] that X and Y are over the same field $\mathbb{K}$ (but this is not precisely stated there). It is not clear why X is assumed to be a non-Archimedean 2-normed space because in the proof of Theorem 2.2 of [52], it is enough to assume that X is a linear space over a field in which $a \neq 0$. Moreover, it is necessary to assume that the characteristic of $\mathbb{K}$ is neither 3 nor 2 (this first property has been used without assuming it in the proof of Theorem 2.1 of [52], which, in turn, has been applied in the proof of Theorem 2.2 of [52]). Moreover, additional reasoning leads to the following improved result.

Theorem 25. *Let X and Y be as depicted above. Assume that $\phi : X^3 \to \mathbb{R}_+$ and $f : X \to Y$ satisfy (97). Then f is additive, i.e., $f(x+y) = f(x) + f(y)$ for every $x, y \in X$.*

Proof. Since the valuation in $\mathbb{K}$ is assumed to be nontrivial (see Definition 3), there exists $b \in \mathbb{K}$ with $|b| > 1$. Therefore, by (97), for every $n \in \mathbb{N}$, $x, y, z \in X$ and $w \in Y$ we get

$$
\begin{aligned}
|b|^n \|f(x) + f(y) + f(az), w\| &= \|f(x) + f(y) + f(az), b^n w\| \\
&\leq \|f(x+y) - f(-az), b^n w\| + \phi(x,y,z) \\
&= |b|^n \|f(x+y) - f(-az), w\| + \phi(x,y,z),
\end{aligned}
$$

which implies that

$$\|f(x) + f(y) + f(az), w\| \leq \|f(x+y) - f(-az), w\| + |b|^{-n}\phi(x,y,z)$$

and consequently (with $n \to \infty$)

$$\|f(x) + f(y) + f(az), w\| \leq \|f(x+y) - f(-az), w\|.$$

Note yet taking $x = y = z = 0$ in the last inequality we have $3f(0) = 0$, whence $f(0) = 0$. Hence, by Theorem 24 with $\phi(x,y,z) = 0$ for all $x, y, z \in X$, we obtain the statement. □

The other stability result in Theorem 2.3 of [52] is similar to Theorem 24, with analogous ambiguities, and it can be improved in a similar way.

In [53], results analogous to Theorem 24 have been obtained for the inequalities

$$\|f(x+y) - f(x) - f(y), w\| \leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), w\right\|,$$

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y), w\right\| \leq \|\rho(f(x+y) - f(x) - f(y)), w\|$$

for functions f mapping a non-Archimedean 2-normed space X into a non-Archimedean 2-Banach space Y over a field $\mathbb{F}$ with a non-Archimedean nontrivial valuation $|\cdot|$, where $\rho \in \mathbb{F}$ is fixed and $|\rho| < 1$ (actually the 2-norm in X seems to be superfluous in the proofs of the main results presented in Theorems 2.3, 2.5, 3.3 and 3.5 of [53], while some norms are necessary in X and Y in the corollaries following them).

Those results given in Theorems 2.3, 2.5, 3.3 and 3.5 of [53] can be improved with the analogous reasoning as in the proof of Theorem 25 and, therefore, we only present those modified versions (without proof) below. Namely, we have the following theorem.

Theorem 26. *Let X and Y be as depicted above. Let $\phi : X^2 \to \mathbb{R}_+$ and $f : X \to Y$ satisfy one of the following two inequalities*

$$\|f(x+y) - f(x) - f(y), w\| \leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), w\right\| + \phi(x,y), \qquad x, y \in X, w \in Y,$$

$$\left\|\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right),w\right\| \le \|\rho(f(x+y)-f(x)-f(y)),w\| + \phi(x,y),$$
$$x,y \in X, w \in Y.$$

Then, f is additive.

