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Modified Fractional Difference Operators Defined Using Mittag-Leffler Kernels

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Abstract: The discrete fractional operators of Riemann–Liouville and Liouville–Caputo are omnipresent due to the singularity of the kernels. Therefore, convexity analysis of discrete fractional differences of these types plays a vital role in maintaining the safe operation of kernels and symmetry of discrete delta and nabla distribution. In their discrete version, the generalized or modified forms of various operators of fractional calculus are becoming increasingly important from the viewpoints of both pure and applied mathematical sciences. In this paper, we present the discrete version of the recently modified fractional calculus operator with the Mittag-Leffler-type kernel. Here, in this article, the expressions of both the discrete nabla derivative and its counterpart nabla integral are obtained. Some applications and illustrative examples are given to support the theoretical results.

Keywords: discrete fractional calculus; discrete Atangana–Baleanu fractional differences; discrete Liouville–Caputo operator; discrete Mittag-Leffler kernels

MSC: 26A48; 26A51; 33B10; 39A12; 39B62

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1. Introduction

Fractional calculus is a 327-year-old interdisciplinary field [1,2] in which the integral and derivatives of a fractional real or complex order have been investigated (see [3–6]) as well as their applications (see [7–10]). It is well-known that there are many definitions of fractional calculus operators, mainly due to the fact that there is no single extension of meaning in this area. We can also mention that the Liouville–Caputo fractional derivative has appeared in several investigations in the history of fractional calculus (see, for example, the works of Abel [11], Liouville [12], Caputo [13], and Dzherbashian and Nersesian [14]). Based on the works of Boltzmann [15] and continuing with the contemporary works, we conclude that finding some generalized (see, for example, [16–18]) or modified versions [19] of some existing operators [20] is an interesting issue in the field of fractional calculus. In passing, we recall Wright's general function $\mathfrak{E}_{\alpha,\beta}(\phi; z)$, which occurred in his study of the asymptotic behavior of a certain Taylor-Maclaurin series, which obviously provides a remarkably deep generalization of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ of two parameters α and β (see [21], p. 424):

$$\mathfrak{E}_{\alpha,\beta}(\phi;z):=\sum_{\ell=0}^{\infty}\frac{\phi(\ell)}{\Gamma(\alpha\,\ell+\beta)}\,z^{\ell}\qquad(\alpha,\beta\in\mathbb{C};\ \mathrm{Re}(\alpha)>0),\tag{1}$$

where $\phi(\tau)$ is a suitably restricted function of τ. Recent works [22,23] (see also [24,25]) provide other historical and important backgrounds in detail regarding an interesting unification of Equation (1) and several multiparameter extensions of many functions happening in analytic number theory. It is defined by

$$\mathcal{E}_{\alpha,\beta}(\varphi;z,s,a):=\sum_{\ell=0}^{\infty}\frac{\varphi(\ell)}{(\ell+a)^{s}\,\Gamma(\alpha\,\ell+\beta)}\,z^{\ell},\tag{2}$$

for $\beta,\alpha\in\mathbb{C}$ with $\mathrm{Re}(\alpha)>0$, where for a suitably restricted function φ, the parameters α, β, s, and a are appropriately constrained. In its very specialized case where

$$\varphi(\ell)=\Gamma(\alpha\,\ell+\beta)\qquad(\ell\in\mathbb{N}_0),$$

the Srivastava function $\mathcal{E}_{\alpha,\beta}(\varphi;z,s,a)$ reduces immediately to the Hurwitz–Lerch zeta function $\Phi(z,s,a)$. More importantly, the Srivastava function $\mathcal{E}_{\alpha,\beta}(\varphi;z,s,a)$, defined by Equation (2), has already been successfully used as the kernel of some general families of Riemann–Liouville-type fractional integrals and fractional derivatives (for details, see [22,23]). Appropriately defined operators of discrete fractional calculus, which are based upon the general operators of fractional calculus with the Srivastava function $\mathcal{E}_{\alpha,\beta}(\varphi;z,s,a)$ in the kernel, can possibly lead to some interesting future investigations.

The discrete fractional calculus [26–29] started to be an interesting part of fractional calculus [30–33] due to its multiple important applications in solving the complex dynamics of several complicated systems [34–36] arising from several fields of science and engineering [37–39]. Aside from that, the possibility of applying the discrete fractional calculus to improve some methods and techniques from artificial intelligence (AI) makes this branch of fractional calculus of great interest to readers. In addition, finding the discrete counterpart of a modified or generalised operator fractional operator is always an interesting topic, mainly because the discrete version of the continuous non-local operators possesses qualitatively different properties. In particular, the fractional continuous and discrete operators involving the Mittag-Leffler kernels (see [17] for the continuous version and [30,40–42] for the discrete version) present their own interest, due to the fact that this special function is considered to be the queen of fractional calculus.

Motivated by the aforementioned works, in the current study, we establish the discrete version of the recently modified fractional calculus operator defined using the Mittag-Leffler kernel in [19]. Furthermore, we obtain its series representation formula and some related examples.

The organization of this manuscript is as follows. Section 2 deals with the basic definitions and properties of the discrete fractional calculus used in this manuscript. Section 2.1 is dedicated to discrete Mittag-Leffler functions, and Section 2.2 is for the discrete Laplace transform, including some of its properties. Section 3 deals with recalling the discrete Atangana–Baleanu derivative of the Liouville–Caputo-type fractional operators and the new finding of discrete modifications of the discrete Atangana–Baleanu derivative of the Liouville–Caputo-type fractional operators. Section 4 contains two illustrative examples, and the last section is devoted to our conclusions.

