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Conjugation Conditions for Systems of Differential Equations of Different Orders on a Star Graph

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Abstract: In this paper, a one-dimensional mathematical model for investigating the vibrations of structures consisting of elastic and weakly curved rods is proposed. The three-dimensional structure is replaced by a limit graph, on each arc of which a system of three differential equations is written out. The differential equations describe the longitudinal and transverse vibrations of an elastic rod, taking into account the influence of longitudinal and transverse vibrations on each other. Describing conjugation conditions at joints of four or more rods is an important problem. This article assumes new conjugation conditions that guarantee the all-around decidability and symmetry of the resulting boundary value problems for systems of differential equations on a star graph. In addition, the paper proposes a physical interpretation of the conjugation conditions found. Thus, the work presents one more area of knowledge where symmetry phenomena occur. The symmetry here is manifested in the preservation of conjugation conditions when passing to the conjugate operator.

Keywords: boundary vertices of a graph; inner vertex of a star graph; boundary problems; Laplace operator

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Citation: Kanguzhin, B.; Auzerkhan, G. Conjugation Conditions for Systems of Differential Equations of Different Orders on a Star Graph. *Symmetry* **2022**, *14*, 1761. https://doi.org/10.3390/sym14091761

Academic Editor: Calogero Vetro

Received: 24 July 2022
Accepted: 19 August 2022
Published: 23 August 2022

1. Introduction

The work is devoted to the description of well-posed boundary value problems for systems of differential equations on graphs. In contrast to well-known works, on each edge of the graph, there is a system of three differential equations with various orders that do not coincide with each other. At the e_n of each arc of star graph G, let there be a system of differential equations at $n = 1, \ldots, m+1$ and $z_n \in \left(-\frac{l_n}{2}, \frac{l_n}{2}\right)$.

$$\begin{cases} \dfrac{d^2}{dz_n^2}\left(\mu_n(z_n)\left(a_n(z_n)\dfrac{d^2w_1^n(z_n)}{dz_n^2} + b_n(z_n)\dfrac{d^2w_2^n(z_n)}{dz_n^2} - d_n(z_n)\dfrac{dw_3^n(z_n)}{dz_n}\right)\right) = F_1^n(z_n), \\ \dfrac{d^2}{dz_n^2}\left(\mu_n(z_n)\left(b_n(z_n)\dfrac{d^2w_1^n(z_n)}{dz_n^2} + c_n(z_n)\dfrac{d^2w_2^n(z_n)}{dz_n^2} - f_n(z_n)\dfrac{dw_3^n(z_n)}{dz_n}\right)\right) = F_2^n(z_n), \\ -\dfrac{d}{dz_n}\left(\mu_n(z_n)\left(-d_n(z_n)\dfrac{d^2w_1^n(z_n)}{dz_n^2} - f_n(z_n)\dfrac{d^2w_2^n(z_n)}{dz_n^2} + \dfrac{dw_3^n(z_n)}{dz_n}\right)\right) = F_3^n(z_n). \end{cases} \quad (1)$$

Assume that the coefficients

$$\{\mu_n(z_n),\ a_n(z_n),\ b_n(z_n),\ d_n(z_n),\ c_n(z_n),\ f_n(z_n),\ n = 1, \ldots, m+1\}$$

give simple real-valued functions. Furthermore, the physical meaning of the coefficients is explained. The required functions $\{w_1^n(z_n),\ w_2^n(z_n),\ w_3^n(z_n),\ n = 1, \ldots, m+1\}$ represent the transverse and longitudinal displacements of rods from their respective axes. The rods are considered to be connected at their ends in one knot. The other ends of the rods are hard-fixed. The hard-fixed ends of the rods in the graph correspond to the boundary vertices of the graph, and the connecting node corresponds to the inner vertex of the star graph. The main question that interests us is: what conjugation conditions can be fulfilled at an interior vertex of the star graph so that the corresponding boundary value problem for the system of differential equations is symmetric? In case the boundary value problem is symmetric, then the eigenfrequencies will be real. As far as physical meaning is concerned, these types of boundary tasks are interesting. The system (1) is introduced in [1] in connection with the asymptotic analysis of one-dimensional equations of deformation of thin weakly curved rods. Thus, this paper presents one more area of knowledge where symmetry plays an important role.

Usually, cross-sections of rods [2] are characterized by space, static moments of plane sections, the center of gravity position, moments of inertia, radii of inertia, and moments of resistances. Let z change along the rod. We denote the transverse cross-section of the rod at point z by $\omega(z)$. The coefficients of the system (1) are then input by the formulas $d(z) = \iint\limits_{\omega(z)} \eta_1 d\eta_1 d\eta_2$, $f(z) = \iint\limits_{\omega(z)} \eta_2 d\eta_1 d\eta_2$, $a(z) = \iint\limits_{\omega(z)} \eta_1^2 d\eta_1 d\eta_2$, $b(z) = \iint\limits_{\omega(z)} \eta_1\eta_2 d\eta_1 d\eta_2$, $c(z) = \iint\limits_{\omega(z)} \eta_2^2 d\eta_1 d\eta_2$.

Then, according to [2], $d(z) = \iint\limits_{\omega(z)} \eta_1 d\eta_1 d\eta_2$, $f(z) = \iint\limits_{\omega(z)} \eta_2 d\eta_1 d\eta_2$ are the static moments of cross-section area $w(z)$ relative to the axes $O\eta_1$ and $O\eta_2$, respectively.

The values $a(z) = \iint\limits_{\omega(z)} \eta_1^2 d\eta_1 d\eta_2$, $c(z) = \iint\limits_{\omega(z)} \eta_2^2 d\eta_1 d\eta_2$ represent the axial moments of inertia of section $\omega(z)$ relative to the axes $O\eta_1$ and $O\eta_2$, respectively. The centrifugal moment of the inertia of cross-section $\omega(z)$ concerning the two co-orthogonal axes $O\eta_1$ and $O\eta_2$ is equal to $b(z) = \iint\limits_{\omega(z)} \eta_1\eta_2 d\eta_1 d\eta_2$. If the diameter of the cross-section $w(z)$ is considered to be a small of order ε, then

1. The static moments of the cross-sectional area $d(z)$, $f(z)$ have an order of smallness ε^3;
2. The cross-sectional axial moments of inertia $a(z)$, $c(z)$ have the order of smallness ε^4;
3. The centrifugal moment of inertia of the section $b(z)$ has an order of smallness ε^4.

Consider separately the third equation of system (1)

$$-\frac{d}{dz_n}\left(\mu_n(z_n)\left(-d_n(z_n)\frac{d^2w_1^n(z_n)}{dz_n^2} - f_n(z_n)\frac{d^2w_2^n(z_n)}{dz_n^2} + \frac{dw_3^n(z_n)}{dz_n}\right)\right) = F_3^n(z_n).$$

As there are coefficients $\varepsilon \to +0$, $d_n(z_n) = O(\varepsilon_3)$, $f_n(z_n) = O(\varepsilon_3)$, the first two terms can be neglected in the last equation. As a result, we have (2)

$$-\frac{d}{dz}\left(\mu(z)\left(\frac{dw_3(z)}{dz}\right)\right) = F_3(z). \tag{2}$$

Similar equations describe the longitudinal vibrations of rods (3). Thus, if the cross-sectional diameter of the rod is $\varepsilon \to 0$, then the influence of the transverse vibrations on the longitudinal vibrations can be neglected.

Under natural simplifying assumptions, this is the equation of longitudinal vibrations of the rod [3]. Therefore, the longitudinal offsets $w_3(z)$ can be found first. The transverse offsets can be defined from the first two equations of system (1).

$$\begin{cases} \dfrac{d^2}{dz^2}\left(\mu(z)\left(a(z)\dfrac{d^2w_1(z)}{dz^2} + b(z)\dfrac{d^2w_2(z)}{dz^2} - d(z)\dfrac{dw_3(z)}{dz}\right)\right) = F_1(z), \\ \dfrac{d^2}{dz^2}\left(\mu(z)\left(b(z)\dfrac{d^2w_1(z)}{dz^2} + c(z)\dfrac{d^2w_2(z)}{dz^2} - f(z)\dfrac{dw_3(z)}{dz}\right)\right) = F_2(z). \end{cases}$$

The last system of equations falls into two systems, and each of them can be solved independently from the other. The fourth-order differential equation for transverse displacement along the Oy axis:

$$\frac{d^2}{dz^2}\left(\mu(z)(b^2(z)-a(z)c(z))\frac{d^2w_2(z)}{dz^2}\right)=\widetilde{F}_2(z), \tag{3}$$

where

$$\widetilde{F}_2(z)=b(z)\left(F_1(z)+\frac{d^2}{dz^2}\left(d(z)\frac{dw_3(z)}{dz}\right)\right)-a(z)\left(F_2(z)+\frac{d^2}{dz^2}\left(f(z)\frac{dw_3(z)}{dz}\right)\right).$$

The fourth-order differential equation for transverse displacement in the $0x$-axis:

$$\frac{d^2}{dz^2}\left(\mu(z)(a(z)c(z)-b^2(z))\frac{d^2w_1(z)}{dz^2}\right)=\widetilde{F}_1(z),$$

where

$$\widetilde{F}_1(z)=c(z)\left(F_1(z)+\frac{d^2}{dz^2}\left(d(z)\frac{dW_3(z)}{dz}\right)\right)-b(z)\left(F_2(z)+\frac{d^2}{dz^2}\left(f(z)\frac{dW_3(z)}{dz}\right)\right).$$

Thus, the transverse displacements in the Ox and Oy axes, in this case, can be computed independently of each other. This confirms Timoshenko's theory of beam bends, which proposes that the bends are determined from fourth-order differential equations. If the section diameter $\omega(z)$ is considered to be a small order of ε, then system (1) confirms Timoshenko's theory of bending beams [4].

In many engineering calculations, it is presumed that movements are separated: transverse vibrations do not affect longitudinal ones and vice versa. However, this division of rod movements is not always justified. Thus, in general, system (1) does not always decompose into equations of (2) and (3) types.

