symmetry

MDPI

Article

UIO-Based Practical Fixed-Time Fault Estimation Observer Design of Nonlinear Systems

Jingping Xia, Bin Jiang * and Ke Zhang

College of Automation Engineering, Nanjing University of Aeronautics and Astronautics, Nanjing 211106, China
* Correspondence: binjiang@nuaa.edu.cn

Abstract: A practical fixed-time fault estimation observer strategy for non-linear systems using an unknown input observer scheme is investigated. The external disturbances of non-linear systems are decoupled by an unknown input observer technique; therefore, the constructed error dynamics do not include these disturbances. The fixed-time fault estimation observer is then constructed. Moreover, a non-linear fault estimator with two power functions is proposed to improve the convergence speed of fault estimation. Finally, simulation results of a non-linear Lorenz chaotic system are provided to demonstrate the feasibility of the presented strategy.

Keywords: fault estimation; fixed time; non-linear observer; unknown input observers

Citation: Xia, J.; Jiang, B.; Zhang, K. UIO-Based Practical Fixed-Time Fault Estimation Observer Design of Nonlinear Systems. *Symmetry* **2022**, *14*, 1618. https://doi.org/10.3390/sym14081618

Academic Editors: Zhirabok Alexey and Sergei D. Odintsov

Received: 4 July 2022
Accepted: 3 August 2022
Published: 6 August 2022

1. Introduction

The issue of fault diagnosis is crucial to improve the safety of control systems. In model-based fault diagnosis, fault estimation can identify the occurrence of faults in a timely and effective manner and provide accurate fault information for active fault-tolerant control design. Therefore, this problem has attracted considerable attention over recent decades [1–4].

The convergence rate is a key performance index of control systems. A finite-time method is proposed to improve the transient response of asymptotic convergence. However, the convergence time of the finite-time method is dependent on the initial value [5,6]. To solve this dilemma, a fixed-time design method is investigated to endow the upper bound of the settling time with independence from the initial values and to further improve the convergence rate of the system [7,8]. In [9], a sliding mode-based attitude stabilisation problem was considered for flexible spacecraft subject to modeling uncertainties and external disturbances. In [10], a fixed-time tracking problem of an unmanned surface vehicle was studied using a fixed-time stability strategy. In [11], an event-triggered method-based distributed fixed-time consensus issue was addressed for time-delay multi-agent systems. At present, most of the fixed-time design methods are applied to controller design. Observer results based on a fixed-time method are rare, and research results obtained for fixed-time fault estimation are limited. Fault information can often be used for active fault-tolerant control to recover system performance quickly [12]. The system has high requirements for rapid fault estimation, which is the motivation for this paper to consider the fault estimation problem using a fixed-time method.

Representative fault estimation observers include sliding mode observers [13–15], adaptive learning observers [16–18] and others. The adaptive learning observer method is widely used because of its simple structure and practicability. However, external disturbances in the design of such observers can easily affect the selection of observer parameters and learning rates. The selection of parameters for a fixed-time design is also easily influenced by these disturbances. In the fixed-time method, a sign function is introduced to suppress the influence of external disturbances. However, the sign function can increase the chattering of the system. This article investigates non-linear fixed-time fault estimation based on disturbance decoupling, which is a challenging task.

The unknown input observer (UIO) technique is an effective method aimed at decoupling external disturbances, which can completely eliminate the influence of unknown inputs. Such an observer has been widely investigated due to its good anti-disturbance ability [19–21]. However, the transient performance in the convergence process is usually ignored in the design of such an observer.

A non-linear fixed-time fault estimation observer using disturbance decoupling is investigated in this paper. This observer not only suppresses the influence of disturbance but also enhances the transient performance of fault estimation. The main contributions of the work are as follows: (i) A UIO-based fixed-time fault estimation observer is proposed to eliminate the influence of external disturbances on fault estimation; (ii) A fault estimation observer and a non-linear fault estimator with two power functions are constructed based on a fixed-time design to improve the rapidity of fault estimation.

The paper is organised as follows: Section 2 describes the non-linear system and the problem investigated. Based on a fixed-time method, Section 3 presents a fault estimation approach using a UIO and a non-linear fault estimator. Section 4 provides the simulation results of a non-linear Lorenz chaotic system, which illustrates the effectiveness of the fixed-time fault estimation. Section 5 provides concluding remarks.

2. Systems Description and Problem Formation

This paper considers a model of non-linear systems as follows:

$$\dot{x}(t) = Ax(t) + \rho(t, x(t)) + Bu_f(t) + Dd(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u_f(t) \in \mathbb{R}^m$ is the control input with actuator faults, $d(t) \in \mathbb{R}^d$ is the external disturbance and is bounded, and $\rho(t, x(t)) \in \mathbb{R}^n$ is the non-linear function; A is the system matrix, B is the input matrix, and D is the disturbance distributed matrix; $\rho(t, x(t))$ satisfies the Lipschitz condition, i.e., $\|\rho(t, x_1(t)) - \rho(t, x_2(t))\|^2 \leq l_p^2 \|x_1(t) - x_2(t)\|^2$, and the Lipschitz constant is $l_p > 0$. It is assumed that B and D are column-full rank.

Remark 1. *For the design of a state observer, many non-linear functions meet Lipschitz conditions in practical systems (at least local Lipschitz conditions).*

The actuator fault is usually described as below

$$u_f(t) = \sigma(t)u(t) + \delta(t), \quad t > t_f \tag{2}$$

where $u(t) \in \mathbb{R}^m$ is the designed control, $\sigma(t) = \mathrm{diag}\{\sigma_1(t), \sigma_2(t), \ldots, \sigma_m(t)\} \in \mathbb{R}^{m \times m}$ denotes the loss rate of actuator effectiveness, and $\sigma_i(t)$ is an unknown value within the range $(0, 1)$; $\delta(t) = [\delta_1(t), \delta_2(t), \ldots, \delta_m(t)]^{\mathrm{T}} \in \mathbb{R}^m$ represents the actuator bias fault.