In [54], the authors considered the Ulam stability of the following system of functional equations

$$f(ax_1+bx_2,y,z)+f(ax_1-bx_2,y,z) = 2af(x_1,y,z), \tag{99}$$
$$f(x,ay_1+by_2,z)+f(x,ay_1-by_2,z) = ab^2(f(x,y_1+y_2,z)+f(x,y_1-y_2,z)) + 2a(a^2-b^2)f(x,y_1,z), \tag{100}$$
$$f(x,y,az_1+bz_2)+f(x,y,az_1-bz_2) = a^2b^2(f(x,y,z_1+z_2)+f(x,y,z_1-z_2)) + 2a^2(a^2-b^2)f(x,y,z_1) - 2b^2(a^2-b^2)f(x,y,z_2) \tag{101}$$

for functions f mapping a non-Archimedean normed space X into a non-Archimedean 2-Banach space Y, where a, b are nonzero integers and $a \neq \pm 1, \pm b$. Namely, under some additionally, rather involved, assumptions on functions $\phi_1, \phi_2, \phi_3 : X^3 \to \mathbb{R}_+$, they have studied the inequalities

$$\|Q_1(x,y_1,y_2,z),w\| \le \phi_1(x_1,x_2,y,z), \qquad x_1,x_2,y,z \in X, w \in Y, \tag{102}$$
$$\|Q_2(x,y_1,y_2,z),w\| \le \phi_2(x,y_1,y_2,z), \qquad x,y_1,y_2,z \in X, w \in Y, \tag{103}$$
$$\|Q_3(x,y,z_1,z_2),w\| \le \phi_3(x,y,z_1,z_2), \qquad x,y,z_1,z_2 \in X, w \in Y, \tag{104}$$

where

$$\begin{aligned}
Q_1(x_1,x_2,y,z) &:= f(ax_1+bx_2,y,z)+f(ax_1-bx_2,y,z)-2af(x_1,y,z),\\
Q_2(x,y_1,y_2,z) &:= f(x,ay_1+by_2,z)+f(x,ay_1-by_2,z)\\
&\quad - ab^2(f(x,y_1+y_2,z)+f(x,y_1-y_2,z))\\
&\quad - 2a(a^2-b^2)f(x,y_1,z),\\
Q_3(x,y,z_1,z_2) &:= f(x,y,az_1+bz_2)+f(x,y,az_1-bz_2)\\
&\quad - a^2b^2(f(x,y,z_1+z_2)+f(x,y,z_1-z_2))\\
&\quad - 2a^2(a^2-b^2)f(x,y,z_1)+2b^2(a^2-b^2)f(x,y,z_2).
\end{aligned}$$

We do not present nor discuss those results here because the reasoning, which we have applied already several times, shows that the following (better than in [54]) outcome is true.

Theorem 27. *Let Y be a non-Archimedean normed space over a field $\mathbb{F}$ with a nontrivial valuation $|\cdot|$. Let $(X,+)$ be a group (not necessarily commutative), a and b be integers, $\phi : X^4 \to \mathbb{R}_+$ and $f : X \to Y$ satisfy the system of inequalities (102)–(104). Then f fulfills Equations (99)–(101) for all $x, x_1, x_2, y, y_1, y_2, z, z_1, z_2 \in X$.*

Proof. Since the valuation in $\mathbb{F}$ is nontrivial, there exists $c \in \mathbb{F}$ with $|c| > 1$. Therefore, by (102), for every $n \in \mathbb{N}$, $x_1, x_2, y, z \in X$ and $w \in Y$ we get

$$|c|^n\|Q_1(x_1,x_2,y,z),w\| = \|Q_1(x_1,x_2,y,z),c^nw\| \le \phi(x_1,x_2,y,z),$$

whence

$$\|Q_1(x_1,x_2,y,z),w\| \le |c|^{-n}\phi(x_1,x_2,y,z),$$

which (with $n \to \infty$) yields $Q_1(x_1, x_2, y, z) = 0$ (see Lemma 1).

Analogously, we show that Equations (100) and (101) hold for all $x, y, y_1, y_2, z, z_1, z_2 \in X$. $\square$

5. Stability in $(2, \beta)$-Normed Spaces

In this section, we consider the Ulam stability results obtained for $(2, \beta)$-normed spaces. Let us start with a suitable definition (see [55,56]).

