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On Geometric Interpretations of Euler's Substitutions

Jan L. Cieśliński * and Maciej Jurgielewicz

Faculty of Physics, University of Bialystok, 15-245 Białystok, Poland
* Correspondence: j.cieslinski@uwb.edu.pl

Abstract: We consider a classical case of integrals containing an irrational integrand in the form of a square root of a quadratic polynomial. It is known that such "irrational integrals" can be expressed in terms of elementary functions by one of three of Euler's substitutions. It is less well known that the Euler substitutions have a geometric interpretation. In the framework of this interpretation, one can see that the number 3 is not the most suitable. We show that it is natural to introduce a fourth Euler substitution. In his original treatise, Leonhard Euler used two substitutions which are sufficient to cover all cases.

Keywords: integral calculus; irrational integrals; conics; rational parameterization; fourth Euler's substitution

1. Introduction

Integrals of rational functions can be expressed in terms of elementary functions. Therefore, a natural method of integration consists of using suitable substitutions and integration by parts to reduce our problem to integration of rational functions.

In this paper, we consider irrational integrals containing the quadratic root of a quadratic polynomial, i.e., integrals of the form

$$\int R(x,y)dx\,, \tag{1}$$

where R is a rational fuction (a quotient of two polynomials) of x and y, and

$$y = \sqrt{ax^2 + bx + c}\,. \tag{2}$$

The subject is, in principle, known. A standard method to deal with such integrals consists of using one of the so-called Euler's substitutions [1–3]. However, there are some details which need to be clarified. We will describe in detail a geometric approach to this problem and explain how many Euler substitutions actually exist.

In fact, to the best of our knowledge, all sources and textbooks mention exactly three types of substitutions in this context. It is not clear who was the first to introduce such classification. Leonhard Euler himself used only two of these substitutions (which is sufficient to cover all cases). Three Euler substitutions are usually introduced and discussed in Russian sources; see, e.g., [4–6] (Leonhard Euler, although of Swiss origin, lived and worked in Saint Petersburg for many years). Surprisingly enough, the three substitutions appeared in an old textbook, published in 1892 by a Harvard professor, William E. Byerly [7], without any reference to Euler.

In our paper, we present a clear geometric intepretation of this problem, shortly mentioned in some sources, mainly of Russian origin [2,8]. The textbook [8] is not translated into English. Another book by the same author [4] does not mention this geometric approach in the section on Euler's substitutions.

The main novelty of this paper is the introduction of the fourth Euler substitution, which is a natural consequence of the geometric approach discussed in our paper.

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2. Three Classical Euler's Substitutions

The main idea of Euler's substitutions consists of expressing $\sqrt{ax^2+bx+c}$ as a linear function of x and a new parameter t in such a way that the resulting equation is linear with respect to x. In this paper, we use the most common numbering of these three substitutions, compare [1,2,4,5,7]. In some sources, a different order is used; see [6,9,10].

2.1. First Euler Substitution

This substitution can be done only in the case $a > 0$:

$$\sqrt{ax^2+bx+c} = \pm x\sqrt{a} + t\,. \tag{3}$$

Squaring both sides we get:

$$ax^2+bx+c = ax^2 \pm 2xt\sqrt{a} + t^2\,.$$

Terms quadratic in x cancel out and the resulting equation is linear in x. Computing x, we get a rational dependence on t:

$$x = \frac{t^2-c}{b \mp 2t\sqrt{a}}\,. \tag{4}$$

Then, from (2) and (3), we get

$$y = \frac{\mp t^2\sqrt{a} + tb \mp c\sqrt{a}}{b \mp 2t\sqrt{a}}\,. \tag{5}$$

2.2. Second Euler Substitution

This substitution can be done only in the case $c > 0$:

$$\sqrt{ax^2+bx+c} = xt \pm \sqrt{c}\,. \tag{6}$$

Squaring both sides we get:

$$ax^2+bx+c = x^2t^2 \pm 2xt\sqrt{c} + c\,. \tag{7}$$

The constant c cancels out and dividing both sides by x we again derive an equation linear in x. Hence, similarly as in the previous case,

$$x = \frac{b \mp 2t\sqrt{c}}{t^2-a}\,, \qquad y = \frac{bt \mp (t^2+a)\sqrt{c}}{t^2-a}\,. \tag{8}$$

2.3. Third Euler Substitution

This substitution can be done only in the case $\Delta > 0$, where

$$\Delta \equiv b^2 - 4ac \tag{9}$$

is the discriminant of the quadratic polynomial. Then the polynomial has two distinct real roots x_1 and x_2, and the third Euler substitution is given by:

$$\sqrt{ax^2+bx+c} = (x-x_1)t\,. \tag{10}$$

Squaring both sides we get:

$$a(x-x_1)(x-x_2) = (x-x_1)^2t^2 \qquad \Rightarrow \qquad a(x-x_2) = (x-x_1)t^2\,. \tag{11}$$

