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Some New Fractional Integral Inequalities Pertaining to Generalized Fractional Integral Operator

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Abstract: Integral inequalities make up a comprehensive and prolific field of research within the field of mathematical interpretations. Integral inequalities in association with convexity have a strong relationship with symmetry. Different disciplines of mathematics and applied sciences have taken a new path as a result of the development of new fractional operators. Different new fractional operators have been used to improve some mathematical inequalities and to bring new ideas in recent years. To take steps forward, we prove various Grüss-type and Chebyshev-type inequalities for integrable functions in the frame of non-conformable fractional integral operators. The key results are proven using definitions of the fractional integrals, well-known classical inequalities, and classical relations.

Keywords: Grüss-type inequalities; Chebyshev-type inequalities; non-conformable fractional operator

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1. Introduction

Fractional calculus theory gained popularity and was employed as a mathematical tool in a variety of pure and practical fields. This approach has previously been used in a variety of industries with some impressive results. It has been used in medicine [1], physics [2], modelling of diseases [3,4], nanotechnology [5], fluid mechanics [6], bioengineering [7], epidemiology [8], economics [9], and control systems [10].

In applied mathematics, inequalities and their applications are crucial. Various fractional operators were used to show a collection of integral inequalities and their generalizations (see [11–17]). To follow this trend, we use a generalized non-conformable fractional integral operator to show an improved version of the Grüss-type inequality. G. Grüss presented the well-known Grüss-type inequality in 1935, which was linked to the Chebyshev's inequality; see [18].

$$\left|\frac{1}{\kappa-\varrho}\int_{\varrho}^{\kappa}\mathfrak{S}(\mathrm{u})\mathfrak{Z}(\mathrm{u})\mathrm{du}-\left(\frac{1}{\kappa-\varrho}\int_{\varrho}^{\kappa}\mathfrak{S}(\mathrm{u})\mathrm{du}\right)\left(\frac{1}{\kappa-\varrho}\int_{\varrho}^{\kappa}\mathfrak{Z}(\mathrm{u})\mathrm{du}\right)\right|\leq\frac{(\mathcal{B}-\mathcal{A})(\mathcal{D}-\mathcal{C})}{4}. \tag{1}$$

Provided that $\mathfrak{S}$ and $\mathfrak{Z}$ are two integrable functions on $[\varrho,\kappa]$, satisfying the condition,

$$\mathcal{A}\leq\mathfrak{S}(\mathrm{u})\leq\mathcal{B},\ \ \mathcal{C}\leq\mathfrak{Z}(\mathrm{u})\leq\mathcal{D},\ \ \mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}\in\mathcal{R},\ \ \mathrm{u}\in[\varrho,\kappa]. \tag{2}$$

For integrable functions, various types of inequalities have been established, but the Grüss inequality has been the focus of many studies as many scholars have examined it extensively. Chaos, bio-sciences, fluid dynamics, engineering, meteorology, biochemistry, vibration analysis, aerodynamics, and many other scientific fields benefit from this inequality. See [19–24] for a steady growth of interest in such a field of study to address the difficulties of various applications of these variants.

$$T(\mathfrak{S},\mathfrak{Z}) = \frac{1}{\kappa-\varrho}\int_{\varrho}^{\kappa}\mathfrak{S}(\mathsf{u})\mathfrak{Z}(\mathsf{u})d\mathsf{u} - \left(\frac{1}{\kappa-\varrho}\int_{\varrho}^{\kappa}\mathfrak{S}(\mathsf{u})d\mathsf{u}\right)\left(\frac{1}{\kappa-\varrho}\int_{\varrho}^{\kappa}\mathfrak{Z}(\mathsf{u})d\mathsf{u}\right), \tag{3}$$

where $\mathfrak{S}$ and $\mathfrak{Z}$ are two integral functions that are synchonous on $[\varrho,\kappa]$, given as

$$(\mathfrak{S}(\mathsf{u})-\mathfrak{S}(\mathsf{y}))(\mathfrak{Z}(\mathsf{u})-\mathfrak{Z}(\mathsf{y}))\geq 0,$$

for any $\mathsf{u}, y \in [\varrho,\kappa]$; then, the Chebyshev inequality states that $T(\mathfrak{S},\mathfrak{Z})\geq 0$.

The interest in inequality (1) has been evoked by the researcher. There are numerous recent studies in the literature on theoretical inequalities. In the approach of unital 2-positive linear maps, Balasubramanian [25] worked on the idea of the Grüss-type inequality. Pecaric [26] looked at certain Grüss inequality extensions and applications using weighted Ozeki's inequality, which is a supplement to the Cauchy–Schwartz inequality. Butt [27] contributed a paper on the Jensen–Grüss-type inequality and its application to the Zipf–Mandelbrot law. The extended generalized Mittag–Leffler function was used by Akdemir [28] to investigate several Grüss-type integral inequalities for fractional integral operators. Akdemir [29] also used the generalized fractional integral operator to analyze several Grüss-type inequalities. Using the generalized Katugampola fractional integral operator, Aljaaidi [30] investigated and proved various Grüss-type inequalities.

Employing the concept of time scales, Sarikaya [31] wrote a remark on Grüss-type inequalities. Pachpatte [32] looked at certain differential function Chebyshev–Grüss inequalities. In the style of a generalized K-fractional integral operator, Noor [33] explained various Grüss-type inequalities. Using the Riemann–Liouville fractional integral operator, Dahmani [34] showed several expansions of the Grüss-type integral inequality. Chinchane [35] presented a paper that used the Hadamard fractional integral operator to create a novel Grüss-type inequality. Sarikaya [36] employed a variation of Pompeiu's mean value theorem to develop a Grüss-type inequality. Kalla [37] investigated Grüss type inequalities for a hypergeometric fractional integral operator. E. Set [38] worked on the novel Grüss type inequalities via conformable fractional integral operator. The Riemann–Liouville fractional integral operator was used to solve the following integral inequality given by Dahmani et al. [34].

Theorem 1. *Let $\mathfrak{S}$ and $\mathfrak{Z}$ be two integrable functions on $(0,\infty)$ satisfying the condition*

$$\mathcal{A}\leq\mathfrak{S}(\mathsf{u})\leq\mathcal{B},\ \ \mathcal{C}\leq\mathfrak{Z}(\mathsf{u})\leq\mathcal{D},\ \ \mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}\in\mathcal{R},\ \ \mathsf{u}\in[\varrho,\kappa],$$

on $(0,\infty)$; then, $\forall\eta>0$, we have

$$\left|\frac{\mathtt{w}^{\eta}}{\Gamma(\eta+1)}\mathfrak{J}^{\eta}\mathfrak{S}\mathfrak{Z}(\mathtt{w})-\mathfrak{J}^{\eta}\mathfrak{S}(\mathtt{w})\mathfrak{J}^{\eta}\mathfrak{Z}(\mathtt{w})\right|\leq\left(\frac{\mathtt{w}^{\eta}}{2\Gamma(\eta+1)}\right)^{2}(\mathcal{B}-\mathcal{A})(\mathcal{D}-\mathcal{C}). \tag{4}$$

The paper is arranged as follows: In Section 2, we give some known concepts. In Section 3, we obtain some Grüss-type fractional integral inequalities on the basis of new lemmas with the help of the Cauchy–Schwarz inequality. In Section 4, we investigate some other fractional integral inequalities involving non-conformable fractional integral operators with the help of Young's inequality. Section 5 deals with Chebyshev-type inequalities. A brief conclusion is given in Section 6.

