Article

Second-Order Approximate Equations of the Large-Scale Atmospheric Motion Equations and Symmetry Analysis for the Basic Equations of Atmospheric Motion

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Abstract: In this paper, symmetry properties of the basic equations of atmospheric motion are proposed. The results on symmetries show that the basic equations of atmospheric motion are invariant under space-time translation transformation, Galilean translation transformations and scaling transformations. Eight one-parameter invariant subgroups and eight one-parameter group invariant solutions are demonstrated. Three types of nontrivial similarity solutions and group invariants are proposed. With the help of perturbation method, we derive the second-order approximate equations for the large-scale atmospheric motion equations, including the non-dimensional equations and the dimensional equations. The second-order approximate equations of the large-scale atmospheric motion equations not only show the characteristics of physical quantities changing with time, but also describe the characteristics of large-scale atmospheric vertical motion.

Keywords: basic equations of atmospheric motion; symmetry; large scale atmospheric motion equations; second-order approximate equations

1. Introduction

Atmospheric dynamics mainly studies the evolution of atmospheric motion and the influence of various dynamic processes on atmospheric motion and state. If the influence of friction is not considered, the atmosphere can be called free atmosphere. The dynamic equations describing the motion of free atmosphere are composed of motion equations, continuity equations and thermodynamic equations. The basic equations of atmospheric motion in the local rectangular coordinate take the form: [1–4]:

$$\frac{\mathrm{d}\,u}{\mathrm{d}\,t} - f\,v = -\frac{1}{\rho}\frac{\partial\,p}{\partial\,x}\,, \tag{1}$$

$$\frac{\mathrm{d}\,v}{\mathrm{d}\,t} + f\,u = -\frac{1}{\rho}\frac{\partial\,p}{\partial\,y}\,, \tag{2}$$

$$\frac{\mathrm{d}\,w}{\mathrm{d}\,t} = -g - \frac{1}{\rho}\frac{\partial\,p}{\partial\,z}\,, \tag{3}$$

$$\frac{\mathrm{d}\,\rho}{\mathrm{d}\,t} + \rho\left(\frac{\partial\,u}{\partial\,x} + \frac{\partial\,v}{\partial\,y} + \frac{\partial\,w}{\partial\,z}\right) = 0\,, \tag{4}$$

$$p = \rho\,R\,T, \tag{5}$$

$$\theta = T\left(\frac{P_0}{p}\right)^{R/c_p}\,, \tag{6}$$

$$c_p\,\frac{\mathrm{d}\,T}{\mathrm{d}\,t} - \frac{1}{\rho}\frac{\mathrm{d}\,p}{\mathrm{d}\,t} = Q\,, \tag{7}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}\,t} \equiv \frac{\partial}{\partial\,t} + u\,\frac{\partial}{\partial\,x} + v\,\frac{\partial}{\partial\,y} + w\,\frac{\partial}{\partial\,z}$$

with $\frac{\mathrm{d}}{\mathrm{d}t}$, $\frac{\partial}{\partial t}$ and $u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$ meaning material derivative, local change and convective change, respectively. Equations (1) and (2) are horizontal motion equations. Equation (3) is a vertical motion equation. Equation (4) is a continuity equation. Equation (5) is an atmospheric state equation. Equation (6) is a potential temperature equation. Equation (7) is related to the first law of thermodynamics. t means time. z, x and y mean atmospheric position in vertical direction, latitudinal direction and longitudinal direction, respectively. u, v and w are wind velocities in x direction, y direction, and z direction. $u > 0$ and $u < 0$ mean west wind and east wind, respectively. $v > 0$ and $v < 0$ denote south wind and north wind, respectively. $w > 0$ and $w < 0$ are linked with ascending motion and descending motion, respectively. f is Coriolis parameters, where $f = 2\sin\phi\,\Omega$ with ϕ meaning latitude and Ω meaning rotational angular velocity of the earth. p is atmospheric pressure, g denotes the gravitational constant, ρ denotes the density and T is atmospheric temperature. c_p is about $1005\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$ and it means specific heat at constant pressure. $R = 287\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$ and it is gas constant of dry air. P_0 is a constant and $P_0 \equiv 1000\,\mathrm{hPa}$. θ is the potential temperature, which is a conserved quantity in adiabatic process. Q is the heat obtained from the outside by a unit mass of air clusters in a unit time. $Q = 0$ denotes atmospheric adiabatic changes.

The physical characteristics of various motions in the atmosphere are mainly determined by the horizontal spatial scale occupied by the motion. Based on this, the atmospheric motion is divided into large-scale motion, mesoscale motion and small-scale motion. The weather systems of large-scale atmospheric motion include long wave, blocking high pressure and large cyclone [5,6]. Mesoscale weather systems include typhoon, regional precipitation, hail and other severe convective weather [7,8]. Small-scale weather systems contain tornadoes, small thunderstorms and cumulus clouds [9,10].

According to the characteristics of different atmospheric motions, some approximate conditions can also be introduced to study the simplified model of the basic equations of atmospheric motion, such as hydrostatic approximation, anelastic approximation and Boussinesq approximation. In the large-scale atmosphere, the static equilibrium is very accurate, so the hydrostatic equation is often applied. Hydrostatic equations include horizontal kinetic energy, elastic potential energy and effective potential energy, but exclude sound waves [11,12]. For a system with small horizontal scale such as cumulus cloud, the static equilibrium is no longer accurate and suitable, and it is necessary to introduce anelastic approximation. The anelastic approximate equations contain kinetic energy and effective potential energy, which also exclude sound waves [13,14]. In the anelastic approximate equations, the atmosphere is compressible. The Boussinesq approximation is corresponding to an incompressible and non-hydrostatic atmosphere. The Boussinesq approximate equations include kinetic energy and effective potential energy, excluding sound waves, and requiring the vertical thickness of motion to be smaller than the elevation of the atmosphere [15–17].

The classic Lie group symmetry analysis is an effective method to solve partial differential equations [18–21]. The symmetry method is also very useful for solving equations related to atmospheric dynamics. By means of Lie Symmetry method, Ref. [22] determined a one-dimensional optimal system for a two-dimensional ideal gas equation. The symmetries of the (2+1)-dimensional nonlinear incompressible non-hydrostatic Boussinesq equations describing atmospheric gravity waves were researched in Ref. [23]. To the best of our knowledge, the symmetry characteristics and group invariant solutions for the basic equations of atmospheric motion have not been researched. The large-scale atmospheric dynamic equations can be expanded by the perturbation method. Neither the zero-order approximate equations nor the first-order approximate equations can describe the convective motion, and only the second-order approximate equations can describe the vertical motion. To our knowledge, the approximate equations of large-scale atmospheric vertical motion have not been reported.

As the above analysis shows, we will concentrate on the symmetries of the and the approximate equations of large-scale atmospheric motion in this paper. The Lie symmetries of the basic equations of atmospheric motion are researched in Section 2. In Section 3, one-parameter group transformations of the basic equations of atmospheric motion are demonstrated. Similarity solutions are addressed in Section 4. Approximate equations of large scale atmospheric vertical motion are derived in Section 5. In the last section, conclusions are concluded.

2. Lie Symmetry for the Basic Equations of Atmospheric Motion

Through the classical Lie point symmetry method, the symmetry property for the basic equations of atmospheric motion can be obtained. The first step of the classical point Lie symmetry method is to find symmetry components. In this section, we will look for symmetric components for the basic equations of atmospheric motion.

