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Quantum Geometry of Spacetime and Quantum Equilibrium

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Abstract: We give a concise review of the properties of quantum geometrodynamics in the pilot-wave quantum cosmology, focusing on the issue of its nonlocal character. We also discuss the problem of the origin of quantum probabilities in this theory with a focus on the ergodic approach to its resolution.

Keywords: quantum gravity; pilot-wave theory

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1. Introduction

The nature of time has remained a philosophical mystery since the ancient epochs [1,2]. Should the spacetime arena and all events in it be regarded as actually existing, while the human spirit is only sliding along its world line in this pre-existing continuum (as first imagined by British writer Herbert Wells [3] and independently by Russian philosopher Mitrofan Aksenov [4])? Or should the world around us be considered a gradual development in time into existence, in accordance with our own experience and perception? It is interesting that both philosophical viewpoints are reflected in the mathematical formalism of the modern relativity theory: the first in the Lagrangian version and the second in the Hamiltonian version.

While both formulations can be used in classical theory, the quantum theory of gravity and cosmology faces specific problems in this regard. There, "the universe as a whole" is described by a wave function of three-dimensional geometry and matter fields, which do not contain any time parameters. How does the observed time-dependent expanding universe emerge in this picture?

One of the interpretational frameworks of quantum theory resolving these issues is the pilot-wave formulation proposed by Bohm [5,6] (see [7,8] for reviews) and based on the pioneering ideas from de Broglie [9,10]. Its basic idea is quite simple. Any physical system evolves deterministically in terms of appropriate configuration variables (for which John Bell coined the term "beables" [11]). These variables can be chosen to be just those of classical physics, i.e., the coordinates of fundamental particles and field spatial configurations. The classical and quantum theories are distinguished by their dynamics. In classical physics, the dynamics follow the principle of extremal local action. In quantum physics, the evolution is guided (piloted, according to de Broglie) by a wave function that obeys the Schrödinger equation. This version of quantum mechanics was termed "ontological interpretation" by its proponent [7]; it is frequently called "Bohmian mechanics" in the literature.

In this paper, we review the main features of the pilot-wave theory applied to the universe. We will see that, in the pilot-wave framework, the arising quantum geometry of spacetime is necessarily described as a development within time. Despite being fully deterministic, the quantum dynamics, nevertheless, fails to be invariant with respect to arbitrary refoliations of spacetime, which means that it is intrinsically nonlocal.

Given the fully deterministic character of the evolution of the universe in the pilot-wave theory, it is then necessary to explore the nature of the quantum probabilities that operate on a smaller scale. The qualitative picture of this probabilistic behaviour is clear: the universe, in the course of its evolution, forms multitudes of identical ensembles of subsystems (such as atoms), in which the positions of constituent particles are not controlled by the observer and, therefore, are random. This explains the randomness in the outcomes of various experiments with such systems. There is, however, an important quantitative problem. The probabilities of the results of the quantum measurements should coincide with those predicted by the standard (Göttingen) interpretation, which is consistent with the experiments. For the pilot-wave theory, this requirement can be satisfied assuming that the configuration variables x in a pure quantum ensemble are distributed according to its wave function $\psi(x)$, with the probability distribution function $p(x) = |\psi(x)|^2$. This condition is usually called "quantum equilibrium": it is preserved in time for an ensemble of closed systems by virtue of the Schrödinger equation. However, in the pilot-wave theory, this distribution cannot be introduced as an independent postulate; its origin must be explained. Several resolutions of this important problem will also be reviewed in this paper, with a focus on the ergodic approach.

2. Classical Geometry of Space and Time

The view of spacetime as the existing "eternity" arises in relativity theory, where it is regarded as a maximally extended four-dimensional manifold, $\mathcal{M}$, endowed with a Lorentzian-signature metric. General-covariant Einstein equations treat space and time as extensions of similar nature. The "evolution" viewpoint is inherent in the Hamiltonian approach to the dynamics of gravitational fields (and matter fields) due to Arnowitt, Deser, and Misner [12]. Here, one considers a foliation of spacetime by spatial hypersurfaces Σ_t, endowed with the induced metric g_{ab}, the lapse function $\mathcal{N}$, and the shift vector $\mathcal{N}^a$, so that these objects completely describe the spacetime metric:

$$ds^2 = -\left(\mathcal{N}^2 - \mathcal{N}^a\mathcal{N}_a\right)dt^2 + 2\mathcal{N}_a dx^a dt + g_{ab}\, dx^a dx^b \,. \tag{1}$$

The (Latin) spatial indices are lowered and raised by g_{ab} and its inverse g^{ab}, respectively.