2. Preliminary Tools

This section recalls some basic concepts of discrete fractional calculus, such as discrete Mittag-Leffler functions and discrete Laplace transformations on the time set $\mathbb{N}_a := \{a, a+1, a+2, ...\}$ for $a \in \mathbb{R}$.

2.1. Riemann–Liouville Fractional Sums and Mittag-Leffler Functions

Definition 1 (see [30,40,43])**.** *For $\kappa \in \mathbb{N}_1$, the κ-rising factorial function can be expressed by*

$$z^{\overline{\kappa}} = \prod_{\kappa=0}^{\kappa-1}(z+\kappa), \quad z^{\overline{0}} = 1.$$

In general, it can be expressed as follows:

$$z^{\overline{\alpha}} = \frac{\Gamma(z+\alpha)}{\Gamma(z)}, \tag{3}$$

for $z, \alpha \in \mathbb{R}$ such that neither z nor $z+\alpha$ is a pole of the Gamma function. One of the major properties of this function is given by

$$\nabla\left(z^{\overline{\alpha}}\right) = \alpha\, z^{\overline{\alpha-1}}.$$

Definition 2 (see [30,40,43])**.** *For $f : \mathbb{N}_a \longrightarrow \mathbb{R}$ and $\alpha > 0$, the nabla left-sided Riemann–Liouville fractional sum of the order α can be expressed as follows:*

$$\left({}_a\nabla^{-\alpha} f\right)(z) = \frac{1}{\Gamma(\alpha)} \sum_{r=a+1}^{z} (z+1-r)^{\overline{\alpha-1}} f(r) \quad (\forall\, z \in \mathbb{N}_{a+1}), \tag{4}$$

In addition, for $f : {}_b\mathbb{N} := \{b, b-1, b-2, ...\} \longrightarrow \mathbb{R}$, the right-sided one can be expressed as follows:

$$\left(\nabla_b^{-\alpha} f\right)(z) = \frac{1}{\Gamma(\alpha)} \sum_{r=z}^{b-1} (r-\rho(z))^{\overline{\alpha-1}} f(r) \quad (\forall\, z \in {}_{b-1}\mathbb{N}). \tag{5}$$

Lemma 1 (see [30,40,43])**.** *Suppose that $a, b, \beta \in \mathbb{R}$, and $\alpha \in \mathbb{R}^+$. Then, one can have*

$$\begin{aligned} {}_a\nabla^{-\alpha} \frac{(z-a)^{\overline{\beta}}}{\Gamma(\beta+1)} &= \frac{1}{\Gamma(\beta+\alpha+1)}(z-a)^{\overline{\beta+\alpha}}, \\ \nabla_b^{-\alpha} \frac{(b-z)^{\overline{\beta}}}{\Gamma(\beta+1)} &= \frac{1}{\Gamma(\beta+\alpha+1)}(b-z)^{\overline{\beta+\alpha}}. \end{aligned}$$

Definition 3 (see [43])**.** *For $\alpha, \beta, \gamma, z \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, the discrete Mittag-Leffler function of a generalized form can be expressed as follows:*

$$\mathrm{E}^{\gamma}_{\overline{\alpha,\beta}}(\lambda, z) := \sum_{\kappa=0}^{\infty} \lambda^{\kappa} \frac{z^{\overline{\kappa\alpha+\beta-1}}(\gamma)_\kappa}{\Gamma(\kappa\alpha+\beta)} \quad (\forall\, \lambda \in \mathbb{R} \text{ such that } |\lambda| < 1), \tag{6}$$

where $(\gamma)_\kappa = \frac{\Gamma(\gamma+\kappa)}{\Gamma(\kappa)}$. Particularly, the discrete Mittag-Leffler function of two parameters can be deduced when $\gamma = 1$ as follows:

$$\mathrm{E}_{\overline{\alpha,\beta}}(\lambda, z) = \mathrm{E}^{1}_{\overline{\alpha,\beta}}(\lambda, z) := \sum_{\kappa=0}^{\infty} \lambda^{\kappa} \frac{z^{\overline{\kappa\alpha+\beta-1}}}{\Gamma(\kappa\alpha+\beta)}, \tag{7}$$

for $\lambda \in \mathbb{R}$, satisfying $|\lambda| < 1$.

Furthermore, the discrete Mittag-Leffler function of one parameter can be deduced when $\beta = \gamma = 1$ *as follows:*

$$\mathrm{E}_{\overline{\alpha}}(\lambda, \mathrm{z}) = \mathrm{E}^{1}_{\overline{\alpha,\beta}}(\lambda, \mathrm{z}) := \sum_{\kappa=0}^{\infty} \lambda^{\kappa} \frac{\mathrm{z}^{\overline{\kappa\alpha}}}{\Gamma(\kappa\alpha + 1)}, \tag{8}$$

where λ *is as explained above.*

Lemma 2 (see [43])**.** *Assume that* $\alpha > 0, \beta > -1, \gamma, \mathrm{z} \in \mathbb{C}$*. Then, it is asserted that*

$${}^{RL}_{a}\nabla^{-\nu}\mathrm{E}^{\gamma}_{\overline{\alpha,\beta}}(\lambda, \mathrm{z}) = \mathrm{E}^{\gamma}_{\overline{\alpha,\beta+\nu}}(\lambda, \mathrm{z}),$$

for $\lambda \in \mathbb{R}$*, satisfying* $|\lambda| < 1$*.*

2.2. Discrete Laplace Transformation

Definition 4 (see [30])**.** *Let* f *and* g *be defined on* $\mathbb{N}_a$*. Then, the discrete Laplace transformation can be expressed as follows:*

$$(\mathfrak{L}_a f(\mathrm{z}))(\mathrm{S}) = \sum_{t=a+1}^{\infty} \frac{f(\mathrm{z})}{(1-\mathrm{S})^{a+1-\mathrm{z}}}. \tag{9}$$