There are seven variable functions in Equations (1)–(7), so there are seven symmetry components, which are $\{\sigma^u, \sigma^v, \sigma^w, \sigma^T, \sigma^\rho, \sigma^p, \sigma^\theta\}$. We can assume that their forms are as follows:

$$\sigma^u = \widehat{x}\frac{\partial u}{\partial x} + \widehat{y}\frac{\partial u}{\partial y} + \widehat{z}\frac{\partial u}{\partial z} + \widehat{t}\frac{\partial u}{\partial t} - \widehat{u}, \tag{8}$$

$$\sigma^v = \widehat{x}\frac{\partial v}{\partial x} + \widehat{y}\frac{\partial v}{\partial y} + \widehat{z}\frac{\partial v}{\partial z} + \widehat{t}\frac{\partial v}{\partial t} - \widehat{v}, \tag{9}$$

$$\sigma^w = \widehat{x}\frac{\partial w}{\partial x} + \widehat{y}\frac{\partial w}{\partial y} + \widehat{z}\frac{\partial w}{\partial z} + \widehat{t}\frac{\partial w}{\partial t} - \widehat{w}, \tag{10}$$

$$\sigma^T = \widehat{x}\frac{\partial T}{\partial x} + \widehat{y}\frac{\partial T}{\partial y} + \widehat{z}\frac{\partial T}{\partial z} + \widehat{t}\frac{\partial T}{\partial t} - \widehat{T}, \tag{11}$$

$$\sigma^\rho = \widehat{x}\frac{\partial \rho}{\partial x} + \widehat{y}\frac{\partial \rho}{\partial y} + \widehat{z}\frac{\partial \rho}{\partial z} + \widehat{t}\frac{\partial \rho}{\partial t} - \widehat{\rho}, \tag{12}$$

$$\sigma^p = \widehat{x}\frac{\partial p}{\partial x} + \widehat{y}\frac{\partial p}{\partial y} + \widehat{z}\frac{\partial p}{\partial z} + \widehat{t}\frac{\partial p}{\partial t} - \widehat{p}, \tag{13}$$

$$\sigma^\theta = \widehat{x}\frac{\partial \theta}{\partial x} + \widehat{y}\frac{\partial \theta}{\partial y} + \widehat{z}\frac{\partial \theta}{\partial z} + \widehat{t}\frac{\partial \theta}{\partial t} - \widehat{\theta}, \tag{14}$$

where $\{\widehat{x}, \widehat{y}, \widehat{z}, \widehat{t}, \widehat{u}, \widehat{v}, \widehat{w}, \widehat{T}, \widehat{\rho}, \widehat{p}, \widehat{\theta}\}$ are all functions of $\{x, y, z, t, u, v, w, T, \rho, p, \theta\}$. The symmetry determinant equations for Equations (1)–(7) are

$$\frac{d\sigma^u}{dt} + \sigma^u\frac{\partial u}{\partial x} + \sigma^v\frac{\partial u}{\partial y} + \sigma^w\frac{\partial u}{\partial z} - f\sigma^v + \frac{1}{\rho}\frac{\partial \sigma^p}{\partial x} - \frac{\sigma^\rho}{\rho^2}\frac{\partial p}{\partial x} = 0, \tag{15}$$

$$\frac{d\sigma^v}{dt} + \sigma^u\frac{\partial v}{\partial x} + \sigma^v\frac{\partial v}{\partial y} + \sigma^w\frac{\partial v}{\partial z} + f\sigma^u + \frac{1}{\rho}\frac{\partial \sigma^p}{\partial y} - \frac{\sigma^\rho}{\rho^2}\frac{\partial p}{\partial y} = 0, \tag{16}$$

$$\frac{d\sigma^w}{dt} + \sigma^u\frac{\partial w}{\partial x} + \sigma^v\frac{\partial w}{\partial y} + \sigma^w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial \sigma^p}{\partial z} - \frac{\sigma^\rho}{\rho^2}\frac{\partial p}{\partial z} = 0, \tag{17}$$

$$\frac{d\sigma^\rho}{dt} + \sigma^u\frac{\partial \rho}{\partial x} + \sigma^v\frac{\partial \rho}{\partial y} + \sigma^w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial \sigma^u}{\partial x} + \frac{\partial \sigma^v}{\partial y} + \frac{\partial \sigma^w}{\partial z}\right)$$
$$+\sigma^\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0, \tag{18}$$

$$\sigma^p = R\left(T\sigma^\rho + \rho\sigma^T\right), \tag{19}$$

$$\left(\frac{P_0}{p}\right)^{\frac{R}{c_p}}\left(R\sigma^p T - \sigma^T c_p p\right) + \sigma^\theta c_p p = 0, \tag{20}$$

$$\frac{\sigma^\rho}{\rho}\frac{dp}{dt} - \frac{d\sigma^p}{dt} + c_p\rho\left(\frac{d\sigma^T}{dt} + \sigma^u\frac{\partial T}{\partial x} + \sigma^v\frac{\partial T}{\partial y} + \sigma^w\frac{\partial T}{\partial z}\right)$$
$$-\left(\sigma^u\frac{\partial p}{\partial x} + \sigma^v\frac{\partial p}{\partial y} + \sigma^w\frac{\partial p}{\partial z}\right) = 0. \tag{21}$$

Substitute Equations (8)–(14) into Equations (15)–(21), and let the coefficients of various order terms on variables $\{u, v, w, T, \rho, p, \theta\}$ in the updated Equations (15)–(21) be zero, we obtain

$$
\begin{cases}
\widehat{x} = C_2 \sin(ft) + C_3 \cos(ft) + C_1, \quad \widehat{y} = C_2 \cos(ft) - C_3 \sin(ft) + C_4, \\
\widehat{z} = tC_5 + C_6, \quad \widehat{t} = C_7, \quad \widehat{T} = 0, \quad \widehat{\rho} = C_8\rho, \quad \widehat{u} = f(C_2 \cos(ft) - C_3 \sin(ft)), \\
\widehat{v} = -f(C_3 \cos(ft) + C_2 \sin(ft)), \quad \widehat{w} = C_5, \quad \widehat{p} = C_8 p, \quad \widehat{\theta} = -\dfrac{C_8\theta\, R}{c_p}.
\end{cases}
\tag{22}
$$

The symmetry components in Equations (8)–(14) turn to

$$
\begin{aligned}
\sigma^u = {}& [\, C_2 \sin(f t) + C_3 \cos(f t) + C_1 \,] \frac{\partial u}{\partial x} + (C_5 t + C_6) \frac{\partial u}{\partial z} + C_7 \frac{\partial u}{\partial t} \\
& + [C_2 \cos(f t) - C_3 \sin(f t) + C_4] \frac{\partial u}{\partial y} - f[C_2 \cos(f t) - C_3 \sin(f t)],
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\sigma^v = {}& [\, C_2 \sin(f t) + C_3 \cos(f t) + C_1 \,] \frac{\partial v}{\partial x} + (C_5 t + C_6) \frac{\partial v}{\partial z} + C_7 \frac{\partial v}{\partial t} \\
& + [C_2 \cos(f t) - C_3 \sin(f t) + C_4] \frac{\partial v}{\partial y} + f[C_2 \cos(f t) - C_3 \sin(f t)],
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\sigma^w = {}& [\, C_2 \sin(f t) + C_3 \cos(f t) + C_1 \,] \frac{\partial w}{\partial x} + (C_5 t + C_6) \frac{\partial w}{\partial z} + C_7 \frac{\partial w}{\partial t} \\
& + [\, C_2 \cos(f t) - C_3 \sin(f t) + C_4 \,] \frac{\partial w}{\partial y} - C_5,
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
\sigma^T = {}& [\, C_2 \sin(f t) + C_3 \cos(f t) + C_1 \,] \frac{\partial T}{\partial x} + (C_5 t + C_6) \frac{\partial T}{\partial z} + C_7 \frac{\partial T}{\partial t} \\
& + [\, C_2 \cos(f t) - C_3 \sin(f t) + C_4 \,] \frac{\partial T}{\partial y},
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
\sigma^\rho = {}& [\, C_2 \sin(f t) + C_3 \cos(f t) + C_1 \,] \frac{\partial \rho}{\partial x} + (C_5 t + C_6) \frac{\partial \rho}{\partial z} + C_7 \frac{\partial \rho}{\partial t} \\
& + [\, C_2 \cos(f t) - C_3 \sin(f t) + C_4 \,] \frac{\partial \rho}{\partial y} - C_8 \rho,
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
\sigma^p = {}& [\, C_2 \sin(f t) + C_3 \cos(f t) + C_1 \,] \frac{\partial p}{\partial x} + (C_5 t + C_6) \frac{\partial p}{\partial z} + C_7 \frac{\partial p}{\partial t} \\
& + [\, C_2 \cos(f t) - C_3 \sin(f t) + C_4 \,] \frac{\partial p}{\partial y} - C_8\, p,
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
\sigma^\theta = {}& [\, C_2 \sin(f t) + C_3 \cos(f t) + C_1 \,] \frac{\partial \theta}{\partial x} + (C_5 t + C_6) \frac{\partial \theta}{\partial z} + C_7 \frac{\partial \theta}{\partial t} \\
& + [\, C_2 \cos(f t) - C_3 \sin(f t) + C_4 \,] \frac{\partial \theta}{\partial y} + \frac{C_8\, \theta\, R}{c_p}.
\end{aligned}
\tag{29}
$$