In active fault-tolerant control, to estimate the actuator fault conveniently, the actuator fault model (2) can be rewritten as

$$u_f(t) = u(t) + f(t) \tag{3}$$

where $f(t) = (\sigma(t) - I)u(t) + \delta(t)$ [22], and it is assumed that $f(t)$ and its derivative $\dot{f}(t)$ are bounded. Therefore, the non-linear system (1) is represented as

$$\dot{x}(t) = Ax(t) + \rho(t, x(t)) + B(u(t) + f(t)) + Dd(t) \tag{4}$$

Remark 2. *In the non-linear system considered in (4), the $f(t)$ considered in (4) denotes actuator fault, which includes loss of effectiveness and bias faults. The presented fault form is general, because loss of actuator effectiveness can be expressed as a special form of bias faults [23]. Note that, in contrast to external disturbances, actuator faults occur in the control input channel, i.e., $B(u(t) + f(t))$ [24].*

Remark 3. *In general, fault-tolerant control methods are divided into passive fault-tolerant control and active fault-tolerant control [25]. Passive fault-tolerant control does not need a fault diagnosis module to obtain fault information, and the influence of a fault on the system is suppressed, based on the idea of robust control. As for active fault-tolerant control, a fault diagnosis module is needed to obtain accurate fault information [26,27].*

Remark 4. *Model-based fault diagnosis is usually divided into the following three steps: fault detection, fault isolation and fault estimation. Fault detection is used to determine the time of fault occurrence, fault isolation can determine the location of fault occurrence, and fault estimation can be used to identify the size and amplitude of faults. Among these, the issue of fault estimation is focused on in this paper. Actuator faults must be estimated online in real-time; the estimated value is used as important information for active fault-tolerant control to recover system performance. Accurate fault estimation effectively increases system performance restoration [28–30]. However, external disturbance* $d(t)$ *needs to be suppressed to reduce its impact on fault estimation. Therefore, decoupling is considered in this paper to eliminate its impact on fault estimation.*

This paper seeks to present a UIO-based fault estimation algorithm with disturbance decoupling characteristics to eliminate the effect of disturbance on fault estimation and to propose a fast non-linear fault estimation algorithm based on a fixed-time strategy.

The useful lemmas are first presented here.

Lemma 1 ([9])**.** *For a continuous-time function* $\dot{x}(t) = F(x,t)$*, the origin is an equilibrium working point. If there is a Lyapunov function* $V(t)$ *satisfying*

$$\dot{V}(t) \leq -\eta_1 V^{h_1}(t) - \eta_2 V^{h_2}(t) + \phi \tag{5}$$

based on Lyapunov stability theory, where $\eta_1 > 0$, $\eta_2 > 0$, $\phi > 0$, $0 < h_1 < 1$, $1 < h_2 < +\infty$*, then the origin of* $\dot{x}(t) = F(x,t)$ *is practical fixed-time stable, and the residual set is*

$$\left\{ x(t) | V(t) \leq \min\left\{ \left(\frac{\phi}{\eta_1(1-\vartheta)} \right)^{\frac{1}{h_1}}, \left(\frac{\phi}{\eta_2(1-\vartheta)} \right)^{\frac{1}{h_2}} \right\} \right\} \tag{6}$$

And the upper bound of the settling time is

$$T_{\max} = \frac{1}{\eta_1 \vartheta (1-h_1)} + \frac{1}{\eta_2 \vartheta (h_2 - 1)} \tag{7}$$

where $0 < \vartheta < 1$.

Lemma 2 ([31])**.** *Let* $\varrho_1, \varrho_2, \ldots, \varrho_N \geq 0$*, then*

$$\sum_{i=1}^{N} \varrho_i^{h_1} \geq \left(\sum_{i=1}^{N} \varrho_i \right)^{h_1}, \quad 0 < h_1 < 1 \tag{8}$$

$$\sum_{i=1}^{N} \varrho_i^{h_2} \geq N^{1-h_2} \left(\sum_{i=1}^{N} \varrho_i \right)^{h_2}, \quad 1 < h_2 < +\infty \tag{9}$$

Lemma 3 ([32])**.** *For* $x, y \in \mathbb{R}$, α, β*, and* λ *are positive scalars, then the following relation holds*

$$|x|^{\alpha} |y|^{\beta} \leq \frac{\alpha}{\alpha+\beta} \lambda |x|^{\alpha+\beta} + \frac{\alpha}{\alpha+\beta} \lambda^{-\frac{\alpha}{\beta}} |y|^{\alpha+\beta} \tag{10}$$

3. Main Results

The design procedure of the non-linear fixed-time fault estimation observer is presented in this section. A stability analysis is provided on the basis of Lyapunov stability theory.

For the non-linear system (4), the following UIO-based fixed-time fault estimation approach can be constructed to decouple external disturbances and accurately identify actuator faults.

$$\dot{z}(t) = Nz(t) + Gu(t) + MB\hat{f}(t) + Lx(t) + M\hat{\rho}(t, \hat{x}(t)) + \Phi(t) \tag{11}$$

$$\hat{x}(t) = z(t) - Fx(t) \tag{12}$$

where $z(t) \in \mathbb{R}^n$ is the observer state, $\hat{f}(t) \in \mathbb{R}^m$ is the fault estimation, $\hat{x}(t) \in \mathbb{R}^n$ is the state estimation, $\hat{\rho}(t, \hat{x}(t)) \in \mathbb{R}^m$ is the estimation of $\rho(t, x(t))$; N, G, M, L, F are matrices with appropriate dimensions to be determined; $\Phi(t)$ is a fixed-time non-linearity to guarantee the convergence speed of state estimation, which is designed in Theorem 1 later.

Remark 5. *For the presented fault estimation scheme, the observer state $z(t)$ is an intermediate variable, whose function is to obtain the state estimation $\hat{x}(t)$ and estimate the unknown fault $f(t)$. Notably, $\hat{f}(t)$ is the online fault estimator; a new non-linear fixed-time estimation algorithm $\hat{f}(t)$ is proposed in Theorem 1.*

Remark 6. *The non-linear term $\hat{\rho}(t, \hat{x}(t))$ has the same structure as $\rho(t, x(t))$, and is constructed by replacing $x(t)$ of $\rho(t, x(t))$ with $\hat{x}(t)$. Instead of the observer state $z(t)$, the observer dynamics (11) reveal that the function $\hat{\rho}(t, \hat{x}(t))$ is related to $\hat{x}(t)$. Therefore, the constructed observer needs the state estimation information $\hat{x}(t)$.*

Remark 7. *In contrast to traditional UIO design, the proposed UIO-based fault estimation strategy adds a fixed-time function $\Phi(t)$, as described in (11), to improve the convergence speed of state and fault vectors; its specific expression is given in the following Theorem 1.*

Denote two error vectors $e(t) = \hat{x}(t) - x(t)$ and $e_f(t) = \hat{f}(t) - f(t)$, then one obtains

$$\begin{aligned} e(t) &= z(t) - Fx(t) - x(t) \\ &= z(t) - Mx(t) \end{aligned} \tag{13}$$

where $M = I + F$.