The classical Hilbert–Einstein action for the metric together with a system of bosonic fields Φ takes the form

$$S = \int_{\mathcal{M}} d^3x\, dt \left(\pi^{ab}\dot{g}_{ab} + \pi_\Phi \dot{\Phi} - \mathcal{N}\mathcal{H} - \mathcal{N}^a\mathcal{H}_a\right), \tag{2}$$

in which $\mathcal{N}$ and $\mathcal{N}^a$ become Lagrange multipliers enforcing the Hamiltonian and momentum constraints $\mathcal{H}$ and $\mathcal{H}_a$, respectively. The symbol π_Φ denotes the system of conjugate momenta of the bosonic fields, and π^{ab} are the conjugate momenta for the metric, g_{ab}. The constraints have the form

$$\mathcal{H} \equiv \frac{1}{2\kappa}\mathcal{G}_{abcd}\pi^{ab}\pi^{cd} + \kappa\sqrt{g}\left(2\Lambda - {}^{(3)}\mathcal{R}\right) + \mathcal{H}^\Phi = 0\,, \tag{3}$$

$$\mathcal{H}_a \equiv -2\nabla_b \pi^b_a + \mathcal{H}^\Phi_a = 0\,, \tag{4}$$

where only the gravitational parts of the constraints have been written explicitly. The Φ-parts $\mathcal{H}^\Phi$ and $\mathcal{H}^\Phi_a$, of the constraints, follow from the respective Lagrangian and will not be specified here. The gravitational constant above is $\kappa = (16\pi G)^{-1}$, where G is Newton's constant, Λ is the cosmological constant, and

$$\mathcal{G}_{abcd} = \frac{1}{\sqrt{g}}(g_{ac}g_{bd} + g_{ad}g_{bc} - g_{ab}g_{cd}) \tag{5}$$

is Wheeler's supermetric. The symbol ∇_a denotes the three-space covariant derivative determined by the metric g_{ab}, with ${}^{(3)}\mathcal{R}$ being its scalar curvature. The classical field equations, including the constraints, are obtained by varying action (2) with respect to all its field variables. Of relevance for our discussion is the equation of motion for the spatial metric:

$$\dot{g}_{ab} = \frac{\mathcal{N}}{\kappa}\mathcal{G}_{abcd}\pi^{cd} + \nabla_a\mathcal{N}_b + \nabla_b\mathcal{N}_a \,. \tag{6}$$

Here and below, the overdot denotes the derivatives with respect to the time coordinate t. In this case, the lapse function $\mathcal{N}$ and shift vector $\mathcal{N}^a$ are not dynamical, they can be specified at will, reflecting the reparametrisation and refoliation freedom in metric (1).

One of the methods of solving the dynamical equations for the metric and matter fields consists in obtaining first a solution to the action functional $S[g_{ab}(x), \Phi(x)]$ that obeys the (time-independent) Einstein–Hamilton–Jacobi equation that follows from the constraint Equation (3) after the substitutions of $\pi^{ab} \to \delta/\delta g_{ab}(x)$ and $\pi^{\Phi} \to \delta/\delta\Phi(x)$:

$$\frac{1}{2\kappa}\delta S \circ \delta S + \kappa\sqrt{g}\left(2\Lambda - {}^{(3)}\mathcal{R}\right) + \mathcal{H}^{\Phi} = 0 \,, \tag{7}$$

where δ symbolises the variational derivative $\delta/\delta g_{ab}(x)$, and the symbol "$\circ$" denotes the scalar product defined by Wheeler's supermetric, Equation (5). Evolution of the spatial metric g_{ab} is then given by Equation (6), in which one should make the substitution

$$\pi^{ab}(x,t) \to \left.\frac{\delta S}{\delta g_{ab}(x)}\right|_{g_{ab}(x) = g_{ab}(x,t)} . \tag{8}$$

The form (6) corresponds to the view of the world's geometry as evolving in time.