3. Invariant Solutions and One-Parameter Invariant Groups for the Basic Equations of Atmospheric Motion

From Formulas (23)–(29), the subvectors in the form of

$$
\begin{cases}
\underline{V_1} = \dfrac{\partial}{\partial x}, \quad \underline{V_2} = \sin(f t)\dfrac{\partial}{\partial x} + \cos(f t)\dfrac{\partial}{\partial y} + f\cos(f t)\dfrac{\partial}{\partial u} - f\sin(f t)\dfrac{\partial}{\partial v}, \\
\underline{V_3} = \cos(f t)\dfrac{\partial}{\partial x} - \sin(f t)\dfrac{\partial}{\partial y} - f\sin(f t)\dfrac{\partial}{\partial u} - f\cos(f t)\dfrac{\partial}{\partial v}, \quad \underline{V_4} = \dfrac{\partial}{\partial y}, \\
\underline{V_5} = t\dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial w}, \quad \underline{V_6} = \dfrac{\partial}{\partial z}, \quad \underline{V_7} = \dfrac{\partial}{\partial t}, \quad \underline{V_8} = \rho\,\partial_\rho + p\dfrac{\partial}{\partial p} - \dfrac{R}{c_p}\theta\dfrac{\partial}{\partial \theta}.
\end{cases}
\tag{30}
$$

can be obtained. $\underline{V_1}$, $\underline{V_4}$ and $\underline{V_6}$ mean translation invariance of atmosphere along the latitudinal direction, longitudinal direction and vertical direction, respectively. $\underline{V_7}$ denotes translation invariance along time translation transformation. $\underline{V_2}$, $\underline{V_3}$ and $\underline{V_5}$ are related to Galilean translation transformations, and $\underline{V_8}$ denotes scaling transformations. These

invariance demonstrate the space-time symmetry of atmospheric motion. Otherwise, from these subvectors, we can obtain some one-parameter invariant groups and one-parameter group invariant solutions. Eight one-parameter invariant groups are concluded in the form of

$$
\{x, y, z, t, u, v, w, T, \rho, p, \theta\} \longrightarrow
$$

$$
\begin{cases}
\{x + \varepsilon, y, z, t, u, v, w, T, \rho, p, \theta\}, \\
\{x + \varepsilon \sin(ft), y + \varepsilon \cos(ft), z, t, u + \varepsilon f \cos(ft), v - \varepsilon f \sin(ft), w, T, \rho, p, \theta\}, \\
\{x + \varepsilon \cos(ft), y - \varepsilon \sin(ft), z, t, u - \varepsilon f \sin(ft), v - \varepsilon f \cos(ft), w, T, \rho, p, \theta\}, \\
\{x, y + \varepsilon, z, t, u, v, w, T, \rho, p, \theta\}, \\
\{x, y, z + \varepsilon t, t, u, v, w + \varepsilon, T, \rho, p, \theta\}, \\
\{x, y, z + \varepsilon, t, u, v, w, T, \rho, p, \theta\}, \\
\{x, y, z, t + \varepsilon, u, v, w, T, \rho, p, \theta\}, \\
\{x, y, z, t, u, v, w, T, \rho\, \mathrm{e}^{\varepsilon}, p\, \mathrm{e}^{\varepsilon}, \theta\, \mathrm{e}^{-\varepsilon \frac{R}{c_p}}\}
\end{cases}
\tag{31}
$$

From the one-parameter subgroups (31), different analytic solutions can be obtained from an analytic solution. Then, we present the one-parameter group invariant solutions.

Theorem 1 (One-parameter group invariant solutions). *If* $\{u(x, y, z, t), v(x, y, z, t), w(x, y, z, t), T(x, y, z, t), \rho(x, y, z, t), p(x, y, z, t), \theta(x, y, z, t)\}$ *is an analytic solution for the basic equations of atmospheric motion, then so are the functions in the form of:*

$$
\begin{cases}
\overline{u}_1 = u(x - \varepsilon, y, z, t), \quad \overline{v}_1 = v(x - \varepsilon, y, z, t), \\
\overline{w}_1 = w(x - \varepsilon, y, z, t), \quad \overline{T}_1 = T(x - \varepsilon, y, z, t), \\
\overline{\rho}_1 = \rho(x - \varepsilon, y, z, t), \quad \overline{p}_1 = p(x - \varepsilon, y, z, t), \quad \overline{\theta}_1 = \theta(x - \varepsilon, y, z, t)
\end{cases},
\tag{32}
$$

$$
\begin{cases}
\overline{u}_2 = u(x - \varepsilon \sin(ft), y - \varepsilon \cos(ft), z, t) + \varepsilon f \cos(ft), \\
\overline{v}_2 = v(x - \varepsilon \sin(ft), y - \varepsilon \cos(ft), z, t) - \varepsilon f \sin(ft), \\
\overline{w}_2 = w(x - \varepsilon \sin(ft), y - \varepsilon \cos(ft), z, t), \\
\overline{T}_2 = T(x - \varepsilon \sin(ft), y - \varepsilon \cos(ft), z, t), \\
\overline{\rho}_2 = \rho(x - \varepsilon \sin(ft), y - \varepsilon \cos(ft), z, t), \\
\overline{p}_2 = p(x - \varepsilon \sin(ft), y - \varepsilon \cos(ft), z, t), \\
\overline{\theta}_2 = \theta(x - \varepsilon \sin(ft), y - \varepsilon \cos(ft), z, t)
\end{cases},
\tag{33}
$$

$$
\begin{cases}
\overline{u}_3 = u(x - \varepsilon \cos(ft), y + \varepsilon \sin(ft), z, t) - \varepsilon f \sin(ft), \\
\overline{v}_3 = v(x - \varepsilon \cos(ft), y + \varepsilon \sin(ft), z, t) - \varepsilon f \cos(ft), \\
\overline{w}_3 = w(x - \varepsilon \cos(ft), y + \varepsilon \sin(ft), z, t), \\
\overline{T}_3 = T(x - \varepsilon \cos(ft), y + \varepsilon \sin(ft), z, t), \\
\overline{\rho}_3 = \rho(x - \varepsilon \cos(ft), y + \varepsilon \sin(ft), z, t), \\
\overline{p}_3 = p(x - \varepsilon \cos(ft), y + \varepsilon \sin(ft), z, t), \\
\overline{\theta}_3 = \theta(x - \varepsilon \cos(ft), y + \varepsilon \sin(ft), z, t)
\end{cases},
\tag{34}
$$

$$
\begin{cases}
\overline{u}_4 = u(x, y - \varepsilon, z, t), \quad \overline{v}_4 = v(x, y - \varepsilon, z, t), \\
\overline{w}_4 = w(x, y - \varepsilon, z, t), \quad \overline{T}_4 = T(x, y - \varepsilon, z, t), \\
\overline{\rho}_4 = \rho(x, y - \varepsilon, z, t), \quad \overline{p}_4 = p(x, y - \varepsilon, z, t), \quad \overline{\theta}_4 = \theta(x, y - \varepsilon, z, t)
\end{cases},
\tag{35}
$$

$$
\begin{cases}
\overline{u}_5 = u(x, y, z - \varepsilon t, t), \quad \overline{v}_5 = v(x, y, z - \varepsilon t, t), \\
\overline{w}_5 = w(x, y, z - \varepsilon t, t) + \varepsilon, \quad \overline{T}_5 = T(x, y, z - \varepsilon t, t), \\
\overline{\rho}_5 = \rho(x, y, z - \varepsilon t, t), \\
\overline{p}_5 = p(x, y, z - \varepsilon t, t), \quad \overline{\theta}_5 = \theta(x, y, z - \varepsilon t, t)
\end{cases},
\tag{36}
$$