2. Preliminaries

Definition 1 ([39]). *For each $\mathfrak{S} \in \mathcal{L}[\xi, \vartheta]$ and $0 < \xi < \vartheta$, we define*

$$ {}_{N_3}\mathfrak{J}^{\eta}_{\mathsf{u}}\mathfrak{S}(\mathsf{x}) = \int_{\mathsf{u}}^{\mathsf{x}} \vartheta^{-\eta}\mathfrak{S}(\vartheta)d\vartheta, $$

for every $\mathsf{x}, \mathsf{u} \in [\xi, \vartheta]$ and $\eta \in \mathcal{R}$.

Definition 2 ([39]). *For each function $\mathfrak{S} \in \mathcal{L}[\xi, \vartheta]$, we define the fractional integrals*

$$ {}_{N_3}\mathfrak{J}^{\eta}_{\xi+}\mathfrak{S}(\mathsf{x}) = \int_{\xi}^{\mathsf{x}} (\mathsf{x} - \vartheta)^{-\eta}\mathfrak{S}(\vartheta)d\vartheta, $$

$$ {}_{N_3}\mathfrak{J}^{\eta}_{\vartheta-}\mathfrak{S}(\mathsf{x}) = \int_{\mathsf{x}}^{\vartheta} (\vartheta - \mathsf{x})^{-\eta}\mathfrak{S}(\vartheta)d\vartheta, $$

for every $\mathsf{x} \in [\xi, \vartheta]$ and $\eta \in \mathcal{R}$.

Remark 1. *In the above definitions, if we put $\eta = 0$ then we have the classical integrals, which are represented by ${}_{N_3}\mathfrak{J}^{\eta}_{\xi+}\mathfrak{S}(\mathsf{x}) = {}_{N_3}\mathfrak{J}^{\eta}_{\vartheta-}\mathfrak{S}(\mathsf{x}) = \int_{\xi}^{\vartheta}\mathfrak{S}(\vartheta)d\vartheta$.*

3. Fractional Inequality of Grüss Type

In this section, first, we prove some new integrable equalities; then, using these equalities and the Cauchy–Schwarz inequality, our main findings are presented.

Lemma 1. *Let the integrable function on $(0, \infty)$ be $\mathfrak{S}$ with $\mathcal{A}, \mathcal{B} \in \mathcal{R}$; then, $\forall \mathtt{w} > 0$ and $\eta > 0$, the following equality holds true:*

$$ \begin{aligned} &\frac{(\mathtt{w} - \varrho)^{\eta}}{\eta}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}\mathfrak{S}^2(\mathtt{w}) + \left({}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}\mathfrak{S}(\mathtt{w})\right)^2 \\ &= \left(\mathcal{B}\frac{(\mathtt{w} - \varrho)^{\eta}}{\eta} - {}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}\mathfrak{S}(\mathtt{w})\right)\left({}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}\mathfrak{S}(\mathtt{w}) - \mathcal{A}\frac{(\mathtt{w} - \varrho)^{\eta}}{\eta}\right) - \frac{(\mathtt{w} - \varrho)^{\eta}}{\eta}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}((\mathcal{B} - \mathfrak{S}(\mathtt{w}))(\mathfrak{S}(\mathtt{w}) - \mathcal{A})). \end{aligned} \tag{5} $$

Proof. Let $\mathcal{A}, \mathcal{B} \in \mathcal{R}$ and $\mathfrak{S}$ be an integrable function on $(0, \infty)$ $\forall\ \mu, \rho \in (0, \infty)$; then, we have

$$ \begin{aligned} &(\mathcal{B} - \mathfrak{S}(\rho))(\mathfrak{S}(\mu) - \mathcal{A}) + (\mathcal{B} - \mathfrak{S}(\mu))(\mathfrak{S}(\rho) - \mathcal{A}) \\ &- (\mathcal{B} - \mathfrak{S}(\mu))(\mathfrak{S}(\mu) - \mathcal{A}) - (\mathcal{B} - \mathfrak{S}(\rho))(\mathfrak{S}(\rho) - \mathcal{A}) \\ &= \mathfrak{S}^2(\mu) + \mathfrak{S}^2\rho + 2\mathfrak{S}(\mu)\mathfrak{S}(\rho). \end{aligned} \tag{6} $$

If we multiply both sides of (6) by $(\mathtt{w} - \mu)^{\eta-1}$ and integrate the resultant equality with respect to μ, we obtain

$$ \begin{aligned} &(\mathcal{B} - \mathfrak{S}(\rho))\left({}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}\mathfrak{S}(\mathtt{w}) - \mathcal{A}\frac{(\mathtt{w} - \varrho)^{\eta}}{\eta}\right) + \left(\mathcal{B}\frac{(\mathtt{w} - \varrho)^{\eta}}{\eta} - {}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}\mathfrak{S}(\mathtt{w})\right)(\mathfrak{S}(\rho) - \mathcal{A}) \\ &- {}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}((\mathcal{B} - \mathfrak{S}(\mathtt{w}))(\mathfrak{S}(\mathtt{w}) - \mathcal{A})) - (\mathcal{B} - \mathfrak{S}(\rho))(\mathfrak{S}(\rho) - \mathcal{A})\frac{(\mathtt{w} - \varrho)^{\eta}}{\eta} \\ &= {}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}\mathfrak{S}^2(\mathtt{w}) + \frac{(\mathtt{w} - \varrho)^{\eta}}{\eta}\mathfrak{S}^2(\rho) + 2\mathfrak{S}(\rho){}_{N_3}\mathfrak{J}^{\eta}_{\varrho+}\mathfrak{S}(\mathtt{w}). \end{aligned} \tag{7} $$

Upon multiplication of both sides of (7) by $(\mathtt{w}-\rho)^{\eta-1}$ and integration of the resultant equality with respect to ρ, we yield

$$\begin{aligned}&\left({}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w})-\mathcal{A}\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}\right)\int_a^{\mathtt{w}}(\mathtt{w}-\rho)^{\eta-1}(\mathcal{B}-\mathfrak{S}(\rho))d\rho\\&+\left(\mathcal{B}\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}-{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w})\right)\int_a^{\mathtt{w}}(\mathtt{w}-\rho)^{\eta-1}(\mathfrak{S}(\rho)-\mathcal{A})d\rho\\&-{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}((\mathcal{B}-\mathfrak{S}(\mathtt{w}))(\mathfrak{S}(\mathtt{w})-\mathcal{A}))\int_a^{\mathtt{w}}(\mathtt{w}-\rho)^{\eta-1}(\mathcal{B}-\mathfrak{S}(\rho))d\rho\\&-\frac{(\mathtt{w}-\eta)^{\eta}}{\eta}\int_a^{\mathtt{w}}(\mathtt{w}-\rho)^{\eta-1}(\mathcal{B}-\mathfrak{S}(\rho))(\mathfrak{S}(\rho)-\mathcal{A})d\rho\\&=\frac{(\mathtt{w}-\eta)^{\eta}}{\eta}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}^2(\mathtt{w})+\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}^2(\mathtt{w})+2{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w}){}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w}),\end{aligned}$$

This led us to the proof of Lemma 1. □

Theorem 2. *Let the integrable functions on* $[0,\infty)$, *be* $\mathfrak{S}$ *and* $\mathfrak{Z}$ *satisfying the condition*

$$\mathcal{A}\le\mathfrak{S}(\mathtt{w})\le\mathcal{B},\ \ \mathcal{C}\le\mathfrak{Z}(\mathtt{w})\le\mathcal{D},\ \ \mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}\in\mathcal{R},\ \ \mathtt{w}\in[0,\infty).$$

Then,

$$\begin{aligned}&\left|\frac{(\mathtt{w}-\eta)^{\eta}}{\eta}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z})(\mathtt{w})-{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w}){}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{Z}(\mathtt{w})\right|\\&\le\left(\frac{(\mathtt{w}-\eta)^{\eta}}{2\eta}\right)^2(\mathcal{B}-\mathcal{A})(\mathcal{D}-\mathcal{C}).\end{aligned}\tag{8}$$

holds true.