Consider a mechanical system (Figure 1) consisting of n rods in three dimensions. The ends of the rods are connected to each other at $X_{\varepsilon,o}$, which is a node. If the parameter ε tend toward zero, we obtain the limit graph (Figure 2).

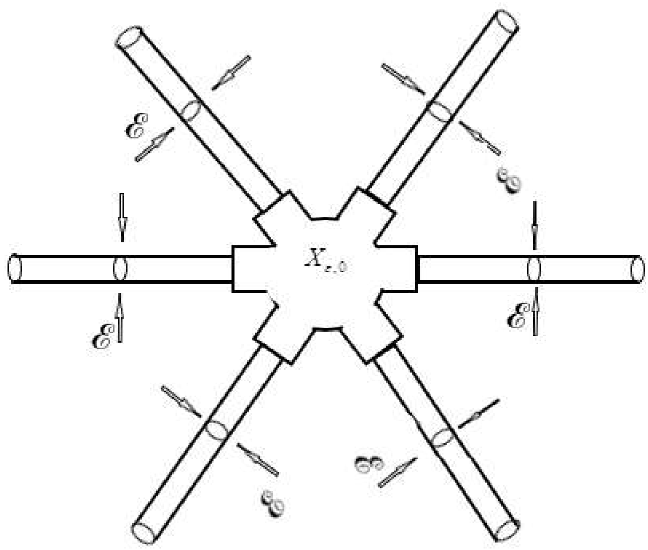

Figure 1. A 2-dimensional graph-like manifold with boundary.

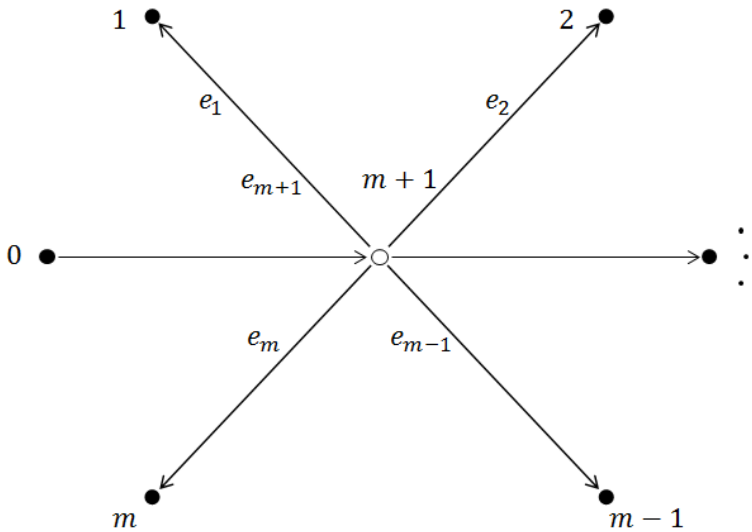

Figure 2. Limit star graph.

The rods are cylinders $X_{\varepsilon,e} = I_e \times \varepsilon Y_e$ with a cross-sectional radius ε of the Y_e variety for each rod $e = 1, 2, \dots, n$. We denote $X_{\varepsilon,o}$ by the central manifold. In the limit, the family $\{X_{\varepsilon,o}, X_{\varepsilon,e}, e = 1, 2, \dots, n\}$ at $\varepsilon \to 0$ converges to the limit graph. The limit operator represents the limits of the differential operators describing the transverse and longitudinal vibrations of the rods. The limit operator depends on the choice of coupling boundary conditions at the central node where the rods connect to the node. The limit operator depends on the choice of boundary conditions and conjugation conditions at the central node where the rods connect to the node.

Different types of rod–node connections correspond to different conjugation conditions. In the literature, there are different types of connections between a rod and a node (Figures 3 and 4). Figure 3 shows the case where the dimensions of the node tend to zero much more slowly than the dimensions of the cross-sections of the rods. Figure 4 shows the case where the dimensions of the node tend to zero much faster than the cross-sectional dimensions of the rods. In [5–9], limiting conjugation conditions corresponding to the two mentioned connections of knots with rods have been investigated. The slowly decaying and borderline cases were introduced in [7] (see also [8]) showing the convergence of the spectrum for compact graphs and manifolds. The notion "plumber's shop" was introduced in the article of Rubinstein and Schatzman [5], where the authors used it for the necessary local estimates of the identification operators from the graph to the graph-like manifold and vice versa. In work [9], the authors extend the analysis here to non-compact spaces and show in particular the quasi-unitary quivalence implying, e.g., the convergence of resolvents and the convergence of the entire spectrum. Other types of boundary conditions for limit operators have been studied in [10–13]. The authors in the work [14] investigate the initial boundary value problem describing the oscillation process on a geometric graph with hysteresis-type boundary conditions.

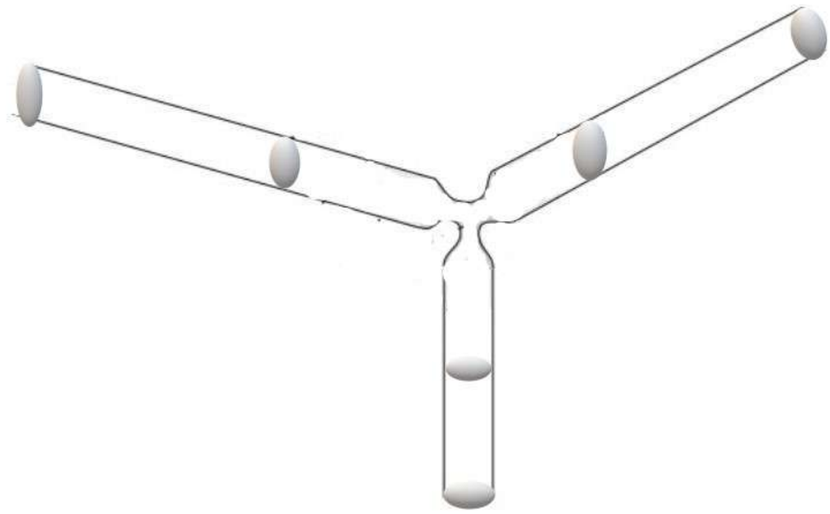

Figure 3. The case where the value of a node tends to zero much more slowly than the cross-sectional values of the rods.

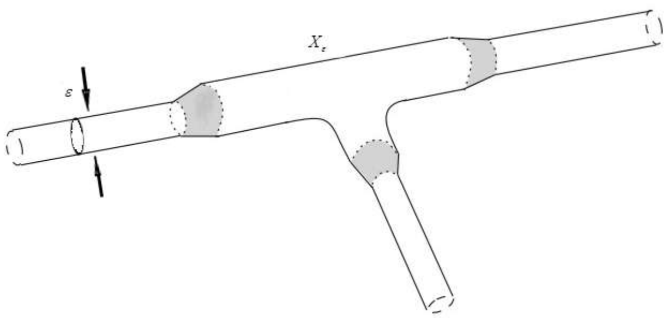

Figure 4. The case where the value of a node tends to zero much faster than the cross-sectional values of the rods.

2. Lagrange Formula on a Star Graph in a Conjugate Condition at the Internal Vertex

Let $\Gamma = \{V, E\}$, a graph, be a star (Figure 2), where V is the set of its vertices numbered from 0 to $m+1$, and the set E means its arcs $e_1, \ldots, e_m$. We introduce a vector function [15,16]

$$Y_j(x_j) = \begin{bmatrix} y_{1j}(x_j) & y_{2j}(x_j) & y_{3j}(x_j) \end{bmatrix}^T.$$

A system of differential equations is given on each arc e_j

$$\begin{aligned} l_{1j}(Y_j) = {} & \frac{d^2}{dx_j^2}\left(\mu_j(x_j)a_j(x_j)\frac{d^2 y_{1j}(x_j)}{dx_j^2}\right) + \frac{d^2}{dx_j^2}\left(\mu_j(x_j)b_j(x_j)\frac{d^2 y_{2j}(x_j)}{dx_j^2}\right) \\ & -\frac{d^2}{dx_j^2}\left(\mu_j(x_j)d_j(x_j)\frac{dy_{3j}(x_j)}{dx_j}\right), \\ l_{2j}(Y_j) = {} & \frac{d^2}{dx_j^2}\left(\mu_j(x_j)b_j(x_j)\frac{d^2 y_{1j}(x_j)}{dx_j^2}\right) + \frac{d^2}{dx_j^2}\left(\mu_j(x_j)c_j(x_j)\frac{d^2 y_{2j}(x_j)}{dx_j^2}\right) \\ & -\frac{d^2}{dx_j^2}\left(\mu_j(x_j)f_j(x_j)\frac{dy_{3j}(x_j)}{dx_j}\right), \end{aligned}$$

$$l_{3j}(Y_j) = \frac{d}{dx_j}\left(\mu_j(x_j)d_j(x_j)\frac{d^2y_{1j}(x_j)}{dx_j^2}\right) + \frac{d}{dx_j}\left(\mu_j(x_j)f_j(x_j)\frac{d^2y_{2j}(x_j)}{dx_j^2}\right)$$
$$-\frac{d}{dx_j}\left(\mu_j(x_j)\frac{dy_{3j}(x_j)}{dx_j}\right).$$

