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An Improved Regularity Criterion for the 3D Magnetic Bénard System in Besov Spaces

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Abstract: This article notably targets the more general (extended) function spaces by investigating the regularity of the weak solutions or turbulent solutions to the Cauchy problem of the 3D magnetic Bénard system by converting it into mathematical symmetric form, in the absence of thermal diffusion, in terms of pressure. In that regard, we successfully improved the results by obtaining sufficient integrable regularity conditions for the pressure and gradient pressure in the homogeneous Besov spaces.

Keywords: integrable regularity conditions; 3D magnetic Bénard system without thermal diffusion; improved regularity criteria; homogenous Besov spaces; weak solutions; pressure

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1. Introduction

In this academic study, we analyze the following magnetic Bénard system:

$$\begin{cases} \frac{\partial \mathcal{U}}{\partial t} + \mathcal{U}\cdot\nabla\mathcal{U} - \beta_1\triangle\mathcal{U} + \nabla\psi - \mathcal{V}\cdot\nabla\mathcal{V} - \theta e_3 = 0, \quad \text{in } \mathbb{R}^3\times\mathbb{R}_+, \\ \frac{\partial \mathcal{V}}{\partial t} + \mathcal{U}\cdot\nabla\mathcal{V} - \beta_2\triangle\mathcal{V} - \mathcal{V}\cdot\nabla\mathcal{U} = 0, \quad \text{in } \mathbb{R}^3\times\mathbb{R}_+, \\ \frac{\partial \theta}{\partial t} + \mathcal{U}\cdot\nabla\theta - \beta_3\triangle\theta - \mathcal{U}\cdot e_3 = 0, \quad \text{in } \mathbb{R}^3\times\mathbb{R}_+, \\ \mathrm{div}\mathcal{U} = 0, \ \mathrm{div}\mathcal{V} = 0, \quad \text{in } \mathbb{R}^3\times\mathbb{R}_+, \\ (\mathcal{U},\mathcal{V},\theta)|_{t=0} = (\mathcal{U}_0,\mathcal{V}_0,\theta_0) \quad \text{in } \mathbb{R}^3, \end{cases} \tag{1}$$

where $\mathcal{U}(x,t)$, $\mathcal{V}(x,t)$, $\theta(x,t)$ are the velocity field vector, magnetic field vector and scalar temperature field, respectively, while $\psi(x,t)$ is the scalar pressure. β_1 and β_2 are the viscosity and diffusivity with β_3 as the thermal diffusion, $e_3 = (0,0,1)$ and θe_3 reports the acting buoyancy force on the fluid motion, $\mathcal{U}\cdot e_3$ imitates the Rayleigh–Bénard convection in a heated inviscid fluid. Equation $(1)_4$ describes the divergence free velocity and magnetic fields with $(1)_5$ tells about the prescribed initial conditions $\mathcal{U}_0$, $\mathcal{V}_0$ and θ_0.

As described by Mulone and Rionero [1] and Nakamura [2], the 3D magnetic Bénard system models the heat convection phenomenon influenced by velocity, magnetic field and temperature. The magnetic Bénard problem has sparked interest due to the thermal instability caused by the magnetic field. Although in 2D, the well-posedness problem has been resolved but the 3D case is still an unresolved issue in the whole space $\mathbb{R}^3$. When we ignore θ system (1) is simplified to MHD system. System (1) is reduced to Boussinesq equations if $\mathcal{V}$ is neglected and to Navier-Stokes equations (NSE) by taking $\mathcal{V} = 0$ and $\theta = 0$. System (1) also studies chemotaxis model, an important biological model, which has been extensively studied by [3–5] in the bounded domains.

In 1934, Leray [6] founded the concept of weak solutions (turbulent solutions), i.e., the solutions with finite kinetic energy belongs to a class $L^\infty(0,T;L^2)\cap L^2(0,T;H^1)$, for the proper definition of weak solution and its properties see [7,8], and the first finite time regularity criteria were given by Serrin [9] for the incompressible NSE, i.e., $\mathcal{U}$ becomes Leray-Hopf weak solution, if

$$\mathcal{U} \in L^m(0,T;L^l(\mathbb{R}^3)), \qquad \frac{2}{m}+\frac{3}{l}=1, \quad 3<l\le\infty, \quad 1<m\le\infty,$$

then smoothness of solution remains in the interval $(0,T]$. Later on, the regularity problem has been extensively explored by establishing various geometrically important constraints on the velocity, vorticity, pressure, strain tensor, etc.

In this paper, our interest is to explore the regularity in pressure terms for the system (1) because pressure controls the solutions of the whole system (1) by taking the divergence by test function, we can decouple velocity, magnetic field, and temperature from pressure. Therefore, it plays a significant role in understanding fluid flows. The NSE's regularity criteria for pressure and its gradient were demonstrated by Chae and Lee [10], Berselli and Galdi [11], and Zhou [12–14], given as

$$\psi \in L^{\frac{2}{2-l}}(0,T,L^{\frac{3}{l}}) \ \text{ with } \ 0<l\le 1,$$

and

$$\nabla\psi \in L^{\frac{2}{3-l}}(0,T,L^{\frac{3}{l}}) \ \text{ with } \ 0<l\le 1.$$

Duan [15] has obtained similar conditions for the MHD system.

For system (1), the global existence problem was addressed by Ma in [16], and the blow-up and regularity problem in terms of $\mathcal{U}$ and $\nabla\mathcal{U}$ in [17] for the multiplier space. The Serrin-type criteria $\psi^{\frac{2}{2-l}}(0.T;L^{\frac{3}{l}})$ with $0<l\le 1$, for the pressure, was given by Liu [18] in Lebesgue space. Recently, Chen et al. [19] established numerous important regularity results for the system (1), without thermal diffusion, based on pressure and its gradient in various function spaces, i.e., in Lebesgue spaces

$$\psi \in L^2(0,T;L^{\frac{3}{l}}) \ \text{ with } \ 0<l\le 1,$$

$$\nabla\psi \in L^{\frac{9-2l}{2l}}(0,T;L^{\frac{3}{l}}) \ \text{ with } \ 0<l\le 1.$$