Furthermore, the discrete convolution of f *and* g *can be expressed as follows:*

$$(f * g)(\mathrm{z}) = \sum_{\kappa=a+1}^{\mathrm{z}} f(\kappa)\, g\big(\mathrm{z} - \rho(\kappa) + a\big). \tag{10}$$

Lemma 3 (see [30])**.** *For any* $\alpha \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$*, it is asserted that*

$$\Big(\mathfrak{L}_a (\mathrm{z}-a)^{\overline{\alpha-1}}\Big)(\mathrm{S}) = \frac{\Gamma(\alpha)}{\mathrm{S}^{\alpha}},$$

provided that $|1 - \mathrm{S}| < 1$*.*

Lemma 4 (see [30])**.** *Let* f *and* g *be defined on* $\mathbb{N}_a$*. Then, it is asserted that*

$$(\mathfrak{L}_a (f * g)(\mathrm{z}))(\mathrm{S}) = (\mathfrak{L}_a f)(\mathrm{S})(\mathfrak{L}_a g)(\mathrm{S}).$$

Lemma 5 (see [30,40,43])**.** *Let* f *be defined on* $\mathbb{N}_a$*. Then, it is asserted that*

$$(\mathfrak{L}_a (\nabla f)(\mathrm{z}))(\mathrm{S}) = \mathrm{S}(\mathfrak{L}_a f)(\mathrm{S}) - f(a).$$

More generally, the following result holds true:

$$\Big(\mathfrak{L}_a (\nabla^n f)(\mathrm{z})\Big)(\mathrm{S}) = \mathrm{S}^n (\mathfrak{L}_a f)(\mathrm{S}) - \sum_{\iota=0}^{n-1} \mathrm{S}^{n-1-\iota} \nabla^{\iota} f(a+1).$$

Lemma 6 (see [30])**.** *Let* α *be any real numbers. Then, it is asserted that*

$$\Big(\mathfrak{L}_a \, {}^{RL}_{a}\nabla^{-\alpha} f(\mathrm{z})\Big)(\mathrm{S}) = \frac{1}{\mathrm{S}^{\alpha}} (\mathfrak{L}_a f)(\mathrm{S}).$$

Lemma 7 (see [43])**.** *For* $\alpha, \beta, \lambda, \mathrm{S}$ *in* $\mathbb{C}$ *with* $\mathrm{Re}(\beta) > 0$*, and if* $|\lambda \mathrm{S}^{-\alpha}| < 1$ *with* $\mathrm{Re}(\mathrm{S}) > 0$*, it is asserted that*

$$\Big(\mathfrak{L}_a \mathrm{E}_{\overline{\alpha,\beta}}(\lambda, \mathrm{z}-a)\Big)(\mathrm{S}) = \mathrm{S}^{-\beta}\big[1 - \lambda \mathrm{S}^{-\alpha}\big]^{-1} \tag{11}$$

and, in particular, that

$$\left(\mathfrak{L}_a \mathrm{E}_{\overline{\alpha},\overline{\alpha}}(\lambda, \mathrm{z}-a)\right)(\mathrm{S}) = \frac{1}{\mathrm{S}^{\alpha}-\lambda}. \tag{12}$$

3. Discrete Atangana–Baleanu and the Modified Atangana–Baleanu of the Liouville–Caputo Fractional Differences

We start this section by briefly recalling the Atangana–Baleanu derivative of the Liouville–Caputo-type fractional difference operators, and then we introduce their modified versions, which are discrete fractional analogues of the continuous case in [19]. For the more salient details on these subjects, see [30,40,43].

Definition 5 (see [30])**.** *For* $\lambda = -\frac{\alpha}{1-\alpha}$ *and* $0 < \alpha < 1/2$*, the left discrete generalized Atangana–Baleanu of the Liouville–Caputo-type fractional difference is given by*

$$\left({}^{ABC}_{\ a}\nabla^{\alpha} f\right)(\mathrm{z}) = \frac{\mathsf{A}(\alpha)}{1-\alpha} \sum_{\mathrm{s}=a+1}^{\mathrm{z}} \mathrm{E}_{\overline{\alpha}}(\lambda, \mathrm{z}-\mathrm{s}+1)(\nabla f)(\mathrm{s}) \quad (\forall\, \mathrm{z} \in \mathbb{N}_a) \tag{13}$$

and the right one is given by

$$\left({}^{ABC}\nabla^{\alpha}_{b} f\right)(\mathrm{z}) = \frac{-\mathsf{A}(\alpha)}{1-\alpha} \sum_{\mathrm{s}=\mathrm{z}}^{b-1} \mathrm{E}_{\overline{\alpha}}(\lambda, \mathrm{s}-\mathrm{z}+1)(\Delta f)(\mathrm{s}) \quad (\forall\, \mathrm{z} \in {}_{b}\mathbb{N}). \tag{14}$$