Definition 6. *Let $\beta \in (0, \infty)$, $\mathbb{K}$ be a field with a valuation $|\cdot| : \mathbb{K} \to \mathbb{R}_+$ and let X be a linear space over $\mathbb{K}$ with a dimension greater than 1.*

We say that a mapping $\|\cdot, \cdot\| : X \times X \to \mathbb{R}_+$ is a $(2, \beta)$-norm in X if, for every $x_1, x_2, x_3 \in X$ and $\alpha \in \mathbb{K}$, conditions (1)–(3) of Definition 3 are fulfilled and

(4′) $\|\alpha x_1, x_2\| = |\alpha|^\beta \|x_1, x_2\|$.

Let $\|\cdot, \cdot\| : X \times X \to \mathbb{R}_+$ be a $(2, \beta)$-norm in X. Then we say that a pair $(X, \|\cdot, \cdot\|)$ is a $(2, \beta)$-normed space. If $\mathbb{K}$ is the field of reals $\mathbb{R}$ and the valuation in $\mathbb{K}$ is the usual absolute value, then we say that $(X, \|\cdot, \cdot\|)$ is a real linear $(2, \beta)$-normed space; if $\mathbb{K}$ is the field of complex numbers $\mathbb{C}$ and the valuation in $\mathbb{K}$ is the usual complex modulus, then we say that $(X, \|\cdot, \cdot\|)$ is a complex $(2, \beta)$-normed space.

The notions of the Cauchy sequence, limit of a sequence, convergent sequence and $(2, \beta)$-Banach space are defined in the same way as for the 2-normed spaces.

Let $\beta \in (0, 1]$ and $(X, \|\cdot, \cdot\|)$ be a 2-normed space. Define $\|\cdot, \cdot\|^* : X^2 \to \mathbb{R}_+$ by

$$\|x, z\|^* = \|x, z\|^\beta, \qquad x, z \in X.$$

Then it is very easy to check that $(X, \|\cdot, \cdot\|^*)$ is a $(2, \beta)$-normed space.

The stability of functional equations in $(2, \beta)$-Banach spaces has been considered in [55,56]. The main result in [55] reads as follows.

Theorem 28. *Let $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$, β be a fixed real number with $0 < \beta \le 1$, $(X, \|\cdot\|_\beta)$ be a $(2, \beta)$-Banach space and let $h_1, h_2 : \mathbb{R}_0 \times X \to \mathbb{R}_+$ be such that the set*

$$\mathcal{M}_0 := \Big\{ n \in N_2 : a_n := 2^\beta s_{1,2}(n^3) + 2^\beta s_{1,2}(n^3 - 1) + 2^\beta s_{1,2}(2n^3 - 1) < 1 \Big\}$$

is nonempty, where $s_{1,2}(n) = s_1(n) s_2(n)$ and

$$s_i(n) := \inf \Big\{ t \in \mathbb{R}_+ : h_i(nx^3, z) \le t\, h_i(x^3, z) \text{ for } x \in \mathbb{R}_0, z \in X \Big\}$$

for $i = 1, 2$ and $n \in \mathbb{N}$. Suppose that $f : \mathbb{R} \to X$ is such that $f(0) = 0$ and

$$\Big\| f\Big(\sqrt[3]{x^3 + y^3}\Big) + f\Big(\sqrt[3]{x^3 - y^3}\Big) - 2f(x) - 2f(y), z \Big\|_\beta \le h_1(x^3, z) h_2(y^3, z)$$

for all $x, y \in \mathbb{R}_0$ and $z \in X$. Then there exists a unique $Q : \mathbb{R} \to X$ such that

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y), \qquad x, y \in \mathbb{R}_0,$$

and

$$\|f(x) - Q(x^3), z\|_\beta \le s_0 h_1(x^3, z) h_2(x^3, z), \qquad (x, z) \in \mathbb{R}_0 \times X,$$

where

$$s_0 := \inf \Big\{ \frac{1}{1 - a_n} s_1(n^3) s_2(n^3 - 1) : n \in \mathcal{M}_0 \Big\}.$$

We can easily obtain a result complementary to this theorem, e.g., arguing analogously as in the proof of Theorem 17.