3. Quantum Geometry of Space and Time

Proceeding to quantisation, we observe that, in the Schrödinger picture, the system described by action (2) is represented by the wave function $\Psi[g_{ab}(x), \Phi(x)]$, which is a solution to the quantum constraint Equations [13,14],

$$\hat{\mathcal{H}}_{\mu}\Psi = 0 \,, \tag{9}$$

where the operators $\hat{\mathcal{H}}_{\mu}$ are obtained from the classical expressions $\mathcal{H}_0 \equiv \mathcal{H}$ and $\mathcal{H}_a$ by replacing the generalised momenta π^{ab} and π_{Φ} with the corresponding operators of variational derivatives (with some operator ordering).

The wave function Ψ, called the "wave function of the universe", does not depend on time t. Since our universe evolves in time, this creates difficulty in interpreting this wave function. This problem is usually resolved in quantum cosmology by assigning the role of time to one of the physical variables and by reducing the phenomenon of time to correlations between this variable and other physical quantities. However, such a choice is always made arbitrarily; even then, time usually fails to be universally defined. Moreover, the usual statistical meaning of the wave function loses its sense in quantum cosmology, making it unclear what measurements and related probabilities it can possibly describe.

A number of interpretations were suggested in order to achieve a coherent picture of the observed reality both on a microscopic quantum and on a macroscopic classical level. Among them is the pilot-wave theory due to Bohm [5–7]. For a non-relativistic system of particles with configuration coordinates $x \equiv \{x_i\}$, conjugate momenta p_i, and wave function $\psi(x,t)$ respecting the Schrödinger equation, the pilot-wave theory describes the evolution as follows. The wave function can always be written in the form $\psi = R\exp(iS/\hbar)$ with real R and S; then, the evolution law is postulated to be

$$p_i(\dot{x}) = \nabla_i S \,, \tag{10}$$

where $p_i(\dot{x})$ is the momentum expressed through the velocity. In the classical limit, phase S satisfies the classical Hamilton–Jacobi equation. Therefore, the pilot-wave formulation can be regarded simply as a quantum "deformation" of the classical dynamics (see [8] for a detailed discussion of this viewpoint). This allows one to hold a special view on the "reality" of the wave function of a physical system: it is no more real than a solution S of the Hamilton–Jacobi equation in classical theory. A generalisation of the guidance Equation (10) to relativistic quantum mechanics and quantum field theory can be found in [7,8].

Pilot-wave treatment of the "wave function of the universe" was first given in [8,15–17] (for recent developments, see [18]). According to the general rules, the wave function is written in the form $\Psi = R\exp(iS/\hbar)$. The momentum constraint $\hat{\mathcal{H}}_a\Psi = 0$ in (9) then implies an invariance of the wave function with respect to spatial coordinate reparametrisations, i.e., its dependence only on the three-geometry ${}^{(3)}\mathcal{G}$ of Σ. The Hamiltonian constraint (or the Wheeler–De Witt Equation [13,14]) $\hat{\mathcal{H}}\Psi = 0$ in (9) gives birth to two equations:

$$\frac{1}{2\kappa}\delta S \circ \delta S + \kappa\sqrt{g}\left(2\Lambda - {}^{(3)}\mathcal{R}\right) - \frac{\hbar^2}{2\kappa}\frac{\delta \circ \delta R}{R} + \frac{\Re\left(\Psi^\dagger \hat{\mathcal{H}}^\Phi \Psi\right)}{R^2} = 0\,, \tag{11}$$

$$\delta \circ \left(R^2 \delta S\right) - \frac{2\kappa}{\hbar}\Im\left(\Psi^\dagger \hat{\mathcal{H}}^\Phi \Psi\right) = 0\,, \tag{12}$$

where by $\Re$ and $\Im$ we denote the real and imaginary parts, respectively. The term proportional to $\hbar^2$ in (11) is called the quantum potential. In the formal limit $\hbar \to 0$, Equation (11) reproduces the classical Einstein–Hamilton–Jacobi Equation (7).