Furthermore, we assume that the functions $\mu_j(x_j)$, $a_j(x_j)$, $b_j(x_j)$, $c_j(x_j)$, $d_j(x_j)$, $f_j(x_j)$ are real continuous on the arc e_j. In this case, $x_j \in [0, l_j]$, and x_j belongs to e_j. The vertex $(m+1) \in V$ is called the internal vertex of the star graph. The vertices $0, 1, \ldots, m$ are called the boundary vertices of the star graph. Denote by

$$L_j(Y_j) = \begin{bmatrix} l_{1j}(Y_j) & l_{2j}(Y_j) & l_{3j}(Y_j) \end{bmatrix}^T$$

at $j = 1, \ldots, m+1$. Introduce formally conjugate differential expressions

$$l_{1j}^{+}(Y_j) = \frac{d^2}{dx_j^2}\left(\mu_j(x_j)a_j(x_j)\frac{d^2y_{1j}(x_j)}{dx_j^2}\right) + \frac{d^2}{dx_j^2}\left(\mu_j(x_j)b_j(x_j)\frac{d^2y_{2j}(x_j)}{dx_j^2}\right)$$
$$+\frac{d}{dx_j}\left(\mu_j(x_j)d_j(x_j)\frac{d^2y_{3j}(x_j)}{dx_j^2}\right),$$
$$l_{2j}^{+}(Y_j) = \frac{d^2}{dx_j^2}\left(\mu_j(x_j)b_j(x_j)\frac{d^2y_{1j}(x_j)}{dx_j^2}\right) + \frac{d^2}{dx_j^2}\left(\mu_j(x_j)c_j(x_j)\frac{d^2y_{2j}(x_j)}{dx_j^2}\right)$$
$$+\frac{d}{dx_j}\left(\mu_j(x_j)f_j(x_j)\frac{d^2y_{3j}(x_j)}{dx_j^2}\right),$$
$$l_{3j}^{+}(Y_j) = -\frac{d^2}{dx_j^2}\left(\mu_j(x_j)d_j(x_j)\frac{dy_{1j}(x_j)}{dx_j}\right) - \frac{d^2}{dx_j^2}\left(\mu_j(x_j)f_j(x_j)\frac{dy_{2j}(x_j)}{dx_j}\right)$$
$$-\frac{d}{dx_j}\left(\mu_j(x_j)\frac{dy_{3j}(x_j)}{dx_j}\right).$$

We will also need a designation $L_j^{+}(Y_j) = \begin{bmatrix} l_{1j}^{+}(Y_j) & l_{2j}^{+}(Y_j) & l_{3j}^{+}(Y_j) \end{bmatrix}^T$ at $j = 1, \ldots, m+1$. Let us also introduce quasi-derivatives according to the formulas at the point ξ :

$$D_{1j}^{(0)}(Y_j;\xi) = [y_{1j}]\Big|_{x_j=\xi},\ D_{2j}^{(0)}(Y_j;\xi) = [y_{2j}]\Big|_{x_j=\xi},\ D_{3j}^{(0)}(Y_j;\xi) = [y_{3j}]\Big|_{x_j=\xi},$$

$$D_{1j}^{(1)}(Y_j;\xi) = [dy_{1j}/dx_j]\Big|_{x_j=\xi},\ D_{2j}^{(1)}(Y_j;\xi) = [dy_{2j}/dx_j]\Big|_{x_j=\xi},$$

$$D_{1j}^{(2)}(Y_j;\xi) = \left[\mu_j a_j\frac{d^2y_{1j}}{dx_j^2} + \mu_j b_j\frac{d^2y_{2j}}{dx_j^2} - \mu_j d_j\frac{dy_{3j}}{dx_j}\right]\Big|_{x_j=\xi},$$

$$D_{2j}^{(2)}(Y_j;\xi) = \left[\mu_j b_j\frac{d^2y_{1j}}{dx_j^2} + \mu_j c_j\frac{d^2y_{2j}}{dx_j^2} - \mu_j f_j\frac{dy_{3j}}{dx_j}\right]\Big|_{x_j=\xi},$$

$$D_{3j}^{(2)}(Y_j;\xi) = \left[\mu_j d_j\frac{d^2y_{1j}}{dx_j^2} + \mu_j f_j\frac{d^2y_{2j}}{dx_j^2} - \mu_j\frac{dy_{3j}}{dx_j}\right]\Big|_{x_j=\xi},$$

$$D_{2j}^{(3)}(Y_j;\xi) = \left[\frac{d}{dx_j}\left(\mu_j b_j \frac{d^2 y_{1j}}{dx_j^2}\right) + \frac{d}{dx_j}\left(\mu_j c_j \frac{d^2 y_{2j}}{dx_j^2}\right) - \frac{d}{dx_j}\left(\mu_j f_j \frac{dy_{3j}}{dx_j}\right)\right]\Big|_{x_j=\xi},$$

$$D_{1j}^{(3)}(Y_j;\xi) = \left[\frac{d}{dx_j}\left(\mu_j a_j \frac{d^2 y_{1j}}{dx_j^2}\right) + \frac{d}{dx_j}\left(\mu_j b_j \frac{d^2 y_{2j}}{dx_j^2}\right) - \frac{d}{dx_j}\left(\mu_j d_j \frac{dy_{3j}}{dx_j}\right)\right]\Big|_{x_j=\xi}.$$

Sometimes, instead of $D_{kj}^{(s)}(Y_j;\xi)$, we will write $D_{kj}^{(s)}(Y_j)$. Denote by

$$L(Y) = \begin{vmatrix} L_1(Y_1) \\ L_2(Y_2) \\ \dots \\ L_{m+1}(Y_{m+1}) \end{vmatrix}, \quad Z = \begin{vmatrix} Z_1 \\ Z_2 \\ \dots \\ Z_{m+1} \end{vmatrix}, \quad Y = \begin{vmatrix} Y_1 \\ Y_2 \\ \dots \\ Y_{m+1} \end{vmatrix}.$$

Introduce a scalar product of

$$< L(Y); Z >_{L_2(\Gamma)} = \sum_{j=1}^{m+1} < L(Y_j); Z_J >_{L_2(e_j)} = \sum_{j=1}^{m+1}\sum_{i=1}^{3} < l_{ij}(Y_j); z_{ij} >_{L_2(e_j)} .$$

Lemma 1. *For any two sets of sufficiently smooth functions $Y = \{Y_j(x_j),\ \ j = 1,\dots,m+1\}$, $Z = \{Z_j(x_j),\ \ j = 1,\dots,m+1\}$, the Lagrange identity is valid*

$$\begin{aligned} &< L(Y); Z >_{L_2(\Gamma)} - < Y, L^+(z) >_{L_2(\Gamma)} \\ &= \sum_{j=1}^{m+1}\Big\{ D_{1j}^{(3)}(Y_j)D_{1j}^{(0)}(\overline{Z_j}) - D_{1j}^{(2)}(Y_j)D_{1j}^{(1)}(\overline{Z_j}) + D_{2j}^{(3)}(Y_j)D_{2j}^{(0)}(\overline{Z_j}) \\ &-D_{2j}^{(2)}(Y_j)D_{2j}^{(1)}(\overline{Z_j}) + D_{3j}^{(2)}(Y_j)D_{3j}^{(0)}(\overline{Z_j}) + D_{1j}^{(1)}(Y_j)D_{1j}^{(2)}(\overline{Z_j}) - D_{1j}^{(0)}(Y_j)D_{1j}^{(3)}(\overline{Z_j}) \\ &+D_{2j}^{(1)}(Y_j)D_{2j}^{(2)}(\overline{Z_j}) - D_{2j}^{(0)}(Y_j)D_{2j}^{(3)}(\overline{Z_j}) - D_{3j}^{(0)}(Y_j)D_{3j}^{(2)}(\overline{Z_j})\Big\}\Big|_0^{l_j}. \end{aligned} \tag{4}$$

Proof of Lemma 1. Consider a scalar product of

$$< L(Y); Z >= \sum_{j=1}^{m+1}\sum_{i=1}^{3}\int_0^{l_j} l_{ij}(Y_j)\cdot \overline{z}_{ij}(x_j)dx_j.$$

Using the fractional integration formula, we obtain the following

$$\begin{aligned} \int_0^{l_j} l_{1j}(Y_j)\cdot \overline{z}_{1j}(x_j)dx_j &= \Big[\frac{d}{dt}\Big(\mu_j(t)\Big(a_j(t)\frac{d^2 y_{1j}(t)}{dt^2} + b_j(t)\frac{d^2 y_{2j}(t)}{dt^2} \\ &-d_j(t)\frac{dy_{3j}(t)}{dt}\Big)\Big)\overline{z_{1j}(t)}\Big]\Big|_0^{l_j} - \Big[\mu_j(t)\Big(a_j(t)\frac{d^2 y_{1j}(t)}{dt^2} + b_j(t)\frac{d^2 y_{2j}(t)}{dt^2} \\ &-d_j(t)\frac{dy_{3j}(t)}{dt}\Big)\frac{d\overline{z_{1j}(t)}}{dt}\Big]\Big|_0^{l_j} + \Big[\frac{dy_{1j}(t)}{dt}\Big(\mu_j(t)a_j(t)\frac{d^2\overline{z_{1j}(t)}}{dt^2}\Big) + \frac{dy_{2j}(t)}{dt}\Big(\mu_j(t)b_j(t)\frac{d^2\overline{z_{1j}(t)}}{dt^2}\Big) \\ &-y_{3j}(t)\Big(\mu_j(t)d_j(t)\frac{d^2\overline{z_{1j}(t)}}{dt^2}\Big)\Big]\Big|_0^{l_j} - \Big[y_{1j}(t)\frac{d}{dt}\Big(\mu_j(t)a_j(t)\frac{d^2\overline{z_{1j}(t)}}{dt^2}\Big) \end{aligned}$$

$$+y_{2j}(t)\frac{d}{dt}\left(\mu_j(t)b_j(t)\frac{d^2\overline{z_{1j}(t)}}{dt^2}\right)\Big]\Big|_0^{l_j}+\int\limits_0^{l_j} y_{1j}(x_j)\cdot l_{1j}^{+}(\overline{z_{1j}})dx_j.$$

The integrals are converted in a similar way $\int\limits_0^{l_j} l_{2j}(Y_j)\cdot\overline{z_{1j}}(x_j)dx_j$, $\int\limits_0^{l_j} l_{3j}(Y_j)\cdot\overline{z_{1j}}(x_j)dx_j$. Using the above relations, we derive identity (4). Lemma 1 is completely proved. From now on, we assume that the boundary vertices of the graph satisfy the rigid anchoring conditions

$$D_{1j}^{(0)}(Y_j;\xi_j)=0,\ D_{2j}^{(0)}(Y_j;\xi_j)=0,\ D_{1j}^{(1)}(Y_j;\xi_j)=0,\ D_{2j}^{(1)}(Y_j;\xi_j)=0,\ D_{3j}^{(0)}(Y_j;\xi_j)=0,\quad j=1,\ldots,m+1. \tag{5}$$