In Morrey-Companato and Multiplier spaces

$$\psi \in L^{\frac{4l}{4l-6}}(0,T;\dot{M}_{l,m}) \ \text{ with } \ \frac{3}{2}<l\le\infty,$$

$$\psi \in L^2(0,T;\dot{X}^{-l}) \ \text{ with } \ 0<l\le 1.$$

In BMO and Besov spaces

$$\nabla\psi \in L^2(0,T;BMO), \tag{2}$$

$$\psi \in L^2(0,T;\dot{B}^{-1}_{\infty,\infty}). \tag{3}$$

Motivated by the above discussions and results, we will present improved integrable regularity conditions for the following 3D magnetic Bénard system with zero thermal diffusion:

$$\begin{cases} \frac{\partial\mathcal{U}}{\partial t}+\mathcal{U}\cdot\nabla\mathcal{U}-\beta_1\triangle\mathcal{U}+\nabla\psi-\mathcal{V}\cdot\nabla\mathcal{V}-\theta e_3=0, \quad \text{in } \mathbb{R}^3\times\mathbb{R}_+, \\ \frac{\partial\mathcal{V}}{\partial t}+\mathcal{U}\cdot\nabla\mathcal{V}-\beta_2\triangle\mathcal{V}-\mathcal{V}\cdot\nabla\mathcal{U}=0, \quad \text{in } \mathbb{R}^3\times\mathbb{R}_+, \\ \frac{\partial\theta}{\partial t}+\mathcal{U}\cdot\nabla\theta=0, \quad \text{in } \mathbb{R}^3\times\mathbb{R}_+, \\ \operatorname{div}\mathcal{U}=0, \ \operatorname{div}\mathcal{V}=0, \quad \text{in } \mathbb{R}^3\times\mathbb{R}_+, \\ (\mathcal{U},\mathcal{V},\theta)|_{t=0}=(\mathcal{U}_0,\mathcal{V}_0,\theta_0) \quad \text{in } \mathbb{R}^3. \end{cases} \tag{4}$$

Remark 1. *We will convert system (4) into mathematical symmetric form by putting $\mathcal{Q}^+ = \mathcal{U} + \mathcal{V}$ and $\mathcal{Q}^- = \mathcal{U} - \mathcal{V}$, as it will be useful in calculations and to apply certain inequalities such as (7) for the prove of our desired regularity conditions.*

The very first log improvement in $\mathcal{U}$ for the 3D NSE system was given by Montgomery-Smith [20]

$$\int_0^T \frac{\|\mathcal{U}\|^l_{L^m}}{1+\ln(e+\|\mathcal{U}\|_{L^m})}dt < \infty, \quad \frac{2}{l} + \frac{3}{m} = 1, \quad 2 < l \le \infty, \text{ and } 3 < m \le \infty. \tag{5}$$

Later on, such types of criteria were enhanced by (see, [21–23]) and also established for other fluid models (see [24,25] and references therein).

Similar to the log-criterion for weak solutions, we established improved logarithmic and double-logarithmic regularity conditions for the system (4) based on pressure and its gradient. Our results naturally generalise the result (5). Throughout the calculations, the non-negative parameters β_1, β_2, and β_3 are taken 1. The following mathematical preliminaries will help prove our main theorems.

Definition 1. *Let $\sigma \in \mathbb{R}$, $1 \le l, m \le \infty$, the homogeneous Besov space $\dot{B}^\sigma_{l,m}(\mathbb{R}^3)$ is defined by the full dyadic decomposition such as*

$$\dot{B}^\sigma_{l,m} = \{f \in \mathbb{Z}'(\mathbb{R}^3); \|f\|_{\dot{B}^\sigma_{l,m}} < \infty\},$$

where

$$\|f\|_{\dot{B}^\sigma_{l,m}} = \|\{2^{j\sigma}\|\Delta_j f\|_{L^l}\}^\infty_{j=-\infty}\|_{l^m}.$$

The details on dyadic decomposition can be found in [26].

Given as follows is the norm of homogeneous Sobolev space:

$$\|f\|_{\dot{H}^\sigma} = \|(-\Delta)^{\frac{\sigma}{2}} f\|_{L^2}.$$

Definition 2 ([27])**.** *Let $l, m, \sigma_1, \sigma_2, \sigma_3 \in [1,\infty]$ with $\sigma_3 \le \min(\sigma_1, \sigma_2)$, $\frac{1}{m} = \frac{1}{l} - \frac{s}{d}$, $1 \le r \le m$, and $\frac{s_1}{d} < \frac{1}{r} - \frac{1}{m} < \frac{s_2}{d}$. Then for $f \in \dot{B}^{s_1}_{r,\sigma_2} \cap \dot{B}^{s_2}_{r,\sigma_2}$, then we have*

$$\|f\|_{\dot{B}^0_{m,\sigma_3}} \le C\big(1 + \|f\|_{\dot{B}^s_{l,\sigma_1}}\big(\log^+\big(\|f\|_{\dot{B}^s_{l,\sigma_1}} + \|f\|_{\dot{B}^s_{l,\sigma_1}}\big)\big)^{\frac{1}{\sigma_3}-\frac{1}{\sigma_1}},$$

here, by choosing $l = m = \sigma_1 = \infty$, $\sigma_3 = r = \sigma_1 = s_2 = 2$ and $s_1 = s = 0$, we have

$$\|f\|_{BMO} \le C\big(1 + \|f\|_{\dot{B}^0_{\infty,\infty}} \log^{\frac{1}{2}}(1 + \|f\|_{H^2})\big). \tag{6}$$

The well-known pressure-velocity relations by the Calderon-Zygmund are given as:

$$\begin{cases} \|\psi\|_{L^\alpha} \le \|\mathcal{U}\|_{L^{2\alpha}}, \\ \|\nabla\psi\|_{L^\alpha} \le \|\mathcal{U}\cdot\nabla\mathcal{U}\|_{L^\alpha}, \\ \|\psi\|_{L^\alpha} \le C\|\mathcal{Q}^+\|_{L^{2\alpha}}\|\mathcal{Q}^-\|_{L^{2\alpha}}, \\ \|\nabla\psi\|_{L^\alpha} \le C\|\mathcal{Q}^+\cdot\nabla\mathcal{Q}^-\|_{L^\alpha}, \\ \|\nabla\psi\|_{L^\alpha} \le C\|\mathcal{Q}^-\cdot\nabla\mathcal{Q}^+\|_{L^\alpha}. \end{cases} \tag{7}$$

2. Main Results and Proofs

This section focuses on the proofs of Theorem 1, Theorem 2, and Theorem 3 using well-known energy methods.