According to the dynamic principle of the pilot-wave theory, the quantum evolution of the metric g_{ab} is described by Equation (6) with substitution (8). The Lagrange multipliers $\mathcal{N}$ and $\mathcal{N}^a$ in Equation (6) do not have evolution equations; therefore, they can be specified arbitrarily, similarly to the case of classical geometrodynamics.

Thus, a family of solutions of the quantum dynamics is obtained by solving the quantum Equations (11) and (12) instead of solving the classical Equation (7). A configuration $g_{ab}(x,\, t)$, $\Phi(x,\, t)$ then represents a solution to the guidance Equation (6) and the similar equation for Φ with arbitrarily specified functions $\mathcal{N}(x,\, t)$ and $\mathcal{N}^a(x,\, t)$. A solution thus obtained will describe a quantum four-geometry with a matter-field configuration Φ.

The procedures described above for obtaining solutions in classical and quantum case are very similar. There is, however, a substantial difference in principle between the classical and quantum geometries. In the classical case, a different specification of $\mathcal{N}$ and $\mathcal{N}^a$ will lead to the same four-geometry, only with a different foliation by a family of hypersurfaces Σ_t and different coordinates on each of these hypersurfaces. In quantum dynamics, due to the reparametrisation-invariance of the wave function, Ψ, the arising four-geometry will remain the same only with respect to the choice of the spatial coordinates x^a on each of the hypersurfaces Σ_t, which is controlled by the shift vector $\mathcal{N}^a$. The role of the lapse function $\mathcal{N}$ is more significant in the quantum case: different specifications of the lapse function $\mathcal{N}$ will result in different quantum four-geometries. This difference becomes negligible only in the classical limit, in which the quantum potential in (11) can be neglected.

This entails another important difference between the classical and quantum geometrodynamics: while the former obeys local differential equations, the latter is essentially nonlocal with a distinguished space-time foliation. The precise meaning of this statement is the following. Given a solution

$${}^{(4)}\mathcal{G} \equiv \left\{g_{ab}(\mathbf{x},\, t)\,,\, \mathcal{N}^\mu(\mathbf{x},\, t)\right\} \tag{13}$$

of the quantum pilot-wave geometrodynamics, an arbitrary change of the space-like foliation of ${}^{(4)}\mathcal{G}$ will lead to the representation of this four-geometry, which fails to be such a solution. This can be proved by a simple argument: because of the non-classical character of the four-geometry (13), by changing the foliation, one generally violates the momentum constraint (4), which should also be satisfied in the quantum case [16].

The nonlocal character is a generic feature of the laws of the pilot-wave theory [7], and this remains to be the case even in the fully covariant formulation based on tensor quantities, including the metric. Quantum geometrodynamics distinguishes foliation (13) as the only one in which its components are evolved according to the guidance of Equation (6) applied to the pilot-wave theory.

4. Quantum Fields and Particles in Curved Spacetime

In this section, we review the way in which the pilot-wave quantum field theory arises on a semiclassical pilot-wave metric background. Consider the usual treatment of the classical limit for quantum gravity where the wave function takes the approximate form [16,19]

$$\Psi\left[{}^{(3)}\mathcal{G},\,\Phi\right] \approx R\left[{}^{(3)}\mathcal{G}\right] \exp\left(\frac{iS\left[{}^{(3)}\mathcal{G}\right]}{\hbar}\right) \chi\left[{}^{(3)}\mathcal{G},\,\Phi\right]. \tag{14}$$

Here, following the semiclassical approximation, we assume that the phase $S\left[{}^{(3)}\mathcal{G}\right]$ is a solution to the classical vacuum Einstein–Hamilton–Jacobi equation (which is just Equation (7) without the matter part), and real $R\left[{}^{(3)}\mathcal{G}\right]$ is chosen so that it obeys the equation

$$\delta \circ \left(R^2 \delta S\right) = 0\,. \tag{15}$$

Assuming a weak dependence of $\chi\left[{}^{(3)}\mathcal{G}, \Phi\right]$ on ${}^{(3)}\mathcal{G}$ and neglecting the quantum potential for gravity, from Equations (11) and (12) one obtains the quantum equation

$$\frac{i\hbar}{\kappa} \delta S \circ \delta \chi = \hat{\mathcal{H}}^{\Phi} \chi\,. \tag{16}$$