□

Here, $\xi_j=l_j$, $j=1,\ldots,m$. At the same time, $\xi_{m+1}=0$. Let for a set of functions $Z=\{Z_j(x_j),\ \ j=1,\ldots,m+1\}$ Equation (5) be also fulfilled. Then, the Lagrange identity follows from Lemma 1.

$$\begin{aligned}
&< L(Y);Z>_{L_2(\Gamma)} - < Y,L^{+}(z)>_{L_2(\Gamma)}\\
&=\Big\{D_{1m+1}^{(3)}(Y_{m+1})D_{1m+1}^{(0)}(\overline{Z_{m+1}})-D_{1m+1}^{(2)}(Y_{m+1})D_{1m+1}^{(1)}(\overline{Z_{m+1}})\\
&+D_{2m+1}^{(3)}(Y_{m+1})D_{2m+1}^{(0)}(\overline{Z_{m+1}})-D_{2m+1}^{(2)}(Y_{m+1})D_{2m+1}^{(1)}(\overline{Z_{m+1}})\\
&+D_{3m+1}^{(2)}(Y_{m+1})D_{3m+1}^{(0)}(\overline{Z_{m+1}})+D_{1m+1}^{(1)}(Y_{m+1})D_{1m+1}^{(2)}(\overline{Z_{m+1}})\\
&-D_{1m+1}^{(0)}(Y_{m+1})D_{1m+1}^{(3)}(\overline{Z_{m+1}})+D_{2m+1}^{(1)}(Y_{m+1})D_{2m+1}^{(2)}(\overline{Z_{m+1}})\\
&-D_{2m+1}^{(0)}(Y_{m+1})D_{2m+1}^{(3)}(\overline{Z_{m+1}})-D_{3m+1}^{(0)}(Y_{m+1})D_{3m+1}^{(2)}(\overline{Z_{m+1}})\Big\}\Big|_{l_{m+1}}\\
&-\sum_{j=1}^{m}\Big\{D_{1j}^{(3)}(Y_j)D_{1j}^{(0)}(\overline{Z_j})-D_{1j}^{(2)}(Y_j)D_{1j}^{(1)}(\overline{Z_j})\\
&+D_{2j}^{(3)}(Y_j)D_{2j}^{(0)}(\overline{Z_j})-D_{2j}^{(2)}(Y_j)D_{2j}^{(1)}(\overline{Z_j})+D_{3j}^{(2)}(Y_j)D_{3j}^{(0)}(\overline{Z_j})+D_{1j}^{(1)}(Y_j)D_{1j}^{(2)}(\overline{Z_j})\\
&-D_{1j}^{(0)}(Y_j)D_{1j}^{(3)}(\overline{Z_j})+D_{2j}^{(1)}(Y_j)D_{2j}^{(2)}(\overline{Z_j})-D_{2j}^{(0)}(Y_j)D_{2j}^{(3)}(\overline{Z_j})-D_{3j}^{(0)}(Y_j)D_{3j}^{(2)}(\overline{Z_j})\Big\}\Big|_0
\end{aligned}\tag{6}$$

At the inner vertex of the graph, we require a continuity condition.

$$D_{sm+1}^{(0)}(\overline{Z_{m+1}}l_{m+1})=D_{sj}^{(0)}(\overline{Z_j},0), D_{sm+1}^{(0)}(Y_{m+1}l_{m+1})=D_{sj}^{(0)}(Y_j,0)\ \ j=1,\ldots,m,\ \ s=1,2 \tag{7}$$

Since it is easier to bend a rod than to stretch it, the longitudinal displacements of the rod are influenced by the first derivatives of the transverse displacements. In particular, such an effect has been highlighted by the authors of the paper [17]. From now on, we will assume that the total longitudinal displacement of the rod end is determined by the expression

$$D_{3j}^{(0)}(Y_j,\xi_j)-\eta_{1j}D_{1j}^{(1)}(Y_j,\xi_j)-\eta_{2j}D_{2j}^{(1)}(Y_j,\xi_j).$$

where ξ_j j are the end rod corresponding to the connection node, η_{1j}, η_{2j} are some constants. Let there now be several rods connected at the same node. Then, three rods can be selected as reference rods (independent), and the total longitudinal displacements of the other rods can be considered dependent on the selected three reference rods. A similar effect only in the case of a flat bar connection has been observed in reference work [18]. Let the numbered i, j, k rods be the base rods.

Then, according to our assumption, the total longitudinal displacement of any rod with a number $s \in \{1, 2, \dots, m+1\} \backslash \{i, j, k\}$ is a linear combination of the total longitudinal displacements of the rods with numbers i, j, k. Thus, in the inner vertex of the graph

$$\begin{aligned} &D_{3s}^{(0)}(Y_s, \xi_s) - \eta_{1s} D_{1s}^{(1)}(Y_s, \xi_s) - \eta_{2s} D_{2s}^{(1)}(Y_s, \xi_s) \\ &= \sum_{t=i,j,k} \alpha_{st} \Big(D_{3t}^{(0)}(Y_t, \xi_t) - \eta_{1t} D_{1t}^{(1)}(Y_t, \xi_t) - \eta_{2t} D_{2t}^{(1)}(Y_t, \xi_t) \Big). \end{aligned} \tag{8}$$

where α_{st} represent certain constants. Consider that $\xi_{m+1} = l_{m+1}, \xi_i = 0, i = \{1, \dots, m\}$.

To record the conjugation conditions at the inner vertex of the star graph, let us introduce the following linear forms:

$$C_j(Y_j, Y_{m+1}) = D_{1m+1}^{(0)}(Y_{m+1}, l_{m+1}) - D_{1j}^{(0)}(Y_j, 0) \quad j = 1, \dots, m,$$

$$C_{j+m}(Y_j, Y_{m+1}) = D_{2m+1}^{(0)}(Y_{m+1}, l_{m+1}) - D_{2j}^{(0)}(Y_j, 0) \quad j = 1, \dots, m,$$

$$\begin{aligned} C_{s+2m}(Y_s, Y_{m-1}, Y_m, Y_{m+1}) &= D_{3s}^{(0)}(Y_s, \xi_s) - \eta_{1s} D_{1s}^{(1)}(Y_s, \xi_s) - \eta_{2s} D_{2s}^{(1)}(Y_s, \xi_s) \\ &- \sum_{t=m-1}^{m+1} \alpha_{st} \Big(D_{3t}^{(0)}(Y_t, \xi_t) - \eta_{1t} D_{1t}^{(1)}(Y_t, \xi_t) - \eta_{2t} D_{2t}^{(1)}(Y_t, \xi_t) \Big), \quad s = 1, \dots, m-2, \end{aligned}$$

$$C_{3m-1}(Y_1, \dots, Y_m, Y_{m+1}) = D_{1\,m+1}^{(3)}(Y_{m+1}, l_{m+1}) - \sum_{j=1}^{m} D_{1j}^{(3)}(Y_j, 0),$$

$$C_{3m}(Y_1, \dots, Y_m, Y_{m+1}) = D_{2\,m+1}^{(3)}(Y_{m+1}, l_{m+1}) - \sum_{j=1}^{m} D_{2j}^{(3)}(Y_j, 0),$$

$$C_{3m+1}(Y_1, \dots, Y_m, Y_{m+1}) = D_{3\,m+1}^{(2)}(Y_{m+1}, l_{m+1}) - \sum_{j=1}^{m-2} \alpha_{jm+1} D_{3j}^{(2)}(Y_j, 0),$$

$$C_{3m+2}(Y_1, \dots, Y_m, Y_{m+1}) = D_{3\,m}^{(2)}(Y_m, 0) + \sum_{j=1}^{m-2} \alpha_{jm} D_{3j}^{(2)}(Y_j, 0),$$

$$C_{3m+3}(Y_1, \dots, Y_m, Y_{m+1}) = D_{3\,m-1}^{(2)}(Y_{m-1}, 0) + \sum_{j=1}^{m-2} \alpha_{jm-1} D_{3j}^{(2)}(Y_j, 0),$$

$$C_{3m+4}(Y_1, \dots, Y_m, Y_{m+1}) = D_{1\,m-1}^{(2)}(Y_{m-1}, 0) + \eta_{1m-1} \sum_{j=1}^{m-2} \alpha_{jm-1} D_{3j}^{(2)}(Y_j, 0),$$

$$C_{3m+5}(Y_1, \dots, Y_m, Y_{m+1}) = D_{1\,m}^{(2)}(Y_m, 0) + \eta_{1m} \sum_{j=1}^{m-2} \alpha_{jm} D_{3j}^{(2)}(Y_j, 0),$$

$$C_{3m+6}(Y_1, \dots, Y_m, Y_{m+1}) = -D_{1\,m+1}^{(2)}(Y_{m+1}, l_{m+1}) + \eta_{1m+1} \sum_{j=1}^{m-2} \alpha_{jm+1} D_{3j}^{(2)}(Y_j, 0),$$

$$C_{3m+7}(Y_1, \dots, Y_m, Y_{m+1}) = D_{2\,m-1}^{(2)}(Y_{m-1}, 0) + \eta_{2m-1} \sum_{j=1}^{m-2} \alpha_{jm-1} D_{3j}^{(2)}(Y_j, 0),$$

$$C_{3m+8}(Y_1, \dots, Y_m, Y_{m+1}) = D_{2\,m}^{(2)}(Y_m, 0) + \eta_{2m} \sum_{j=1}^{m-2} \alpha_{jm} D_{3j}^{(2)}(Y_j, 0),$$

$$C_{3m+9}(Y_1,\ldots,Y_m,Y_{m+1}) = -D^{(2)}_{2\,m+1}(Y_{m+1},l_{m+1}) + \eta_{2m+1}\sum_{j=1}^{m-2}\alpha_{jm+1}D^{(2)}_{3j}(Y_j,0),$$

$$C_{3m+9+j}(Y_1,\ldots,Y_m,Y_{m+1}) = \eta_{1j}D^{(2)}_{3j}(Y_j,0) - D^{(2)}_{1j}(Y_j,0),\ \ j=1,\ldots,m-2,$$

$$C_{4m+7+j}(Y_1,\ldots,Y_m,Y_{m+1}) = \eta_{2j}D^{(2)}_{3j}(Y_j,0) - D^{(2)}_{2j}(Y_j,0),\ \ j=1,\ldots,m-2.$$