Theorem 1. *Assume that* $(\mathcal{U}_0, \mathcal{V}_0, \theta_0) \in H^1(\mathbb{R}^3)$ *with* $\nabla \cdot \mathcal{U}_0 = 0$, $\nabla \cdot \mathcal{V}_0 = 0$ *in the sense of distributions. Let* $T > 0$ *and* $(\mathcal{U}, \mathcal{V}, \theta)$ *is a weak solution of system (1) in the interval* $(0, T]$. *If pressure* ψ *satisfies*

$$\int_0^T \frac{\|\psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\left(1+\ln\left(e+\|\psi\|_{\dot{B}^{-1}_{\infty,\infty}}\right)\right.} dt < \infty, \tag{8}$$

then $(\mathcal{U}, \mathcal{V}, \theta)$ *remians its smoothness on* $\mathbb{R}^3 \times (0, T]$, *and there are no moving singular points or blow-ups in the area under consideration, i.e, the interval (0, T].*

Proof of Theorem 1. Firstly, we will convert the system (4) into a symmetric form:

$$\begin{cases} \frac{\partial \mathcal{Q}^+}{\partial t} + \mathcal{Q}^- \cdot \nabla \mathcal{Q}^+ - \triangle \mathcal{Q}^+ + \nabla \psi - \theta e_3 = 0, \\ \frac{\partial \mathcal{Q}^-}{\partial t} + \mathcal{Q}^+ \cdot \nabla \mathcal{Q}^- - \triangle \mathcal{Q}^- + \nabla \psi - \theta e_3 = 0, \\ \frac{\partial \theta}{\partial t} + \frac{1}{2}(\mathcal{Q}^+ + \mathcal{Q}^-) \cdot \nabla \theta = 0, \\ \operatorname{div} \mathcal{Q}^+ = 0, \ \ \operatorname{div} \mathcal{Q}^- = 0, \\ (\mathcal{Q}^+, \mathcal{Q}^-, \theta)|_{t=0} = (\mathcal{Q}^+_0, \mathcal{Q}^-_0, \theta_0). \end{cases} \tag{9}$$

Now, testing $(9)_1$ with $\mathcal{Q}^+|\mathcal{Q}^+|^2$, $(9)_2$ with $\mathcal{Q}^-|\mathcal{Q}^-|^2$ and $(9)_3$ with $\theta|\theta|^2$, integrating over $\mathbb{R}^3$, adding all the equations, we finally get an L^4-estimates for $\mathcal{Q}^+$, $\mathcal{Q}^-$ and for θ, given as

$$\begin{aligned} &\frac{1}{4}\frac{d}{dt}\left(\|\mathcal{Q}^+\|^4_{L^4} + \|\mathcal{Q}^-\|^4_{L^4} + \|\theta\|^4_{L^4}\right) + \frac{1}{2}\left(\|\nabla|\mathcal{Q}^+|^2\|^2_{L^2} + \|\nabla|\mathcal{Q}^-|^2\|^2_{L^2}\right) + \\ &\left(\||\mathcal{Q}^+||\nabla \mathcal{Q}^+|\|^2_{L^2} + \||\mathcal{Q}^-||\nabla \mathcal{Q}^-|\|^2_{L^2}\right) \\ &= -\int_{\mathbb{R}^3} \nabla \Psi(\mathcal{Q}^+|\mathcal{Q}^+|^2 + \mathcal{Q}^-|\mathcal{Q}^-|^2)dx + \int_{\mathbb{R}^3} \theta e_3 \mathcal{Q}^+|\mathcal{Q}^+|^2 dx + \int_{\mathbb{R}^3} \theta e_3 \mathcal{Q}^-|\mathcal{Q}^-|^2 dx \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{10}$$

For I_2 and I_3, we derive that

$$I_2 \leq C\|\theta\|^4_{L^4} + \|\mathcal{Q}^+\|^4_{L^4}.$$
$$I_3 \leq C\|\theta\|^4_{L^4} + \|\mathcal{Q}^-\|^4_{L^4}.$$

I_1 is estimated as in (5.2) by Chen et al. [19].

Putting all the estimates in (10), using $\|\mathcal{Q}^+\|^4_{L^4} + \|\mathcal{Q}^-\|^4_{L^4} = \|\mathcal{U}\|^4_{L^4} + \|\mathcal{V}\|^4_{L^4}$, we get

$$\begin{aligned} &\frac{1}{4}\frac{d}{dt}\left(\|\mathcal{U}\|^4_{L^4} + \|\mathcal{V}\|^4_{L^4} + 1\right) + \frac{1}{4}\left(\|\nabla|\mathcal{U}|^2\|^2_{L^2} + \|\nabla|\mathcal{V}|^2\|^2_{L^2}\right) + \frac{1}{2}(\|\mathcal{U}\cdot\nabla\mathcal{U}\|^2_{L^2} \\ &+\|\mathcal{V}\cdot\nabla\mathcal{U}\|^2_{L^2} + \|\mathcal{U}\cdot\nabla\mathcal{V}\|^2_{L^2} + \|\mathcal{V}\cdot\nabla\mathcal{V}\|^2_{L^2}) \\ &\leq C(\|\psi\|^2_{\dot{B}^{-1}_{\infty,\infty}} + 1)\left(\|\mathcal{U}\|^4_{L^4} + \|\mathcal{V}\|^4_{L^4} + 1\right) \\ &\leq C\left(1 + \frac{\|\psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{1+\ln(e+\|\psi\|_{\dot{B}^{-1}_{\infty,\infty}})}\right)(1+\ln(e+\|\psi\|_{\dot{B}^{-1}_{\infty,\infty}})\left(\|\mathcal{U}\|^4_{L^4} + \|\mathcal{V}\|^4_{L^4} + 1\right). \end{aligned} \tag{11}$$

Using inequality $(7)_1$, we deduce

$$\leq C\left(1 + \frac{\|\psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{1+\ln(e+\|\psi\|_{\dot{B}^{-1}_{\infty,\infty}})}\right)(1+\ln(e+\|\mathcal{U}\|^2_{L^6})\left(\|\mathcal{U}\|^4_{L^4} + \|\mathcal{V}\|^4_{L^4} + 1\right).$$

$$\leq C\left(1 + \frac{\|\psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{1+\ln(e+\|\psi\|_{\dot{B}^{-1}_{\infty,\infty}})}\right)(1+\ln(e+Z(t))\left(\|\mathcal{U}\|^4_{L^4} + \|\mathcal{V}\|^4_{L^4} + 1\right).$$

$\forall\, t \in [T_*, T]$, define $Z(t) := \sup_{T_* \le s \le t} \|\Lambda^3 \mathcal{U}\|_{L^2}^2 + \|\Lambda^3 \mathcal{V}\|_{L^2}^2 + \|\Lambda^3 \theta\|_{L^2}^2$.
Applying Gronwall's lemma on the interval $[T_*, t]$, we have

$$\Big(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4 + 1\Big) \le C_0 \exp\Big(C \int_{T_*}^t \Big(1 + \frac{\|\psi\|_{\dot{B}^{-1}_{\infty,\infty}}^2}{1 + \ln(e + \|\psi\|_{\dot{B}^{-1}_{\infty,\infty}})}\Big) ds (1 + \ln(e + Z(t))\Big),$$

$$\text{where } C_0 = \Big(\|\mathcal{U}(\cdot, T_*)\|_{L^4}^4 + \|\mathcal{V}(\cdot, T_*)\|_{L^4}^4 + 1\Big).$$

$$\Big(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4 + 1\Big) \le C_0 \exp(2C\epsilon \ln(e + Z(t))) \le C_0 (e + Z(t))^{2C\epsilon}. \tag{12}$$

If there were a sufficiently small constant $\epsilon > 0$, $\exists\, T_* < T$, such that

$$\int_{T_*}^T \Big(1 + \frac{\|\psi\|_{\dot{B}^{-1}_{\infty,\infty}}^2}{1 + \ln(e + \|\psi\|_{\dot{B}^{-1}_{\infty,\infty}})}\Big) dt < \epsilon.$$

Now, we get bounds for $Z(t)$.