The four-geometry arises as a solution to the guidance Equations (6) and (8), with the quantum potential neglected, it respects the classical Einstein equations. The functional $\chi[t, \Phi] \equiv \chi\left[{}^{(3)}\mathcal{G}(t), \Phi\right]$ then evolves on the background of this four-geometry according to the usual Schrödinger's equation,

$$i\hbar\dot{\chi} = \int_{\Sigma} d^3x\, \mathcal{N}^{\mu} \hat{\mathcal{H}}^{\Phi}_{\mu}\, \chi\,, \tag{17}$$

which follows from (3)–(6) and (16). This represents the equation of the quantum field theory on a classical geometric background, which naturally arises in a semiclassical limit of the pilot-wave geometrodynamics. We note that the local physical time is related to the coordinate time t as $dt_{\text{phys}} = \mathcal{N}(t)dt$.

Thus far, we have been dealing with bosonic fields, Φ. Fermions with their unusual statistics require different treatments in the pilot-wave theory. Such a complete theory can be constructed in many ways, choosing different entities as "beables". One choice consists in treating fermions as point particles and demanding that the arising wave function be antisymmetric with respect to the permutation of the coordinates and spin indices of identical fermions. In relativistic quantum mechanics, there arises the problem of the negative-energy states of particles. A possible resolution of this problem in the pilot-wave theory is based on Dirac's idea that all such energy states are occupied [7,16]. The wave function, in this case, should have an infinite number of arguments for each fermionic specie, and this number will be countable assuming the spatial geometry is compact, for example, topologically a three-sphere. This, then, will lead us to wave functions of the form [16]

$$\Psi_{\alpha_1 \ldots \alpha_n \ldots}\left[{}^{(3)}\mathcal{G},\, \Phi,\, x_1,\, \ldots,\, x_n,\, \ldots\right], \tag{18}$$

which are also multispinors with a countable set of particle arguments x_i and corresponding spinor indices α_i. The wave function will presumably be a solution to a constraint equation of the form

$$\left(\hat{\mathcal{H}} + \sum_n \hat{\mathcal{H}}_D^{(n)} + \hat{\mathcal{H}}_{\text{int}}\right)\Psi = 0\,, \tag{19}$$

where $\hat{\mathcal{H}}$ is the Hamiltonian operator constraint for the bosonic fields, acting on the field arguments of the wave function, $\hat{\mathcal{H}}_D^{(n)}$ is the Dirac Hamiltonian constraint acting on the particle coordinate x_n and on the corresponding spinor index α_n in (18), and $\hat{\mathcal{H}}_{\text{int}}$ is the interaction Hamiltonian acting both on the field variables and on the particle coordinates. The interactions will enable particle transitions between negative-energy and positive-energy levels, which will describe the creation and annihilation of fermionic pairs. The pilot-wave guidance condition for a Dirac particle is given by [7,16]

$$\frac{dx_n^a}{dt} = \frac{\Psi^\dagger \left(\gamma^0 \gamma^a\right)_n \Psi}{\Psi^\dagger \Psi}\,, \tag{20}$$

where Dirac's gamma matrices, γ^μ, act on the corresponding spinor index, α_n.

5. Cosmological Implications

The fact that the pilot wave theory describes causal evolution on all scales has important implications for cosmology and gravitational physics. First, there is no issue of a quantum-to-classical transition and collapse of the wave function in the pilot-wave inflationary quantum cosmology. The primordial "quantum fluctuations" evolve in a causal deterministic way, similarly to classical configurations of fields all over the history of the universe, and the field configuration in the universe can be assumed to be "lumpy" from the very beginning even though its wave function is symmetric with respect to spatial translations. Inhomogeneities become classical once their corresponding quantum potential (the last term in (11) in the case of metric field) becomes small. Observationally, we deal with only one realisation of the cosmological evolution, and its statistical properties should be interpreted in the ergodic sense, as averages over large spatial regions. This circumstance, I believe, allows one to assert that the pilot-wave theory makes the same predictions for cosmological observables as the usual quantum approach, contrary to some expectations in the literature (see [20]). The unique realisation of the pilot-wave cosmological history is the instance of cosmic variance. While its origin in the usual quantum cosmology remains to be mysterious and is connected with the notion of collapse of wave function or with many-worlds interpretation, in the pilot-wave quantum cosmology it arises most naturally as just a given deterministic realisation.