The derivative of the error dynamics (13) is

$$\begin{aligned} \dot{e}(t) &= \dot{z}(t) - M\dot{x}(t) \\ &= Nz(t) + Gu(t) + MB\hat{f}(t) + Lx(t) + M\hat{\rho}(t, \hat{x}(t)) + \Phi(t) \\ &\quad - MAx(t) - MBu(t) - MBf(t) - M\rho(t, x(t)) - MDd(t) \\ &= Ne(t) + M(\hat{\rho}(t, \hat{x}(t)) - \rho(t, x(t))) + MBe_f(t) + (NM + L - MA)x(t) \\ &\quad + (G - MB)u(t) - MDd(t) + \Phi(t) \end{aligned} \tag{14}$$

If the following conditions hold

$$NM + L - MA = 0, \tag{15}$$

$$G = MB, \tag{16}$$

$$MD = 0, \tag{17}$$

then the error dynamics (14) are expressed as

$$\dot{e}(t) = Ne(t) + M\Psi(t, e(t)) + MBe_f(t) + \Phi(t) \tag{18}$$

where $\Psi(t, e(t)) = \hat{\rho}(t, \hat{x}(t)) - \rho(t, x(t))$.

Remark 8. *Since the condition $MD = 0$ is introduced, we can see that the error Equation (18) is completely unaffected by the disturbance, which will facilitate the design of a fault estimation observer scheme.*

Theorem 1. *If there is a symmetric positive definite matrix P, matrices $\bar{Y}, \bar{K}, N, G, M, L, F$, and positive scalars $\epsilon, \chi, \nu_1, \nu_2, \mu_1, \mu_2, 0 < h_1, p_1 < 1, 1 < h_2, p_2 < +\infty$ such that equalities (15)–(17) and the following condition are satisfied*

$$\begin{bmatrix} (P(I+U)A + \bar{Y}VA - \bar{K}) + (P(I+U)A + \bar{Y}VA - \bar{K})^{\mathrm{T}} + \epsilon l_p^2 & P(I+U) + \bar{Y}V \\ * & -\epsilon I \end{bmatrix} < 0 \tag{19}$$

then the fixed-time non-linear fault estimator

$$\dot{\hat{f}}(t) = -\chi \hat{f}(t) - \Gamma B^{\mathrm{T}}(\mu_1 \mathrm{sig}^{p_1}(e) + \mu_2 \mathrm{sig}^{p_2}(e) + M^{\mathrm{T}} P e) \tag{20}$$

where $\Gamma > 0$ is the learning rate, and the error dynamics (18) with the non-linear term $\Phi(t)$ such that the errors $e(t)$ and $e_f(t)$ are practical fixed-time stable and converge to the residual set (37) within a fixed time (38), where $\Phi(t) = \nu_1 P^{-0.5}\mathrm{sig}^{h_1}(P^{0.5}e(t)) + \nu_2 P^{-0.5}\mathrm{sig}^{h_2}(P^{0.5}e(t))$, and $\mathrm{sig}^{h_1}(P^{0.5}e(t)) = [\mathrm{sign}(P_1^{0.5}e(t))|P_1^{0.5}e(t)|^{h_1}, \mathrm{sign}(P_2^{0.5}e(t))|P_2^{0.5}e(t)|^{h_1}, \ldots, \mathrm{sign}(P_n^{0.5}e(t))|P_n^{0.5}e(t)|^{h_1}]^{\mathrm{T}}$, sign$(\cdot)$ is the sign function, $P_i^{0.5}$ represents the ith row of matrix $P^{0.5}$, and $|\cdot|$ is the absolute value of a scalar.

Proof. Step 1. The Lyapunov function is chosen as:

$$V(t) = e^{\mathrm{T}}(t)Pe(t) + e_f^{\mathrm{T}}(t)\Gamma e_f(t) \tag{21}$$

The derivative of the Lyapunov function $V(t)$ along the error dynamics (18) is

$$\begin{aligned} \dot{V}(t) =& e^{\mathrm{T}}(t)[PN + N^{\mathrm{T}}P]e(t) + 2e^{\mathrm{T}}(t)PM\Psi(t, e(t)) + 2e^{\mathrm{T}}(t)PMBe_f(t) \\ & - 2\nu_1 e^{\mathrm{T}}(t)P^{0.5}\mathrm{sig}^{h_1}(P^{0.5}e(t)) - 2\nu_2 e^{\mathrm{T}}(t)P^{0.5}\mathrm{sig}^{h_2}(P^{0.5}e(t)) \\ & - 2\chi e_f^{\mathrm{T}}(t)\Gamma^{-1}\hat{f}(t) - 2e_f^{\mathrm{T}}(t)B^{\mathrm{T}}(\mu_1\mathrm{sig}^{p_1}(e(t)) + \mu_2\mathrm{sig}^{p_2}(e(t)) + M^{\mathrm{T}}Pe(t)) \\ & - 2e_f^{\mathrm{T}}(t)\Gamma^{-1}\dot{f}(t) \end{aligned} \tag{22}$$

Based on the non-linearity $\rho(t, x(t))$ meeting the Lipschitz constraint, one obtains

$$\begin{aligned} & e^{\mathrm{T}}(t)[PN + N^{\mathrm{T}}P]e(t) + 2e^{\mathrm{T}}(t)PM\Psi(t, e(t)) \\ \leq & e^{\mathrm{T}}(t)[PN + N^{\mathrm{T}}P]e(t) + \frac{1}{\epsilon}e^{\mathrm{T}}(t)PMM^{\mathrm{T}}Pe(t) + \epsilon l_p^2 e^{\mathrm{T}}(t)e(t) \end{aligned} \tag{23}$$

According to the Schur complement lemma, if condition

$$\begin{bmatrix} PN + N^{\mathrm{T}}P + \epsilon l_p^2 I & PM \\ * & -\epsilon I \end{bmatrix} < 0 \tag{24}$$

holds, we can derive

$$e^{\mathrm{T}}(t)[PN + N^{\mathrm{T}}P]e(t) + \frac{1}{\epsilon}e^{\mathrm{T}}(t)PMM^{\mathrm{T}}Pe(t) + \epsilon l_p^2 e^{\mathrm{T}}(t)e(t) < 0 \tag{25}$$

After that, we have

$$\begin{aligned} \dot{V}(t) \leq & - 2\nu_1 e^{\mathrm{T}}(t)P^{0.5}\mathrm{sig}^{h_1}(P^{0.5}e(t)) - 2\nu_2 e^{\mathrm{T}}(t)P^{0.5}\mathrm{sig}^{h_2}(P^{0.5}e(t)) \\ & - 2\chi e_f^{\mathrm{T}}(t)\Gamma^{-1}\hat{f}(t) - 2e_f^{\mathrm{T}}(t)B^{\mathrm{T}}(\mu_1\mathrm{sig}^{p_1}(e(t)) + \mu_2\mathrm{sig}^{p_2}(e(t))) \\ & - 2e_f^{\mathrm{T}}(t)\Gamma^{-1}\dot{f}(t) \end{aligned} \tag{26}$$