$$
\begin{cases}
\overline{u}_6 = u(x, y, z - \varepsilon, t), \quad \overline{v}_6 = v(x, y, z - \varepsilon, t), \\
\overline{w}_6 = w(x, y, z - \varepsilon, t), \quad \overline{T}_6 = T(x, y, z - \varepsilon, t), \\
\overline{\rho}_6 = \rho(x, y, z - \varepsilon, t), \quad \overline{p}_6 = p(x, y, z - \varepsilon, t), \quad \overline{\theta}_6 = \theta(x, y, z - \varepsilon, t)
\end{cases},
\tag{37}
$$

$$\begin{cases} \overline{u}_7 = u(x,y,z,t-\varepsilon), \overline{v}_7 = v(x,y,z,t-\varepsilon), \\ \overline{w}_7 = w(x,y,z,t-\varepsilon), \overline{T}_7 = T(x,y,z,t-\varepsilon), \\ \overline{\rho}_7 = \rho(x,y,z,t-\varepsilon), \ \overline{p}_7 = p(x,y,z,t-\varepsilon), \ \overline{\theta}_7 = \theta(x,y,z,t-\varepsilon) \end{cases}. \tag{38}$$

$$\begin{cases} \overline{u}_8 = u(x,y,z,t), \overline{v}_8 = v(x,y,z,t), \\ \overline{w}_8 = w(x,y,z,t), \overline{T}_8 = T(x,y,z,t), \\ \overline{\rho}_8 = \rho(x,y,z,t)\,\mathrm{e}^\varepsilon, \ \overline{p}_8 = p(x,y,z,t)\,\mathrm{e}^\varepsilon, \ \overline{\theta}_8 = \theta(x,y,z,t)\,\mathrm{e}^{-\varepsilon\frac{R}{c_p}} \end{cases}. \tag{39}$$

4. Similarity Solutions and Symmetry Reduction Equations for the Basic Equations of Atmospheric Motion

From the symmetries of the basic equations of atmospheric motion, one can obtain similarity solutions. The following three types of nontrivial similarity solutions and group invariants are obtained.

Case 1 $C_7 \neq 0$

The first case is the most general condition. In this case, the group invariants read

$$\begin{aligned} \xi &= -z + \frac{C_5 t^2 + 2 C_6 t}{2 C_7}, \\ \eta &= x + \frac{-C_1 f t + C_2 \cos(f t) - C_3 \sin(f t)}{C_7 f}, \\ \zeta &= y - \frac{C_4 f t + C_3 \cos(f t) + C_2 \sin(f t)}{C_7 f}. \end{aligned}$$

The corresponding similarity solutions are

$$u = \widetilde{u}(\xi,\eta,\zeta) + \frac{C_2 \sin(ft) + C_3 \cos(ft)}{C_7}, \tag{40}$$

$$v = \widetilde{v}(\xi,\eta,\zeta) + \frac{C_2 \cos(ft) - C_3 \sin(ft)}{C_7}, \tag{41}$$

$$w = \widetilde{w}(\xi,\eta,\zeta) + \frac{C_5 t}{C_7}, \tag{42}$$

$$T = \widetilde{T}(\xi,\eta,\zeta), \tag{43}$$

$$\rho = \widetilde{\rho}(\xi,\eta,\zeta)\mathrm{e}^{\frac{C_8 t}{C_7}}, \tag{44}$$

$$p = \widetilde{p}(\xi,\eta,\zeta)\mathrm{e}^{\frac{C_8 t}{C_7}}, \tag{45}$$

$$\theta = \widetilde{\theta}(\xi,\eta,\zeta)\mathrm{e}^{\frac{C_8 R t}{C_7 C_p}}, \tag{46}$$

where $\{\widetilde{u},\widetilde{v},\widetilde{w},\widetilde{T},\widetilde{\rho},\widetilde{p},\widetilde{\theta}\}$ are all functions of the group invariants $\{\xi,\eta,\zeta\}$. The reduction equations are very lengthy, and we will not list them here.

Case 2 $C_1{}^2 + C_2{}^2 + C_3{}^2 \neq 0$ and $C_7 = 0$

The formula $C_7 = 0$ means that t is one of the three group invariants. The three group invariants are given as

$$\begin{aligned} \xi &= y - \frac{x[C_2 \cos(f t) - C_3 \sin(f t) + C_4]}{C_2 \sin(f t) + C_3 \cos(ft) + C_1}, \\ \eta &= z - \frac{x(tC_5 + C_6)}{C_2 \sin(f t) + C_3 \cos(f t) + C_1}, \\ \zeta &= t. \end{aligned}$$

In this case, the similarity solutions are given by

$$u = \frac{f\left[C_2\cos(f\,t) - C_3\sin(f\,t)\right]x}{C_2\sin(f\,t) + C_3\cos(ft) + C_1} + \widetilde{u}(\xi,\eta,\zeta)\,, \tag{47}$$

$$v = -\frac{f[C_2\sin(f\,t) + C_3\cos(f\,t)]x}{C_2\sin(f\,t) + C_3\cos(f\,t) + C_1} + \widetilde{v}(\xi,\eta,\zeta)\,, \tag{48}$$

$$w = \frac{C_5\,x}{C_2\sin(f\,t) + C_3\cos(f\,t) + C_1} + \widetilde{w}(\xi,\eta,\zeta)\,, \tag{49}$$

$$T = \widetilde{T}(\xi,\eta,\zeta)\,, \tag{50}$$

$$\rho = \widetilde{\rho}(\xi,\eta,\zeta)e^{\frac{C_8 x}{C_2\sin(ft)+C_3\cos(ft)+C_1}}\,, \tag{51}$$

$$p = \widetilde{p}(\xi,\eta,\zeta)e^{\frac{C_8 x}{C_2\sin(f\,t)+C_3\cos(f\,t)+C_1}}\,, \tag{52}$$

$$\theta = \widetilde{\theta}(\xi,\eta,\zeta)e^{\frac{C_8 x R}{C_p\,(C_2\sin(f\,t)+C_3\cos(f\,t)+C_1)}}\,. \tag{53}$$

The reduction equations are also very lengthy in this case, and it is not suitable for listing them here.

Case 3 $C_2 = 0, C_3 = 0, C_1 \neq 0$ and $C_7 \neq 0$

When $C_2 = 0$ and $C_3 = 0$, the Formula (23) degenerates to

$$\sigma^u = C_1\,u_x + C_4\,u_y + (C_5\,t + C_6)\,u_z + C_7\,u_t\,, \tag{54}$$

$$\sigma^v = C_1\,v_x + C_4\,v_y + (C_5\,t + C_6)\,v_z + C_7\,v_t\,, \tag{55}$$

$$\sigma^w = C_1\,w_x + C_4\,w_y + (C_5\,t + C_6)\,w_z + C_7\,w_t - C_5\,, \tag{56}$$

$$\sigma^T = C_1\,T_x + C_4\,T_y + (C_5\,t + C_6)\,T_z + C_7\,T_t\,, \tag{57}$$

$$\sigma^\rho = C_1\,\rho_x + C_4\,\rho_y + (C_5\,t + C_6)\,\rho_z + C_7\,\rho_t - C_8\rho\,, \tag{58}$$

$$\sigma^p = C_1\,p_x + C_4\,p_y + (C_5\,t + C_6)\,p_z + C_7\,p_t - C_8 p\,, \tag{59}$$

$$\sigma^\theta = C_1\,\theta_x + C_4\,\theta_y + (C_5\,t + C_6)\,\theta_z + C_7\,\theta_t + \frac{C_8\,\theta\,R}{c_p}\,. \tag{60}$$