Proof. Let $\mathfrak{S}$ and $\mathfrak{Z}$ be two given integrable function on $[0,\infty)$, with the condition

$$\mathcal{A}\le\mathfrak{S}(\mathtt{w})\le\mathcal{B},\ \ \mathcal{C}\le\mathfrak{Z}(\mathtt{w})\le\mathcal{D},\ \ \mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}\in R,\ \ \mathtt{w}\in[0,\infty).$$

If we define

$$H(\mu,\rho)=(\mathfrak{S}(\mu)-\mathfrak{S}(\rho))(\mathfrak{Z}(\mu)-\mathfrak{Z}(\rho)).$$

It readily follows that

$$H(\mu,\rho)=(\mathfrak{S}(\mu)\mathfrak{Z}(\mu)-\mathfrak{S}(\mu)\mathfrak{Z}(\rho)-\mathfrak{S}(\rho)\mathfrak{Z}(\mu)+\mathfrak{S}(\rho)\mathfrak{Z}(\rho).$$

Then, multiplying the above equality by $(\mathtt{w}-\mu)^{\eta-1}$ and integrating the resultant equality with respect to μ, we have

$$\begin{aligned}&\int_a^{\mathtt{w}}(\mathtt{w}-\mu)^{\eta-1}H(\mu,\rho)d\mu\\&={}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}\mathfrak{Z}(\mathtt{w})-\mathfrak{S}(\rho){}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{Z}(\mathtt{w})-\mathfrak{Z}(\rho){}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w})+\mathfrak{S}(\rho)\mathfrak{Z}(\rho)\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}.\end{aligned}\tag{9}$$

Again, multiplying the above equality by $(\mathtt{w}-\rho)^{\eta-1}$ and then integrating the with respect to ρ, we have

$$\int_a^{\mathtt{w}}\int_a^{\mathtt{w}}(\mathtt{w}-\mu)^{\eta-1}(\mathtt{w}-\rho)^{\eta-1}H(\mu,\rho)d\mu d\rho=2\left(\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}\mathfrak{Z}(\mathtt{w})-{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w}){}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{Z}(\mathtt{w})\right).$$

Employing the Cauchy–Schwarz inequality, we obtain

$$\left(\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{S}\mathfrak{Z}(\mathtt{w})-{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w}){}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{Z}(\mathtt{w})\right)^2$$
$$\leq\left(\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{S}^2(\mathtt{w})-\left({}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w})\right)^2\right)\left(\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{Z}^2(\mathtt{w})-\left({}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{Z}(\mathtt{w})\right)^2\right). \quad (10)$$

Since, $(\mathcal{B}-\mathfrak{S}(\mathtt{w}))(\mathfrak{S}(\mathtt{w})-\mathcal{A})\geq 0$ and $(\mathcal{D}-\mathfrak{Z}(\mathtt{w}))(\mathfrak{Z}(\mathtt{w})-\mathcal{C})\geq 0$, we consequently have

$$\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}((\mathcal{B}-\mathfrak{S}(\mathtt{w}))(\mathfrak{S}(\mathtt{w})-\mathcal{A}))\geq 0, \quad (11)$$

and

$$\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}((\mathcal{D}-\mathfrak{Z}(\mathtt{w}))(\mathfrak{Z}(\mathtt{w})-\mathcal{C}))\geq 0. \quad (12)$$

Thus,

$$\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{S}^2(\mathtt{w})-\left({}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w})\right)^2$$
$$\leq\left(\frac{\mathcal{B}(\mathtt{w}-\varrho)^{\eta}}{\eta}-{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w})\right)\left({}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w})-\frac{\mathcal{A}(\mathtt{w}-\varrho)^{\eta}}{\eta}\right). \quad (13)$$

Additionally,

$$\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{Z}^2(\mathtt{w})-\left({}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{Z}(\mathtt{w})\right)^2$$
$$\leq\left(\frac{\mathcal{D}(\mathtt{w}-\varrho)^{\eta}}{\eta}-{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{Z}(\mathtt{w})\right)\left({}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{Z}(\mathtt{w})-\frac{\mathcal{C}(\mathtt{w}-\varrho)^{\eta}}{\eta}\right). \quad (14)$$

From Lemma 1 and the above inequalities (10) and (14), we can conclude that

$$\left(\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{S}\mathfrak{Z}(\mathtt{w})-{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w}){}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{Z}(\mathtt{w})\right)^2$$
$$\leq\left(\frac{\mathcal{B}(\mathtt{w}-\varrho)^{\eta}}{\eta}-{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w})\right)\left({}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w})-\frac{\mathcal{A}(\mathtt{w}-\varrho)^{\eta}}{\eta}\right)$$
$$\times\left(\frac{\mathcal{D}(\mathtt{w}-\varrho)^{\eta}}{\eta}-{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{Z}(\mathtt{w})\right)\left({}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{Z}(\mathtt{w})-\frac{\mathcal{C}(\mathtt{w}-\varrho)^{\eta}}{\eta}\right). \quad (15)$$

Now, using the inequality $4bc\leq(b+c)^2$, $b,c\in\mathcal{R}$, we obtain

$$4\left(\frac{\mathcal{B}(\mathtt{w}-\varrho)^{\eta}}{\eta}-{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w})\right)\left({}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w})-\frac{\mathcal{A}(\mathtt{w}-\varrho)^{\eta}}{\eta}\right)\leq\left(\frac{(\mathtt{w}-\eta)^{\eta}}{\eta}(\mathcal{B}-\mathcal{A})\right)^2, \quad (16)$$

and

$$4\left(\frac{\mathcal{D}(\mathtt{w}-\varrho)^{\eta}}{\eta}-{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{Z}(\mathtt{w})\right)\left({}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{Z}(\mathtt{w})-\frac{\mathcal{C}(\mathtt{w}-\varrho)^{\eta}}{\eta}\right)\leq\left(\frac{(\mathtt{w}-\eta)^{\eta}}{\eta}(\mathcal{D}-\mathcal{C})\right)^2. \quad (17)$$

The proof of Theorem 7 is completed from the above developments (Equations (15)–(17)). □