Multiply $\Lambda^3 = (-\Delta)^{\frac{3}{2}}$ with $(9)_1$ and taking the inner product with $\Lambda^3 \mathcal{Q}^+$, Multiply Λ^3 with $(9)_2$ and taking the inner product with $\Lambda^3 \mathcal{Q}^-$, Multiply Λ^3 with $(9)_3$ and taking the inner product with $\Lambda^3 \theta$, and using $(4)_4$, adding all the equations. We finally obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Lambda^3 \mathcal{Q}^+\|_{L^2}^2 + \|\Lambda^3 \mathcal{Q}^-\|_{L^2}^2 + \|\Lambda^3 \theta\|_{L^2}^2) + \|\Lambda^4 \mathcal{Q}^+\|_{L^2}^2 + \|\Lambda^4 \mathcal{Q}^-\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^3} (\Lambda^3(\mathcal{Q}^- \cdot \nabla \mathcal{Q}^+)\Lambda^3 \mathcal{Q}^+)) dx - \int_{\mathbb{R}^3} (\Lambda^3(\mathcal{Q}^+ \cdot \nabla \mathcal{Q}^-)\Lambda^3 \mathcal{Q}^-)) dx + \int_{\mathbb{R}^3} \Lambda^3(\theta e_3)\Lambda^3 \mathcal{Q}^+ dx \\
&\quad + \int_{\mathbb{R}^3} \Lambda^3(\theta e_3)\Lambda^3 \mathcal{Q}^- dx - \int_{\mathbb{R}^3} \Lambda^3((\mathcal{Q}^+ + \mathcal{Q}^-) \cdot \nabla \theta)\Lambda^3 \theta dx. \\
&= P_1 + P_2 + P_3 + P_4 + P_5,
\end{aligned} \tag{13}$$

where we used integration by parts, $\Lambda^s = (-\Delta)^{\frac{s}{2}}$ for $s \in \mathbb{R}$, and property of differentiating distributions. Now, we get estimate for $P_3 + P_4$

$$P_3 + P_4 = \int_{\mathbb{R}^3} \Lambda^3(\theta e_3)\Lambda^3 \mathcal{U} dx$$

$$\le C(\|\Lambda^3 \theta\|_{L^2}^2 + \|\Lambda^3 \mathcal{U}\|_{L^2}^2) \le C(e + \|\Lambda^3 \theta\|_{L^2}^2 + \|\Lambda^3 \mathcal{U}\|_{L^2}^2 + + \|\Lambda^3 \mathcal{V}\|_{L^2}^2) \le C_1 (e + Z(t))^2,$$

where C_1 is a positive constant.

Similarly,

$$\begin{aligned}
|P_5| &= \int_{\mathbb{R}^3} \Lambda^3(\mathcal{U} \cdot \nabla \theta)\Lambda^3 \theta dx \\
&\le C(\|\Lambda^4 \mathcal{U}\|_{L^2}^2 + \|\Lambda^4 \theta\|_{L^2}^2) + C_1 (e + Z(t))^{\frac{3}{2} + \frac{13}{2} C\epsilon},
\end{aligned}$$

here we use $\mathcal{Q}^+ + \mathcal{Q}^- = \mathcal{U}$.

For P_1 and P_2, Due to Kato and Ponce [28], we shall utilize the commutator estimate that follows:

$$\|\nabla^\alpha (fg) - f \nabla^\alpha g\|_{L^l} \le C(\|\Lambda^{\alpha-1} g\|_{L^{m_1}} \|\nabla f\|_{L^{l_1}} + \|\Lambda^\alpha f\|_{L^{l_2}} \|g\|_{L^{m_2}}), \tag{14}$$

for $\alpha > 1$ and $\frac{1}{l} = \frac{1}{l_1} + \frac{1}{m_1} = \frac{1}{l_2} + \frac{1}{m_2}$.

$$\begin{aligned}
|P_1 + P_2| \le \Big| &\int_{\mathbb{R}^3} (\Lambda^3(\mathcal{Q}^- \cdot \nabla \mathcal{Q}^+) - \mathcal{Q}^- \cdot \nabla \Lambda^3 \mathcal{Q}^+))\Lambda^3 \mathcal{Q}^+ dx \\
&+ \int_{\mathbb{R}^3} (\Lambda^3(\mathcal{Q}^+ \cdot \nabla \mathcal{Q}^-) - \mathcal{Q}^+ \cdot \nabla \Lambda^3 \mathcal{Q}^-))\Lambda^3 \mathcal{Q}^- dx\Big|.
\end{aligned}$$

Using (14) with these inequalities

$$\|\nabla\mathcal{U}\|_{L^3} \le C\|\nabla\mathcal{U}\|_{L^2}^{\frac{3}{4}}\|\nabla\Delta\mathcal{U}\|_{L^2}^{\frac{1}{4}}, \quad \|\nabla\Delta\mathcal{U}\|_{L^3} \le C\|\nabla\mathcal{U}\|_{L^2}^{\frac{1}{6}}\|\Delta^2\mathcal{U}\|_{L^2}^{\frac{5}{6}},$$

we deduce the final estimate that is given as

$$\begin{aligned}
|P_1+P_2| &\le C\big(\|\nabla\mathcal{Q}^-\|_{L^3}\|\Lambda^3\mathcal{Q}^+\|_{L^3}^2 + \|\nabla\mathcal{Q}^+\|_{L^3}\|\Lambda^3\mathcal{Q}^+\|_{L^3}\|\Lambda^3\mathcal{Q}^-\|_{L^3}\big)\\
&+C\big(\|\nabla\mathcal{Q}^+\|_{L^3}\|\Lambda^3\mathcal{Q}^-\|_{L^3}^2 + \|\nabla\mathcal{Q}^-\|_{L^3}\|\Lambda^3\mathcal{Q}^+\|_{L^3}\|\Lambda^3\mathcal{Q}^-\|_{L^3}\big)\\
&\le C\big(\|\nabla\mathcal{Q}^+\|_{L^2}^{\frac{13}{2}} + \|\nabla\mathcal{Q}^+\|_{L^2}^2\|\nabla\mathcal{Q}^-\|_{L^2}^{\frac{9}{2}} + \|\nabla\mathcal{Q}^+\|_{L^2}^{\frac{9}{2}}\|\nabla\mathcal{Q}^-\|_{L^2}^2 + \|\nabla\mathcal{Q}^-\|_{L^2}^{\frac{13}{2}}\big)\\
&\cdot\big(\|\Lambda^3\mathcal{Q}^-\|_{L^2}^{\frac{3}{2}} + \|\Lambda^3\mathcal{Q}^+\|_{L^2}^{\frac{3}{2}}\big) + \frac{1}{2}\big(\|\Lambda^3\nabla\mathcal{Q}^-\|_{L^2}^2 + \|\Lambda^3\nabla\mathcal{Q}^+\|_{L^2}^2\big)\\
&\le \frac{1}{2}\big(\|\Lambda^4\mathcal{Q}^+\|_{L^2}^2 + \|\Lambda^4\mathcal{Q}^-\|_{L^2}^2\big) + C\big(\|\nabla\mathcal{Q}^+\|_{L^2}^2 + \|\nabla\mathcal{Q}^-\|_{L^2}^2\big)^{\frac{13}{4}} Z^{\frac{3}{2}}(t).
\end{aligned}$$