Then, by substituting relations (7) and (8) in the Lagrange identity (6), we obtain the equality

$$
\begin{aligned}
&< L(Y);Z >_{L_2(\Gamma)} - < Y, L^{+}(z) >_{L_2(\Gamma)}\\
&= -D^{(0)}_{1\,m+1}(Y_{m+1},l_{m+1})C_{3m-1}(Z_1,\ldots,Z_m,Z_{m+1}) - D^{(0)}_{2\,m+1}(Y_{m+1},l_{m+1})C_{3m}(Z_1,\ldots,Z_m,Z_{m+1})\\
&+D^{(0)}_{1\,m+1}(Z_{m+1},l_{m+1})C_{3m-1}(Y_1,\ldots,Y_m,Y_{m+1}) + D^{(0)}_{2\,m+1}(Z_{m+1},l_{m+1})C_{3m}(Y_1,\ldots,Y_m,Y_{m+1})\\
&+D^{(0)}_{3\,m+1}(Z_{m+1},l_{m+1})C_{3m+1}(Y_1,\ldots,Y_m,Y_{m+1}) - D^{(0)}_{3\,m+1}(Y_{m+1},l_{m+1})C_{3m+1}(Z_1,\ldots,Z_m,Z_{m+1})\\
&-D^{(0)}_{3\,m}(Z_m,0)C_{3m+2}(Y_1,\ldots,Y_m,Y_{m+1}) + D^{(0)}_{3\,m}(Y_m,0)C_{3\,m+2}(Z_1,\ldots,Z_m,Z_{m+1})\\
&-D^{(0)}_{3\,m-1}(Z_{m-1},0)C_{3m+3}(Y_1,\ldots,Y_m,Y_{m+1}) + D^{(0)}_{3\,m-1}(Y_{m-1},0)C_{3m+3}(Z_1,\ldots,Z_m,Z_{m+1})\\
&+\sum_{j=1}^{m-2}D^{(1)}_{1j}(Y_j,0)C_{3m+9+j}(Z_1,\ldots,Z_m,Z_{m+1}) - \sum_{j=1}^{m-2}D^{(1)}_{1j}(Z_j,0)C_{3m+9+j}(Y_1,\ldots,Y_m,Y_{m+1})\\
&+\sum_{j=1}^{m-2}D^{(1)}_{2j}(Y_j,0)C_{4m+7+j}(Z_1,\ldots,Z_m,Z_{m+1}) - \sum_{j=1}^{m-2}D^{(1)}_{2j}(Z_j,0)C_{4m+7+j}(Y_1,\ldots,Y_m,Y_{m+1})\\
&-D^{(1)}_{1\,m-1}(Y_{m-1},0)C_{3m+4}(Z_1,\ldots,Z_m,Z_{m+1}) - D^{(1)}_{1\,m}(Y_m,0)C_{3m+5}(Z_1,\ldots,Z_m,Z_{m+1})\\
&-D^{(1)}_{1\,m+1}(Y_{m+1},l_{m+1})C_{3m+6}(Z_1,\ldots,Z_m,Z_{m+1}) - D^{(1)}_{1\,m-1}(Y_{m-1},0)C_{3m+7}(Z_1,\ldots,Z_m,Z_{m+1})\\
&-D^{(1)}_{1\,m}(Y_m,0)C_{3m+8}(Z_1,\ldots,Z_m,Z_{m+1}) - D^{(1)}_{1\,m+1}(Y_{m+1},l_{m+1})C_{3m+9}(Z_1,\ldots,Z_m,Z_{m+1})\\
&+D^{(1)}_{1\,m-1}(Z_{m-1},0)C_{3m+4}(Y_1,\ldots,Y_m,Y_{m+1}) + D^{(1)}_{1\,m}(Z_m,0)C_{3m+5}(Y_1,\ldots,Y_m,Y_{m+1})\\
&+D^{(1)}_{1\,m+1}(Z_{m+1},l_{m+1})C_{3m+6}(Y_1,\ldots,Y_m,Y_{m+1}) + D^{(1)}_{1\,m-1}(Z_{m-1},0)C_{3m+7}(Y_1,\ldots,Y_m,Y_{m+1})\\
&+D^{(1)}_{1\,m}(Z_m,0)C_{3m+8}(Y_1,\ldots,Y_m,Y_{m+1}) + D^{(1)}_{1\,m+1}(Z_{m+1},l_{m+1})C_{3m+9}(Y_1,\ldots,Y_m,Y_{m+1}).
\end{aligned}
\tag{9}
$$

Taking into account the last identity, we introduce the main object of study of the article. We also consider the inhomogeneous operator equation $L(Y) = F$ with conjugation conditions in the interior vertex $C_j(Y) = 0$, $j = 1,\ldots,5m+5$ and with rigid fixing conditions in the boundary vertices (5) on the star graph Γ. The number of boundary conditions is also $5m+5$. The operator corresponding to the above inhomogeneous problem is denoted by B.

Theorem 1. *The domains of definition of operators B and B^* coincide, i.e., they are given by the same conjugation conditions and fixation conditions.*

The proof of Theorem 1 immediately follows from the Lagrange relationship, which is written in the form (9).

Remark 1. *The meaning of Theorem 1 is that the operators B and B^* are self-adjoint in the sense of definitions.*

Remark 2. *The mechanical interpretation of a part of conjugation conditions $C_j(Y) = 0$, $j = 1,\ldots,3m-2$ is given above when introducing relations (7) and (8). The conjugation condi-*

tions $C_j(Y) = 0$, $j = 3m-1, \dots, 5m+5$ also have a mechanical interpretation, indicating the distribution of forces and moments between the rods in the function. The conjugation conditions given in Theorem 1 seem to be new.

In fact, Theorem 1 is one of the main results of this paper. It states that we have found self-adjoint conjugate conditions in the interior vertex of the graph. The conjugation conditions we have found generalize the well-known Kirchhoff conditions, which are written for second-order differential equations. We have been able to generalize Kirchhoff conditions for systems of differential equations consisting of differential equations of different orders. In [10–13], different variants of Kirchhoff analogs conditions of the graph's inner vertex for the second-order differential equations are given. In a sense, Theorem 1 generalizes the results of [10–13] for systems of differential equations on a star graph.

3. The Reversibility of Operator B

The operator B depends on the real parameters η_{1j}, η_{2j}, α_{st} at all valid indexes. For operator B to be reversible, it is necessary to impose constraints on the specified parameters. In this section, we will find out the values of the above parameters that make possible the existence of the inverse operator B^{-1}. Let $\xi_{m+1} = 0$, $\xi_j = b_j$, $j = 1, \dots, m$. Denote by

$$p_j(t_j) = \mu_j(t_j)\begin{vmatrix} a_j(t_j) & b(t_j) & d_j(t_j) \\ b(t_j) & c_j(t_j) & f_j(t_j) \\ d_j(t_j) & f_j(t_j) & 1 \end{vmatrix}, \quad q_{1j}^1(t_j) = \begin{vmatrix} c_j(t_j) & f_j(t_j) \\ f_j(t_j) & 1 \end{vmatrix},$$

$$q_{1j}^2(t_j) = \begin{vmatrix} b_j(t_j) & d_j(t_j) \\ f_j(t_j) & 1 \end{vmatrix}, \quad q_{1j}^3(t_j) = \begin{vmatrix} b_j(t_j) & d_j(t_j) \\ c_j(t_j) & f_j(t_j) \end{vmatrix},$$

$$q_{2j}^1(t_j) = \begin{vmatrix} b_j(t_j) & f_j(t_j) \\ d_j(t_j) & 1 \end{vmatrix}, \quad q_{2j}^2(t_j) = \begin{vmatrix} a_j(t_j) & d_j(t_j) \\ d_j(t_j) & 1 \end{vmatrix}, \quad q_{2j}^3(t_j) = \begin{vmatrix} a_j(t_j) & d_j(t_j) \\ b_j(t_j) & f_j(t_j) \end{vmatrix},$$

$$q_{3j}^1(t_j) = \begin{vmatrix} b_j(t_j) & c_j(t_j) \\ d_j(t_j) & f_j(t_j) \end{vmatrix}, \quad q_{3j}^2(t_j) = \begin{vmatrix} a_j(t_j) & b_j(t_j) \\ d_j(t_j) & f_j(t_j) \end{vmatrix}, \quad q_{3j}^3(t_j) = \begin{vmatrix} a_j(t_j) & d_j(t_j) \\ b_j(t_j) & c_j(t_j) \end{vmatrix}.$$

Then, the solutions of the homogeneous system of differential equations

$$l_{1j}(Y_j) = 0, \ l_{2j}(Y_j) = 0, \ l_{3j}(Y_j) = 0 \tag{10}$$

with rigid anchoring conditions at the boundary vertices

$$D_{1j}^{(0)}(Y_j; \xi_j) = 0, \ D_{2j}^{(0)}(Y_j; \xi_j) = 0, \ D_{1j}^{(1)}(Y_j; \xi_j) = 0, \ D_{2j}^{(1)}(Y_j; \xi_j) = 0, \ D_{3j}^{(0)}(Y_j; \xi_j) = 0$$

at $j = 1, \dots, m+1$ will be

$$y_{1j}(x_j) = K_{1j}\int_{\xi_j}^{x_j}(t_j - \xi_j)t_j\frac{q_{1j}^1(t_j)}{p_j(t_j)}dt_j + N_{1j}\int_{\xi_j}^{x_j}(t_j - \xi_j)\frac{q_{1j}^1(t_j)}{p_j(t_j)}dt_j$$

$$-K_{2j}\int_{\xi_j}^{x_j}(t_j - \xi_j)t_j\frac{q_{1j}^2(t_j)}{p_j(t_j)}dt_j - N_{2j}\int_{\xi_j}^{x_j}(t_j - \xi_j)\frac{q_{1j}^2(t_j)}{p_j(t_j)}dt_j + K_{3j}\int_{\xi_j}^{x_j}(t_j - \xi_j)\frac{q_{1j}^3(t_j)}{p_j(t_j)}dt_j,$$