Secondly, since the pilot-wave evolution is nonclassical in the quantum domain of the wave function (in which the quantum potential term in (11) is important), this may, in some cases, prevent such a universe from forming cosmological singularities. For these and related features of the pilot-wave cosmology, see [18,21].

6. Origin of Quantum Randomness

6.1. The Problem of Quantum Equilibrium

In the pilot-wave theory, the wave function (18) describes a single object, the universe, and the fields and particles in this universe are piloted according to deterministic laws. As we have pointed out, on very large scales, such a universe does not allow for any fundamental statistical description. Indeed, in the pilot-wave theory, the wave function of the universe is devoid of any probabilistic meaning; in particular, it need not be normalisable or be a member of any Hilbert space. Its role in the formalism is quite similar to the role of the Jacobi functional, S, in classical theory: it just pilots the configuration variables.

However, on smaller scales and in microphysical experiments, we deal with unpredictability of the results of measurements and with quantum probabilities. Thus, the origin of such quantum probabilities in the cosmological framework of the pilot-wave theory begs for explanation. This will be the topic of the present section.

The evolution guided by the universal wave function (18) will occasionally lead to the formation of naturally or artificially prepared quantum ensembles of identical systems piloted by the same wave function. That is where the laws of the usual quantum mechanics start to apply, and we should understand why and how this happens in the present formalism. In the pilot-wave theory, the measurement process is treated as a partial case of the generic deterministic evolution [7]. The probabilistic character of the measurement outcomes is caused by a random distribution of the actual (initial) values of the microscopic particle and field configurations in each system of an ensemble, as well as in the measuring apparatus.

Consider a system with configuration variables x_S and a typical ideal measurement of an observable Λ with discrete eigenvalues Λ_n, and the corresponding normalised eigenstates $\psi_n(x_S)$. Before the measurement, the measuring apparatus, described by configuration variables x_A, has wave function $\phi(x_A)$. Thus, the initial total wave function is

$$\Psi_i(x_S, x_A) = \psi(x_S)\phi(x_A) = \sum_n c_n \psi_n(x_S)\phi(x_A) \,. \tag{21}$$

Interaction between the measuring apparatus and the system causes evolution of the total wave function into

$$\Psi_f(x_S, x_A) = \sum_n c_n \psi_n(x_S)\phi_n(x_A) \,. \tag{22}$$

The measurement is efficient if the normalised states ϕ_n of the measuring apparatus are non-overlapping and macroscopically distinct. According to the pilot-wave theory, the configuration variables X_S and X_A, respectively, of the system and apparatus, evolving in a deterministic way, eventually get into a localisation region of only one of the states ϕ_n. The actual position of the configuration variables X_S and X_A uniquely determines the macroscopic state of the apparatus, hence, the measurement outcome in each particular experiment. This provides a solution to the "measurement problem" in the pilot-wave theory.

According to the established laws of quantum mechanics, the probability with which the result Λ_n appears in an ensemble of measurements with the initial wave function (21) is given by $p_n = |c_n|^2$. In order that this result be valid in the pilot wave theory, it is necessary to assume that the configuration variables x in a quantum ensemble with wave function $\psi(x)$ are distributed as $p(x) = |\psi(x)|^2$. This condition is termed "quantum equilibrium" [20,22–25]: as a consequence of the Schrödinger equation, it is preserved in time for a closed system. In the the pilot-wave theory, the origin of such a distribution begs for explanation. In this section, we will consider this important question (for a review of different approaches to this problem, see [26]).

In his pioneering works [5,6,27], Bohm provided a qualitative solution to this problem (see also [7]). He argued that quantum equilibrium will be established by the complicated motions of interacting particles. If $p(x)$ is the real particle distribution, then it is easy to show that the ratio $p(x)/|\psi(x)|^2$ is conserved along the quantum pilot-wave trajectories. Bohm then conjectured that the complicated mixing character of the pilot-wave dynamics will cause the coarse-grained value $\overline{p}(x)/\overline{|\psi(x)|^2}$ to approach unity, thus establishing the quantum equilibrium.