Then, the similarity solutions are

$$u = \widetilde{u}(\xi,\eta,\zeta)\,, \tag{61}$$

$$v = \widetilde{v}(\xi,\eta,\zeta)\,, \tag{62}$$

$$w = \frac{C_5 x}{C_1} + \widetilde{w}(\xi,\eta,\zeta)\,, \tag{63}$$

$$T = \widetilde{T}(\xi,\eta,\zeta)\,, \tag{64}$$

$$\rho = \widetilde{\rho}(\xi,\eta,\zeta)e^{\frac{C_8 x}{C_1}}\,, \tag{65}$$

$$p = \widetilde{p}(\xi,\eta,\zeta)e^{\frac{C_8 x}{C_1}}\,, \tag{66}$$

$$\theta = \widetilde{\theta}(\xi,\eta,\zeta)e^{\frac{C_8 R x}{C_1 c_p}}\,, \tag{67}$$

where ξ, η, ζ are group invariants, which read

$$\xi = -\frac{xC_4}{C_1} + y\,,$$

$$\eta = t - \frac{xC_7}{C_1}\,,$$

$$\zeta = z - \frac{C_5 x\,t}{C_1} + \frac{x^2\,C_5 C_7}{2\,C_1{}^2} - \frac{xC_6}{C_1}\,.$$

The reduction equations are in the form

$$(C_1\,\widetilde{w} - \eta_1\widetilde{u}_1)\frac{\partial\,\widetilde{u}}{\partial\,\zeta} - C_7\,\widetilde{u}\,\frac{\partial\,\widetilde{u}}{\partial\,\eta} + (C_1\widetilde{v} - \widetilde{u}_1\,C_4)\frac{\partial\,\widetilde{u}}{\partial\,\xi} - f\,\widetilde{v}\,C_1$$

$$+\frac{C_8\,\widetilde{p}}{\widetilde{\rho}} - \frac{C_7}{\widetilde{\rho}}\frac{\partial\,\widetilde{p}}{\partial\,\eta} - \frac{C_4}{\widetilde{\rho}}\frac{\partial\,\widetilde{p}}{\partial\,\xi} - \frac{\eta_1}{\widetilde{\rho}}\frac{\partial\,\widetilde{p}}{\partial\,\zeta} = 0, \tag{68}$$

$$\frac{C_1}{\widetilde{\rho}}\frac{\partial\,\widetilde{p}}{\partial\,\xi} - C_7\widetilde{u}\frac{\partial\,\widetilde{v}}{\partial\,\eta} + (C_1\widetilde{v} - C_4\,\widetilde{u}_1)\frac{\partial\,\widetilde{v}}{\partial\,\xi} + (C_1\widetilde{w} - \eta_1\widetilde{u}_1)\frac{\partial\,\widetilde{v}}{\partial\,\zeta} + C_1 f\,\widetilde{u} = 0, \tag{69}$$

$$(C_1\,\widetilde{v} - C_4\,\widetilde{u}_1)\frac{\partial\,\widetilde{w}}{\partial\,\xi} - C_7\,\widetilde{u}\,\frac{\partial\,\widetilde{w}}{\partial\,\eta}$$

$$+(C_1\widetilde{w} - \eta_1\widetilde{u}_1\,)\frac{\partial\,\widetilde{w}}{\partial\,\zeta} + C_1\,g + C_5\widetilde{u}_1 + \frac{C_1}{\widetilde{\rho}}\frac{\partial\,\widetilde{p}}{\partial\,\zeta} = 0, \tag{70}$$

$$C_7\frac{\partial\,\widetilde{u}}{\partial\,\eta} + C_4\,\frac{\partial\,\widetilde{u}}{\partial\,\xi} + \eta_1\,\frac{\partial\,\widetilde{u}}{\partial\,\zeta} - C_1\,\frac{\partial\,\widetilde{v}}{\partial\,\xi} - C_1\,\frac{\partial\,\widetilde{w}}{\partial\,\zeta} - C_8\,\widetilde{u}_1 + C_7\,\widetilde{u}\,\frac{\partial\,\ln\widetilde{\rho}}{\partial\,\eta}$$

$$+(C_4\,\widetilde{u}_1 - C_1\widetilde{v})\frac{\partial\,\ln\widetilde{\rho}}{\partial\,\xi} + (\eta_1\widetilde{u}_1 - C_1\widetilde{w})\frac{\partial\,\ln\widetilde{\rho}}{\partial\,\zeta} = 0, \tag{71}$$

$$\widetilde{p} = \widetilde{\rho}\,R\,\widetilde{T}, \tag{72}$$

$$(C_4\widetilde{u}_1 - C_1\,\widetilde{v})\left(\frac{\partial\,\widetilde{p}}{\partial\,\xi} - c_p\,\widetilde{\rho}\,\frac{\partial\,\widetilde{T}}{\partial\,\xi}\right) + (\eta_1\widetilde{u}_1 - C_1\widetilde{w})\left(\frac{\partial\,\widetilde{p}}{\partial\,\zeta} - c_p\,\widetilde{\rho}\,\frac{\partial\,\widetilde{T}}{\partial\,\zeta}\right)$$

$$+C_7\,\widetilde{u}\left(\frac{\partial\,\widetilde{p}}{\partial\,\eta} - c_p\,\frac{\partial\,\widetilde{T}}{\partial\,\eta}\,\widetilde{\rho}\right) - C_1\,Q\widetilde{\rho} - C_8\,\widetilde{u}_1\,\widetilde{p} = 0, \tag{73}$$

$$\widetilde{\theta} = \widetilde{T}\left(\frac{P_0}{\widetilde{p}}\right)^{R/c_p}. \tag{74}$$

where $\widetilde{u}_1 = \widetilde{u} + \frac{C_1}{C_7}$ and $\eta_1 = \eta\,C_5 + C_6$.

Solving Equations (68)–(74), one can obtain some analytic solutions of Equations (68)–(74). The combination of the analytic solutions for Equations (68)–(74) and the similarity solution (61)–(67) leads to the analytic solutions of the basic equations of atmospheric motion. Analytical solutions of Equations (68)–(74) are very rich and need to be discussed in different condition, so we don't discuss this problem here.

5. Second-Order Approximate Equations of the Large-Scale Atmospheric Motion Equations

Given the conditions of large-scale atmospheric motion, the basic equations of atmospheric motion will degenerate to large-scale atmospheric motion equations. By perturbation method, large-scale atmospheric motion equations can be expanded as zero-order, first-order and second-order and higher order approximate equations. The zero-order approximate equations reflect the basic characteristics of atmospheric large-scale motion, namely geostrophic equilibrium, static equilibrium and horizontal nondivergence [3]. The first-order approximate equations reflect the quasi geostrophic equilibrium of large-scale atmospheric motion, but there are still no convection term [3]. In order to consider convective motion, we need to study the second-order approximate equations. In this section, we aims to derive the second-order approximate equations of the large-scale atmospheric motion equations.

We define the state of the static atmosphere as the background state of the atmosphere. Suppose that in the atmosphere of this background state, there is a small deviation p', ρ', T', θ' and perturbation velocity u', v', w'. Suppose

$$u = u', v = v', w = w', p = p_0(z) + p', \rho = \rho_0(z) + \rho',$$

$$T = T_0(z) + T', \theta = \theta_0(z) + \theta', \tag{75}$$

and

$$p^{'} \ll p_0(z), \; \rho^{'} \ll \rho_0(z), \; T^{'} \ll T_0(z), \; \theta^{'} \ll \theta_0(z) \tag{76}$$

Atmospheric motion is consider to be frictionless and adiabatic. Under the above assumptions, the basic equations of atmospheric motion (1)–(7) turn to [3]

$$\frac{\mathrm{d}\,u}{\mathrm{d}\,t} - f\,v = -\frac{1}{\rho_0}\frac{\partial p^{'}}{\partial x}\,, \tag{77}$$

$$\frac{\mathrm{d}\,v}{\mathrm{d}\,t} + f\,u = -\frac{1}{\rho_0}\frac{\partial p^{'}}{\partial y}\,, \tag{78}$$

$$\frac{\mathrm{d}\,w}{\mathrm{d}\,t} = -\frac{1}{\rho_0}\frac{\partial p^{'}}{\partial z} - g\frac{\rho^{'}}{\rho_0}\,, \tag{79}$$

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\rho^{'}}{\rho_0}\right) + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho_0}\frac{\partial(\rho_0 w)}{\partial z} = 0\,, \tag{80}$$

$$\frac{\theta^{'}}{\theta_0} = \frac{1}{\gamma\,g\,H}\frac{p^{'}}{\rho_0} - \frac{\rho^{'}}{\rho_0}\,, \tag{81}$$

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\theta^{'}}{\theta_0}\right) + \frac{N^2\,w}{g} = 0. \tag{82}$$

Equation (81) is derived from the combination of the state Equation (5), potential temperature equation (6) and adiabatic conditions, so the number of Equations (77)–(82) is one less than that of Equations (1)–(7). Here, N satisfies $\frac{\mathrm{d}^2 z}{\mathrm{d} t^2} + N^2 z = 0$. $N^2 > 0$, $N^2 = 0$ and $N^2 < 0$ indicate that stratification is stable, neutral and instability, respectively. In the atmosphere with stable stratification, N has the characteristics of frequency, so it is also called buoyancy frequency or Brunt-Väisälä frequency.