In [56], the author has proven a stability result for $(2, \beta)$-Banach spaces very similar to Theorem 19, and we can easily obtain an improved version of it analogous to Theorem 20.

In [45], the authors have investigated the stability of the functional equations

$$f\left(\sqrt{ax^2 + by^2}\right) = af(x) + bf(y), \tag{105}$$

$$f\left(\sqrt{cx^2 + cy^2}\right) + f\left(\sqrt{|cx^2 - dy^2|}\right) = 2c^2 f(x) + 2d^2 f(y), \tag{106}$$

with fixed positive reals a, b, c, d such that $a + b \neq 1$ and $c^2 + d^2 \neq 1$ and for mappings f from $\mathbb{R}$ into a $(2, \beta)$-normed space Y. They have presented several interesting outcomes, but with similar imperfections as depicted in connection with Theorem 11.

6. Stability in Random 2-Normed Spaces

Finally, let us mention that in [57,58] the authors provided some Ulam stability results in random 2-normed spaces. Their forms are quite involved and go beyond the scope of this article, so we omit them. Let us only add that in [57], the stability of the pexiderized quadratic functional equation of the form

$$f(x + y) + f(x - y) = 2g(x) + 2h(y)$$

has been studied and in [58], the authors presented some stability results for the cubic functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x).$$

7. Conclusions

It can roughly be said that an equation (e.g., difference, differential, functional, integral) is Ulam stable if every function satisfying it approximately (in a given sense) must be (in some way) close to an accurate solution of the equation. Since the notions of 'approximate solution' and 'closeness of two functions' can be understood in various ways (see, e.g., [67]), such stability can also be considered in 2-normed spaces.

In this paper, we have presented and discussed the results on Ulam stability in 2-normed spaces provided in articles [40–58]. In this way, we complement the paper [23], where the results from [24–39] have been surveyed. We have shown how to supplement or improve several of these results. We also have pointed to various traps and mistakes that we have noticed in some of these papers.

Finally, let us mention that a natural generalization of the 2-normed space is the n-normed space. In our future work, we will prepare a similar survey (on Ulam stability in n-normed spaces) of the outcomes contained in [68–82] and in any papers on this subject still to be published.

Author Contributions: Conceptualization, J.B. and E.-s.E.-h.; methodology, J.B. and E.-s.E.-h.; software, J.B. and E.-s.E.-h.; validation, J.B. and E.-s.E.-h.; formal analysis, J.B.; investigation, J.B.; data curation, J.B. and E.-s.E.-h.; writing—original draft preparation, J.B. and E.-s.E.-h.; writing—review and editing, J.B. and E.-s.E.-h.; visualization, J.B.; supervision, J.B.; project administration, J.B. and E.-s.E.-h.; funding acquisition, E.-s.E.-h. All authors have read and agreed to the published version of the manuscript.