The corresponding discrete Atangana–Baleanu fractional sum is given by

$$\left({}^{AB}_{\ a}\nabla^{-\alpha} f\right)(\mathrm{z}) = \frac{1-\alpha}{\mathsf{A}(\alpha)} f(\mathrm{z}) + \frac{\alpha}{\mathsf{A}(\alpha)} \left({}_{a}\nabla^{-\alpha} f\right)(\mathrm{z}) \quad (\forall\, \mathrm{z} \in \mathbb{N}_a), \tag{15}$$

where $\mathsf{A}(\alpha) > 0$ *such that* $\mathsf{A}(0) = \mathsf{A}(1) = 1$.

It is worth mentioning that the above definition is the discrete analogue of the Atangana–Baleanu fractional operators in [20]. Now, we will proceed to obtain the modified version of Definition 5.

Considering the definition in Equation (13), we see that

$$\begin{aligned}
&\left({}^{ABC}_{\ a}\nabla^{\alpha} f\right)(\mathrm{z}) \\
&= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left\{ \sum_{\mathrm{s}=a+1}^{\mathrm{z}} \mathrm{E}_{\overline{\alpha}}(\lambda, \mathrm{z}-\mathrm{s}+1) f(\mathrm{s}) - \sum_{\mathrm{s}=a+1}^{\mathrm{z}} \mathrm{E}_{\overline{\alpha}}(\lambda, \mathrm{z}-\mathrm{s}+1) f(\mathrm{s}-1) \right\} \\
&= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left\{ \sum_{\mathrm{s}=a+1}^{\mathrm{z}} \mathrm{E}_{\overline{\alpha}}(\lambda, \mathrm{z}-\mathrm{s}+1) f(\mathrm{s}) - \sum_{\mathrm{s}=a}^{\mathrm{z}-1} \mathrm{E}_{\overline{\alpha}}(\lambda, \mathrm{z}-\mathrm{s}) f(\mathrm{s}) \right\} \\
&= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left\{ f(\mathrm{z}) - \mathrm{E}_{\overline{\alpha}}(\lambda, \mathrm{z}-a) f(a) + \sum_{\mathrm{s}=a+1}^{\mathrm{z}} \left\{ \mathrm{E}_{\overline{\alpha}}(\lambda, \mathrm{z}-\mathrm{s}+1) - \mathrm{E}_{\overline{\alpha}}(\lambda, \mathrm{z}-\mathrm{s}) \right\} f(\mathrm{s}) \right\} \\
&= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left\{ f(\mathrm{z}) - \mathrm{E}_{\overline{\alpha}}(\lambda, \mathrm{z}-a) f(a) + \sum_{\mathrm{s}=a+1}^{\mathrm{z}} \nabla \mathrm{E}_{\overline{\alpha}}(\lambda, \mathrm{z}-\mathrm{s}+1) f(\mathrm{s}) \right\} \\
&= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left\{ f(\mathrm{z}) - \mathrm{E}_{\overline{\alpha}}(\lambda, \mathrm{z}-a) f(a) + \lambda \sum_{\mathrm{s}=a+1}^{\mathrm{z}} \mathrm{E}_{\overline{\alpha},\overline{\alpha}}(\lambda, \mathrm{z}-\mathrm{s}+1) f(\mathrm{s}) \right\},
\end{aligned}$$

where we have used (see [31])

$$\mathrm{E}_{\overline{\alpha}}(\lambda, 1) = 1 \quad \text{and} \quad \nabla\, \mathrm{E}_{\overline{\alpha}}(\lambda, \mathrm{z}) = \lambda\, \mathrm{E}_{\overline{\alpha},\overline{\alpha}}(\lambda, \mathrm{z}).$$

Therefore, we can define the left discrete modified Atangana–Baleanu of the Liouville–Caputo-type fractional difference for $\alpha \in \left(0, \frac{1}{2}\right)$ as follows:

Definition 6. *For $\lambda = -\frac{\alpha}{1-\alpha}$ and $0 < \alpha < 1/2$, the left discrete modified Atangana–Baleanu of the Liouville–Caputo-type fractional difference is given by*

$$\begin{aligned}&\left({}^{MABC}_{\ \ a}\nabla^{\alpha} f\right)(\mathrm{z}) \\ &= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left[f(\mathrm{z}) - \mathrm{E}_{\overline{\alpha}}(\lambda, \mathrm{z}-a)f(a) + \lambda \sum_{\mathrm{s}=a+1}^{\mathrm{z}} \mathrm{E}_{\overline{\alpha,\alpha}}(\lambda, \mathrm{z}-\mathrm{s}+1)f(\mathrm{s})\right]\end{aligned} \tag{16}$$

for each z in $\mathbb{N}_a$. Furthermore, by applying the action of the $\mathcal{Q}$ operator to Equation (16) (for further information on this action, see [32,37]), we can deduce the right one as follows:

$$\begin{aligned}&\left({}^{MABC}\nabla^{\alpha}_{b} f\right)(\mathrm{z}) \\ &= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left[f(\mathrm{z}) - \mathrm{E}_{\overline{\alpha}}(\lambda, b-\mathrm{z})f(b) + \lambda \sum_{\mathrm{s}=\mathrm{z}}^{b-1} \mathrm{E}_{\overline{\alpha,\alpha}}(\lambda, \mathrm{s}-\mathrm{z}+1)f(\mathrm{s})\right],\end{aligned} \tag{17}$$

for each z in ${}_{b}\mathbb{N}$.