We see solution $y_{2j}(x_j)$

$$y_{2j}(x_j) = -K_{1j}\int_{\xi_j}^{x_j}(t_j - \xi_j)t_j\frac{q_{2j}^1(t_j)}{p_j(t_j)}dt_j - N_{1j}\int_{\xi_j}^{x_j}(t_j - \xi_j)\frac{q_{2j}^1(t_j)}{p_j(t_j)}dt_j$$

$$+K_{2j}\int\limits_{\xi_j}^{x_j}(t_j-\xi_j)t_j\frac{q_{2j}^2(t_j)}{p_j(t_j)}dt_j+N_{2j}\int\limits_{\xi_j}^{x_j}(t_j-\xi_j)\frac{q_{2j}^2(t_j)}{p_j(t_j)}dt_j-K_{3j}\int\limits_{\xi_j}^{x_j}(t_j-\xi_j)\frac{q_{2j}^3(t_j)}{p_j(t_j)}dt_j,$$

$$y_{3j}(x_j)=-K_{1j}\int\limits_{\xi_j}^{x_j}t_j\frac{q_{3j}^1(t_j)}{p_j(t_j)}dt_j-N_{1j}\int\limits_{\xi_j}^{x_j}\frac{q_{3j}^1(t_j)}{p_j(t_j)}dt_j$$

$$+K_{2j}\int\limits_{\xi_j}^{x_j}t_j\frac{q_{3j}^2(t_j)}{p_j(t_j)}dt_j+N_{2j}\int\limits_{\xi_j}^{x_j}\frac{q_{3j}^2(t_j)}{p_j(t_j)}dt_j-K_{3j}\int\limits_{\xi_j}^{x_j}\frac{q_{3j}^3(t_j)}{p_j(t_j)}dt_j.$$

where $K_{1j},\ N_{1j},\ K_{2j},\ N_{2j},\ K_{3j}$ are arbitrary constants.

In accordance with the above representation of the solution, let us introduce the following solutions to the homogeneous system of differential Equation (10):

$$\theta_{sj}^1(x_j)=(-1)^{s-1}\int\limits_{\xi_j}^{x_j}(t_j-\xi_j)t_j\frac{q_{sj}^1(t_j)}{p_j(t_j)}dt_j,\ \ \theta_{sj}^2(x_j)=(-1)^{s-1}\int\limits_{\xi_j}^{x_j}(t_j-\xi_j)\frac{q_{sj}^1(t_j)}{p_j(t_j)}dt_j,$$

$$\theta_{sj}^3(x_j)=(-1)^{s}\int\limits_{\xi_j}^{x_j}(t_j-\xi_j)t_j\frac{q_{sj}^2(t_j)}{p_j(t_j)}dt_j,\ \ \theta_{sj}^4(x_j)=(-1)^{s}\int\limits_{\xi_j}^{x_j}(t_j-\xi_j)\frac{q_{sj}^2(t_j)}{p_j(t_j)}dt_j,$$

$$\theta_{sj}^5(x_j)=(-1)^{s-1}\int\limits_{\xi_j}^{x_j}(t_j-\xi_j)\frac{q_{sj}^3(t_j)}{p_j(t_j)}dt_j,\ \ s=1,2,$$

$$\theta_{sj}^1(x_j)=(-1)^{s}\int\limits_{\xi_j}^{x_j}t_j\frac{q_{sj}^1(t_j)}{p_j(t_j)}dt_j,\ \ \theta_{sj}^2(x_j)=(-1)^{s}\int\limits_{\xi_j}^{x_j}\frac{q_{sj}^1(t_j)}{p_j(t_j)}dt_j,$$

$$\theta_{sj}^3(x_j)=(-1)^{s-1}\int\limits_{\xi_j}^{x_j}t_j\frac{q_{sj}^2(t_j)}{p_j(t_j)}dt_j,\ \ \theta_{sj}^4(x_j)=(-1)^{s-1}\int\limits_{\xi_j}^{x_j}\frac{q_{sj}^2(t_j)}{p_j(t_j)}dt_j,$$

$$\theta_{sj}^5(x_j)=(-1)^{s}\int\limits_{\xi_j}^{x_j}\frac{q_{sj}^3(t_j)}{p_j(t_j)}dt_j,\ \ s=3.$$

Denote by $\Theta_j^s(x_j)$ the following vector

$$\Theta_j^s(x_j)=[\theta_{1j}^s(x_j),\ \theta_{2j}^s(x_j),\ \theta_{3j}^s(x_j)]^T.$$

For further purposes, it is convenient to introduce a matrix $M=[m_{ik}]$ with the following elements

$$m_{1\ 5(j-1)+s}=-D_{1j}^{(3)}(\Theta_j^s,0),\ \ j=1,\ldots,m-2,$$

$$m_{1\ 5(j-1)+s}=0,\ \ j=m-1,m,\ \ m_{1\ 5m+s}=D_{1\ m+1}^{(3)}(\Theta_{m+1}^s,l_{m+1}),\ \ s=1,\ldots,5,$$

Hereon in, consider that index j changes only after the index s.

$$m_{2\ 5(j-1)+s}=-D_{2j}^{(3)}(\Theta_j^s,0),\ \ j=1,\ldots,m-2,$$

$$m_{2\ 5(j-1)+s}=0,\ \ j=m-1,m,\ \ m_{1\ 5m+s}=D_{2m+1}^{(3)}(\Theta_{m+1}^s,l_{m+1}),\ \ s=1,\ldots,5,$$

$$m_{3\ 5(j-1)+s} = -\alpha_{jm+1} D_{3j}^{(2)}(\Theta_j^s, 0),\ \ j = 1, \ldots, m-2,$$

$$m_{3\ 5(j-1)+s} = 0,\ \ j = m-1, m,\ \ m_{3\ 5m+s} = D_{3\ m+1}^{(3)}(\Theta_{m+1}^s, l_{m+1}),\ \ s = 1, \ldots, 5,$$

$$m_{4\ 5(j-1)+s} = \alpha_{jm} D_{3j}^{(2)}(\Theta_j^s, 0),\ \ j = 1, \ldots, m-2,$$

$$m_{4\ 5(j-1)+s} = 0,\ \ j = m-1, m,\ \ m_{4\ 5m+s} = D_{3\ m}^{(3)}(\Theta_{m+1}^s, l_{m+1}),\ \ s = 1, \ldots, 5,$$

$$m_{5\ 5(j-1)+s} = \alpha_{jm-1} D_{3j}^{(2)}(\Theta_j^s, 0),\ \ j = 1, \ldots, m-2,$$

$$m_{5\ 5(j-1)+s} = 0,\ \ j = m-1, m,\ \ m_{5\ 5m+s} = D_{3\ m-1}^{(3)}(\Theta_{m+1}^s, l_{m+1}),\ \ s = 1, \ldots, 5,$$

$$m_{6\ 5(j-1)+s} = \eta_{1m-1}\alpha_{jm-1} D_{3j}^{(2)}(\Theta_j^s, 0),\ \ j = 1, \ldots, m-2,$$

$$m_{6\ 5(j-1)+s} = 0,\ \ j = m+1, m,\ \ m_{6\ 5(m-2)+s} = D_{1\ m-1}^{(2)}(\Theta_{m-1}^s, 0),\ \ s = 1, \ldots, 5,$$

$$m_{7\ 5(j-1)+s} = \eta_{1m}\alpha_{jm} D_{3j}^{(2)}(\Theta_j^s, 0),\ \ j = 1, \ldots, m-2,$$

$$m_{7\ 5(j-1)+s} = 0,\ \ j = m+1, m-1,\ \ m_{7\ 5(m-1)+s} = D_{1\ m}^{(2)}(\Theta_m^s, 0),\ \ s = 1, \ldots, 5,$$

$$m_{8\ 5(j-1)+s} = \eta_{1m+1}\alpha_{jm+1} D_{3j}^{(2)}(\Theta_j^s, 0),\ \ j = 1, \ldots, m-2,$$

$$m_{8\ 5(j-1)+s} = 0,\ \ j = m, m-1,\ \ m_{8\ 5m+s} = D_{1\ m+1}^{(2)}(\Theta_{m+1}^s, l_{m+1}),\ \ s = 1, \ldots, 5,$$

$$m_{9\ 5(j-1)+s} = \eta_{2m}\alpha_{jm} D_{3j}^{(2)}(\Theta_j^s, 0),\ \ j = 1, \ldots, m-2,$$

$$m_{9\ 5(j-1)+s} = 0,\ \ j = m+1, m,\ \ m_{9\ 5(m-2)+s} = D_{2\ m-1}^{(2)}(\Theta_{m-1}^s, 0),\ \ s = 1, \ldots, 5,$$

$$m_{10\ 5(j-1)+s} = \eta_{2m}\alpha_{jm} D_{3j}^{(2)}(\Theta_j^s, 0),\ \ j = 1, \ldots, m-2,$$

$$m_{10\ 5(j-1)+s} = 0,\ \ j = m+1, m-1,\ \ m_{10\ 5(m-1)+s} = D_{2\ m}^{(2)}(\Theta_m^s, 0),\ \ s = 1, \ldots, 5,$$

$$m_{11\ 5(j-1)+s} = \eta_{2m+1}\alpha_{jm+1} D_{3j}^{(2)}(\Theta_j^s, 0),\ \ j = 1, \ldots, m-2,$$

$$m_{11\ 5(j-1)+s} = 0,\ \ j = m, m-1,\ \ m_{11\ 5m+s} = D_{2\ m+1}^{(2)}(\Theta_{m+1}^s, l_{m+1}),\ \ s = 1, \ldots, 5,$$