The weather system of large-scale atmospheric motion covers thousands of kilometers horizontally and the whole troposphere vertically. Its life history is generally about 5 days. The wind speed is about $10\,\mathrm{m/s}$ and the vertical speed is about $(1-5)\times 10^{-2}\,\mathrm{m/s}$. The basic scales of large-scale atmospheric motion are horizontal distance scale $L = 10^6\,\mathrm{m}$, vertical distance scale $D = 10^4\,\mathrm{m}$, horizontal wind speed scale $U = 10\,\mathrm{m}\cdot\mathrm{s}^{-1}$ and time scale $\tau = L/U = 10^5\,\mathrm{s}$.

The scales of Coriolis parameter f, Rossby parameter β, N^2 and the height are $f_0 = 10^{-4}\,\mathrm{s}^{-1}$, $\beta_0 = 10^{-11}\,\mathrm{m}^{-1}\cdot\mathrm{s}^{-1}$, $N^2 = 10^{-4}\,\mathrm{s}^{-2}$ and $H = 10^4\,\mathrm{m}$, respectively. Some other dimensionless parameters are [3]

$$\begin{cases} \alpha_0 \equiv \dfrac{N^2 H}{g} = 10^{-1},\; \mu_0^2 = \dfrac{f_0^2 L^2}{g\,H} = 10^{-1}, \\[3mm] \mu_1^2 = \dfrac{f_0^2 L^2}{N^2 H^2} = \dfrac{\mu_0^2}{\alpha_0} = 1,\; \beta_1 \equiv \dfrac{\beta_0 L^2}{U} = 1,\; \delta \equiv \dfrac{D}{L} = 10^{-2}. \end{cases} \tag{83}$$

Introduce a parameter Ro, which is defined as

$$Ro \equiv \frac{U}{f_0 L} = \frac{U^2/L}{f_0 U} = \frac{f_0^{-1}}{L/U} = \frac{U/L}{f_0} \tag{84}$$

The above formula means that Ro represents the ratio of horizontal inertial force to Coriolis force, the ratio of inertial characteristic time to advection time, and the ratio of relative vorticity to implicated vorticity. Therefore, Ro is a very important dimensionless parameter, which is often used to judge the scale of atmospheric motion. In large-scale atmospheric motion,

$$Ro = \frac{U}{f_0 \, L} = \frac{10}{10^{-4} \times 10^6} = 10^{-1}, \tag{85}$$

so we choose Ro as a small parameter. In the atmospheric motion with a frictionless and adiabatic static atmosphere as the background,

$$\sigma_0 \equiv \frac{\partial \rho_0}{\partial z} = \frac{N^2}{g} + \frac{1}{\gamma \, H}. \tag{86}$$

Its dimensionless quantity can be defined as

$$\sigma_1 \equiv \frac{\partial \rho_0}{\partial z_1} = D\sigma_0 = \frac{N^2 D}{g} + \frac{D}{\gamma \, H} = \alpha_0 + \frac{1}{\gamma}. \tag{87}$$

The physical quantities can be transformed into the corresponding dimensionless quantities, the concrete forms are

$$\begin{cases} x = L \, x_1, \ y = L \, y_1, \ z = D \, z_1, \ t = \frac{L}{U} \, t_1, \\ u = U \, u_1, \ v = U \, v_1, \ w = Ro \, \frac{D \, U}{L} \, w_1, \ f = f_0 \, f_1, \\ p' = \rho_0 \, f_0 \, U \, L \, p_1', w_1, \ \rho' = \rho_0 \, \mu_0^2 \, Ro \, \rho_1', \ \theta' = \theta_0 \, \mu_0^2 \, Ro \, \theta_1'. \end{cases} \tag{88}$$

The physical quantities marked with "1" are dimensionless, and their orders of magnitude are all 1.

The substitution of Formulas (88) into Formulas Equations (77)–(82) leads to [3]

$$Ro \left(\frac{\partial}{\partial t_1} + u_1 \frac{\partial}{\partial x_1} + v_1 \frac{\partial}{\partial y_1} + Ro \, w_1 \frac{\partial}{\partial z_1} \right) u_1 - f_1 \, v_1 = -\frac{\partial p_1'}{\partial x_1}, \tag{89}$$

$$Ro \left(\frac{\partial}{\partial t_1} + u_1 \frac{\partial}{\partial x_1} + v_1 \frac{\partial}{\partial y_1} + Ro \, w_1 \frac{\partial}{\partial z_1} \right) v_1 + f_1 \, u_1 = -\frac{\partial p_1'}{\partial y_1}, \tag{90}$$

$$\delta^2 \, Ro^2 \left(\frac{\partial}{\partial t_1} + u_1 \frac{\partial}{\partial x_1} + v_1 \frac{\partial}{\partial y_1} + Ro \, w_1 \frac{\partial}{\partial z_1} \right) w_1 = -\frac{\partial p_1'}{\partial z_1} + \sigma_1 p_1' - \rho_1', \tag{91}$$

$$\mu_0^2 \, Ro \left(\frac{\partial}{\partial t_1} + u_1 \frac{\partial}{\partial x_1} + v_1 \frac{\partial}{\partial y_1} + Ro \, w_1 \frac{\partial}{\partial z_1} \right) \rho_1' + \frac{\partial u_1}{\partial x_1}$$

$$+ \frac{\partial v_1}{\partial y_1} + Ro \, \frac{1}{\rho_0} \frac{\partial (\rho_0 w_1)}{\partial z_1} = 0, \tag{92}$$

$$\theta_1' = \frac{1}{\gamma} \, p_1' - \rho_1', \tag{93}$$

$$Ro \left[\left(\frac{\partial}{\partial t_1} + u_1 \frac{\partial}{\partial x_1} + v_1 \frac{\partial}{\partial y_1} + Ro \, w_1 \frac{\partial}{\partial z_1} \right) \theta_1' + \frac{\alpha_0}{\mu_0^2} \, w_1 \right] = 0. \tag{94}$$

The left term of (91) $\delta^2 \, Ro^2 = 10^{-6} \ll 1$, which can be accurately discarded. Substituting (93) into (91) and (92), and using the Formula (87), we can rewrite Formulas (89)–(94) as

$$Ro\left(\frac{\partial}{\partial t_1} + u_1\frac{\partial}{\partial x_1} + v_1\frac{\partial}{\partial y_1} + Ro\,w_1\frac{\partial}{\partial z_1}\right)u_1 - f_1\,v_1 = -\frac{\partial p_1{}'}{\partial x_1}, \tag{95}$$

$$Ro\left(\frac{\partial}{\partial t_1} + u_1\frac{\partial}{\partial x_1} + v_1\frac{\partial}{\partial y_1} + Ro\,w_1\frac{\partial}{\partial z_1}\right)v_1 + f_1\,u_1 = -\frac{\partial p_1{}'}{\partial y_1}, \tag{96}$$

$$\frac{\partial p_1{}'}{\partial z_1} - \alpha_0\,p_1{}' - \theta_1{}' = 0, \quad (\alpha_0 = 10^{-1} = Ro), \tag{97}$$

$$\mu_0{}^2\,Ro\left(\frac{\partial}{\partial t_1} + u_1\frac{\partial}{\partial x_1} + v_1\frac{\partial}{\partial y_1} + Ro\,w_1\frac{\partial}{\partial z_1}\right)\left(\frac{1}{\gamma}\,p_1{}' - \theta_1{}'\right)$$
$$+\frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial y_1} + Ro\,\frac{1}{\rho_0}\frac{\partial(\rho_0 w_1)}{\partial z_1} = 0 \tag{98}$$

$$Ro\left[\left(\frac{\partial}{\partial t_1} + u_1\frac{\partial}{\partial x_1} + v_1\frac{\partial}{\partial y_1} + Ro\,w_1\frac{\partial}{\partial z_1}\right)\theta_1{}' + \frac{\alpha_0}{\mu_0{}^2}\,w_1\right] = 0, \tag{99}$$