Furthermore, according to the Young's inequality and Lemma 2, one gets

$$-2\nu_1 e^{\mathrm{T}}(t)P^{0.5}\mathrm{sig}^{h_1}(P^{0.5}e(t)) \leq -2\nu_1(e^{\mathrm{T}}(t)Pe(t))^{\frac{h_1+1}{2}} \tag{27}$$

$$-2\nu_2 e^{\mathrm{T}}(t)P^{0.5}\mathrm{sig}^{h_2}(P^{0.5}e(t)) \leq -2\nu_2 n^{\frac{1-h_2}{2}}(e^{\mathrm{T}}(t)Pe(t))^{\frac{h_2+1}{2}} \tag{28}$$

$$-2\mu_1 e_f^{\mathrm{T}}(t)B^{\mathrm{T}}\mathrm{sig}^{p_1}(e) \leq \mu_1\sigma e_f^{\mathrm{T}}(t)\Gamma^{-1}e_f(t) + \frac{\mu_1}{\sigma}(\mathrm{sig}^{p_1}(e))^{\mathrm{T}}B\Gamma B^{\mathrm{T}}\mathrm{sig}^{p_1}(e) \tag{29}$$

$$-2\mu_2 e_f^{\mathrm{T}}(t)B^{\mathrm{T}}\mathrm{sig}^{p_2}(e) \leq \mu_2\psi e_f^{\mathrm{T}}(t)\Gamma^{-1}e_f(t) + \frac{\mu_2}{\psi}(\mathrm{sig}^{p_2}(e))^{\mathrm{T}}B\Gamma B^{\mathrm{T}}\mathrm{sig}^{p_2}(e) \tag{30}$$

$$\begin{aligned}-2\chi e_f^{\mathrm{T}}(t)\Gamma^{-1}\hat{f}(t) =& -2\chi e_f^{\mathrm{T}}(t)\Gamma^{-1}(e_f(t)+f(t))\\ \leq& -\chi e_f^{\mathrm{T}}(t)\Gamma^{-1}e_f(t) + \chi f^{\mathrm{T}}(t)\Gamma^{-1}f(t)\end{aligned} \tag{31}$$

$$-2e_f^{\mathrm{T}}(t)\Gamma^{-1}\dot{f}(t) \leq \varsigma e_f^{\mathrm{T}}(t)\Gamma^{-1}e_f(t) + \frac{1}{\varsigma}\dot{f}^{\mathrm{T}}(t)\Gamma^{-1}\dot{f}(t) \tag{32}$$

If $\chi - \mu_1\sigma - \mu_2\psi - \varsigma > 0$ and let $\tau = \chi - \mu_1\sigma - \mu_2\psi - \varsigma$, then one obtains

$$\begin{aligned}\dot{V}(t) \leq& -2\nu_1(e^{\mathrm{T}}(t)Pe(t))^{\frac{h_1+1}{2}} - 2\nu_2 n^{\frac{1-h_2}{2}}(e^{\mathrm{T}}(t)Pe(t))^{\frac{h_2+1}{2}}\\ &-\tau e_f^{\mathrm{T}}(t)\Gamma^{-1}e_f(t) + \xi_1(t)\end{aligned} \tag{33}$$

where

$$\begin{aligned}\xi_1(t) = &\frac{\mu_1}{\sigma}(\mathrm{sig}^{p_1}(e))^{\mathrm{T}}B\Gamma B^{\mathrm{T}}\mathrm{sig}^{p_1}(e) + \frac{\mu_2}{\psi}(\mathrm{sig}^{p_2}(e))^{\mathrm{T}}B\Gamma B^{\mathrm{T}}\mathrm{sig}^{p_2}(e) +\\ &\chi f^{\mathrm{T}}(t)\Gamma^{-1}f(t) + \frac{1}{\varsigma}\dot{f}^{\mathrm{T}}(t)\Gamma^{-1}\dot{f}(t).\end{aligned}$$

Moreover, we have

$$\begin{aligned}\dot{V}(t) \leq& -2\nu_1(e^{\mathrm{T}}(t)Pe(t))^{\frac{h_1+1}{2}} - 2\nu_2 n^{\frac{1-h_2}{2}}(e^{\mathrm{T}}(t)Pe(t))^{\frac{h_2+1}{2}} - (\tau e_f^{\mathrm{T}}(t)\Gamma^{-1}e_f(t))^{\frac{h_1+1}{2}}\\ &-(\tau e_f^{\mathrm{T}}(t)\Gamma^{-1}e_f(t))^{\frac{h_2+1}{2}} + (\tau e_f^{\mathrm{T}}(t)\Gamma^{-1}e_f(t))^{\frac{h_1+1}{2}} + (\tau e_f^{\mathrm{T}}(t)\Gamma^{-1}e_f(t))^{\frac{h_2+1}{2}}\\ &-\tau e_f^{\mathrm{T}}(t)\Gamma^{-1}e_f(t) + \xi_1(t)\end{aligned} \tag{34}$$

Based on Lemma 3, the following inequality is satisfied

$$(\tau e_f^{\mathrm{T}}(t)\Gamma^{-1}e_f(t))^{\frac{h_1+1}{2}} - \tau e_f^{\mathrm{T}}(t)\Gamma^{-1}e_f(t) \leq (\frac{h_1+1}{2})^{\frac{1+h_1}{1-h_1}} - (\frac{h_1+1}{2})^{\frac{2}{1-h_1}} \tag{35}$$

Let $\alpha_1 = \min\{2\nu_1, \tau^{\frac{h_1+1}{2}}\}$, $\alpha_2 = \min\{2\nu_2 n^{\frac{1-h_2}{2}}, \tau^{\frac{h_2+1}{2}}\}$, then it yields

$$\begin{aligned}\dot{V}(t) \leq& -\alpha_1(V(t))^{\frac{h_1+1}{2}} - \alpha_2 2^{\frac{1-h_2}{2}}(V(t))^{\frac{h_2+1}{2}} + \xi_1(t) + \xi_2(t)\\ \leq& -\alpha_1(V(t))^{\frac{h_1+1}{2}} - \alpha_2 2^{\frac{1-h_2}{2}}(V(t))^{\frac{h_2+1}{2}} + \xi\end{aligned} \tag{36}$$

where $\xi_2(t) = (\tau e_f^{\mathrm{T}}(t)\Gamma^{-1}e_f(t))^{\frac{h_2+1}{2}} + (\frac{h_1+1}{2})^{\frac{1+h_1}{1-h_1}} - (\frac{h_1+1}{2})^{\frac{2}{1-h_1}}$, ξ is the upper bound of $\xi_1(t) + \xi_2(t)$, i.e., $\xi \geq \xi_1(t) + \xi_2(t)$. Since both $\xi_1(t)$ and $\xi_2(t)$ are positive and bounded, the ξ does exist.