A quantitative justification of Bohm's conjecture was proposed by Valentini [20,22–24] by introduction of the quantity $H = -\int \overline{p} \log\left(\overline{p}/\overline{|\psi|^2}\right) dx$ called "subquantum entropy." By analogy with Boltzmann's H-theorem in statistical mechanics, it is suggested in [20,22–24] that the "subquantum entropy" increases in time approaching its maximum value of zero, thereby establishing the coarse-grained quantum equilibrium, $\overline{p} = \overline{|\psi|^2}$. This is substantiated by the "subquantum H-theorem" [20,22–24] stating that the coarse-grained "entropy" H reaches its local minimum under the conditions of "no fine-grained microstructure": $p = \overline{p}$ and $|\psi|^2 = \overline{|\psi|^2}$. Obviously, some specification of the property for a system to be sufficiently "complicated" is required here.

A different solution to the quantum equilibrium problem was proposed by Dürr, Goldstein and Zanghì [25]. Here, the key role is played by the notion of *typicality* applied to measurable subsets of the configuration space of a closed system. The measure density of typicality is taken to be the equivariant expression $|\Psi|^2$ based on the system wave function. It is then shown that the set of initial conditions that produce the usual quantum-mechanical statistical outcomes (to certain precision) of various measurements in the history of this system has measure of typicality close to one. A drawback of this solution is the specific dependence of the typicality measure on the wave function itself, which makes the argument look circular [20]. Indeed, as noted by the authors of this approach (Section 7 [25]), a different choice of the typicality measure in the space of initial conditions would result in different predicted probability distributions.

6.2. Ergodicity Argument

In [28], we justified the quantum equilibrium hypothesis of the pilot-wave dynamics using the ideas of ergodic theory [29–31]. Similarly to some approaches to the classical statistical mechanics [32], this theory represents the equilibrium ensemble averages of various functions of dynamical variables by their time averages. The dynamical space of a system in this theory is regarded as a measure space, and its temporal evolution as a one-parameter group of measure-preserving transformations. A dynamical system is called ergodic if the measure of any subset which is left invariant by all these transformations, or of its complement set, is equal to zero. A central role in the ergodic theory belongs to the Birkhoff–Khinchin theorem, which states that, for almost all initial conditions, the fraction of time spent by an ergodic system in a measurable region of its dynamical variables is proportional to the invariant measure of this region. In classical statistical mechanics, one is talking about the Liouville measure in the phase space, and the assumption of ergodicity of the Hamiltonian dynamics can be used to justify the microcanonical equilibrium distribution.

Applying the ergodicity argument to the pilot-wave quantum dynamics, we can assume that this dynamics is ergodic with respect to the measure density $|\Psi|^2$. One of the difficulties of this approach is that $|\Psi|^2$ is, in general, time-dependent [24] (p. 40). This difficulty is removed if we restrict ourselves to systems in stationary quantum states. Incidentally, this is the case with the universal wave function (18), which, as we have seen in the previous section, does not depend on time. A sufficiently "disentangled" large subsystem of such a universe will also be stationary, and the ergodicity argument can be applied to it as well.

We then follow the reasoning of [28] with a slight improvement. For a system in a stationary ergodic state, the pilot-wave dynamics preserves the measure with density $|\Psi|^2$, and the average time spent in any configuration region is given by the measure of that region with measure density $|\Psi|^2$, as required. Note that the ergodicity property does not single out a specific measure, it depends only on the equivalence class of measures. (Two measures with common domain are said to be equivalent if they have common system of sets of measure zero.) However, the invariant measure with density $|\Psi|^2$ is unique (modulo normalisation) for an ergodic system, which makes it relevant to the objective probabilities.

Consider in more detail the process of preparation for a subsystem and the emergence of time-dependent wave functions. We split the configuration variables $z = (x, y)$ of the total system into the coordinates x of the subsystem of interest and the coordinates y of the environment. For the x subsystem, we can introduce the conditional time-dependent wave function (see [25]) $\psi_t(x) = \Psi(x, Y(t))$, where $Y(t)$ is the actual pilot-wave dynamics of the environment variables. The x subsystem is "well prepared" in the domain Ω, if its pilot-wave dynamics in this domain does not affect that of the environment; in this case, the conditional wave function $\psi_t(x)$ is its effective wave function in this domain. (For example, in an experiment where electrons are diffracted by a narrow slit, the domain Ω may be the space behind the slit.) Suppose that this takes place in some small time interval containing t_0 and that the effective wave function $\psi(x) \approx \psi_{t_0}(x)$, $x \in \Omega$, in this time interval is sufficiently closely approximated by $\Psi(x, Y)$ when $Y \in \Gamma$.