Now, testing $(9)_1$ with $-\Delta\mathcal{Q}^+$ and $(9)_2$ with $-\Delta\mathcal{Q}^-$, the weak form is derived as

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|\nabla\mathcal{Q}^+\|_{L^2}^2 + \|\nabla\mathcal{Q}^-\|_{L^2}^2\big) + \|\Delta\mathcal{Q}^+\|_{L^2}^2 + \|\Delta\mathcal{Q}^-\|_{L^2}^2\\
&= -\int_{\mathbb{R}^3}(\mathcal{Q}^-\cdot\nabla\mathcal{Q}^+)\cdot\Delta\mathcal{Q}^+dx + \int_{\mathbb{R}^3}\theta e_3\cdot\Delta\mathcal{Q}^+dx - \int_{\mathbb{R}^3}(\mathcal{Q}^+\cdot\nabla\mathcal{Q}^-)\cdot\Delta\mathcal{Q}^-dx + \int_{\mathbb{R}^3}\theta e_3\cdot\Delta\mathcal{Q}^-dx.\\
&\le \|\Delta\mathcal{Q}^+\|_{L^2}^2 + \|\Delta\mathcal{Q}^-\|_{L^2}^2 + \frac{1}{2}\big(\|\Delta\mathcal{Q}^+\|_{L^2}^2 + \|\Delta\mathcal{Q}^-\|_{L^2}^2\big) + C\big(\|\mathcal{Q}^+\|_{L^6}^8 + \|\mathcal{Q}^-\|_{L^6}^8\big),
\end{aligned} \tag{15}$$

where we employed the following maximum principle frequently used and presented in [19] for system (9)

$$\|\theta\|_{L^l} \le \|\theta_0\|_{L^l} \le 1, \quad \text{where} \quad 1 < l \le \infty. \tag{16}$$

Integrating (15) in $[T_*, t]$, we deduce that

$$\big(\|\nabla\mathcal{Q}^+\|_{L^2}^2 + \|\nabla\mathcal{Q}^-\|_{L^2}^2\big) \le C(1+Z(t))^{\frac{4C\epsilon}{3}}(t-T_*) + \|\nabla\mathcal{Q}^+(T_*)\|_{L^2}^2 + \|\nabla\mathcal{Q}^-(T_*)\|_{L^2}^2. \tag{17}$$

Putting all the estimates into (13), absorbing dissipative terms together with (17) we have final H^3-bounds by applying Gronwall's inequality providing that ϵ must be sufficiently small. We get

$$\|\Lambda^3\mathcal{Q}^+\|_{L^2}^2 + \|\Lambda^3\mathcal{Q}^-\|_{L^2}^2 + \|\Lambda^3\theta\|_{L^2}^2 \le C. \tag{18}$$

Bounds (18) and (17) together with (12) implies that

$$\Big(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4 + 1\Big) \le C.$$

Thus, by providing sufficient estimates that ensure the smoothness up to time T of our solutions. Hence, Theorem 1 is proved. □

Corollary 1. *One of the foremost outcomes of above theroem is the result (3).*

Theorem 2. *Suppose that $(\mathcal{U}_0, \mathcal{V}_0, \theta_0) \in H^3(\mathbb{R}^3)$ with $\nabla\cdot\mathcal{U}_0 = 0$, $\nabla\cdot\mathcal{V}_0 = 0$ in distributional sense. For $T > 0$, $(\mathcal{U}, \mathcal{V}, \theta)$ is a weak solution of system (1). If pressure ψ satisfies an integrable regularity condition*

$$\int_0^T \frac{\|\nabla\psi\|_{\dot{B}^0_{\infty,\infty}}^{\frac{2}{3}}}{\big(1+\ln(e+\|\nabla\psi\|_{\dot{B}^0_{\infty,\infty}})\big)^{\frac{3}{2}}}dt < \infty, \tag{19}$$

then $(\mathcal{U},\mathcal{V},\theta)$ shows its smoothness in the interval $\mathbb{R}^3\times(0,T]$, and there are no moving singular points or blow-ups in the area under consideration, i.e, the interval (0,T].

Proof of Theorem 2. To prove this theorem we established *a priori* estimate for the weakly formulated equation (10).

For I_3

$$I_3 \le \|\theta\|_{L^4}\|\mathcal{Q}^+\|_{L^4}\|\mathcal{Q}^+\|^2_{L^4} \le \frac{1}{2}(\|\theta\|^2_{L^4}\|\mathcal{Q}^+\|^2_{L^4}) + C\|\mathcal{Q}^+\|^4_{L^4}$$

$$\frac{1}{4}\|\theta\|^4_{L^4} + C\|\mathcal{Q}^+\|^4_{L^4} + C\|\mathcal{Q}^+\|^4_{L^4} \le \frac{1}{4}\|\theta\|^4_{L^4} + C\|\mathcal{Q}^+\|^4_{L^4}. \tag{20}$$

Similarly,

$$I_2 \le \frac{1}{4}\|\theta\|^4_{L^4} + C\|\mathcal{Q}^-\|^4_{L^4}. \tag{21}$$

$$I_1 = -\int_{\mathbb{R}^3}\nabla\Psi(\mathcal{Q}^+|\mathcal{Q}^+|^2 + \mathcal{Q}^-|\mathcal{Q}^-|^2)dx = -\int_{\mathbb{R}^3}\nabla\Psi(\mathcal{Q}^+|\mathcal{Q}^+|^2)dx - \int_{\mathbb{R}^3}\nabla\Psi(\mathcal{Q}^-|\mathcal{Q}^-|^2)dx$$
$$= P_1 + P_2. \tag{22}$$

$$|P_1| \le |-\int_{\mathbb{R}^3}\nabla\psi\cdot\mathcal{Q}^+|\mathcal{Q}^+|^2dx| \le \|\nabla\psi\|_{L^4}\|\mathcal{Q}^+\|^3_{L^4} \le C\|\nabla\psi\|^{\frac{1}{2}}_{L^2}\|\nabla\psi\|^{\frac{1}{2}}_{BMO}\|\mathcal{Q}^+\|^3_{L^4}.$$