According to Lemma 1, the errors $e(t)$ and $e_f(t)$ are practical fixed-time stable and converge to the residual set

$$\left\{\{e(t), e_f(t)\}|V(t) \leq \min\left\{\left(\frac{\xi}{\alpha_1(1-\vartheta)}\right)^{\frac{2}{h_1+1}}, \left(\frac{\xi}{\alpha_2 2^{\frac{1-h_2}{2}}(1-\vartheta)}\right)^{\frac{2}{h_2+1}}\right\}\right\} \tag{37}$$

within a fixed time. The settling time is

$$\begin{aligned} T &\leq \frac{1}{\alpha_1\vartheta(1-\frac{h_1+1}{2})} + \frac{1}{\alpha_2 2^{\frac{1-h_2}{2}}\vartheta(\frac{h_2+1}{2}-1)} \\ &= \frac{1}{\alpha_1\vartheta(\frac{1-h_1}{2})} + \frac{1}{\alpha_2 2^{\frac{1-h_2}{2}}\vartheta(\frac{h_2-1}{2})} \end{aligned} \tag{38}$$

where $0 < \vartheta < 1$.

Step 2. It follows from (17) that

$$MD = (I+F)D = 0 \tag{39}$$

then we can obtain the solution of the unknown matrix F

$$F = U + YV \tag{40}$$

where $U = -DD^+$, $V = (I - DD^+)$, $D^+ = (D^{\mathrm{T}}D)^{-1}D^{\mathrm{T}}$, and Y is an any matrix with appropriate dimension.

Let $N = MA - K$ and $K = L + NF$ satisfy condition (15), then

$$\begin{aligned} PN &= PMA - PK \\ &= P(I+U+YV)A - PK \\ &= P(I+U)A + \bar{Y}VA - \bar{K} \end{aligned} \tag{41}$$

$$PM = P(I+U) + \bar{Y}V \tag{42}$$

where $\bar{Y} = PY$ and $\bar{K} = PK$, then, substituting expressions PN and PM into (25), we can obtain condition (19). This is the end of this proof. □

Remark 9. *The non-linear fixed-time fault estimator is presented in (20), where Γ is the learning rate. The estimator is used to identify the unknown actuator fault based on Lyapunov stability theory. Meanwhile, the learning rate Γ is chosen to meet the performance requirements of fault estimation.*

Remark 10. *The fault estimation algorithm depends on the state estimation error; thus, the convergence speed of the state estimation error is also crucial. From (11) and (20), the observer and non-linear fault estimation algorithm contain double-power terms. The function $\Phi(t)$ in (11) is used to accelerate the convergence of the state estimation error, whilst power functions in (20) can improve the speed of fault estimation. The given double-power functions are the signum functions of state estimation error. If $\Phi(t) = 0$, then the convergence speed of state estimation errors would slow down, which also affects the performance of fault estimation.*

Remark 11. *For the first step in this proof, the matrices to be designed in the fault estimation observer cannot be obtained directly from (25); therefore, Step 2 is necessary to calculate these matrices.*

Remark 12. *From Theorem 1, matrices P, $Y = P^{-1}\bar{Y}$, and $K = P^{-1}\bar{K}$ can be calculated directly. Afterwards, matrices by $F = U + YV$, $M = I + F$, $G = MB$, and $N = MA - K$ can be obtained in the UIO design (11) and (12).*

Remark 13. *In contrast to existing fixed-time results [32,33], the proposed method includes not only two power functions in the fault estimation observer, but also in the fault estimator. Moreover, the two groups of power functions are different, which increases the degrees of freedom and improves the convergence speed of fault estimation.*

4. Simulation Results

Consider the non-linear Lorenz chaotic system to verify the effectiveness of the presented design method [34], whose system parameters are as follows:

$$A = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{bmatrix}, \quad \rho(t, x(t)) = \begin{bmatrix} 0 \\ -x_1(t)x_3(t) \\ x_1(t)x_2(t) \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

It is verified that matrices B and D are column full rank. According to the disturbance matrix D, we can obtain matrices U and V in (40).

$$U = -DD^{+} = \begin{bmatrix} -1/3 & -1/3 & -1/3 \\ -1/3 & -1/3 & -1/3 \\ -1/3 & -1/3 & -1/3 \end{bmatrix}, V = (I - DD^{+}) = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}.$$

For the non-linear function $\rho(t, x(t))$, a local Lipschitz constant is selected as $l_p = 20$. Then the following solutions can be obtained by solving condition (19) in Theorem 1.

$$P = \begin{bmatrix} 38.5460 & 0.0894 & 0.5412 \\ 0.0894 & 38.0065 & 0.7700 \\ 0.5412 & 0.7700 & 38.5166 \end{bmatrix}, \quad Y = \begin{bmatrix} -0.5035 & 0.1073 & 0.3961 \\ 0.2416 & -0.6319 & 0.3903 \\ 0.3219 & 0.3258 & -0.6477 \end{bmatrix},$$

$$K = \begin{bmatrix} 4.3876 & 0.6881 & -0.1951 \\ 0.7036 & 7.1243 & -0.2910 \\ -0.1674 & -0.2914 & 7.4154 \end{bmatrix}, \quad \epsilon = 1.1948.$$

Based on Remark 11, the unknown matrices in UIO-based fault estimation scheme (11) and (12) are calculated as

$$F = \begin{bmatrix} -0.8368 & -0.2260 & 0.0628 \\ -0.0918 & -0.9652 & 0.0570 \\ -0.0114 & -0.0075 & -0.9810 \end{bmatrix}, \quad M = \begin{bmatrix} 0.1632 & -0.2260 & 0.0628 \\ -0.0918 & 0.0348 & 0.0570 \\ -0.0114 & -0.0075 & 0.0190 \end{bmatrix},$$

$$G = \begin{bmatrix} 0.1632 & 0.0628 \\ -0.0918 & 0.0570 \\ -0.0114 & 0.0190 \end{bmatrix}, \quad N = \begin{bmatrix} -12.3484 & 1.1701 & 0.0276 \\ 1.1884 & -8.0769 & 0.1390 \\ 0.0702 & 0.1849 & -7.4660 \end{bmatrix},$$

$$L = \begin{bmatrix} -5.8375 & -0.9732 & 0.5408 \\ 0.9583 & -0.4019 & 0.2311 \\ -0.1769 & -0.1534 & 0.0760 \end{bmatrix}.$$

The other parameters of the fixed-time fault estimation observer are selected as $h_1 = 0.1, h_2 = 2, \nu_1 = 0.1$, and $\nu_2 = 3$. Therefore, the non-linear term in the UIO-based fault estimation observer (11) is $\Phi(t) = 0.1P^{-0.5}\mathrm{sig}^{0.1}(P^{0.5}e(t)) + 3P^{-0.5}\mathrm{sig}^{2}(P^{0.5}e(t))$. The parameters of the fault estimator are $p_1 = 0.01, p_2 = 2, \mu_1 = 0.1, \mu_2 = 4$, and $\chi = 0.001$.