Lemma 2. *Let the integrable functions on* $[0,\infty)$ *be* $\mathfrak{S}$ *and* $\mathfrak{Z}$*; then, the following identity for all* $\mathtt{w}\leq 0$, $\eta\leq 0$ *and* $\beta\leq 0$ *holds true:*

$$\begin{aligned}
&\left[\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}{}_{N_3}\mathfrak{J}^{\beta}_{\varrho^+}\mathfrak{S}\mathfrak{Z}(\mathtt{w})+\frac{(\mathtt{w}-\varrho)^{\beta}}{\beta}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}\mathfrak{Z}(\mathtt{w})-{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w}){}_{N_3}\mathfrak{J}^{\beta}_{\varrho^+}\mathfrak{Z}(\mathtt{w})-{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{Z}(\mathtt{w}){}_{N_3}\mathfrak{J}^{\beta}_{\varrho^+}\mathfrak{S}(\mathtt{w})\right]^2\\
&\le\left(\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}{}_{N_3}\mathfrak{J}^{\beta}_{\varrho^+}\mathfrak{S}^2(\mathtt{w})+\frac{(\mathtt{w}-\varrho)^{\beta}}{\beta}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}^2(\mathtt{w})-2{}_{N_3}\mathfrak{J}^{\beta}_{\varrho^+}\mathfrak{S}(\mathtt{w}){}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w})\right)\\
&\times\left(\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}{}_{N_3}\mathfrak{J}^{\beta}_{\varrho^+}\mathfrak{Z}^2(\mathtt{w})+\frac{(\mathtt{w}-\varrho)^{\beta}}{\beta}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{Z}^2(\mathtt{w})-2{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{Z}(\mathtt{w}){}_{N_3}\mathfrak{J}^{\beta}_{\varrho^+}\mathfrak{Z}(\mathtt{w})\right).
\end{aligned}\tag{18}$$

Proof. Multiplying (9) by $(\mathtt{w}-\rho)^{\beta-1}$, then integrating the resultant with respect to ρ, and applying the Cauchy–Schwarz inequality, we have the desired inequality (18). □

Lemma 3. *Let the integrable functions on $[0,\infty)$ be $\mathfrak{S}$ and $\mathcal{A},\mathcal{B}\in\mathcal{R}$; then, for all $\mathtt{w}\le 0$, $alpha\le 0$ and $\beta\le 0$, the following equality holds true:*

$$\begin{aligned}
&\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}{}_{N_3}\mathfrak{J}^{\beta}_{\varrho^+}\mathfrak{S}^2(\mathtt{w})+\frac{(\mathtt{w}-\varrho)^{\beta}}{\beta}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}^2(\mathtt{w})+2{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w}){}_{N_3}\mathfrak{J}^{\beta}_{\varrho^+}\mathfrak{S}(\mathtt{w})\\
&=\left(\mathcal{B}\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}-{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w})\right)\left({}_{N_3}\mathfrak{J}^{\beta}_{\varrho^+}\mathfrak{S}(\mathtt{w})-\mathcal{A}\frac{(\mathtt{w}-\varrho)^{\beta}}{\beta}\right)\\
&+\left(\mathcal{B}\frac{(\mathtt{w}-\varrho)^{\beta}}{\beta}-{}_{N_3}\mathfrak{J}^{\beta}_{\varrho^+}\mathfrak{S}(\mathtt{w})\right)\left({}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w})-\mathcal{A}\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}\right)\\
&-\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}-{}_{N_3}\mathfrak{J}^{\beta}_{\varrho^+}((\mathcal{B}-\mathfrak{S}(\mathtt{w}))(\mathfrak{S}(\mathtt{w})-\mathcal{A}))-\frac{(\mathtt{w}-\varrho)^{\beta}}{\beta}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}((\mathcal{B}-\mathfrak{S}(\mathtt{w}))(\mathfrak{S}(\mathtt{w})-\mathcal{A})).
\end{aligned}$$

Proof. Multiplying (7) by $(\mathtt{w}-\rho)^{\beta-1}$ and then integrating the resulting identity with respect to ρ, we have

$$\begin{aligned}
&\left({}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w})-\mathcal{A}\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}\right)\int_a^{\mathtt{w}}(\mathtt{w}-\rho)^{(\beta-1)}(\mathcal{B}-\mathfrak{S}(\rho))d\rho\\
&+\left(\mathcal{B}\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}-{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w})\right)\int_a^{\mathtt{w}}(\mathfrak{S}(\rho)-\mathcal{A})(\mathtt{w}-\rho)^{(\beta-1)}d\rho\\
&-{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}((\mathcal{B}-\mathfrak{S}(\mathtt{w})(\mathfrak{S}(\mathtt{w})-\mathcal{A}))(\mathtt{w}-\rho)^{(\beta-1)}d\rho\\
&-\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}\int_a^{\mathtt{w}}(\mathcal{B}-\mathfrak{S}(\rho))(\mathfrak{S}(\rho)-\mathcal{A})d\rho\\
&=\frac{(\mathtt{w}-\varrho)^{\beta}}{\beta}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}^2(\mathtt{w})+\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}{}_{N_3}\mathfrak{J}^{\beta}_{\varrho^+}\mathfrak{S}^2(\mathtt{w})+2{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w}){}_{N_3}\mathfrak{J}^{\beta}_{\varrho^+}\mathfrak{S}(\mathtt{w}).
\end{aligned}$$

The above developments completes the proof of Lemma 3. □

Theorem 3. *Let $\mathfrak{S}$ and $\mathfrak{Z}$ be two integrable function on $[0,\infty)$ satisfying the condition*

$$\mathcal{A}\le\mathfrak{S}(\mathtt{w})\le\mathcal{B},\ \ \mathcal{C}\le\mathfrak{Z}(\mathtt{w})\le\mathcal{D},\ \ \mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}\in R,\ \ \mathtt{w}\in[0,\infty),$$

we have

$$\left[\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}{}_{N_3}\mathfrak{I}^{\beta}_{\varrho^+}\mathfrak{S}\mathfrak{Z}(\mathtt{w})+\frac{(\mathtt{w}-\varrho)^{\beta}}{\beta}{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{S}\mathfrak{Z}(\mathtt{w})-{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w}){}_{N_3}\mathfrak{I}^{\beta}_{\varrho^+}\mathfrak{Z}(\mathtt{w})-{}_{N_3}\mathfrak{I}^{\beta}_{\varrho^+}\mathfrak{S}(\mathtt{w}){}_{N_3}\mathfrak{I}^{\beta}_{\varrho^+}\mathfrak{Z}(\mathtt{w})\right]^2$$
$$\leq\left(\mathcal{B}\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}-{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w})\right)\left({}_{N_3}\mathfrak{I}^{\beta}_{\varrho^+}\mathfrak{S}(\mathtt{w})-\mathcal{A}\frac{(\mathtt{w}-\varrho)^{\beta}}{\beta}\right)$$
$$+\left(\mathcal{B}\frac{(\mathtt{w}-\varrho)^{\beta}}{\beta}-{}_{N_3}\mathfrak{I}^{\beta}_{\varrho^+}\mathfrak{S}(\mathtt{w})\right)\left({}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w})-\mathcal{A}\frac{(\mathtt{w}-\varrho)^{\beta}}{\beta}\right)$$
$$\times\left(\mathcal{D}\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}-{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{Z}(\mathtt{w})\right)\left({}_{N_3}\mathfrak{I}^{\beta}_{\varrho^+}\mathfrak{Z}(\mathtt{w})-\mathcal{C}\frac{(\mathtt{w}-\varrho)^{\beta}}{\beta}\right)$$
$$+\left(\mathcal{D}\frac{(\mathtt{w}-\varrho)^{\beta}}{\beta}-{}_{N_3}\mathfrak{I}^{\beta}_{\varrho^+}\mathfrak{Z}(\mathtt{w})\right)\left({}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{Z}(\mathtt{w})-\mathcal{C}\frac{(\mathtt{w}-\varrho)^{\beta}}{\beta}\right).$$