Similarly,

$$|P_2| \le |-\int_{\mathbb{R}^3}\nabla\psi\cdot\mathcal{Q}^-|\mathcal{Q}^-|^2dx| \le \|\nabla\psi\|_{L^4}\|\mathcal{Q}^-\|^3_{L^4} \le C\|\nabla\psi\|^{\frac{1}{2}}_{L^2}\|\nabla\psi\|^{\frac{1}{2}}_{BMO}\|\mathcal{Q}^-\|^3_{L^4}.$$

Putting estimates (20), (21) and for (22) into (10), and using $\|\mathcal{Q}^+\|^3_{L^4} + \|\mathcal{Q}^-\|^3_{L^4} = \|\mathcal{U}\|^3_{L^4} + \|\mathcal{V}\|^3_{L^4}$, we are down to

$$\frac{1}{4}\frac{d}{dt}\Big(\|\mathcal{U}\|^4_{L^4} + \|\mathcal{V}\|^4_{L^4} + 1\Big) + \frac{1}{4}\Big(\|\nabla|\mathcal{U}|^2\|^2_{L^2} + \|\nabla|\mathcal{V}|^2\|^2_{L^2}\Big) + \frac{1}{2}(\|\mathcal{U}\cdot\nabla\mathcal{U}\|^2_{L^2} + \|\mathcal{V}\cdot\nabla\mathcal{U}\|^2_{L^2}$$
$$+\|\mathcal{U}\cdot\nabla\mathcal{V}\|^2_{L^2} + \|\mathcal{V}\cdot\nabla\mathcal{V}\|^2_{L^2})$$
$$\le \|\mathcal{U}\|^3_{L^4}(\|\mathcal{U}\cdot\nabla\mathcal{U}\|^{\frac{1}{2}}_{L^2}\|\nabla\psi\|^{\frac{1}{2}}_{BMO}) + \|\mathcal{V}\|^3_{L^4}(\|\mathcal{U}\cdot\nabla\mathcal{U}\|^{\frac{1}{2}}_{L^2}\|\nabla\psi\|^{\frac{1}{2}}_{BMO}) + (\|\mathcal{U}\|^4_{L^4} + \|\mathcal{V}\|^4_{L^4} + \|\theta\|^4_{L^4})$$
$$\le \frac{1}{2}\||\mathcal{U}||\nabla\mathcal{U}|\|^2_{L^2} + C\|\nabla\psi\|^{\frac{2}{3}}_{BMO}\|\mathcal{U}\|^4_{L^4} + \frac{1}{2}\||\mathcal{U}||\nabla\mathcal{U}|\|^2_{L^2} + C\|\nabla\psi\|^{\frac{2}{3}}_{BMO}\|\mathcal{V}\|^4_{L^4}$$
$$+(\|\mathcal{U}\|^4_{L^4} + \|\mathcal{V}\|^4_{L^4} + \|\theta\|^4_{L^4})$$
$$\le C(\|\mathcal{U}\|^4_{L^4} + \|\mathcal{V}\|^4_{L^4} + \|\theta\|^4_{L^4})(1 + \|\nabla\psi\|^{\frac{2}{3}}_{BMO}).$$

Using (6) for $\nabla\psi$, we get that

$$\le C(\|\mathcal{U}\|^4_{L^4} + \|\mathcal{V}\|^4_{L^4} + \|\theta\|^4_{L^4})(1 + \|\nabla\psi\|^{\frac{2}{3}}_{\dot{B}^0_{\infty,\infty}}\ln^{\frac{1}{3}}(1 + \|\nabla\psi\|_{H^2}))$$

$$\le (\|\mathcal{U}\|^4_{L^4} + \|\mathcal{V}\|^4_{L^4} + \|\theta\|^4_{L^4})\Big(1 + \frac{\|\nabla\psi\|^{\frac{2}{3}}_{\dot{B}^0_{\infty,\infty}}}{(1 + \ln(1 + \|\nabla\psi\|_{\dot{B}^0_{\infty,\infty}})^{\frac{2}{3}}}\Big)\ln(1 + \|\Lambda^3\mathcal{U}\|_{L^2})). \tag{23}$$

For θ we use (16), which implies that

$$\le (\|\mathcal{U}\|^4_{L^4} + \|\mathcal{V}\|^4_{L^4} + 1)\Big(1 + \frac{\|\nabla\psi\|^{\frac{2}{3}}_{\dot{B}^0_{\infty,\infty}}}{(1 + \ln(1 + \|\nabla\psi\|_{\dot{B}^0_{\infty,\infty}})^{\frac{2}{3}}}\Big)\ln(1 + \kappa(t)).$$

Because of (19), $\exists\, T_* < T$, such that

$$\int_{T_*}^{T} \frac{\|\nabla\psi\|_{\dot{B}^0_{\infty,\infty}}^{\frac{2}{3}}}{1+\ln(1+\|\nabla\psi\|_{\dot{B}^0_{\infty,\infty}})^{\frac{2}{3}}} < \epsilon.$$

We set

$$\kappa(t) := (\|\Lambda^3\mathcal{U}\|_{L^2} + \|\Lambda^3\mathcal{V}\|_{L^2} + \|\Lambda^3\theta\|_{L^2}).$$

$\kappa(t)$ is bounded by the same process as $Z(t)$.

Due to the application of Gronwall's Lemma to (23), we obtain

$$\sup_{T_*<t\leq T} (\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4 + 1) \leq C_*(e+\kappa(t))^{C\epsilon}$$

This proves Theorem 2. □

Corollary 2. *The continuous embedding BMO $\hookrightarrow \dot{B}^0_{\infty,\infty}$ results in very important consequence of Theorem 2 that is the condition*

$$\nabla\psi \in L^{\frac{2}{3}}(0,T;\dot{B}^0_{\infty,\infty}),$$

which improves the criteria (2) by taking it from BMO (Bounded mean oscillations) space to larger Besov space $\dot{B}^0_{\infty,\infty}$.

Theorem 3. *Suppose that $(\mathcal{U}_0,\mathcal{V}_0,\theta_0) \in H^1(\mathbb{R}^3)$ with $\nabla\cdot\mathcal{U}_0 = 0$, $\nabla\cdot\mathcal{V}_0 = 0$ in the sense of distributions. Let $T>0$ and $(\mathcal{U},\mathcal{V},\theta)$ is a weak solution of system (1) on the interval $(0,T]$. If pressure ψ satisfies*

$$\int_0^T \frac{\|\psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\left(e+\ln\left(e+\|\psi\|_{\dot{B}^{-1}_{\infty,\infty}}\right)\ln\left(e+\ln\left(e+\|\psi\|_{\dot{B}^{-1}_{\infty,\infty}}\right)\right)\right.} dt < \infty, \tag{24}$$

then $(\mathcal{U},\mathcal{V},\theta)$ is a regular solution on $\mathbb{R}^3\times(0,T]$, and there are no moving singular points or blow-ups in the area under consideration, i.e, the interval (0, T].