The above can be generalized by the same technique used for $0 < \alpha < 1$, and we can deduce the following:

Definition 7. *For $\lambda_{\ell} = -\frac{\alpha-\ell}{\ell+1-\alpha}$ with $\ell < \alpha < \ell + \frac{1}{2}$ and $\ell \in \mathbb{N}_0$, the left discrete modified Atangana–Baleanu of the Liouville–Caputo-type fractional difference of a higher order can be expressed as follows:*

$$\begin{aligned}\left({}^{MABC}_{\ \ a}\nabla^{\alpha} f\right)(\mathrm{z}) &= \left({}^{ABC}_{\ \ a}\nabla^{\alpha-\ell}\, \nabla^{\ell} f\right)(\mathrm{z}) \\ &= \frac{\mathsf{A}(\alpha-\ell)}{\ell+1-\alpha}\Bigg[\left(\nabla^{\ell} f\right)(\mathrm{z}) - \mathrm{E}_{\overline{\alpha-\ell}}(\lambda_{\ell}, \mathrm{z}-a)\left(\nabla^{\ell} f\right)(a) \\ &\quad + \lambda_{\ell} \sum_{\mathrm{s}=a+1}^{\mathrm{z}} \mathrm{E}_{\overline{\alpha-\ell,\alpha-\ell}}(\lambda_{\ell}, \mathrm{z}-\mathrm{s}+1)\left(\nabla^{\ell} f\right)(\mathrm{s})\Bigg]\end{aligned} \tag{18}$$

for each z in $\mathbb{N}_a$. Additionally, the right one can be expressed as follows:

$$\begin{aligned}\left({}^{MABC}\nabla^{\alpha}_{b} f\right)(\mathrm{z}) &= \frac{\mathsf{A}(\alpha-\ell)}{\ell+1-\alpha}\Bigg[\left(\nabla^{\ell} f\right)(\mathrm{z}) - \mathrm{E}_{\overline{\alpha-\ell}}(\lambda_{\ell}, b-\mathrm{z})\left(\nabla^{\ell} f\right)(b) \\ &\quad + \lambda_{\ell} \sum_{\mathrm{s}=\mathrm{z}}^{b-1} \mathrm{E}_{\overline{\alpha-\ell,\alpha-\ell}}(\lambda_{\ell}, \mathrm{s}-\mathrm{z}+1)\left(\nabla^{\ell} f\right)(\mathrm{s})\Bigg],\end{aligned} \tag{19}$$

for each z in ${}_{b}\mathbb{N}$.

Throughout the remainder of this article, we consider the left discrete modified Atangana–Baleanu of the Liouville–Caputo-type fractional differences. Interested readers can use the $\mathcal{Q}$ operator action on the left one's results, and they will be able to obtain the corresponding results for the right one.

Theorem 1. *For $\alpha \in \left(0, \frac{1}{2}\right)$, each of the following results holds true*:

$$\left({}^{AB}_{\ a}\nabla^{-\alpha}\ {}^{MABC}_{\ \ a}\nabla^{\alpha} f\right)(\mathrm{z}) = f(\mathrm{z}) - f(a) \tag{20}$$

and

$$\left({}^{MABC}_{\quad a}\nabla^{\alpha}\ {}^{AB}_{\ a}\nabla^{-\alpha}f\right)(\mathrm{z}) = f(\mathrm{z}) - f(a)\mathrm{E}_{\overline{\alpha}}(\lambda, \mathrm{z}-a), \tag{21}$$

for z *in* $\mathbb{N}_{a+1}$.

Proof. Denote the following:

$$\left({}^{MABC}_{\quad a}\nabla^{\alpha}f\right)(\mathrm{z}) := u(\mathrm{z}), \quad \text{for all } \mathrm{z} \in \mathbb{N}_{a+1}.$$

Take the Laplace transform $\mathfrak{L}_a$ on both sides to obtain

$$\begin{aligned}\left(\mathfrak{L}_a\, {}^{MABC}_{\quad a}\nabla^{\alpha}f\right)(\mathrm{S}) &\underset{\text{Lemma 7}}{\overset{by}{=}} \frac{\mathsf{A}(\alpha)}{1-\alpha}\left[F(\mathrm{S}) - \frac{\mathrm{S}^{\alpha-1}(1-\mathrm{S})^{a}}{\mathrm{S}^{\alpha}-\lambda}f(a) + \frac{\lambda}{\mathrm{S}^{\alpha}-\lambda}F(\mathrm{S})\right] \\ &= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left[\frac{\mathrm{S}^{\alpha}}{\mathrm{S}^{\alpha}-\lambda}F(\mathrm{S}) - \frac{\mathrm{S}^{\alpha-1}(1-\mathrm{S})^{a}}{\mathrm{S}^{\alpha}-\lambda}f(a)\right] = U(\mathrm{S}),\end{aligned} \tag{22}$$

where $F(\mathrm{S}) := \left(\mathfrak{L}_a f\right)(\mathrm{S})$ and $U(\mathrm{S}) := \left(\mathfrak{L}_a u\right)(\mathrm{S})$. By solving for $F(\mathrm{S})$, we find that