Proof. Since $(\mathcal{B}-\mathfrak{S}(\mathtt{w}))(\mathfrak{S}(\mathtt{w})-\mathcal{A})\geq 0$ and $(\mathcal{D}-\mathfrak{Z}(\mathtt{w}))(\mathfrak{Z}(\mathtt{w})-\mathcal{C})\geq 0$, then we can write

$$-\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}{}_{N_3}\mathfrak{I}^{\beta}_{\varrho^+}((\mathcal{B}-\mathfrak{S}(\mathtt{w}))(\mathfrak{S}(\mathtt{w})-\mathcal{A}))-\frac{(\mathtt{w}-\varrho)^{\beta}}{\beta}{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}((\mathcal{B}-\mathfrak{S}(\mathtt{w}))(\mathfrak{S}(\mathtt{w})-\mathcal{A}))\leq 0, \tag{19}$$

and

$$-\frac{(\mathtt{w}-\varrho)^{\eta}}{\eta}{}_{N_3}\mathfrak{I}^{\beta}_{\varrho^+}((\mathcal{D}-\mathfrak{Z}(\mathtt{w}))(\mathfrak{Z}(\mathtt{w})-\mathcal{C}))-\frac{(\mathtt{w}-\varrho)^{\beta}}{\beta}{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}((\mathcal{D}-\mathfrak{Z}(\mathtt{w}))(\mathfrak{Z}(\mathtt{w})-\mathcal{C}))\leq 0. \tag{20}$$

If we apply Lemma 3 for $\mathfrak{S}$ and $\mathfrak{Z}$, with Lemma 2 and Equations (19) and (20), we have the desired Theorem 3. □

4. Certain New Fractional Integral Inequalities

Here, we present some new type of inequalities (Theorems 4–6) pertaining to non-conformable fractional integral operator.

Theorem 4. *Let the positive functions defined on $[0,\infty)$ be $\mathfrak{S}$ and $\mathfrak{Z}$. Then, the inequalities holds:*

(i) $\frac{1}{p}{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})^p+\frac{1}{q}{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})^q\geq\left[\frac{(\mathtt{w}-\eta)^{\eta}}{\eta}\right]^{-1}{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}){}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})$.

(ii) $\frac{1}{p}{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})^p{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})^p+\frac{1}{q}{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})^q{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})^q\geq\left({}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z})\right)^2$.

(iii) $\frac{1}{p}{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})^p{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})^q+\frac{1}{q}{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})^q{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})^p\geq{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z}^{p-1}){}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z}^{q-1})$.

(iv) ${}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})^p{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})^q\geq{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z}){}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}^{p-1}\mathfrak{Z}^{q-1})$, *where for* $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$,

Proof. From Young's inequality, we have

$$\frac{1}{p}\mathsf{u}^p+\frac{1}{q}v^q\geq\mathsf{uv},\quad\text{for all}\quad\mathsf{u},\mathsf{v}\geq 0,\quad p,q>1,\quad\frac{1}{p}+\frac{1}{q}=1.$$

If we choose, $u=\mathfrak{S}(\mu)$ and $v=\mathfrak{Z}(\rho)$, $\mu,\rho>o$, then

$$\frac{1}{p}(\mathfrak{S}(\mu))^p+\frac{1}{q}(\mathfrak{Z}(\rho))^q\geq\mathfrak{S}(\mu)\mathfrak{Z}(\rho),\ \ \forall\ \mathfrak{S}(\mu)\mathfrak{Z}(\rho)\geq 0. \tag{21}$$

Multiplication of inequality (21) by $(\mathtt{w}-\mu)^{\eta-1}$, and integrating the resultant inequality with respect to μ, we get

$$\frac{1}{p}\int_a^{\mathtt{w}}(\mathtt{w}-\mu)^{\eta-1}(\mathfrak{S}(\mu))^p d\mu+\frac{1}{q}\mathfrak{Z}(\rho)^q\int_a^{\mathtt{w}}(\mathtt{w}-\mu)^{\eta-1}d\mu\geq\mathfrak{Z}(\rho)\int_a^{\mathtt{w}}(\mathtt{w}-\mu)^{\eta-1}\mathfrak{S}(\mu)d\mu. \tag{22}$$

Consequently,

$$\frac{1}{p}{}_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}(\mathtt{w}))^p + \frac{(\mathtt{w}-\eta)^{\eta}}{q\eta}\mathfrak{Z}(\rho)^q \geq \mathfrak{Z}(\rho)_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}(\mathtt{w})). \tag{23}$$

Analogously, multiplying inequality (23) by $(\mathtt{w}-\rho)^{\eta-1}$ and integrating the obtained identity, we obtain

$$\frac{(\mathtt{w}-\eta)^{\eta}}{p\eta}{}_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}(\mathtt{w}))^p + \frac{(\mathtt{w}-\eta)^{\eta}}{q\eta}{}_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z}(\mathtt{w}))^q \geq {}_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w})_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{Z}(\mathtt{w}).$$

This readily follows:

$$\frac{(\mathtt{w}-\eta)^{\eta}}{\eta}\left[\frac{1}{p}{}_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}(\mathtt{w}))^p + \frac{1}{q}{}_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z}(\mathtt{w}))^q\right] \geq {}_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w})_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{Z}(\mathtt{w}). \tag{24}$$

Additionally,

$$\frac{1}{p}{}_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}(\mathtt{w}))^p + \frac{1}{q}{}_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z}(\mathtt{w}))^q \geq \left[\frac{(\mathtt{w}-\eta)^{\eta}}{\eta}\right]^{-1}{}_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{S}(\mathtt{w})_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}\mathfrak{Z}(\mathtt{w}), \tag{25}$$

which implies (i). Similarly, we can prove the rest of the inequalities by making the correct choice of parameters as follows:

For (ii) $\quad U = \mathfrak{S}(\mu)\mathfrak{Z}(\rho), \quad \mathsf{v} = \mathfrak{S}(\rho)\mathfrak{Z}(\mu).$

For (iii) $\quad U = \frac{\mathfrak{S}(\mu)}{\mathfrak{Z}(\mu)}, \quad \mathsf{v} = \frac{\mathfrak{S}(\rho)}{\mathfrak{Z}(\rho)}, \quad \mathfrak{Z}(\mu), \mathfrak{Z}(\rho) \neq 0.$

For (iv) $\quad U = \frac{\mathfrak{S}(\rho)}{\mathfrak{S}(\mu)}, \quad \mathsf{v} = \frac{\mathfrak{Z}(\rho)}{\mathfrak{Z}(\mu)}, \quad \mathfrak{Z}(\mu), \mathfrak{Z}(\rho) \neq 0. \quad \square$