Funding: This work was funded by the Deanship of Scientific Research at Jouf University under grant No (DSR-2021-03-03158).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Hayes, W.; Jackson, K.R. A survey of shadowing methods for numerical solutions of ordinary differential equations. *Appl. Numer. Math.* **2005**, *53*, 299–321. [CrossRef]
2. Hyers, D.H. On the stability of the linear functional equation. *Proc. Nat. Acad. Sci. USA* **1941**, *27*, 222–224. [CrossRef] [PubMed]
3. Hyers, D.H.; Isac, G.; Rassias, T. *Stability of Functional Equations in Several Variables*; Springer: Birkhauser, Basel, 1998.
4. Jung, S.-M. *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*; Springer: New York, NY, USA, 2011.
5. Brzdęk, J.; Popa, D.; Raşa, I.; Xu, B. *Ulam Stability of Operators, Mathematical Analysis and its Applications*, 1st ed.; Academic Press: Oxford, UK, 2018.
6. Aoki, T. On the stability of the linear transformation in Banach spaces. *J. Math. Soc. Jpn.* **1950**, *2*, 64–66. [CrossRef]
7. Rassias, T.M. On the stability of the linear mapping in Banach spaces. *Proc. Am. Math. Soc.* **1978**, *72*, 297–300. [CrossRef]
8. Rassias, T.M. On a modified Hyers-Ulam sequence. *J. Math. Anal. Appl.* **1991**, *158*, 106–113. [CrossRef]
9. Gajda, Z. On stability of additive mappings. *Int. J. Math. Math. Sci.* **1991**, *14*, 431–434. [CrossRef]
10. Brzdęk, J.; Fechner, W.; Moslehian, M.S.; Sikorska, J. Recent developments of the conditional stability of the homomorphism equation. *Banach J. Math. Anal.* **2015**, *9*, 278–326. [CrossRef]
11. Brzdęk, J. A note on stability of additive mappings. In *Stability of Mappings of Hyers-Ulam Type*; Rassias, T.M., Tabor, J., Eds.; Hadronic Press: Palm Harbor, FL, USA, 1994; pp. 19–22.
12. Brzdęk, J. Hyperstability of the Cauchy equation on restricted domains. *Acta Math. Hungar.* **2013**, *141*, 58–67. [CrossRef]
13. Brzdęk, J.; Fošner, A. Remarks on the stability of Lie homomorphisms. *J. Math. Anal. Appl.* **2013**, *400*, 585–596. [CrossRef]
14. Badora, R.; Brzdęk, J.; Ciepliński, K. Applications of Banach limit in Ulam stability. *Symmetry* **2021**, *13*, 841. [CrossRef]
15. Badora, R.; Brzdęk, J. Banach limit in the stability problem of a linear functional equation. *Results Math.* **2021**, *76*, 51. [CrossRef]
16. Rassias, J.M. On approximation of approximately linear mappings by linear mappings. *J. Funct. Anal.* **1982**, *46*, 126–130. [CrossRef]
17. Rassias, J.M. On a new approximation of approximately linear mappings by linear mappings. *Discuss. Math.* **1985**, *7*, 193–196.
18. Brzdęk, J. A hyperstability result for the Cauchy equation. *Bull. Austral. Math. Soc.* **2014**, *89*, 33–40. [CrossRef]
19. Xu, B.; Brzdęk, J.; Zhang, W. Fixed point results and the Hyers-Ulam stability of linear equations of higher orders. *Pac. J. Math.* **2015**, *273*, 483–498. [CrossRef]
20. Gähler, S. 2-metrisch Räume und ihre topologische Struktur. *Math. Nachr.* **1963**, *26*, 115–148. [CrossRef]