$$\begin{aligned}F(\mathrm{S}) &= \frac{1-\alpha}{\mathsf{A}(\alpha)}U(\mathrm{S}) + \frac{\alpha}{\mathsf{A}(\alpha)}\frac{1}{\mathrm{S}^{\alpha}}U(\mathrm{S}) + \frac{(1-\mathrm{S})^{a}}{\mathrm{S}}f(a) \\ &= \left(\mathfrak{L}_a\left({}^{AB}_{\ a}\nabla^{-\alpha}f\right)(\mathrm{z})\right)(\mathrm{S}) + \frac{(1-\mathrm{S})^{a}}{\mathrm{S}}f(a),\end{aligned} \tag{23}$$

where we used the following (see [30], Equation (32)):

$$\left(\mathfrak{L}_a\left({}^{AB}_{\ a}\nabla^{-\alpha}f\right)(\mathrm{z})\right)(\mathrm{S}) = \frac{1-\alpha}{\mathsf{A}(\alpha)}\left(\mathfrak{L}_a f\right)(\mathrm{S}) + \frac{\alpha}{\mathsf{A}(\alpha)}\frac{1}{\mathrm{S}^{\alpha}}\left(\mathfrak{L}_a f\right)(\mathrm{S}). \tag{24}$$

By taking the inverse Laplace transform $\mathfrak{L}_a^{-1}$ on both sides of (23), we obtain

$$f(\mathrm{z}) = \left({}^{AB}_{\ a}\nabla^{-\alpha}u\right)(\mathrm{z}) + f(a),$$

which rearranges to the desired result of Equation (20).

To prove the second part, we set

$$\left({}^{AB}_{\ a}\nabla^{-\alpha}f\right)(\mathrm{z}) := g(\mathrm{z}), \quad \text{for all } \mathrm{z} \in \mathbb{N}_{a+1}.$$

Then, by taking the Laplace transform $\mathfrak{L}_a$ on $\left({}^{MABC}_{\quad a}\nabla^{\alpha}g\right)(\mathrm{z})$, we obtain

$$\begin{aligned}\left(\mathfrak{L}_a\left({}^{MABC}_{\quad a}\nabla^{\alpha}g\right)(\mathrm{z})\right)(\mathrm{S}) &\underset{(22)}{\overset{by}{=}} \frac{\mathsf{A}(\alpha)}{1-\alpha}\left[\frac{\mathrm{S}^{\alpha}}{\mathrm{S}^{\alpha}-\lambda}G(\mathrm{S}) - \frac{\mathrm{S}^{\alpha-1}(1-\mathrm{S})^{a}}{\mathrm{S}^{\alpha}-\lambda}g(a)\right] \\ &= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left[\frac{\mathrm{S}^{\alpha}}{\mathrm{S}^{\alpha}-\lambda}\left(\frac{1-\alpha}{\mathsf{A}(\alpha)}F(\mathrm{S}) + \frac{\alpha}{\mathsf{A}(\alpha)}\frac{1}{\mathrm{S}^{\alpha}}F(\mathrm{S})\right)\right. \\ &\qquad \left. - \frac{\mathrm{S}^{\alpha-1}(1-\mathrm{S})^{a}}{\mathrm{S}^{\alpha}-\lambda}\frac{1-\alpha}{\mathsf{A}(\alpha)}f(a)\right] \\ &= F(\mathrm{S}) - \frac{\mathrm{S}^{\alpha-1}(1-\mathrm{S})^{a}}{\mathrm{S}^{\alpha}-\lambda}f(a),\end{aligned} \tag{25}$$

where $G(\mathrm{S}) := \big(\mathfrak{L}_a g\big)(\mathrm{S})$, and we first used

$$G(\mathrm{S}) \overset{by}{\underset{(24)}{=}} \frac{1-\alpha}{\mathsf{A}(\alpha)} F(\mathrm{S}) + \frac{\alpha}{\mathsf{A}(\alpha)} \frac{1}{\mathrm{S}^\alpha} F(\mathrm{S}),$$

Later, we used

$$g(a) = \left({}^{AB}_{\;a}\nabla^{-\alpha} f\right)(a) \overset{by}{\underset{(15)}{=}} \frac{1-\alpha}{\mathsf{A}(\alpha)} f(a).$$

By taking the Laplace transform $\mathfrak{L}_a$ on both sides of (25), we obtain

$$\begin{aligned}\left({}^{MABC}_{\quad a}\nabla^\alpha g\right)(\mathrm{z}) &= f(\mathrm{z}) - f(a)\mathfrak{L}_a^{-1}\left\{\frac{\mathrm{S}^{\alpha-1}(1-\mathrm{S})^a}{\mathrm{S}^\alpha - \lambda}\right\} \\ &\overset{by}{\underset{(12)}{=}} f(\mathrm{z}) - f(a)\mathrm{E}_{\overline{\alpha}}(\lambda, \mathrm{z}-a),\end{aligned}$$

which ends the second part our proof. The proof of Theorem 1 is thus completed. □

Theorem 2. *For $\alpha \in \left(0, \frac{1}{2}\right)$, the following result provides an alternative series representation of the discrete modified Atangana–Baleanu of the Liouville–Caputo-type fractional difference*:

$$\left({}^{MABC}_{\quad a}\nabla^\alpha f\right)(\mathrm{z}) = \frac{\mathsf{A}(\alpha)}{1-\alpha}\left[f(\mathrm{z}) - \mathrm{E}_{\overline{\alpha}}(\lambda, \mathrm{z}-a)f(a) + \sum_{\kappa=0}^{\infty} \lambda^{\kappa+1}\left({}_a\nabla^{-(\alpha\kappa+\alpha)} f\right)(\mathrm{z})\right], \quad (26)$$

for z *in* $\mathbb{N}_{a+1}$.