Theorem 5. *Let the positive functions defined on* $[0,\infty)$ *be* $\mathfrak{S}$ *and* $\mathfrak{Z}$*. Then, the following inequalities hold true:*

(i)

$$\frac{1}{p}{}_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})^p{}_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})^2 + \frac{1}{q}{}_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})^2{}_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})^q \geq {}_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z})_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}^{\frac{2}{q}}\mathfrak{Z}^{\frac{2}{p}}).$$

(ii)

$$\frac{1}{p}{}_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})^2{}_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})^q + \frac{1}{q}{}_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})^q{}_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})^2 \geq {}_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}^{\frac{2}{q}}\mathfrak{Z}^{\frac{2}{p}})_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}^{p-1}\mathfrak{Z}^{q-1}).$$

(iii)

$${}_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})^2{}_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}\left(\frac{1}{p}\mathfrak{Z}^q + \frac{1}{q}\mathfrak{Z}^p\right) \geq {}_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}^{\frac{2}{p}}\mathfrak{Z})_{\mathrm{N}_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}^{\frac{2}{q}}\mathfrak{Z}),$$

for $p,q > 1$ *satisfying* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We can prove the results following similar procedures as in the previous Theorem 4 with an appropriate choice of parameters:

(i)

$$U = \mathfrak{S}(\mu)\mathfrak{Z}^{\frac{2}{p}}(\rho), V = \mathfrak{S}^{\frac{2}{q}}(\rho)\mathfrak{Z}(\mu).$$

(ii)

$$U = \frac{\mathfrak{S}^{\frac{2}{p}}(\mu)}{\mathfrak{S}(\rho)}, V = \frac{\mathfrak{Z}^{\frac{2}{q}}(\mu)}{\mathfrak{Z}(\rho)}, \mathfrak{S}(\rho), \mathfrak{Z}(\rho) \neq 0.$$

(*iii*)

$$U = \frac{\mathfrak{S}^{\frac{2}{p}(\mu)}}{\mathfrak{Z}(\rho)}, V = \frac{\mathfrak{S}^{\frac{2}{q}}(\mu)}{\mathfrak{Z}(\rho)}, \mathfrak{Z}(\rho) \neq 0.$$

□

Theorem 6. *Let the positive functions defined on* $[0,\infty)$ *be* $\mathfrak{S}$ *and* $\mathfrak{Z}$*. For* $\mathtt{w} > 0$ *with conditions*

$$\mathcal{A} \leq \min_{0\leq\mu\leq\mathtt{w}} \frac{\mathfrak{S}(\mu)}{\mathfrak{Z}(\mu)}, \quad \mathcal{B} = \max_{0\leq\mu\leq\mathtt{w}} \frac{\mathfrak{S}(\mu)}{\mathfrak{Z}(\mu)}, \tag{26}$$

the following inequalities hold true:
(*i*)

$$0 \leq {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})^2 {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})^2 \leq \frac{(\mathcal{A}+\mathcal{B})^2}{4\mathcal{A}\mathcal{B}} \Big({}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z})\Big)^2.$$

(*ii*)

$$0 \leq \sqrt{{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})^2 {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})^2} - {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z}) \leq \frac{(\sqrt{\mathcal{B}}-\sqrt{\mathcal{A}})^2}{4\mathcal{A}\mathcal{B}} \Big({}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z})\Big).$$

(*iii*)

$$0 \leq {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})^2 {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})^2 - {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z})^2 \leq \frac{(\mathcal{B}-\mathcal{A})^2}{4\mathcal{A}\mathcal{B}} \Big({}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z})\Big)^2.$$

Proof. From (26) and

$$\left(\frac{\mathfrak{S}(\mu)}{\mathfrak{Z}(\mu)} - \mathcal{A}\right)\left(\mathcal{B} - \frac{\mathfrak{S}(\mu)}{\mathfrak{Z}(\mu)}\right)\mathfrak{Z}^2(\mu) \geq 0, \quad 0 \leq \mu \leq \mathtt{w}, \tag{27}$$

we have

$$\mathfrak{S}^2(\mu) + \mathcal{A}\mathcal{B}\mathfrak{Z}^2(\mu) \leq (\mathcal{A}+\mathcal{B})\mathfrak{S}(\mu)\mathfrak{Z}(\mu). \tag{28}$$

Multiplying the above inequality (28) by $(\mathtt{w}-\mu)^{\eta-1}$ and then integrating the obtained result with respect to μ, we get

$$\int_a^{\mathtt{w}} (\mathtt{w}-\mu)^{\eta-1}\mathfrak{S}^2(\mu)d\mu + \mathcal{A}\mathcal{B}\int_a^{\mathtt{w}} (\mathtt{w}-\mu)^{\eta-1}\mathfrak{Z}^2(\mu)d\mu \leq (\mathcal{A}+\mathcal{B})\int_a^{\mathtt{w}} (\mathtt{w}-\mu)^{\eta-1}\mathfrak{S}(\mu)\mathfrak{S}(\mu).$$

This implies

$${}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})^2(\mathtt{w}) + \mathcal{A}\mathcal{B}{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})^2(\mathtt{w}) \leq (\mathcal{A}+\mathcal{B}){}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z})(\mathtt{w}). \tag{29}$$

On the other hand, it follows from
$\mathcal{A}\mathcal{B} > 0$, and $\left(\sqrt{{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})^2} - \sqrt{\mathcal{A}\mathcal{B}{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})^2}\right)^2 \geq 0$

$$\Rightarrow 2\left(\sqrt{{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})^2}\sqrt{\mathcal{A}\mathcal{B}{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})^2}\right) \leq {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})^2 + \mathcal{A}\mathcal{B}{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})^2. \tag{30}$$

Then, from the last two inequalities (29) and (30), we obtain

$$4\mathcal{A}\mathcal{B}{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})^2 {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})^2 \leq (\mathcal{A}+\mathcal{B})^2\Big({}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z})\Big)^2,$$

which readily follows (i); using the same operations as of (i), we can prove (ii) and (iii). □

5. Chebyshev-Type Inequalities

Theorem 7. *Let the integrable functions be* $\mathcal{L}^+_{\eta,0}[\varrho,\kappa]$*, which are synchronous on* $[\varrho,\kappa]$*. Then,*

$${}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z})(\kappa) \geq \left[\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}\right]^{-1} {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})(\kappa){}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})(\kappa). \tag{31}$$

Proof. Since $\mathfrak{S}$ and $\mathfrak{Z}$ are synchronous on $[\varrho, \kappa]$, we have

$$(\mathfrak{S}(a) - \mathfrak{S}(b))(\mathfrak{Z}(a) - \mathfrak{Z}(b)) \geq 0 \quad a, b \in [\varrho, \kappa],$$

or equivalently

$$\mathfrak{S}(a)\mathfrak{Z}(a) + \mathfrak{S}(b)\mathfrak{Z}(b) \geq \mathfrak{S}(a)\mathfrak{Z}(b) + \mathfrak{S}(b)\mathfrak{Z}(a).$$

If we multiply both sides of the above inequality by $(\kappa - a)^{-\eta}$, we have

$$\mathfrak{S}(a)\mathfrak{Z}(a)(\kappa - a)^{-\eta} + \mathfrak{S}(b)\mathfrak{Z}(b)(\kappa - a)^{-\eta} \geq \mathfrak{S}(a)\mathfrak{Z}(b)(\kappa - a)^{-\eta} + \mathfrak{S}(b)\mathfrak{Z}(a)(\kappa - a)^{-\eta}.$$