21. Gähler, S. Lineare 2-normierte Räumen. *Math. Nachr.* **1964**, *28*, 1–43. [CrossRef]
22. Misiak, A. n-inner product spaces. *Math. Nachr.* **1989**, *140*, 299–319. [CrossRef]
23. Bahyrycz, A; Brzdęk, J.; El-Hady, E.-S.; Leśniak, Z. On Ulam stability of functional equations in 2-normed spaces—A survey. *Symmetry* **2021**, *13*, 2200. [CrossRef]
24. Gao, J. On the stability of the linear mapping in $2-$normed spaces. *Nonlinear Funct. Anal. Appl.* **2009**, *14*, 801–807.
25. Chung, S.C.; Park, W.G. Hyers-Ulam stability of functional equations in 2-Banach spaces. *Int. J. Math. Anal. (Ruse)* **2012**, *6*, 951–961.
26. Cho, Y.J.; Park, C.; Eshaghi Gordji, M. Approximate additive and quadratic mappings in 2-Banach spaces and related topics. *Int. J. Nonlin. Anal. Appl.* **2012**, *3*, 75–81.
27. Almahalebi, M.; Kabbaj, S. A fixed point approach to stability of the quartic equation in 2-Banach spaces. *J. Math. Comput. Sci.* **2013**, *3*, 972–984.
28. Arunkumar, M.; Bodaghi, A.; Rassias, J.M.; Sathya, E. The general solution and approximations of a decic type functional equation in various normed spaces. *J. Chungcheong Math. Soc.* **2016**, *29*, 287–328. [CrossRef]
29. Arunkumar, M. Stability of $n-$dimensional additive functional equation in generalized 2-normed space. *Demonstr. Math.* **2016**, *49*, 319–330. [CrossRef]
30. Rassias, J.M.; Arunkumar, M.; Sathya, E.; Kumar, N.M. Solution and stability of a acq functional equation in generalized 2-normed spaces. *Intern. J. Fuzzy Math. Arch.* **2015**, *7*, 213–224.
31. Almahalebi, M.; Chahbi, A.; Kabbaj, S. A fixed point approach to the stability of a bi-cubic functional equations in 2-Banach spaces. *Palest. J. Math.* **2016**, *5*, 220–227.
32. Ciepliński, K. Approximate multi-additive mappings in 2-Banach spaces. *Bull. Iranian Math. Soc.* **2015**, *41*, 785–792.
33. Ciepliński, K.; Xu, T.Z. Approximate multi-Jensen and multi-quadratic mappings in 2-Banach spaces. *Carpath. J. Math.* **2013**, *29*, 159–166. [CrossRef]
34. Park, W.G. Approximate additive mappings in 2-Banach spaces and related topics. *J. Math. Anal. Appl.* **2011**, *376*, 193–202. [CrossRef]
35. Patel, B.M.; Patel, A.B. Stability of quadratic functional equations in 2-Banach space. *Gen. Math. Notes* **2013**, *15*, 1–13.
36. Kim, C.I.; Jung, K.H. Stability of a cubic functional equation in 2-normed spaces. *J. Appl. Math. Infor.* **2014**, *32*, 817–825. [CrossRef]
37. Eivani, S.; Ostadbashi, S. Approximate mixed additive and quadratic functional in 2-Banach spaces. *Casp. J. Math. Sci. (CJMS)* **2015**, *4*, 167–173.
38. Murthy, S.; Govindhan, V.; Velan, M.S.S. Solution and stability of two types of n-dimensional quartic functional equation in generalized 2-normed spaces. *Int. J. Pure Appl. Math.* **2016**, *111*, 249–272. [CrossRef]
39. Arunkumar, M.; Murthy, S.; Namachivayam, T.; Ganapathy, G. Stability of $n-$dimensional quartic functional equation in generalized $2-$normed spaces using two different methods. *Malaya J. Math.* **2015**, *1*, 242–251.