Then, whenever the variable Y evolves in the domain Γ, the variable X is piloted in the domain Ω by the wave function $\psi(x)$. Under the condition that Y is in the domain Γ, the chance of finding X in a domain $\omega \subset \Omega$ is given by the time ratio

$$\mathbb{P}(X \in \omega \,|\, Y \in \Gamma) = \lim_{T\to\infty} \frac{\int_0^T \chi_{\omega\times\Gamma}(Z(t))\, dt}{\int_0^T \chi_{\Omega\times\Gamma}(Z(t))\, dt}, \tag{23}$$

where by χ_M we denote the characteristic function of a set M. Since the total system piloted by $\Psi(z)$ is assumed to be ergodic, the Birkhoff–Khinchin ergodic theorem [29–31] ensures the existence of the limit in Equation (23) for almost all initial values of Z, with the result

$$\mathbb{P}(X \in \omega \,|\, Y \in \Gamma) = \frac{\mu_\Psi(\omega \times \Gamma)}{\mu_\Psi(\Omega \times \Gamma)} = \mu_\psi(\omega)\,, \tag{24}$$

where μ_Ψ and μ_ψ are the measures in the domains of z and of x, respectively, with measure densities given by the corresponding normalised wave functions. Equality (24) constitutes the justification of the standard quantum probabilities.

We have already noted that the measure density $|\Psi|^2$ of the ergodic approach is objectively singled out on the ground of its invariance. We should only comment on the Lebesgue integration measure for the time parameter t in Equation (23). This parameter encodes the time translation symmetry of the quantum dynamics and can be identified with local physical time. The Lebesgue integration measure in Equation (23) is then the only natural measure in the context of stationarity.

Certainly, there are quantum states that do not exhibit ergodic pilot-wave evolution; such are, for example, states with real wave functions, for which the phase $S = 0$, and the pilot-wave dynamics is trivial. Therefore, ergodicity should be regarded as our specification of complexity of quantum states. The conditions under which the pilot-wave evolution becomes ergodic can be studied in various particular cases. Recent progress in this direction can be found in [33], and some simple examples are given in [28].

The ergodic approach implies that quantum equilibrium for a whole quantum system is reached on the recurrence timescale of this system. In the case of a macroscopically large system, this time is usually astronomically large. However, since we do not make quantum experiments with large systems, it is sufficient to assume ergodicity to take place for small enough quantum subsystems evolving on small timescales. It looks plausible that conditions of this sort do really occur in nature.

7. Discussion

Pilot-wave interpretation of the "wave function of the universe" produces deterministic quantum geometry of spacetime in the form of foliation (13). Unlike in classical general relativity, evolution of this geometry is governed by a functional S obeying the quantum Hamilton–Jacobi Equation (11) with a nonlocal quantum potential. As a result, its foliation is distinguished by the quantum pilot-wave dynamics: formally changing the foliation results in a new four-geometry that cannot be obtained as a solution in the pilot-wave theory. This is the precise meaning in which the quantum pilot-wave dynamics is nonlocal. This nonlocality disappears in the classical limit, in which the effect of the quantum potential is negligible.

It is remarkable that, in the pilot-wave quantum theory, the wave function of the "universe as a whole" need not be regarded as a member of a Hilbert space, in particular, it does not require square-integrability with respect to all its arguments (see also the general discussion in [7]). The only role of the wave function in this interpretation is to provide the time evolution for the particle and field configuration variables, which does not require its normalisation. However, this same fact calls for a justification of the "quantum equilibrium" hypothesis for subsystems, i.e., the property that their configuration variables are distributed according to $|\psi|^2$ in pure quantum ensembles. We have reviewed several approaches to these problems and have shown how such a justification can be given by an appeal to ergodicity of quantum dynamics.

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