Upon integrating the inequality obtained with respect to a, one has

$$\begin{aligned} &\int_{\varrho}^{\kappa} (\kappa - a)^{-\eta}\mathfrak{S}(a)\mathfrak{Z}(a)da + \mathfrak{S}(b)\mathfrak{Z}(b)\int_{\varrho}^{\kappa} (\kappa - a)^{-\eta}da \\ &\qquad \geq \mathfrak{Z}(b)\int_{\varrho}^{\kappa} (\kappa - a)^{-\eta}\mathfrak{S}(a)da + \mathfrak{S}(b)\int_{\varrho}^{\kappa} (\kappa - a)^{-\eta}\mathfrak{Z}(a)da. \end{aligned}$$

From the above developments, we have

$$_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z})(\kappa) + \left[\frac{(\kappa - \varrho)^{1-\eta}}{1-\eta}\right]\mathfrak{S}(b)\mathfrak{Z}(b) \geq \mathfrak{Z}(b)\,{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S})(\kappa) + \mathfrak{S}(b)\,{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{Z})(\kappa). \quad (32)$$

Multiplying inequality (32) by $(\kappa - b)^{-\eta}$ and integrating the resultant inequality with respect to b, we obtain

$$\begin{aligned} &_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z})(\kappa)\int_{\varrho}^{\kappa} (\kappa - b)^{-\eta}db + \left[\frac{(\kappa - \varrho)^{1-\eta}}{1-\eta}\right]\int_{\varrho}^{\kappa} (\kappa - b)^{-\eta}\mathfrak{S}(b)\mathfrak{Z}(b)db \\ &\qquad \geq {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}(\kappa)\int_{\varrho}^{\kappa} (\kappa - b)^{-\eta}\mathfrak{Z}(b)db + {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{Z}(\kappa)\int_{\varrho}^{\kappa} (\kappa - b)^{-\eta}\mathfrak{S}(b)db. \end{aligned}$$

This readily gives

$$2\left[\frac{(\kappa - \varrho)^{1-\eta}}{1-\eta}\right]{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z})(\kappa) \geq 2\,{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{S}(\kappa)\,{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\mathfrak{Z}(\kappa).$$

and we have the desired inequality ((31)). □

Remark 2. *Let $\mathfrak{S}, \mathfrak{Z} \in \mathcal{L}^{-}_{\eta,0}$ be synchronous functions on $[\varrho, \kappa]$; then, we have*

$$_{N_3}\mathfrak{J}^{\eta}_{\kappa^-}(\mathfrak{S}\mathfrak{Z})(\varrho) \geq \left[\frac{(\kappa - \varrho)^{1-\eta}}{1-\eta}\right]^{-1}{}_{N_3}\mathfrak{J}^{\eta}_{\kappa^-}\mathfrak{S}(\varrho)\,{}_{N_3}\mathfrak{J}^{\eta}_{\kappa^-}\mathfrak{Z}(\varrho). \quad (33)$$

Remark 3. *If we take $\eta = 0$ in the above Theorem 7 (or in Remark 2), then the inequality (31) or inequality (33) reduces to the classical Chebyshev inequality.*

Theorem 8. *Let $\mathfrak{S}$ and $\mathfrak{Z}$ be two function from $\mathcal{L}^{+}_{\eta,0}[\varrho, \kappa] \bigcap \mathcal{L}^{+}_{\beta,0}[\varrho, \kappa]$, which are synchronous on $[\varrho, \kappa]$; then, the following inequality holds true:*

$$\begin{aligned} &\frac{(\kappa - \varrho)^{1-\beta}}{1-\beta}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z})(\kappa) + \frac{(\kappa - \varrho)^{1-\eta}}{1-\eta}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z})(\kappa) \\ &\geq {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S})(\kappa)\,{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{Z})(\kappa) + {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{Z})(\kappa)\,{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S})(\kappa). \end{aligned} \quad (34)$$

Proof. Multiplying the inequality (32) by $(\kappa-b)^{-\beta}$ yields

$$(\kappa-b)^{-\beta}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z})(\kappa)+\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}(\kappa-b)^{-\beta}\mathfrak{S}(b)\mathfrak{Z}(b)$$
$$\geq {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S})(\kappa)(\kappa-b)^{-\beta}\mathfrak{Z}(b)+{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{Z})(\kappa)(\kappa-b)^{-\beta}\mathfrak{S}(b).$$

Integrating the above inequality with respect to b yields inequality (34). □

Remark 4. *If we take $\eta=\beta$, then we obtain Theorem 7.*

Theorem 9. *Let $\{\mathfrak{S}_i\}_{i=1,2,3,4,\ldots,n}$ be a positive function $\mathcal{L}^{+}_{\eta,0}[\varrho,\kappa]$; then, we have*

$$\left[{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\left(\prod_{i=1}^{n}\mathfrak{S}_i\mathfrak{S}_{n+1}\right)\right]\geq\left[\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}\right]^{-n}\left(\prod_{i=1}^{n+1}\mathfrak{S}_i\right)(\kappa). \tag{35}$$

Proof. The theorem can be proven by the method of induction on $n\in\mathbb{N}$. For $n=1$, the above inequality trivially holds. For $n=2$, since $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are synchronous and positive functions and by the hypothesis of theorem 7, the inequality (35) readily follows. Now, let us assume that the inequality (35) holds true for $n\in\mathbb{N}$. Let $\mathfrak{S}=\prod_{i=1}^{n}\mathfrak{S}_i$ and $\mathfrak{Z}=\mathfrak{S}_{n+1}$, as $\mathfrak{S}$ and $\mathfrak{Z}$ be increasing functions on $[\varrho,\kappa]$; therefore, under the assumption of the inequality (31) and induction hypothesis, we have

$${}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\left(\prod_{i=1}^{n}\mathfrak{S}_i\mathfrak{S}_{n+1}\right)\geq\left[\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}\right]^{-1}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\left(\prod_{i=1}^{n}\mathfrak{S}_i\right)(\kappa){}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S}_{n+1})(\kappa)$$
$$\geq\left[\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}\right]^{-1}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}\left(\prod_{i=1}^{n+1}\mathfrak{S}_i\right)(\kappa).$$

This concludes the desired proof. □

Theorem 10. *Let $\mathfrak{S},\mathfrak{Z}:[0,\infty)\to\mathcal{R}$ and $\mathfrak{S},\mathfrak{Z}\in\mathcal{L}^{+}_{\varrho}[\varrho,\kappa]$, be increasing and differentiable functions, respectively. $\mathfrak{Z}'$ is bounded below by $m=\inf_{\mathtt{w}\in[0,\infty)}\mathfrak{Z}'(t)$; then, we have*

$${}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z})(\kappa)\geq\left[\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}\right]^{-1}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S})(\kappa)\,{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{Z})(\kappa)$$
$$-m\left[\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}\right]^{-1}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S})(\kappa)\,{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathtt{w})(\kappa)+m{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathtt{w}\mathfrak{S})(\kappa),$$

where $\mathtt{w}(x)=x$ is the identity function.