40. Eskandani, G.Z.; Gavruta, P. Hyers-Ulam-Rassias stability of Pexiderized Cauchy functional equation in 2-Banach spaces. *J. Nonlin. Sci. Appl.* **2012**, *5*, 459–465. [CrossRef]
41. Brzdęk, J.; Ciepliński, K. On a fixed point theorem in 2-Banach spaces and some of its applications. *Acta Math. Sci.* **2018**, *38*, 377–390. [CrossRef]
42. Brzdęk, J.; El-hady, E.-S. On approximately additive mappings in 2-Banach spaces. *Bull. Aust. Math. Soc.* **2020**, *101*, 299–310. [CrossRef]
43. Aiemsomboon, L.; Sintunavarat, W. On new approximations for generalized Cauchy functional equations using Brzdęk and Ciepliński's fixed point theorems in 2-Banach spaces. *Acta Math. Sci.* **2020**, *40*, 824–834. [CrossRef]
44. EL-Fassi, I.-I. Approximate solution of radical quartic functional equation related to additive mapping in 2-Banach spaces. *J. Math. Anal. Appl.* **2017**, *455*, 2001–2013. [CrossRef]
45. Khodaei, H.; Eshaghi Gordji, M.; Kim, S.S.; Cho, Y.J. Approximation of radical functional equations related to quadratic and quartic mappings. *J. Math. Anal. Appl.* **2012**, *395*, 284–297. [CrossRef]
46. AL-Ali, S.A.; Almahalebi, M.; Elkettani, Y. Stability of a general p-radical functional equation related to additive mappings in 2-Banach spaces. *Proyecciones (Antofagasta)* **2021**, *40*, 49–71. [CrossRef]
47. Sayar, K.Y.N.; Bergam, A. Approximate solutions of a quadratic functional equation in 2-Banach spaces using fixed point theorem. *J. Fixed Point Theory Appl.* **2020**, *22*, 3. [CrossRef]
48. Park, W.G.; Bae, J.H. Hyers-Ulam stability of quadratic forms in 2-normed spaces. *Demonstr. Math.* **2019**, *52*, 496–502. [CrossRef]
49. El-hady, E.-s. On stability of the functional equation of p-Wright affine functions in 2-Banach spaces. In *Ulam Type Stability*; Brzdęk, J., Popa, D., Rassias, T.M., Eds.; Springer: Cham, Switzerland, 2019; pp. 131–141.
50. Ghali, R.E.; Kabbaj, S. 2-Banach stability results for the radical cubic functional equation related to quadratic mapping. *J. Lin. Topol. Algebra (JLTA)* **2020**, *9*, 35–51.
51. Sayar, K.Y.N.; Bergam, A. Some hyperstability results for a Cauchy-Jensen type functional equation in 2-Banach spaces. *Proyecciones (Antofagasta)* **2020**, *39*, 73–89. [CrossRef]
52. Park, S.; Kim, C. The generalized Hyers-Ulam stability of additive functional inequalities in non-Archimedean 2-normed space. *Korean J. Math.* **2014**, *22*, 339–348.
53. Wang, Z.; Park, C.; Shin, D.Y. Additive ρ-functional inequalities in non-Archimedean 2-normed spaces. *AIMS Math.* **2021**, *6*, 1905–1919. [CrossRef]
54. Park, C.; Eshaghi Gordji, M.; Ghaemi, M.B.; Majani, H. Fixed points and approximately octic mappings in non-Archimedean 2-normed spaces. *J. Inequal. Appl.* **2012**, *2012*, 289. [CrossRef]
55. EL-Fassi, I.-I. New stability results for the radical sextic functional equation related to quadratic mappings in $(2,\beta)$-Banach spaces. *J. Fixed Point Theory Appl.* **2018**, *20*, 138. [CrossRef]
56. El-hady, E.-S. On stability of the functional equation of p-Wright affine functions in $(2,\alpha)$-Banach spaces. *J. Egypt. Math. Soc.* **2019**, *27*, 21. [CrossRef]
57. Alghamdi, M.A. Stability of Pexiderized quadratic functional equation in random 2-normed spaces. *J. Funct. Spaces* **2015**, *2015*, 828967. [CrossRef]
58. Alotaibi, A.; Mohiuddine, S.A. On the stability of a cubic functional equation in random 2-normed spaces. *Adv. Differ. Equ.* **2012**, *2012*, 39. [CrossRef]
59. Aczél, J. *Lectures on Functional Equations and Their Applications*; Academic Press: New York, NY, USA, 1966.
60. Aczél, J.; Dhombres, J. *Functional Equations in Several Variables. Volume 31 of Encyclopedia of Mathematics and Its Applications*; Cambridge University Press: Cambridge, UK, 1989.
61. Kuczma, M. *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, 2nd ed.; Birkhäuser: Basel, Switzerland, 2009.
62. Kuczma, M.; Choczewski, B.; Ger, R. *Iterative Functional Equations*; Cambridge University Press: Cambridge, UK, 1990.
63. Freese, R.W.; Cho, Y.J. *Geometry of Linear 2-Normed Spaces*; Nova Science Publishers, Inc.: Hauppauge, NY, USA, 2001.
64. Brzdęk, J. On Ulam's type stability of the Cauchy additive equation. *Sci. World J.* **2014**, *2014*, 540164. [CrossRef]
65. Brzdęk, J., Schwaiger, J. Remarks on solutions to a generalization of the radical functional equations. *Aeq. Math.* **2018**, *92*, 975–991. [CrossRef]
66. Brzdęk, J. Remarks on solutions to the functional equations of the radical type. *Adv. Theory Nonlinear Anal. Appl.* **2017**, *1*, 125–135. [CrossRef]
67. Brzdęk, J.; Popa, D.; Raşa, I. Hyers–Ulam stability with respect to gauges. *J. Math. Anal. Appl.* **2017**, *453*, 620–628. [CrossRef]
68. Yang, X.; Chang, L.; Liu, G.; Shen, G. Stability of functional equations in (n,β)-normed spaces. *J. Inequal. Appl.* **2015**, *2015*, 112. [CrossRef]
69. Chu, H.-Y.; Kim, A.; Park, J. On the Hyers-Ulam stabilities of functional equations on n-Banach spaces. *Math. Nachr.* **2016**, *289*, 1177–1188. [CrossRef]
70. Xu, T.Z. Approximate multi-Jensen, multi-Euler-Lagrange additive and quadratic mappings in n-Banach spaces. *Abstr. Appl. Anal.* **2013**, *2013*, 648709. [CrossRef]
71. Xu, T.Z.; Rassias, J.M. On the Hyers-Ulam stability of a general mixed additive and cubic functional equation in n-Banach spaces. *Abstr. Appl. Anal.* **2012**, *2012*, 926390. [CrossRef]