The term $\mu_0{}^2\,Ro = 10^{-2}$ in (98), which is ignored in Ref. [3], since Ref. [3] only considers the zero-order and first-order approximate equations. In this paper, we aim to derive the second-order approximate equation, then the first three terms $\mu_0{}^2\,Ro\left(\frac{\partial}{\partial t_1} + u_1\frac{\partial}{\partial x_1} + v_1\frac{\partial}{\partial y_1} + Ro\,w_1\frac{\partial}{\partial z_1}\right)\left(\frac{1}{\gamma}\,p_1{}' - \theta_1{}'\right)$ are very important, and we retain them. This is a difference between this paper and Ref. [3].

From Formulas (95)–(99), we can see that $Ro\,w_1$ is formally equivalent to u_1 and v_1, so the physical variables $\{u_1, v_1, w_1, p_1{}', \theta_1{}'\}$ can be expanded as follows:

$$u_1 = u_1{}^{(0)} + Ro\,u_1{}^{(1)} + Ro^2\,u_1{}^{(2)} + Ro^3\,u_1{}^{(3)} + \cdots, \tag{100}$$

$$v_1 = v_1{}^{(0)} + Ro\,v_1{}^{(1)} + Ro^2\,v_1{}^{(2)} + Ro^3\,v_1{}^{(3)} + \cdots, \tag{101}$$

$$w_1 = w_1{}^{(1)} + Ro\,w_1{}^{(2)} + Ro^2\,w_1{}^{(3)} + \cdots, \tag{102}$$

$$p_1{}' = p_1{}^{(0)} + Ro\,p_1{}^{(1)} + Ro^2\,p_1{}^{(2)} + Ro^3\,p_1{}^{(3)} + \cdots, \tag{103}$$

$$\theta_1{}' = \theta_1{}^{(0)} + Ro\,\theta_1{}^{(1)} + Ro^2\,\theta_1{}^{(2)} + Ro^3\,\theta_1{}^{(3)} + \cdots \tag{104}$$

where the upper right corners (0), (1), (2) and (3) represent the zero-order, first-order, second-order and third-order approximation, respectively. f_1 can be expanded as

$$f_1 \equiv \frac{f_0 + \beta_0\,y}{f_0} = 1 + Ro\,\beta_1\,y_1. \tag{105}$$

Substitute Equations (100)–(105) into the Equations (95)–(99), and compare the coefficients of different power terms for Ro, we can obtain different order approximate equations. The coefficients of Ro^0 and Ro^1 lead to zero-order and first-order approximate equations of Equations (95)–(99), respectively. From the coefficients of Ro^2, we derive the second-order approximate equations in the form of

$$\left(\frac{\partial}{\partial t_1} + u_1^{(0)} \frac{\partial}{\partial x_1} + v_1^{(0)} \frac{\partial}{\partial y_1}\right) u_1^{(1)} - \beta_1 y_1 v_1^{(1)} - v_1^{(2)}$$

$$+ \left(u_1^{(1)} \frac{\partial}{\partial x_1} + v_1^{(1)} \frac{\partial}{\partial y_1} + w_1^{(1)} \frac{\partial}{\partial z_1}\right) u_1^{(0)} = -\frac{\partial p_1^{(2)}}{\partial x_1}, \qquad (106)$$

$$\left(\frac{\partial}{\partial t_1} + u_1^{(0)} \frac{\partial}{\partial x_1} + v_1^{(0)} \frac{\partial}{\partial y_1}\right) v_1^{(1)} + \beta_1 y_1 u_1^{(1)} + u_1^{(2)}$$

$$+ \left(u_1^{(1)} \frac{\partial}{\partial x_1} + v_1^{(1)} \frac{\partial}{\partial y_1} + w_1^{(1)} \frac{\partial}{\partial z_1}\right) v_1^{(0)} = -\frac{\partial p_1^{(2)}}{\partial y_1}, \qquad (107)$$

$$\frac{\partial p_1^{(2)}}{\partial z_1} = \theta_1^{(2)} + \alpha_0 Ro^{-1} p_1^{(1)} \quad (\alpha_0 Ro^{-1} = 1), \qquad (108)$$

$$\frac{\partial u_1^{(2)}}{\partial x_1} + \frac{\partial v_1^{(2)}}{\partial y_1} + \frac{1}{\gamma}\left(\frac{\partial}{\partial t_1} + u_1^{(0)} \frac{\partial}{\partial x_1} + v_1^{(0)} \frac{\partial}{\partial y_1}\right) p_1^{(0)}$$

$$+ \frac{1}{\rho_0} \frac{\partial(\rho_0 w_1^{(2)})}{\partial z_1} - \left(\frac{\partial}{\partial t_1} + u_1^{(0)} \frac{\partial}{\partial x_1} + v_1^{(0)} \frac{\partial}{\partial y_1}\right) \theta_1^{(0)} = 0, \qquad (109)$$

$$\left(\frac{\partial}{\partial t_1} + u_1^{(0)} \frac{\partial}{\partial x_1} + v_1^{(0)} \frac{\partial}{\partial y_1}\right) \theta_1^{(1)}$$

$$+ \left(u_1^{(1)} \frac{\partial}{\partial x_1} + v_1^{(1)} \frac{\partial}{\partial y_1} + w_1^{(1)} \frac{\partial}{\partial z_1}\right) \theta_1^{(0)} + \frac{\alpha_0}{\mu_0^2} w_1^{(2)} = 0. \qquad (110)$$

The corresponding dimensional forms are

$$\left(\frac{\partial}{\partial t} + u^{(0)} \frac{\partial}{\partial x} + v^{(0)} \frac{\partial}{\partial y}\right) u^{(1)} - \beta_0 y v^{(1)} - f_0 v^{(2)}$$

$$+ \left(u^{(1)} \frac{\partial}{\partial x} + v^{(1)} \frac{\partial}{\partial y} + w^{(1)} \frac{\partial}{\partial z}\right) u^{(0)} = -\frac{1}{\rho_0} \frac{\partial p^{(2)}}{\partial x}, \qquad (111)$$

$$\left(\frac{\partial}{\partial t} + u^{(0)} \frac{\partial}{\partial x} + v^{(0)} \frac{\partial}{\partial y}\right) v^{(1)} + \beta_0 y u^{(1)} + f_0 u^{(2)}$$

$$+ \left(u^{(1)} \frac{\partial}{\partial x} + v^{(1)} \frac{\partial}{\partial y} + w^{(1)} \frac{\partial}{\partial z}\right) v^{(0)} = -\frac{1}{\rho_0} \frac{\partial p^{(2)}}{\partial y}, \qquad (112)$$

$$\frac{\partial}{\partial z}\left(\frac{p^{(2)}}{\rho_0}\right) = g \frac{\theta^{(2)}}{\theta_0} + \frac{N^2}{g} \frac{p^{(1)}}{\rho_0}, \qquad (113)$$

$$\frac{\partial u^{(2)}}{\partial x} + \frac{\partial v^{(2)}}{\partial y} + \frac{1}{\gamma} \frac{1}{g H}\left(\frac{\partial}{\partial t} + u^{(0)} \frac{\partial}{\partial x} + v^{(0)} \frac{\partial}{\partial y}\right) \frac{p^{(0)}}{\rho_0}$$

$$+ \frac{1}{\rho_0} \frac{\partial(\rho_0 w^{(2)})}{\partial z} - \left(\frac{\partial}{\partial t} + u^{(0)} \frac{\partial}{\partial x} + v^{(0)} \frac{\partial}{\partial y}\right) \frac{\theta^{(0)}}{\theta_0} = 0, \qquad (114)$$

$$\left(\frac{\partial}{\partial t} + u^{(0)} \frac{\partial}{\partial x} + v^{(0)} \frac{\partial}{\partial y}\right)\left(g \frac{\theta^{(1)}}{\theta_0}\right)$$

$$+ \left(u^{(1)} \frac{\partial}{\partial x} + v^{(1)} \frac{\partial}{\partial y} + w^{(1)} \frac{\partial}{\partial z}\right)\left(g \frac{\theta^{(0)}}{\theta_0}\right) + N^2 w^{(2)} = 0. \qquad (115)$$