In simulation, it is supposed that the actuator fault $f(t) = [f_1(t), f_2(t)]^{\mathrm{T}}$ is as follows:

$$f_1(t) = \begin{cases} 0 & t \leq 8s \\ 4 & t > 8s \end{cases}, \quad f_2(t) = \begin{cases} 0 & t \leq 10s \\ 2\sin(0.5t) & t > 10s \end{cases}$$

Herein, actuator bias and time-varying faults, which can be used to describe loss of actuator effectiveness, are both considered. The simulation results are illustrated in Figure 1. This figure shows that the presented fault estimation method can identify all faults of each control channel simultaneously and is unaffected by external disturbances.

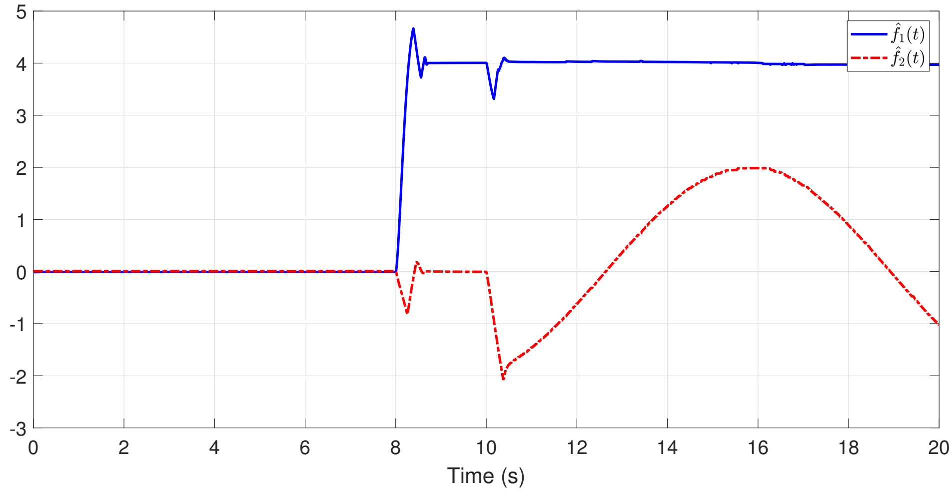

Figure 1. Actuator fault estimation.

The real fault value is compared with the estimated value, as shown in Figures 2 and 3, to verify the effect of fault estimation. The proposed fixed fault estimator has a rapid fault estimation effect, regardless of a constant or a time-varying fault.

When $\Phi(t) = 0$ in (11), the convergence speed of state estimation errors will slow down, which will influence the fault estimation effect, as shown in Figure 4. Comparison reveals that the introduction of $\Phi(t)$ can improve the performance of fault estimation by accelerating the convergence speed of the fault estimation error.

When $\mu_1, \mu_2 = 0$ in (20), the presented fault algorithm strategy (20) becomes the conventional one. Figure 5 shows that the proposed method in this paper has superior performance for time-varying fault estimation despite the positive effect of constant fault estimation. Therefore, the proposed fault estimation algorithm in this paper shows good fault estimation performance for constant and time-varying faults.

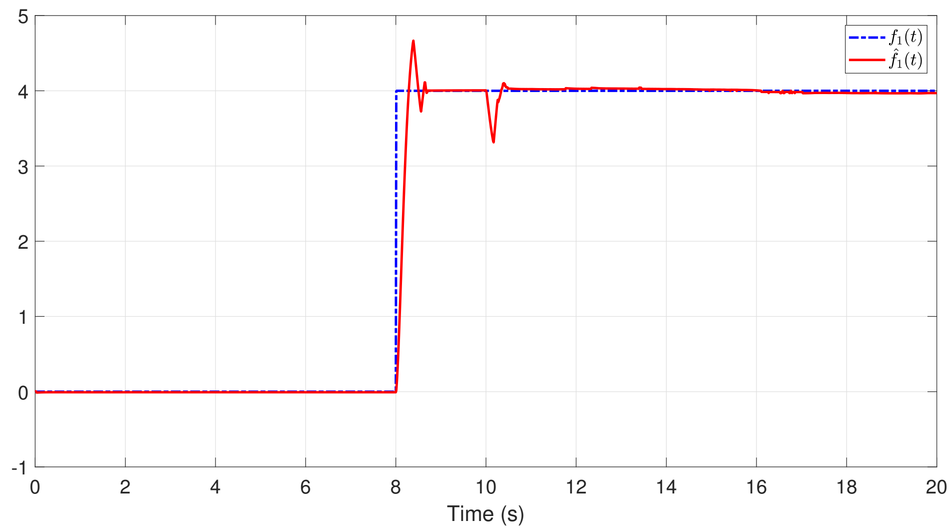

Figure 2. Fault $f_1(t)$ estimation comparison.

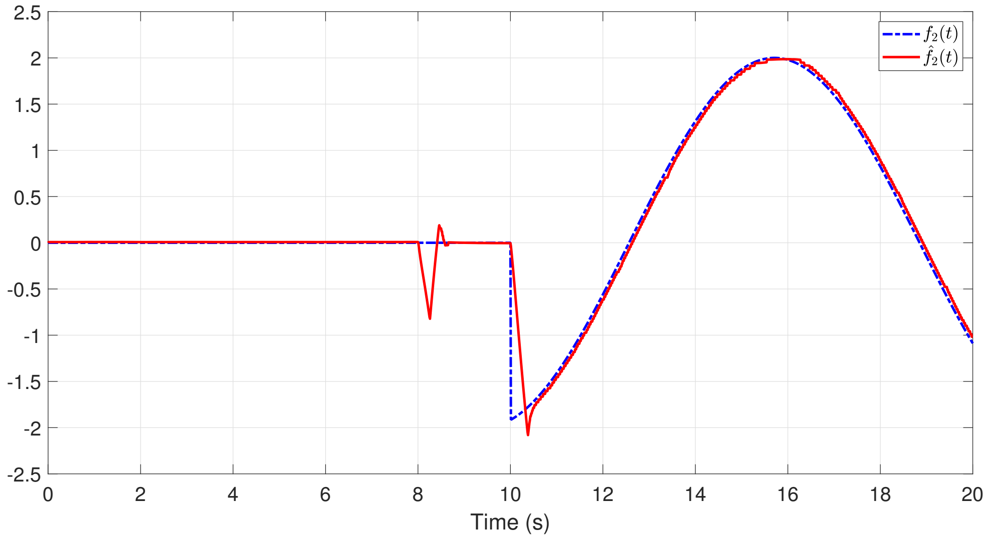

Figure 3. Fault $f_2(t)$ estimation comparison.