$$m_{11+p\ 5(j-1)+s} = 0,\ \ j = 1, \ldots, p-1, p+1, \ldots, m+1,$$

$$m_{11+p\ 5(p-1)+s} = \eta_{1p} D_{3p}^{(2)}(\Theta_p^s, 0) - D_{1p}^{(2)}(\Theta_p^s, 0),\ \ s = 1, \ldots, 5,\ \ p = 1, \ldots, m-2,$$

$$m_{9+m+p\ 5(j-1)+s} = 0,\ \ j = 1, \ldots, p-1, p+1, \ldots, m+1,$$

$$m_{9+m+p\ 5(p-1)+s} = \eta_{2p} D_{3p}^{(2)}(\Theta_p^s, 0) - D_{2p}^{(2)}(\Theta_p^s, 0),\ \ s = 1, \ldots, 5,\ \ p = 1, \ldots, m-2,$$

$$m_{7+2m+p\ 5(j-1)+s} = 0,\ \ j = 1, \ldots, p-1, p+1, \ldots, m,\ \ m_{7+2m+p\ 5(p-1)+s} = -D_{1p}^{(0)}(\Theta_p^s, 0),$$

$$m_{7+2m+p\ 5m+s} = D_{1\ m+1}^{(0)}(\Theta_{m+1}^s, l_{m+1}),\ \ s = 1, \ldots, 5,\ \ p = 1, \ldots, m.$$

$$m_{7+3m+p\ 5(j-1)+s} = 0,\ \ j = 1, \ldots, p-1, p+1, \ldots, m,\ \ m_{7+3m+p\ 5(p-1)+s} = -D_{2p}^{(0)}(\Theta_p^s, 0),$$

$$m_{7+3m+p\ 5m+s} = D_{2\ m+1}^{(0)}(\Theta_{m+1}^s, l_{m+1}),\ \ s = 1, \ldots, 5,\ \ p = 1, \ldots, m.$$

$$m_{7+4m+p\ 5(j-1)+s} = 0,\ \ j = 1, \ldots, p-1, p+1, \ldots, m-2,$$

$$m_{7+4m+p\ 5(p-1)+s} = D_{3p}^{(0)}(\Upsilon_s, \xi_s) - \eta_{1p} D_{1p}^{(1)}(\Theta_p^s, 0) - \eta_{2p} D_{2p}^{(1)}(\Theta_p^s, 0),$$

$$m_{7+4m+p\ 5(t-1)+s} = -\alpha_{pt}(D_{3t}^{(0)}(\Theta_t^s, 0) - \eta_{1t} D_{1t}^{(1)}(\Theta_t^s, 0) - \eta_{2t} D_{2t}^{(1)}(\Theta_t^s, 0)),\ \ t = m-1, m,$$

$$m_{7+4m+p\ 5m+s} = -\alpha_{pm+1}(D_{3m+1}^{(0)}(\Theta_{m+1}^s, l_{m+1}) - \eta_{1m+1} D_{1m+1}^{(1)}(\Theta_{m+1}^s, l_{m+1}) -$$

$$-\eta_{2m+1}D^{(1)}_{2m+1}(\Theta^s_{m+1},l_{m+1})),\ \ s=1,\ldots,5,\ \ p=1,\ldots,m-2.$$

We can now formulate the theorem on the reversibility of operator B.

Theorem 2. *If* $\det M\neq 0$, *then a bounded operator* B^{-1} *exists in space* $L_2(\Gamma)$.

Proof of the Theorem 2. Consider the inhomogeneous operator equation $L(Y)=F$ with conjugation conditions in the interior vertex $C_j(Y)=0$, $j=1,\ldots,5m+5$ and with rigid fixing conditions in the boundary vertices (5) on the star graph Γ. To prove Theorem 2, it suffices to prove the single-valued solvability of this boundary value problem for any right-hand side $F\in L_2(\Gamma)$.

It is known that the following system of equations

$$\begin{cases} D^{(2)}_{1j}(Y_j;\xi)=\int\limits_{\xi_j}^{x_j}(t_j-\xi_j)f_{1j}(t_j)dt_j,\\ D^{(2)}_{2j}(Y_j;\xi)=\int\limits_{\xi_j}^{x_j}(t_j-\xi_j)f_{2j}(t_j)dt_j,\\ D^{(2)}_{3j}(Y_j;\xi)=\int\limits_{\xi_j}^{x_j}f_{3j}(t_j)dt_j. \end{cases}.$$

with initial conditions

$$D^{(0)}_{1j}(Y_j;\xi_j)=0,\ \ D^{(0)}_{2j}(Y_j;\xi_j)=0,\ \ D^{(1)}_{1j}(Y_j;\xi_j)=0,\ \ D^{(1)}_{2j}(Y_j;\xi_j)=0,\ \ D^{(0)}_{3j}(Y_j;\xi_j)=0.$$

is uniquely solvable. Denote the only solution to the above problem by $Y^0_j(x_j)$, $j=1,\ldots,m+1$. It is a partial solution of the inhomogeneous operator equation $L(Y)=F$. The solution to the required problem is found at $j=1,\ldots,m+1$ as follows

$$Y_j(x_j)=Y^0_j(x_j)+K_{1j}\Theta^1_j(x_j)+N_{1j}\Theta^2_j(x_j)+K_{2j}\Theta^3_j(x_j)+N_{2j}\Theta^4_j(x_j)+K_{3j}\Theta^5_j(x_j).$$

We have to prove that the numbers $K_{1j},\ N_{1j},\ K_{2j},\ N_{2j},\ K_{3j},\ j=1,\ldots,m+1$ are determined from the conjugation conditions in the inner vertex. If the conditions of Theorem 2 are fulfilled, a vector h is uniquely found from a system of linear algebraic equations $M\ h=g$ for any $g\in C^{5m+5}$. The elements of vector h can be interpreted as numbers $K_{1j},\ N_{1j},\ K_{2j},\ N_{2j},\ K_{3j},\ j=1,\ldots,m+1$. In this case, g is the numeric column that depends on partial solutions $Y^0_j(x_j)$. Thus, the existence of a solution to the required problem is proved. The uniqueness of the solution follows from general statements about systems of linear differential equations. Since the coefficients in the system of linear differential equations represent continuous functions, it follows that the corresponding a priori estimates are correct. This means that the boundedness of the inverse operator B^{-1}. Theorem 2 is fully proved. □

Theorem 2 immediately implies the following corollary.

Corollary 1. *Let* $\det M\neq 0$. *Then, for any* $F\in L_2(\Gamma)$ *and arbitrary constants* φ_i, $i=1,\ldots,5m+5$, $\psi_{5(j-1)+s}$, $s=1,\ldots,5$, $j=1,\ldots,m+1$ *the solution of the problem*

$$L(Y)=F,\ C_j(Y)=\varphi_j,\ \ j=1,\ldots,5m+5,$$

$$D^{(0)}_{1j}(Y_j;\xi_j)=\psi_{5(j-1)+1},\ \ D^{(0)}_{2j}(Y_j;\xi_j)=\psi_{5(j-1)+2},$$

$$D^{(1)}_{1j}(Y_j;\xi_j)=\psi_{5(j-1)+3},\ \ D^{(1)}_{2j}(Y_j;\xi_j)=\psi_{5(j-1)+4},$$

$$D^{(0)}_{3j}(Y_j;\xi_j)=\psi_{5(j-1)+5},\ \ j=1,\ldots,m+1$$

exists and is unique.

4. Everywhere Solvable Reversible Boundary Value Problems for Systems of Differential Equations on a Star Graph

In this section, we describe all possible universally solvable reversible boundary value problems for the equation $L(Y) = F$. In the previous paragraphs, we wrote out conjugation conditions using linear forms $C_1(\cdot), \ldots, C_{5m+5}(\cdot)$ and fixation conditions using linear forms $D^{(0)}_{1j}(0)$, $D^{(0)}_{2j}(0)$, $D^{(1)}_{1j}(0)$, $D^{(1)}_{2j}(0)$, $D^{(0)}_{3j}$, $j = 1, \ldots, m+1$. Under the above boundary conditions, the solvability and reversibility of the corresponding problem are proved everywhere.

How do we obtain new conjugation conditions and new anchoring conditions that guarantee all-around solvability and reversibility? This paragraph answers this question.

Let us choose arbitrarily two sets of linear bounded functionals in the function space $L_2(\Gamma)$. The first set of linear functionals is denoted by

$$\varphi_j(\cdot), \;\; j = 1, \ldots, 5m+5.$$

It is convenient to denote the second set of linear continuous functionals by

$$\psi_{5(j-1)+s}(\cdot), \;\; s = 1, \ldots, 5, \;\; j = 1, \ldots, m+1.$$

Let us write a new boundary value problem corresponding to the chosen sets of linear continuous functionals.

Consider the inhomogeneous operator equation $L(Y) = F$ with conjugation conditions in the interior vertex $C_j(Y) = \varphi_j(L(Y))$, $j = 1, \ldots, 5m+5$ and with fixation conditions in the boundary vertices on the star graph Γ

$$\begin{aligned}
&D^{(0)}_{1j}(Y_j;\xi_j) = \psi_{5(j-1)+1}(L(Y)), \;\; D^{(0)}_{2j}(Y_j;\xi_j) = \psi_{5(j-1)+2}(L(Y)),\\
&D^{(1)}_{1j}(Y_j;\xi_j) = \psi_{5(j-1)+3}(L(Y)), \;\; D^{(1)}_{2j}(Y_j;\xi_j) = \psi_{5(j-1)+4}(L(Y)),\\
&D^{(0)}_{3j}(Y_j;\xi_j) = \psi_{5(j-1)+5}(L(Y)), \;\; j = 1, \ldots, m+1.
\end{aligned} \tag{11}$$

The written boundary value problem is everywhere solvable in the space $L_2(\Gamma)$. The inversion of the problem also follows from Theorem 2. Indeed, if the expression $L(Y)$ is replaced by F, in the boundary conditions, we find ourselves in the conditions of Theorem 2. Therefore, a solution to the new problem exists and is unique. The boundedness of the inverse operator in $L_2(\Gamma)$ space follows from the boundedness of the chosen functionals $\varphi_j(\cdot)$ and $\psi_{5(j-1)+s}(\cdot)$. Thus, we have proved the conclusion.