72. Brzdęk, J.; El-hady, E.-S. On hyperstability of the Cauchy functional equation in n-Banach spaces. *Mathematics* **2020**, *8*, 1886. [CrossRef]
73. Chu, H.-Y.; Lee, K.; Park, C.-G. On the Aleksandrov problem in linear n-normed spaces. *Nonlinear Anal.* **2004**, *59*, 1001–1011.
74. Dutta, H. On some n-normed linear space valued difference sequences. *J. Franklin Inst.* **2011**, *348*, 2876–2883. [CrossRef]
75. Eghbali, N.; Rassias, J.M.; Taheri, M. On the stability of a k-cubic functional equation in intuitionistic fuzzy n-normed spaces. *Res. Math.* **2016**, *70*, 233–248. [CrossRef]
76. Choy, J.; Chu, H.Y.; Kim, A. A Remark for the Hyers-Ulam stabilities on n-Banach spaces. *Axioms* **2021**, *10*, 2. [CrossRef]
77. EL-Fassi, I.-I.; Elqorachi, E.; Khodaei, H. A fixed point approach to stability of k-th radical functional equation in non-archimedean (n, β)-Banach spaces. *Bull. Iran. Math. Soc.* **2021**, *47*, 487–504. [CrossRef]
78. El-hady, E.-S.; El-Fassi, I.-I. Stability of the equation of q-Wright affine functions in non-Archimedean (n, β)-Banach spaces. *Symmetry* **2022**, *14*, 633. [CrossRef]
79. Ciepliński, K. On Ulam stability of a functional equation. *Results Math.* **2020**, *75*, 151. [CrossRef]
80. Brzdęk, J.; Ciepliński, K. A fixed point theorem in n-Banach spaces and Ulam stability. *J. Math. Anal. Appl.* **2019**, *470*, 632–646. [CrossRef]
81. Chen, X.Y.; Song, M.M. Characterizations on isometries in linear n-normed spaces. *Nonlinear Anal.* **2010**, *72*, 1895–1901. [CrossRef]
82. Gunawan, H.; Neswan, O.; Sukaesih, E. Fixed point theorems in bounded sets in an n-normed space. *J. Math. Anal.* **2015**, *6*, 51–58.