Proof of Theorem 3. To prove this theorem, we will continue from inequality (11), taking the Gronwall's lemma into consideration for (11), we can show that

$$\left(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4 + 1\right) \leq \left(\|\mathcal{U}_0\|_{L^4}^4 + \|\mathcal{V}_0\|_{L^4}^4 + 1\right)\exp\Big(C\int_0^T(\|\Psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}+1)\Big). \tag{25}$$

Now, testing $(4)_1$ with $\Delta\mathcal{U}$

$$\int_{\mathbb{R}^3}\partial_t\mathcal{U}\cdot\Delta\mathcal{U}dx + \int_{\mathbb{R}^3}(\mathcal{U}\cdot\nabla\mathcal{U})\Delta\mathcal{U}dx - \int_{\mathbb{R}^3}\Delta\mathcal{U}\cdot\Delta\mathcal{U}dx + \int_{\mathbb{R}^3}\nabla\psi\cdot\Delta\mathcal{U}dx - \int_{\mathbb{R}^3}(\mathcal{V}\cdot\nabla\mathcal{V})\Delta\mathcal{U}dx$$
$$-\int_{\mathbb{R}^3}\theta e_3\cdot\Delta\mathcal{U}dx = 0. \tag{26}$$

Testing $(4)_2$ with $\Delta\mathcal{V}$

$$\int_{\mathbb{R}^3}\partial_t\mathcal{V}\cdot\Delta\mathcal{V}dx + \int_{\mathbb{R}^3}(\mathcal{U}\cdot\nabla\mathcal{V})\Delta\mathcal{V}dx - \int_{\mathbb{R}^3}\Delta\mathcal{V}\cdot\Delta\mathcal{V}dx - \int_{\mathbb{R}^3}(\mathcal{V}\cdot\nabla\mathcal{U})\Delta\mathcal{V}dx = 0. \tag{27}$$

Testing $(4)_3$ with $\Delta\theta$

$$\int_{\mathbb{R}^3}\partial_t\theta\cdot\Delta\theta dx + \int_{\mathbb{R}^3}(\mathcal{U}\cdot\nabla\theta)\cdot\Delta\theta dx = 0. \tag{28}$$

Adding (26), (27) and (28), we derive that

$$\frac{1}{2}\frac{d}{dt}(\|\nabla\mathcal{U}\|_{L^2}^2+\|\nabla\mathcal{V}\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)+(\|\Delta\mathcal{U}\|_{L^2}^2+\|\Delta\mathcal{V}\|_{L^2}^2)=-\int_{\mathbb{R}^3}(\mathcal{U}\cdot\nabla\mathcal{U})\Delta\mathcal{U}dx$$
$$+\int_{\mathbb{R}^3}(\mathcal{V}\cdot\nabla\mathcal{V})\Delta\mathcal{U}dx-\int_{\mathbb{R}^3}(\mathcal{U}\cdot\nabla\mathcal{V})\Delta\mathcal{V}dx+\int_{\mathbb{R}^3}(\mathcal{V}\cdot\nabla\mathcal{U})\Delta\mathcal{V}dx+\int_{\mathbb{R}^3}\theta e_3\cdot\Delta\mathcal{U}dx$$
$$-\int_{\mathbb{R}^3}(\mathcal{U}\cdot\nabla\theta)\cdot\Delta\theta dx$$
$$\leq\|\nabla\mathcal{U}\|_{L^3}^3+3\|\nabla\mathcal{U}\|_{L^3}\|\nabla\mathcal{V}\|_{L^3}^2+\|\nabla\mathcal{U}\|_{L^2}\|\nabla\theta\|_{L^2}+\|\nabla\mathcal{U}\|_{L^3}\|\nabla\theta\|_{L^3}^2.$$

Using Gagliardo-Nirenberg inequality, we get

$$\frac{1}{2}\frac{d}{dt}(\|\nabla\mathcal{U}\|_{L^2}^2+\|\nabla\mathcal{V}\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)+(\|\Delta\mathcal{U}\|_{L^2}^2+\|\Delta\mathcal{V}\|_{L^2}^2)$$
$$\leq\frac{1}{2}(\|\Delta\mathcal{U}\|_{L^2}^2+\|\Delta\mathcal{V}\|_{L^2}^2)+C\|\mathcal{U}\|_{L^4}^{12}+C\|\mathcal{V}\|_{L^4}^{12}+C\|\theta\|_{L^4}^{12}+\|\nabla\mathcal{U}\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2. \tag{29}$$

Integrating (29) on the interval $(0,t]$

$$(\|\nabla\mathcal{U}\|_{L^2}^2+\|\nabla\mathcal{V}\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)+\int_0^t(\|\Delta\mathcal{U}\|_{L^2}^2+\|\Delta\mathcal{V}\|_{L^2}^2)d\tau$$
$$\leq(\|\nabla\mathcal{U}_0\|_{L^2}^2+\|\nabla\mathcal{V}_0\|_{L^2}^2+\|\nabla\theta_0\|_{L^2}^2)+C\int_0^t(\|\mathcal{U}\|_{L^4}^{12}+\|\mathcal{V}\|_{L^4}^{12}+\|\theta\|_{L^4}^{12})d\tau. \tag{30}$$

Now, by Sobolev embedding theorem $\dot{H}^1(\mathbb{R}^3)\subset L^6(\mathbb{R}^3)$, $(7)_1$, (30) and (25), we obtain

$$e+\|\psi(\cdot,t)\|_{L^3}\leq e+C\|\mathcal{U}\|_{L^6}^2\leq e+C(\|\nabla\mathcal{U}\|_{L^2}^2+\|\nabla\mathcal{V}\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)$$
$$\leq e+C(\|\nabla\mathcal{U}_0\|_{L^2}^2+\|\nabla\mathcal{V}_0\|_{L^2}^2+\|\nabla\theta_0\|_{L^2}^2)+C\int_0^t(1+\|\mathcal{U}(\cdot,\tau)\|_{L^4}^{12}+\|\mathcal{V}(\cdot,\tau)\|_{L^4}^{12}+\|\theta(\cdot,\tau)\|_{L^4}^{12})d\tau$$
$$\leq C(e+\|\nabla\mathcal{U}_0\|_{L^2}^2+\|\nabla\mathcal{V}_0\|_{L^2}^2+\|\nabla\theta_0\|_{L^2}^2)(e+t)\sup_{0\leq\tau\leq t}(1+\|\mathcal{U}(\cdot,\tau)\|_{L^4}^{12}+\|\mathcal{V}(\cdot,\tau)\|_{L^4}^{12}+\|\theta(\cdot,\tau)\|_{L^4}^{12})$$
$$\leq C_0(e+t)\exp\left(C\int_0^t(1+\|\psi\|_{\dot{B}^{-1}_{\infty,\infty}}^2)d\tau\right).$$