Proof. If h is differentiable and increasing on $[0,\infty)$ with $P(\mathrm{u})=m\mathrm{u}$ and $h(\mathrm{u})=\mathfrak{Z}(\mathrm{u})-P(\mathrm{u})$. Then, applying the results of Theorem 31, we have

$${}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S}h)(\kappa)\geq\left[\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}\right]^{-1}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S})(\kappa)\,{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(h)(\kappa)$$
$$=\left[\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}\right]^{-1}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S})(\kappa)\,{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{Z})(\kappa)-\left[\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}\right]^{-1}{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S})(\kappa)\,{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(P)(\kappa). \tag{36}$$

since ${}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(P)(\kappa)=m{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(t)(\kappa)$ and ${}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S}P)(\kappa)=m{}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathtt{w}\mathfrak{S})(\kappa)$.

From the above developments, we have

$$
\begin{aligned}
&{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z})(\kappa) = {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}h)(\kappa) + {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}P)(\kappa) \\
&\geq \left[\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}\right]^{-1} {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})(\kappa)\, {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})(\kappa) - \left[\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}\right]^{-1} {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})(\kappa)_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(P)(\kappa) + {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}P)(\kappa) \\
&\geq \left[\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}\right]^{-1} {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})(\kappa)\, {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})(\kappa) - m\left[\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}\right]^{-1} {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})(\kappa)\, {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(t)(\kappa) + m_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(t\mathfrak{S})(\kappa).
\end{aligned}
$$

This completes the desired proof. □

Theorem 11. *Let $\mathfrak{S}, \mathfrak{Z} : [0,\infty) \to \mathcal{R}$ and $\mathfrak{S}, \mathfrak{Z} \in \mathcal{L}^{+}_{\varrho}[\varrho,\kappa]$, be increasing and differentiable functions, respectively. $\mathfrak{Z}'$ is bounded below by $m = \inf_{\mathtt{w}\in[0,\infty)} \mathfrak{Z}'(t)$; then, we have*

Let $\mathfrak{S}, \mathfrak{Z} : [0,\infty) \to \mathcal{R}$ and $\mathfrak{S}, \mathfrak{Z} \in \mathcal{L}^{+}_{\varrho}[\varrho,\kappa]$ be two differentiable functions. If $\mathfrak{S}'$ bounded below by $m_1 = \inf_{\mathtt{w}\in[0,\infty)} \mathfrak{S}'(\mathtt{w})$ and $\mathfrak{Z}'$ bounded below by $m_2 = \inf_{\mathtt{w}\in[0,\infty)} \mathfrak{Z}'(\mathtt{w})$. Then we have

$$
\begin{aligned}
&{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z})(\kappa) \geq \left[\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}\right]^{-1} {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})(\kappa)\, {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})(\kappa) - m_2\left[\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}\right]^{-1} {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})(\kappa)\, {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathtt{w})(\kappa) \\
&- m_1\left[\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}\right]^{-1} {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})(\kappa)\, {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathtt{w})(\kappa) + m_1 m_2\left[\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}\right]^{-1} {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathtt{w})(\kappa)\, {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathtt{w})(\kappa) \\
&+ m_2\, {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathtt{w}\mathfrak{S})(\kappa) + m_1\, {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathtt{w}\mathfrak{Z})(\kappa) - m_1 m_2\, {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathtt{w}^2)(\kappa),
\end{aligned}
$$

where $\mathtt{w}(\mathtt{u}) = \mathtt{u}$ is the identity function.

Proof. Let h_1 and h_2 be differentiable and increasing functions on $[0,\infty)$ with $P_1(\mathtt{u}) = m_1\mathtt{u}$ and $h_1(\mathtt{u}) = \mathfrak{Z}(\mathtt{u}) - P_1(\mathtt{u})$; similarly, $P_2(\mathtt{u}) = m_2\mathtt{u}$ and $h_2(\mathtt{u}) = \mathfrak{Z}(\mathtt{u}) - P_2(\mathtt{u})$. Then, applying the results of Theorem 7, we have

$$
\begin{aligned}
&{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(h_1h_2)(\kappa) \geq \left[\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}\right]^{-1} {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(h_1)(\kappa)\, {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(h_2)(\kappa) \\
&\geq \left[\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}\right]^{-1} \left[{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})(\kappa) - {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(P_1)(\kappa)\right]\left[{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})(\kappa) - {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(P_2)(\kappa)\right] \\
&\geq \left[\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}\right]^{-1} {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})(\kappa)\, {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})(\kappa) - m_2\left[\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}\right]^{-1} {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{S})(\kappa)\, {}_{O^+}I_{\varrho}(\mathtt{w})(\kappa) \\
&- m_1\left[\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}\right]^{-1} {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathfrak{Z})(\kappa)\, {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathtt{w})(\kappa) - m_1 m_2\left[\frac{(\kappa-\varrho)^{1-\eta}}{1-\eta}\right]^{-1} {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathtt{w})(\kappa)\, {}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathtt{w})(\kappa).
\end{aligned} \tag{37}
$$

Moreover,

$$
{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(h_1P_2)(\kappa) = m_{2N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathtt{w}h_1)(\kappa) = m_{2N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathtt{w}\mathfrak{S})(\kappa) - m_1 m_{2N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathtt{w}^2)(\kappa). \tag{38}
$$

Similarly,

$$
{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(h_2P_1)(\kappa) = m_{1N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathtt{w}\mathfrak{Z})(\kappa) - m_1 m_{2N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathtt{w}^2)(\kappa), \tag{39}
$$

and

$$
{}_{N_3}\mathfrak{I}^{\eta}_{\varrho^+}(P_1P_2)(\kappa) = m_1 m_{2N_3}\mathfrak{I}^{\eta}_{\varrho^+}(\mathtt{w}^2)(\kappa). \tag{40}
$$

From the equality,

$$
\mathfrak{S}\mathfrak{Z} = (h_1 + P_1)(h_2 + P_2) = h_1h_2 + h_1P_2 + h_2P_1 + P_1P_2,
$$

we have

$$ {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(\mathfrak{S}\mathfrak{Z})(\kappa) = {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(h_1 h_2)(\kappa) + {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(h_1 P_2)(\kappa) + {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(P_1 h_2)(\kappa) + {}_{N_3}\mathfrak{J}^{\eta}_{\varrho^+}(P_1 P_2)(\kappa), $$

and this equality together with (37)–(40) implies the required result. □

Remark 5. *If we take $m_1 = 0$, then we obtain Theorem 10.*

Remark 6. *If we consider $-\mathfrak{S}$ and $\mathfrak{Z}$, or $\mathfrak{S}$ and $-\mathfrak{Z}$ instead of $\mathfrak{S}$ and $\mathfrak{Z}$, under the assumptions of the synchronous functions, we will have new results with changes in the direction of the inequalities.*

6. Conclusions

The Grüss inequality and the Chebyshev inequality have been extensively studied, and numerous generalizations, extensions, and variants of these two valuable inequalities have been established. Using a generalized integral operator, namely the non-conformable operator, several generalizations of the Grüss inequality as well as the Chebyshev-type inequality are presented in this paper. The findings provide novel approaches to the Grüss inequality thanks to the peculiarities of the fractional operator and some inequalities employed in the proofs. In future research work, different forms of fractional integral operators can be used to enhance the outcomes of researchers working on this topic.

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