Proof. According to Definitions 3 and 6, we can have

$$\begin{aligned}&\left({}^{MABC}_{\quad a}\nabla^\alpha f\right)(\mathrm{z}) \\ &= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left[f(\mathrm{z}) - \mathrm{E}_{\overline{\alpha}}(\lambda, \mathrm{z}-a)f(a) + \lambda \sum_{\mathrm{s}=a+1}^{\mathrm{z}} \mathrm{E}_{\overline{\alpha,\alpha}}(\lambda, \mathrm{z}-\mathrm{s}+1)f(\mathrm{s})\right] \\ &= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left[f(\mathrm{z}) - \mathrm{E}_{\overline{\alpha}}(\lambda, \mathrm{z}-a)f(a) + \lambda \sum_{\mathrm{s}=a+1}^{\mathrm{z}} \sum_{\kappa=0}^{\infty} \lambda^\kappa \frac{(\mathrm{z}-\mathrm{s}+1)^{\overline{\alpha\kappa+\alpha-1}}}{\Gamma(\alpha\kappa+\alpha)} f(\mathrm{s})\right] \\ &= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left[f(\mathrm{z}) - \mathrm{E}_{\overline{\alpha}}(\lambda, \mathrm{z}-a)f(a) + \sum_{\kappa=0}^{\infty} \lambda^{\kappa+1} \frac{1}{\Gamma(\alpha\kappa+\alpha)} \sum_{\mathrm{s}=a+1}^{\mathrm{z}} (\mathrm{z}-\mathrm{s}+1)^{\overline{\alpha\kappa+\alpha-1}} f(\mathrm{s})\right] \\ &= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left[f(\mathrm{z}) - \mathrm{E}_{\overline{\alpha}}(\lambda, \mathrm{z}-a)f(a) + \sum_{\kappa=0}^{\infty} \lambda^{\kappa+1}\left({}_a\nabla^{-(\alpha\kappa+\alpha)} f\right)(\mathrm{z})\right],\end{aligned}$$

which is the result asserted by Theorem 2. □

4. Applications

This section presents some specific illustrative examples to verify the results which we obtained in the preceding sections. In addition, each of these examples shows the applicability of the alternative discrete modified Atangana–Baleanu of the Liouville–Caputo-type fractional difference series representation.

Example 1. *Let $f(z) = \ell$ be any constant function. We can see from Definition 6 that*

$$\begin{aligned}
\left({}^{MABC}_{\ \ a}\nabla^{\alpha}1\right)(z) &= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left[\ell - \ell\,E_{\overline{\alpha}}(\lambda, z-a) + \ell\,\lambda \sum_{s=a+1}^{z} E_{\overline{\alpha},\overline{\alpha}}(\lambda, z-s+1)\right] \\
&= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left[\ell - \ell\,E_{\overline{\alpha}}(\lambda, z-a) + \ell \sum_{s=a+1}^{z} \nabla E_{\overline{\alpha}}(\lambda, z-s+1)\right] \\
&= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left[\ell - \ell\,E_{\overline{\alpha}}(\lambda, z-a) + \ell \int_{a}^{z} \nabla E_{\overline{\alpha}}(\lambda, z-s)\nabla s\right] \\
&= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left[\ell - \ell\,E_{\overline{\alpha}}(\lambda, z-a) - \ell\,E_{\overline{\alpha}}(\lambda, z-s)\Big|_{s=a}^{s=z}\right] = 0.
\end{aligned}$$

On the other hand, by using the series representation theorem (Theorem 2), we have

$$\begin{aligned}
\left({}^{MABC}_{\ \ a}\nabla^{\alpha}1\right)(z) &= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left[\ell - \ell\,E_{\overline{\alpha}}(\lambda, z-a) + \ell \sum_{\kappa=0}^{\infty} \lambda^{\kappa+1}\left({}_{a}\nabla^{-(\alpha\kappa+\alpha)}1\right)(z)\right] \\
&= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left[\ell - \ell\,E_{\overline{\alpha}}(\lambda, z-a) + \ell \sum_{\kappa=0}^{\infty} \lambda^{\kappa+1} \underbrace{\frac{(z-a)^{\overline{\alpha\kappa+\alpha}}}{\Gamma(\alpha\kappa+\alpha+1)}}_{\text{by Lemma 1}}\right] \\
&= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left[-\ell\,E_{\overline{\alpha}}(\lambda, z-a) + \ell \sum_{\kappa=0}^{\infty} \lambda^{\kappa} \frac{(z-a)^{\overline{\alpha\kappa}}}{\Gamma(\alpha\kappa+1)}\right], \\
&= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left[-\ell\,E_{\overline{\alpha}}(\lambda, z-a) + \ell\,E_{\overline{\alpha}}(\lambda, z-a)\right] = 0,
\end{aligned}$$

where we used $(z-a)^{\overline{0}} = 1$.