Now, using $L^3(\mathbb{R}^3)\subset\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$

$$e+\|\psi\|_{\dot{B}^{-1}_{\infty,\infty}}\leq C(e+t)\exp\left(C\int_0^t(1+\|\psi\|_{\dot{B}^{-1}_{\infty,\infty}}^2)d\tau\right).$$

Applying ln on both sides

$$\ln\left(e+\|\psi\|_{\dot{B}^{-1}_{\infty,\infty}}\right)\leq\ln(C(e+t))+\left(C\int_0^t(1+\|\psi\|_{\dot{B}^{-1}_{\infty,\infty}}^2)d\tau\right). \tag{31}$$

For ease in the calculations, we let

$$\Upsilon(t)=\ln(e+\|\psi\|_{\dot{B}^{-1}_{\infty,\infty}}). \tag{32}$$

$$\Phi(t)=\ln(C(e+t))+\left(C\int_0^t(1+\|\psi\|_{\dot{B}^{-1}_{\infty,\infty}}^2)d\tau\right). \tag{33}$$

By (32) and (33), Inequality (31) implies that

$$0<\Upsilon\leq\Phi$$

It results in

$$(e+\Upsilon)(\ln(e+\Upsilon)) \leq (e+\Phi)(\ln(e+\Phi)).$$

On the other hand, to prove our result we take time derivative of $\ln(e+\Phi)$, and obtain

$$\begin{aligned}\tfrac{d}{dt}\ln(e+\Phi) &= \tfrac{1}{(e+\Phi)}\left(\tfrac{1}{e+t}+C(1+\|\psi\|^2_{\dot{B}^{-1}_{\infty,\infty}})\right)\\ &\leq \tfrac{1}{e^2}+C\tfrac{1+\|\psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{e+\Phi}\\ &= \tfrac{1}{e^2}+C\tfrac{1+\|\psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{(e+\Phi)\ln(e+\Phi)}\ln(e+\Phi)\\ &\leq \tfrac{1}{e^2}+C\tfrac{1+\|\psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{(e+\Upsilon)\ln(e+\Upsilon)}\ln(e+\Phi).\end{aligned}$$

Apply the Gronwall's lemma to $\ln(e+\Phi)$, we get that

$$\ln(e+\Phi(t)) \leq \ln(e+\Phi(0))\exp\left(\frac{T}{e^2}+C\int_0^t \frac{1+\|\psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{(e+\Upsilon(\tau))\ln(e+\Upsilon(\tau))}d\tau\right),$$

resulting as

$$(e+\Phi(t)) \leq (e+\Phi(0))^{\exp\left(\frac{T}{e^2}+C\int_0^t \frac{1+\|\psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{(e+\Upsilon(\tau))\ln(e+\Upsilon(\tau))}d\tau\right)},$$

and from (33) we deduce that

$$\int_0^t (1+\|\psi\|^2_{\dot{B}^{-1}_{\infty,\infty}})d\tau \leq (e+\Phi(0))^{\exp\left(\frac{T}{e^2}+\frac{1}{C}\int_0^t \frac{1+\|\psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{(e+\Upsilon(\tau))\ln(e+\Upsilon(\tau))}d\tau\right)} < \infty. \tag{34}$$

Estimate (34) together with (29) ensures the regularity of weak solutions in the interval $C^\infty(\mathbb{R}^3\times[0,T))$. Thus, completing the prove of Theorem 3. $\square$

The other very important aspect of the non-linear differential system (1), i.e., the 3D ma- gnetic Bénard system, is the occurrence of movable singularities, i.e., starting from smooth initial data, the solution becomes infinite in finite time due to the cumulative effect of the nonlinearities. Such types of singularity formations in non-linear differential systems are also known as blow-ups. In the framework of the regularity theory of weak solutions, the blow-up or singularity occurs if the solution becomes infinite at some (or many) points as t approaches a certain finite time T. The singularity or blow-up problem states that the sol- ution with some smooth initial data is well-defined in some function space for some time $0<t<T$. Such type of singularities explicitly depend upon the type of function space and time. The alternative interpretation of conditions (8), (19), and (24) is let $T=T^\dagger<\infty$ is the maximal time for the existence of a smooth solutions, then the solution blows up (also called the first time blow up) to create finite time singularity, and condition (8) takes the form shown as

$$\int_0^{T^\dagger} \frac{\|\psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\left(1+\ln\left(e+\|\psi\|_{\dot{B}^{-1}_{\infty,\infty}}\right)\right)}dt=\infty,$$

similarly, the condition (19) becomes

$$\int_0^{T^\dagger} \frac{\|\nabla\psi\|^{\frac{2}{3}}_{\dot{B}^{0}_{\infty,\infty}}}{(1+\ln(e+\|\nabla\psi\|_{\dot{B}^{0}_{\infty,\infty}}))^{\frac{3}{2}}}dt=\infty,$$

and the regularity condition (24) appears as

$$\int_0^{T^\dagger} \frac{\|\psi\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\left(e+\ln\left(e+\|\psi\|_{\dot{B}^{-1}_{\infty,\infty}}\right)\ln\left(e+\ln\left(e+\|\psi\|_{\dot{B}^{-1}_{\infty,\infty}}\right)\right)\right.} dt = \infty.$$

Therefore, the blow-up is exactly the inability to continue the weak solution up to or past a given time.

Remark 2. *The integrable regularity condition (8) improves the regularity criteria (3), and result (19) is the improvement of criteria (2). Result (24) is the optimal in the sense that it refines all the previous results for pressure terms in the largest scale invariant double logarithmic Besov spaces.*

3. Conclusions

In this work, the mathematical significance of results (8) and (24) lies in wider spaces, i.e., Besov spaces of a negative index. Such spaces are important due to their criticality defined by their scale invariance because the local regularity results by using scale invariance pro- perty could be taken to global regularity results. The criteria (19) replace *BMO* space with larger space, i.e., $\dot{B}^0_{\infty,\infty}$, consequently, improving the regularity of solutions. Our results that are proved in the finite-time interval $C^\infty(\mathbb{R}^3 \times [0,T))$ constitute vital work on the millennium clay mathematical problem [29] which requires the solutions to be regular in $C^\infty(\mathbb{R}^3 \times [0,\infty))$ i.e., for all time. We use pressure, which has remarkable properties, to control the solutions of the system (4) by imposing sufficient integrable regularity conditions that improve numerous previously established results.

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