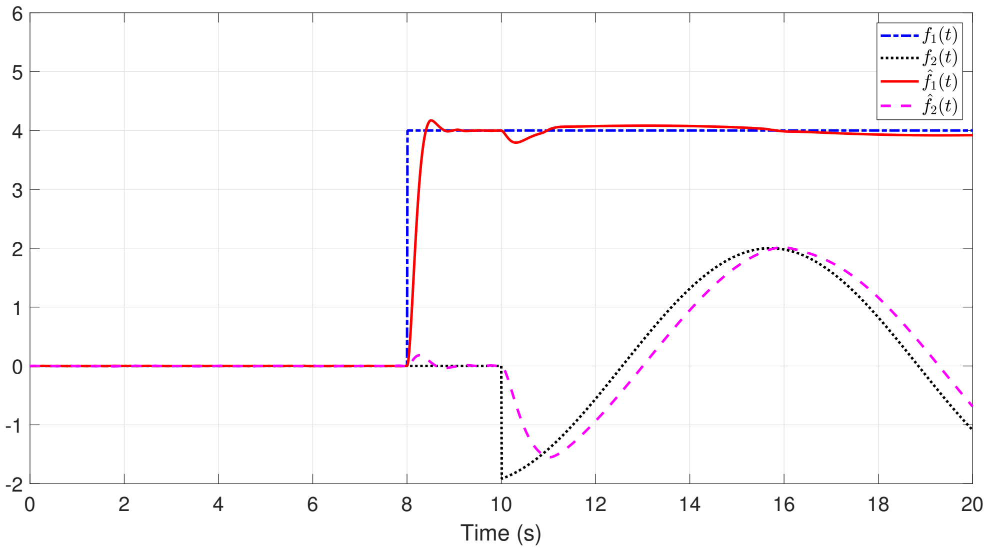

Figure 4. Fault estimation under $\Phi(t) = 0$ in (11).

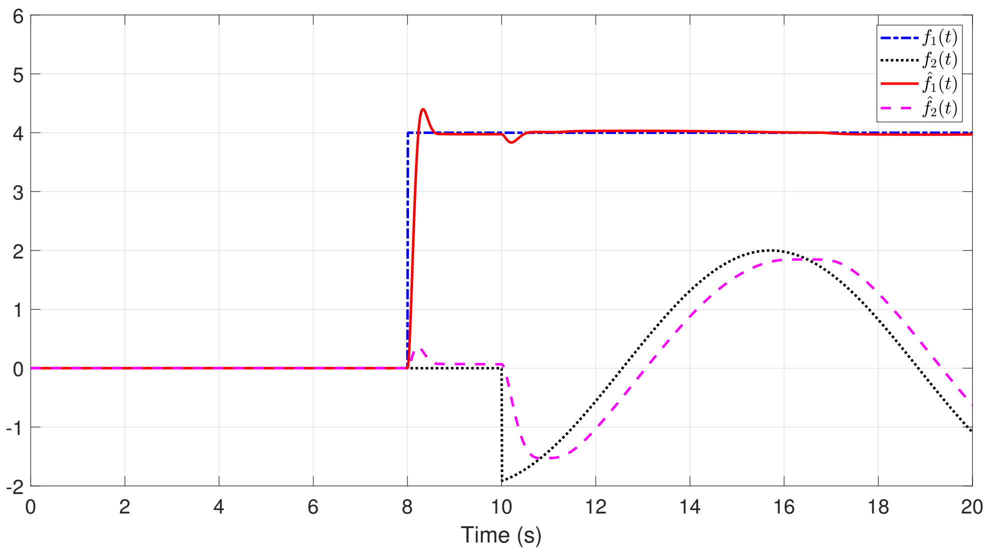

Figure 5. Fault estimation under $\mu_1, \mu_2 = 0$ in (20).

5. Conclusions

A practical fixed-time fault estimation observer method is proposed on the basis of a UIO approach. The external disturbance is completely decoupled, which facilitates the fault estimation design. The fault estimation observer and the non-linear fault estimation algorithm contain two power functions to enhance the convergence rate of fault estimation. Finally, simulation results of a non-linear Lorenz chaotic system are provided to certify the performance of the method. The fixed-time scheme can only obtain the upper bound of the convergence time but cannot accurately calculate the exact time, which will be addressed in future work. In addition, active fault-tolerant control using the obtained accurate fault estimation will also be investigated.

Author Contributions: Conceptualization, J.X. and B.J.; methodology, J.X., B.J. and K.Z.; software, J.X.; validation, J.X., B.J. and K.Z.; formal analysis, J.X., B.J. and K.Z.; investigation, J.X., B.J. and K.Z.; resources, J.X., B.J. and K.Z.; data curation, J.X.; writing—original draft preparation, J.X.; writing—review and editing, B.J. and K.Z.; visualization, J.X., B.J. and K.Z.; supervision, B.J. and K.Z.; project administration, B.J. and K.Z.; funding acquisition, B.J. and K.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the National Natural Science Foundation of China under Grants 62020106003 and 62173180, the Outstanding Youth Foundation of Jiangsu Province of China under Grant BK20200015, the Qing Lan Project of Jiangsu Province of China, and the Fundamental Research Funds for the Central Universities, under Grants NC2022003 and NE2022002.