Theorem 3. *Let $\varphi_j(\cdot)$, $j = 1, \ldots, 5m+5$ and $\psi_{5(j-1)+s}(\cdot)$, $s = 1, \ldots, 5$, $j = 1, \ldots, m+1$ represent arbitrary sets of linear bounded functionals in space $L_2(\Gamma)$. Then, the operator K, given by the differential expression $KY = L(Y)$ on the domain of definition*

$$D(K) = \Big\{ Y \in L_2(\Gamma),\; L(Y) \in L_2(\Gamma) \;:\; C_j(Y) = \varphi_j(L(Y)),$$

$$j = 1, \ldots, 5m+5, \;\; \textit{conditions} \;\; (11) \Big\}.$$

has a bounded inverse in $L_2(\Gamma)$ space.

Remark 3. *The converse (in a sense) to Theorem 3 is true. However, we do not clarify this issue in this paper.*

5. Examples

In this paragraph, we will give illustrative examples to show the meaning of Theorem 3. To do this, let us write down the general form of a linear bounded functional in the space $L_2(\Gamma)$.

$$\varphi_k(F) =< F,\ \Phi_j >_{L_2(\Gamma)}= \sum_{s=1}^{m+1} \int_0^{l_s} F(x_s)\overline{\Phi_k(x_s)}dx_s$$

We substitute F with $L(Y)$ and transform according to the Lagrange formula. In this case, the functions Y are considered to satisfy the rigid fixing conditions (5). Functionals are selected so that the ratios are true

$$L^+(\Phi_k) = 0,\ \ k = 1, \dots, 5m + 5.$$

We also assume that the functions Φ_k, $k = 1, \dots, 5m + 5$ also satisfy the rigid anchoring conditions (5). In this case, the value of the functional $\varphi_k(F)$ will be the same as the right-hand side of equality (9), where the vector of function Z should be replaced by the vector of function Φ_k. According to Corollary 1, for each fixed k, we can arbitrarily choose values of linear forms $C_j(\Phi_k)$, $j = 1, \dots, 5m + 5$. Let $C_j(\Phi_k) = 0$, $j, k = 1, \dots, 5m + 5$. Thus, the unambiguous choice of functions Φ_k, $k = 1, \dots, 5m + 5$ is fulfilled. As a result, we have

$$\begin{gathered}
\varphi_k(L(Y)) = D^{(0)}_{1\ m+1}(\Phi_{km+1}, l_{m+1})C_{3m-1}(Y_1, \dots, Y_m, Y_{m+1}) \\
+D^{(0)}_{2\ m+1}(\Phi_{km+1}, l_{m+1})C_{3m}(Y_1, \dots, Y_m, Y_{m+1}) \\
-D^{(0)}_{3\ m}(\Phi_{km}, 0)C_{3m+2}(Y_1, \dots, Y_m, Y_{m+1}) \\
-D^{(0)}_{3\ m-1}(\Phi_{km-1}, 0)C_{3m+3}(Y_1, \dots, Y_m, Y_{m+1}) \\
-\sum_{j=1}^{m-2} D^{(1)}_{1j}(\Phi_{kj}, 0)C_{3m+9+j}(Y_1, \dots, Y_m, Y_{m+1}) \\
-\sum_{j=1}^{m-2} D^{(1)}_{2j}(\Phi_{kj}, 0)C_{4m+7+j}(Y_1, \dots, Y_m, Y_{m+1}) \\
+D^{(1)}_{1\ m-1}(\Phi_{km-1}, 0)C_{3m+4}(Y_1, \dots, Y_m, Y_{m+1}) \\
+D^{(1)}_{1\ m}(\Phi_{km}, 0)C_{3m+5}(Y_1, \dots, Y_m, Y_{m+1}) \\
+D^{(1)}_{1\ m+1}(\Phi_{km+1}, l_{m+1})C_{3m+6}(Y_1, \dots, Y_m, Y_{m+1}) \\
+D^{(1)}_{1\ m-1}(\Phi_{km-1}, 0)C_{3m+7}(Y_1, \dots, Y_m, Y_{m+1}) \\
+D^{(1)}_{1\ m}(\Phi_{km}, 0)C_{3m+8}(Y_1, \dots, Y_m, Y_{m+1}) \\
+D^{(1)}_{1\ m+1}(\Phi_{km+1}, l_{m+1})C_{3m+9}(Y_1, \dots, Y_m, Y_{m+1}).
\end{gathered}$$

Here, some constants are denoted by $D^{(0)}_{1\ m+1}(\Phi_{km+1}, l_{m+1}) + D^{(0)}_{2\ m+1}(\Phi_{km+1}, l_{m+1})$ and so on. That is, the value of the functional $\varphi_k(L(Y))$ represents a linear combination of the values of $C_j(Y)$, $j = 1, \dots, 5m + 5$.

Thus, the conjugation condition $C_k(Y) - \varphi_k(L(Y)) = 0$ at a fixed k after making the aforesaid choice of functions Φ_k, $k = 1, \dots, 5m + 5$ will be

$$\sum_{j=1}^{5m+5} a_{k\ j}C_j(Y) = 0,$$

where $a_{k\ j}$ are some constants.

Given the above reasoning, Theorem 3 will become as follows.

Theorem 4. *Let the set of numbers $a_{k\,j}$, $k, j = 1, \ldots, 5m+5$ be implemented in the above-mentioned way. Then, on the star graph Γ, the inhomogeneous operator equation $L(Y) = F$ with conjugation conditions at the inner vertex*

$$\sum_{j=1}^{5m+5} a_{k\,j} C_j(Y) = 0, \;\; k = 1, \ldots, 5m+5$$

and with fixing conditions (5) in the boundary vertices is uniquely solvable for any function F from the space $L_2(\Gamma)$.

Thus, this example generalizes the conjugation conditions under fixed anchoring conditions. Now, we give another example where the anchoring conditions are generalized.

Let the vector function Y also satisfy the conjugation conditions.

$$C_j(Y) = 0, \;\; j = 1, \ldots, 5m+5, \tag{12}$$

and otherwise be arbitrary. The set of functions Φ_k, $k = 1, \ldots, 5m+5$ satisfy the relation $L^+(\Phi_k) = 0$ and the same conjugation conditions (12) and rigid anchoring conditions (5). In this case, the function Φ_k is uniquely defined. That is, the function Φ_k, $k = 1, \ldots, 5m+5$ is the same as in Example 1.

In this case, the value of the functional $\varphi_k(L(Y))$ is a linear combination of the form values.

$$D_{1j}^{(0)}(Y_j;\xi_j), \;\; D_{2j}^{(0)}(Y_j;\xi_j), \;\; D_{1j}^{(1)}(Y_j;\xi_j), \;\; D_{2j}^{(1)}(Y_j;\xi_j), \;\; D_{3j}^{(0)}(Y_j;\xi_j), \;\; j = 1, \ldots, m+1.$$

As a result, a statement can be formulated.

Theorem 5. *There exists a set of numbers b_{kj}^s that depends on boundary values of functions Φ_k, $k = 1, \ldots, 5m+5$. Then, on the star graph Γ, the inhomogeneous operator equation $L(Y) = F$ with anchoring conditions in the boundary vertices*

$$\sum_{j=1}^{m+1} \Big(b_{kj}^1 D_{1j}^{(0)}(Y_j;\xi_j) + b_{kj}^2 D_{2j}^{(0)}(Y_j;\xi_j) + b_{kj}^3 D_{1j}^{(1)}(Y_j;\xi_j)$$
$$+ b_{kj}^4 D_{2j}^{(1)}(Y_j;\xi_j) + b_{kj}^5 D_{3j}^{(0)}(Y_j;\xi_j) \Big) = 0, \;\; k = 1, \ldots, 5m+5.$$

and with conjugation conditions (12) in the interior vertex is uniquely solvable for any function F from the space $L_2(\Gamma)$.

6. Conclusions

The paper presents boundary value problems for systems of differential equations on a star graph. A distinctive feature of this paper is that the system consists of differential equations of different orders. In this process, it was necessary to work out a technique for making conjugation conditions at the nodes where several spatial rods are joined.

The result of this work can be used to calculate eigenvalues and eigenforms of spatially connected rod systems. Up to now, only plane eigenforms separately for longitudinal and for transverse vibrations have been used in engineering practice. In the present study, the determination of spatial eigenforms of vibrations of a rod system is proposed.

Furthermore, it is necessary to evaluate how much the eigenfrequencies and spatial eigenforms of the model problem on a graph approximate the original three-dimensional problem for a structure consisting of several rods. Similar asymptotic problems for the Laplace operator have been studied in [7,11]. In our case, instead of a Laplace operator, we consider equations of linear elasticity theory for a system of rods connected in a node.

Similar tasks have been investigated in [19–23]. However, the results require further progress.

Note that this paper obtains new classes of conjugacy conditions in the inner vertex of a star graph in the case of systems of differential equations. In this case, the boundary conditions in the boundary vertices of the star graph are considered fixed. In particular, only the case of rigidly fixed boundary vertices is considered. In this case, the boundary conditions in the boundary vertices of the star graph are considered fixed. In particular, only the case of rigidly fixed boundary vertices is considered. We have not investigated other types of boundary fixings. Other types of boundary clauses for second-order differential equations are investigated in [10–13]. In the future, for systems of differential equations on graphs, we should study all possible kinds of boundary clauses.

Author Contributions: Introduction and canonical problem, B.K. and G.A.; proof theorem, B.K. and G.A.; writing—review and editing, G.A. and B.K.; project administration, B.K. All authors have read and agreed to the published version of the manuscript.

Funding: This work was financially supported by the Ministry of Education and Science of the Republic of Kazakhstan (Project AP08855402).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors acknowledge the financial support provided by the Ministry of Education and Science of the Republic of Kazakhstan.

Conflicts of Interest: The authors declare no conflict of interest.

Sample Availability: Samples of the compounds are available from the authors.

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