Example 2. *Consider*

$$f(z) = (z-a)^{\overline{\beta}},$$

for $\beta \in \mathbb{R}$ and $\alpha \in \mathbb{R}^{+}$. Then, from, Equation (26), we have

$$\begin{aligned}
&\left({}^{MABC}_{\ \ a}\nabla^{\alpha}(z-a)^{\overline{\beta}}\right)(z) \\
&= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left[(z-a)^{\overline{\beta}} - E_{\overline{\alpha}}(\lambda, z-a)(0)^{\overline{\beta}} + \sum_{\kappa=0}^{\infty} \lambda^{\kappa+1}\left({}_{a}\nabla^{-(\alpha\kappa+\alpha)}(z-a)^{\overline{\beta}}\right)(z)\right] \\
&= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left[(z-a)^{\overline{\beta}} - E_{\overline{\alpha}}(\lambda, z-a)(0)^{\overline{\beta}} + \Gamma(\beta+1)\sum_{\kappa=0}^{\infty} \lambda^{\kappa+1} \frac{(z-a)^{\overline{\alpha\kappa+\alpha+\beta}}}{\Gamma(\alpha\kappa+\alpha+\beta+1)}\right] \\
&= \frac{\mathsf{A}(\alpha)}{1-\alpha}\left[(z-a)^{\overline{\beta}} + \Gamma(\beta+1)\sum_{\kappa=1}^{\infty} \lambda^{\kappa} \frac{(z-a)^{\overline{\alpha\kappa+\beta}}}{\Gamma(\alpha\kappa+\beta+1)}\right] \\
&= \frac{\Gamma(\beta+1)\mathsf{A}(\alpha)}{1-\alpha}\sum_{\kappa=0}^{\infty} \lambda^{\kappa} \frac{(z-a)^{\overline{\alpha\kappa+\beta}}}{\Gamma(\alpha\kappa+\beta+1)} \\
&= \frac{\Gamma(\beta+1)\mathsf{A}(\alpha)}{1-\alpha} E_{\overline{\alpha,\beta+1}}(\lambda, z-a),
\end{aligned}$$

where we used $(0)^{\overline{\beta}} = 0$.

In particular, for

$$f(z) = (1-\alpha)\frac{(z-a)^{\overline{\alpha-1}}}{A(\alpha)\Gamma(\alpha)},$$

it becomes

$$\left({}^{MABC}_{\quad a}\nabla^{\alpha}\frac{(1-\alpha)(z-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)A(\alpha)}\right)(z) = E_{\overline{\alpha},\overline{\alpha}}(\lambda, z-a).$$

This has been shown in Figure 1 for $\alpha = 0.1, 0.2, 0.3, 0.4$, *and* $z \in \mathbb{N}_a^{a+20}$.

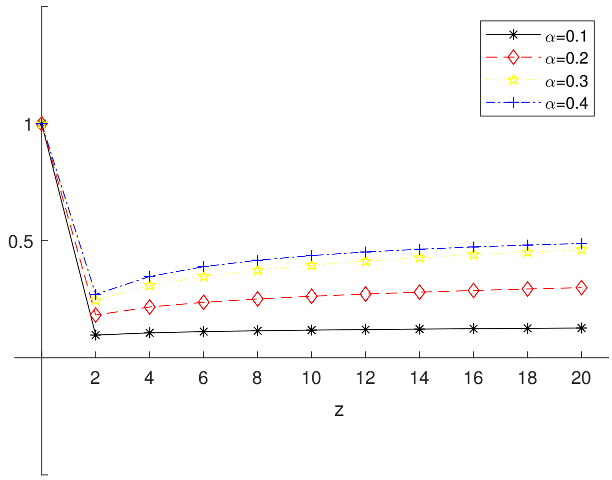

Figure 1. Graph of $E_{\overline{\alpha},\overline{\alpha}}(\lambda, z-a)$ for different values of α.

5. Concluding Remarks

Continuous fractional calculus, as an extension of its meaning, has a profound impact on its discrete version counterpart. During recent years, several singular and nonsingular fractional calculus operators were deeply scrutinized from the perspective and viewpoints of both pure and applied mathematical sciences. In particular, discrete fractional calculus has huge potential for applications in treating the extraction and modeling of hidden aspects from complicated real-world problems. Typically, the results provided by discrete fractional calculus can be easily adapted to improve and extend the classical methods and techniques within the artificial intelligence field. In particular, the operator possessing the Mittag-Leffler-type kernel, in the Liouville–Caputo sense, was extended in several ways in order to bypass the standard initialization procedure. In this paper, we constructed the modified nabla version of the Atangana–Baleanu discrete operator of the Liouville–Caputo type in Definitions 6 and 7, which was derived from the original discrete Atangana–Baleanu operator as defined in Definition 5. Furthermore, we proved the corresponding commutation relations and its series representation formula in Theorem 2. Aside from that, some examples were provided to see the similarities and differences with the classical discrete counterpart. We believe that the new expressions presented in this paper can be applied successfully in the modeling of complicated systems from various branches of engineering and science. We also briefly indicated some potential directions for further research by using much more general families of the appropriately defined operators of discrete fractional calculus, which are based upon the continuous fractional calculus operators with the kernel involving the Srivastava function $\mathcal{E}_{\alpha,\beta}(\varphi; z, s, a)$ defined by Equation (2) (for details, see [22,23]).

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