Data Availability Statement: Data sharing not applicable to this article as no data sets were generated or analyzed during the current article.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Koenig, D. Unknown input proportional multiple-integral observer design for linear descriptor systems: Application to state and fault estimation. *IEEE Trans. Autom. Control* **2005**, *50*, 212–217. [CrossRef]
2. Liu, M.; Shi, P. Sensor fault estimation and tolerant control for Itô stochastic systems with simultaneous input and output disturbances. *Automatica* **2013**, *49*, 1242–1250. [CrossRef]
3. Zhang, K.; Jiang, B.; Shi, P.; Cocquempot, V. *Observer-Based Fault Estimation Techniques*; Springer Publishing Company: Berlin/Heidelberg, Germany, 2018.
4. Liu, Z.; Wang, Z.; Wang, Y.; Ji, Z. Sensor fault estimation based on the constrained zonotopic Kalman filter. *Int. J. Robust Nonlinear Control* **2021**, *31*, 5984–6006. [CrossRef]
5. Zhao, Y.; Duan, Z.; Wen, G.; Chen, G. Distributed finite-time tracking for a multi-agent system under a leader with bounded unknown acceleration. *Syst. Control Lett.* **2015**, *81*, 8–13. [CrossRef]
6. Yu, X.; Liu, Z.; Zhang, Y. Fault-Tolerant Flight Control Design With Finite-Time Adaptation Under Actuator Stuck Failures. *IEEE Trans. Control Syst. Technol.* **2017**, *25*, 1431–1440. [CrossRef]
7. Wu, Y.; Wang, Z.; Huang, Z. Distributed fault detection for non-linear multi-agent systems under fixed-time observer. *J. Frankl. Inst.* **2019**, *356*, 7515–7532. [CrossRef]
8. Xu, C.; Wu, B.; Wang, D.; Zhang, Y. Distributed Fixed-Time Output-Feedback Attitude Consensus Control for Multiple Spacecraft. *IEEE Trans. Aerosp. Electron. Syst.* **2020**, *56*, 4779–4795. [CrossRef]
9. Cao, L.; Xiao, B.; Golestani, M.; Ran, D. Faster Fixed-Time Control of Flexible Spacecraft Attitude Stabilization. *IEEE Trans. Ind. Inform.* **2020**, *16*, 1281–1290. [CrossRef]
10. Song, S.; Park, J.H.; Zhang, B.; Song, X. Event-Triggered Adaptive Practical Fixed-Time Trajectory Tracking Control for Unmanned Surface Vehicle. *IEEE Trans. Circuits Syst. II Express Briefs* **2021**, *68*, 436–440. [CrossRef]
11. Ai, X.; Wang, L. Distributed fixed-time event-triggered consensus of linear multi-agent systems with input delay. *Int. J. Robust Nonlinear Control* **2021**, *31*, 2526–2545. [CrossRef]
12. Liu, M.; Zhang, A.; Xiao, B. Velocity-Free State Feedback Fault-Tolerant Control for Satellite with Actuator and Sensor Faults. *Symmetry* **2022**, *14*, 157. [CrossRef]
13. Chen, L.; Edwards, C.; Alwi, H. Sensor fault estimation using LPV sliding mode observers with erroneous scheduling parameters. *Automatica* **2019**, *101*, 66–77. [CrossRef]
14. Li, R.; Yang, Y. Sliding-Mode Observer-Based Fault Reconstruction for T-S Fuzzy Descriptor Systems. *IEEE Trans. Syst. Man Cybern. Syst.* **2021**, *51*, 5046–5055. [CrossRef]
15. Zhang, Y.; Nie, Y.; Chen, L. Adaptive Fuzzy Fault-Tolerant Control against Time-Varying Faults via a New Sliding Mode Observer Method. *Symmetry* **2021**, *13*, 1615. [CrossRef]

16. Zhang, K.; Jiang, B.; Cocquempot, V. Adaptive observer-based fast fault estimation. *Int. J. Control Autom. Syst.* **2008**, *6*, 320–326.
17. Chen, L.; Shi, P.; Liu, M. Fault reconstruction for Markovian jump systems with iterative adaptive observer. *Automatica* **2019**, *105*, 254–263. [CrossRef]
18. Yan, J.J.; Yang, G.H.; Li, X.J. Adaptive Observer-Based Fault-Tolerant Tracking Control for T–S Fuzzy Systems With Mismatched Faults. *IEEE Trans. Fuzzy Syst.* **2020**, *28*, 134–147. [CrossRef]
19. Sundaram, S.; Hadjicostis, C.N. Partial state observers for linear systems with unknown inputs. *Automatica* **2008**, *44*, 3126–3132. [CrossRef]
20. Ichalal, D.; Mammar, S. On Unknown Input Observers for LPV Systems. *IEEE Trans. Ind. Electron.* **2015**, *62*, 5870–5880. [CrossRef]
21. Gao, Z.; Liu, X.; Chen, M.Z.Q. Unknown Input Observer-Based Robust Fault Estimation for Systems Corrupted by Partially Decoupled Disturbances. *IEEE Trans. Ind. Electron.* **2016**, *63*, 2537–2547. [CrossRef]
22. Shen, Q.; Yue, C.; Goh, C.H.; Wang, D. Active Fault-Tolerant Control System Design for Spacecraft Attitude Maneuvers with Actuator Saturation and Faults. *IEEE Trans. Ind. Electron.* **2019**, *66*, 3763–3772. [CrossRef]
23. Wang, T.; Lu, M.; Zhu, X.; Patton, R.J. Aggressive maneuver oriented robust actuator fault estimation of a 3-DOF helicopter prototype considering measurement noises. *IEEE/ASME Trans. Mechatron.* **2022**, *27*, 1672–1682. [CrossRef]
24. Yu, C.; Verhaegen, M. Data-driven fault estimation of non-minimum phase LTI systems. *Automatica* **2018**, *92*, 181–187. [CrossRef]
25. Edwards, C.; Lombaerts, T.; Smaili, H. *Fault Tolerant Flight Control: A Benchmark Challenge*; Springer Publishing Company, Inc.: Berlin/Heidelberg, Germany, 2010.
26. Stoustrup, J.; Niemann, H.H. Fault estimation-a standard problem approach. *Int. J. Robust Nonlinear Control* **2002**, *12*, 649–673. [CrossRef]
27. Saberi, A.; Stoorvogel, A.A.; Sannuti, P. *Filtering Theory with Applications to Fault Detection, Isolation and Estimation*; Birkhauser: Boston, MA, USA, 2007.
28. Ding, S.X. *Model-Based Fault Diagnosis Techniques*; Springer Publishing Company, Inc.: Berlin/Heidelberg, Germany, 2013.
29. Gao, Z. Fault estimation and fault tolerant control for discrete-time dynamic systems. *IEEE Trans. Ind. Electron.* **2015**, *62*, 3874–3884. [CrossRef]
30. Lan, J.; Patton, R.J. A new strategy for integration of fault estimation within fault-tolerant control. *Automatica* **2016**, *69*, 48–59. [CrossRef]
31. Polyakov, A. Nonlinear feedback design for fixed-time stabilization of linear control systems. *IEEE Trans. Autom. Control* **2012**, *57*, 2106–2110. [CrossRef]
32. Cui, G.; Yang, W.; Yu, J.; Li, Z.; Tao, C. Fixed-time prescribed performance adaptive trajectory tracking control for a QUAV. *IEEE Trans. Circuits Syst. II Express Briefs* **2022**, *69*, 494–498. [CrossRef]
33. Hu, X.; Li, Y.X.; Hou, Z. Event-triggered fuzzy adaptive fixed-time tracking control for non-linear systems. *IEEE Trans. Cybern.* **2020**, *52*, 7206–7217. doi: 10.1109/TCYB.2020.3035779. [CrossRef]
34. Xu, Y.; Wu, X.; Mao, B.; Xie, C. A unified finite-/fixed-time synchronization approach to multi-layer networks. *IEEE Trans. Circuits Syst. II Express Briefs* **2021**, *68*, 311–315. [CrossRef]