The second-order approximate equations not only show the characteristics of physical quantities changing with time, but also establish the relationship between zero-order approximation, first-order approximation and second-order approximation. Moreover, the

second-order approximate equations are no longer static equations, but reflect the motion characteristics of convection.

6. Conclusions and Discussion

In this manuscript, symmetry properties for the basic equations of atmospheric motion are studied. The basic equations of atmospheric motion are invariant under space-time translation transformation, Galilean translation transformations and scaling transformations. From these results, we can know the symmetries for spatiotemporal variation of atmospheric motion.

Symmetry method is an effective means to obtain new solutions of the researched equations. Eight one-parameter invariant subgroups are listed. Eight one-parameter group invariant solutions are produced. Several types of nontrivial group invariants, similarity solutions and symmetry reduction equations are obtained. If further calculations are carried out, we can obtain some analytic solutions of these reduction equations. Combing the similarity solutions and the analytic solutions of the reduction equations, some analytic solutions of the basic equations of atmospheric motion can be obtained. These results are helpful for us to understand the law of atmospheric motion and provide guidance for weather forecasting.

By means of perturbation method, large-scale atmospheric motion equations can be expanded as equations of different orders. The zero-order approximate equations reflect the basic characteristics of atmospheric large-scale motion, namely geostrophic equilibrium, static equilibrium and horizontal nondivergence. The first-order approximate equations reflect the quasi geostrophic equilibrium of large-scale atmospheric motion, but there are still no convection term. In this paper, we develop a system of second-order approximate equations for the large scale atmospheric motion equations. The second-order approximate equations not only show the characteristics of physical quantities changing with time, but also establish the relationship between zero-order approximation, first-order approximation and second-order approximation. It is worth noting that the second-order approximate equations reflect the characteristics of atmospheric vertical motion. The second-order approximate equations include the non-dimensional equations and the dimensional equations. In the future, we will consider how to obtain the second-order approximate equations for large-scale atmospheric motion equations if the viscous terms are added. In addition, we will study analytic solutions, physical property and the corresponding atmospheric dynamics of the second-order approximate equations in the future.

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References

1. McWilliams, J.C. *Fundamentals of Geophysical Fluid Dynamics*; Cambridge University Press: London, UK, 2006.
2. Holton, J.R. *An Introduction to Dynamic Meteorogy*; Academic Press: Pittsburgh, PA, USA, 1992.
3. Liu, S.K.; Liu, S.D. *Atmospheric Dynamics*; Beijing University Press: Beijing, China, 2011.
4. Dutton, J. *Dynimic of Atmospheric Motion*; Dover Publication Inc.: New York, NY, USA, 1995.
5. Zhao, H.; Lu, Y.; Jiang, X.; Klotzbach, P.J.; Wu, L.; Cao, J. A statistical intraseasonal prediction model of extended boreal summer western north pacific tropical cyclone genesis. *J. Clim.* **2022**, *35*, 2459–2478. [CrossRef]

6. Chen, T.; Chen, S.; Zhou, M.; Tu, C. Northward shift in landfall locations of tropical cyclones over the western north pacific during the Last four decades. *Adv. Atmos. Sci.* **2022**, *39*, 304–319. [CrossRef]
7. Coniglio, M.C.; Jewell, R.E. SPC mesoscale analysis compared to field-project soundings: Implications for supercell environment studies. *Mon. Weather Rev.* **2022**, *150*, 567–588. [CrossRef]
8. Werth, D.; Buckley, R. The application of a genetic algorithm to the optimization of a mesoscale Model for emergency response. *J. Appl. Meteorol. Climatol.* **2022**, *61*, 329–343. [CrossRef]
9. Thompson, R.L.; Mead, C.M.; Edwards, R. Effective storm-relative helicity and bulk shear in supercell thunderstorm environments. *Weather Forecast.* **2007**, *22*, 102–115. [CrossRef]
10. Hua, Z.; Anderson-Frey, A.K. Self-organizing maps for the classification of spatial and temporal variability of tornado-favorable parameters. *Mon. Weather Rev.* **2022**, *150*, 393–407. [CrossRef]
11. Azerad, P.; Guill, F. Mathematical justification of the hydrostatic approximation in the primitive equations of geophysical fluid dynamics. *SIAM J. Math. Anal.* **2001**, *33*, 847–859. [CrossRef]
12. Szoeke, R.D.; Samelson, R.M. The duality between the Boussinesq and non-Boussinesq hydrostatic equations of Motion. *J. Phys. Ocean.* **2002**, *32*, 2194–2203. [CrossRef]
13. Feng, Y.; Miranda-Fuentes, J.; Jacob, J.; Sagaut, P. Hybrid lattice Boltzmann model for atmospheric flows under anelastic approximation. *Phys. Fluids* **2021**, *33*, 036607. [CrossRef]
14. Calkins, M.A.; Julien, K.; Marti, P. Onset of rotating and non-rotating convection in compressible and anelastic ideal gases. *Geophys. Astrophys. Fluid Dyn.* **2015**, *109*, 422–449. [CrossRef]
15. Liu, P.; Zeng, B.Q.; Deng, B.B.; Yang, J.R. Exact solutions of atmospheric (3+1)-dimensional nonlinear incompressible nonhydrostatic Boussinesq equations with viscosity. *AIP Adv.* **2015**, *5*, 087162. [CrossRef]
16. Liu, P.; Wang, Y.; Ren, B.; Li, J.H. Exact solutions of atmospheric (2+1)-dimensional nonlinear incompressible non-hydrostatic Boussinesq Equations. *Commun. Theor. Phys.* **2016**, *66*, 595–608. [CrossRef]
17. Liu, P.; Li, Z.L. Exact solutions of (3+1)-dimensional nonlinear incompressible non-hydrostatic Boussinesq equations. *Chin. Phys. B* **2013**, *22*, 050204. [CrossRef]
18. Olver, P. *Applications of Lie Group to Differential Equations*; Spring: New York, NY, USA, 1986.
19. Liu, P.; Huang, B.; Ren, B.; Yang, J.R. Consistent Riccati expansion solvability, symmetries, and analytic solutions of a forced variable-coefficient extended Korteveg-de Vries equation in fluid dynamics of internal solitary waves. *Chin. Phys. B* **2021**, *30*, 080203. [CrossRef]
20. Lou, S.Y. Full reversal symmetric multiple soliton solutions for integrable systems. *Acta Phys. Sin.* **2020**, *69*, 010503. [CrossRef]
21. Gao, X.N.; Lou, S.Y.; Tang, X.Y. Bosonization, singularity analysis, nonlocal symmetry reductions and exact solutions of supersymmetric KdV equation. *J. High Energy Phys.* **2013**, *5*, 29. [CrossRef]
22. Paliathanasis, A. One-Dimensional Optimal System for 2D Rotating Ideal Gas. *Symmetry* **2019**, *11*, 1115. [CrossRef]
23. Liu, P.; Gao, X.N. Symmetry analysis of nonlinear incompressible non-hydrostatic Boussinesq equations. *Commun. Theor. Phys.* **2010**, *53*, 609–614.