Computing x from the resulting equation and then using (10) and (2) we obtain

$$x = \frac{t^2 x_1 - a x_2}{t^2 - a}\,, \qquad y = \frac{(x_1 - x_2)at}{t^2 - a}\,, \tag{12}$$

where, of course,

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,. \tag{13}$$

2.4. Original Euler's Approach

It is interesting that Leonhard Euler himself, in his famous monograph, used only two of these substitutions, see [11]. He considered two cases: $\Delta > 0$ and $\Delta < 0$. In the first case ($\Delta > 0$) he proposed the substitution (6), while in the second case ($\Delta < 0$) he proposed the substitution (3) in a slightly modified form:

$$\sqrt{ax^2 + bx + c} = x\sqrt{a} - t\sqrt{c}\,. \tag{14}$$

Obviously, the case $\Delta = 0$ is not included because then the quadratic polynomial is a square of the linear function in x and y is linear is x as well. Hence, the integrand in (1) is rational in x from the very beginning.

3. Geometric Interpretation

It is convenient to square both sides of (2) resulting in the equation of a quadratic curve

$$y^2 = ax^2 + bx + c\,. \tag{15}$$

We will denote this curve (a conic section) by Q_{abc}, i.e., $(x, y) \in Q_{abc}$.

3.1. Elliptic Case: $a < 0$

The canonical form of the quadratic polynomial yields:

$$y^2 + |a|\left(x - \frac{b}{2|a|}\right)^2 = c - \frac{b^2}{4a}\,. \tag{16}$$

We can distinguish three cases, depending on the sign of the discriminant Δ:

$$\Delta < 0 \quad \Longrightarrow \quad Q_{abc} = \emptyset \quad \text{(empty set)}\,, \tag{17}$$

$$\Delta = 0 \quad \Longrightarrow \quad Q_{abc} = \left\{\left(\frac{b}{2|a|}, 0\right)\right\} \quad \text{(single point)}\,, \tag{18}$$

$$\Delta > 0 \quad \Longrightarrow \quad Q_{abc} \text{ is an ellipse}\,. \tag{19}$$

Only in the last case, we get a non-degenerated quadratic curve.

3.2. Parabolic Case: $a = 0$

For $a = 0$ (and $b \neq 0$) the conic Q_{abc} is a parabola with the symmetry axis $y = 0$.

3.3. Hyperbolic Case: $a > 0$

The canonical form of the quadratic polynomial yields:

$$y^2 - a\left(x + \frac{b}{2a}\right)^2 = c - \frac{b^2}{4a}\,. \tag{20}$$

We can distinguish three cases, depending on the sign of the discriminant Δ:

$$\Delta < 0 \quad \Longrightarrow \quad Q_{abc} \text{ is a hyperbola with vertices at the line } x = -\frac{b}{2a}\,, \tag{21}$$

$$\Delta = 0 \quad \Longrightarrow \quad Q_{abc} \text{ is a pair of intersection lines}\,, \tag{22}$$

$$\Delta > 0 \quad \Longrightarrow \quad Q_{abc} \text{ is a hyperbola with vertices at } x \text{ axis}\,. \tag{23}$$

Therefore, for $\Delta \neq 0$ we get a non-degenerated quadratic curve.

3.4. Rational Parameterization: Standard Approach

The key idea leading to a rational parameterization consists of fixing an arbitrary point $P_0 = (x_0, y_0)$ on the conic Q_{abc} and assigning to any other point $P = (x, y)$ of this conic the line P_0P. Taking as a parameter t the slope of this line, we obtain a rational parameterization of the conic Q_{abc} [2,8]. Thus, we have the system of three equations:

$$\begin{aligned} &y^2 = ax^2 + bx + c\,, \\ &y_0^2 = ax_0^2 + bx_0 + c\,, \\ &y - y_0 = t(x - x_0)\,. \end{aligned} \tag{24}$$

The points (x, y) and (x_0, y_0) belong to the conic Q_{abc} and t is the slope of the straight line passing through (x, y) and (x_0, y_0). Subtracting the second equation from the first one we get:

$$\begin{aligned} &(y - y_0)(y + y_0) = (x - x_0)(a(x + x_0) + b)\,, \\ &y_0^2 = ax_0^2 + bx_0 + c\,, \\ &y - y_0 = t(x - x_0)\,. \end{aligned} \tag{25}$$

Substituting the last equation into the first one we obtain:

$$\begin{aligned} &(t(y + y_0) - a(x + x_0) - b)(x - x_0) = 0\,, \\ &y_0^2 = ax_0^2 + bx_0 + c\,, \\ &y - y_0 = t(x - x_0)\,. \end{aligned} \tag{26}$$

Assuming $x \neq x_0$, we get

$$\begin{aligned} &t(y + y_0) = a(x + x_0) + b\,, \\ &y_0^2 = ax_0^2 + bx_0 + c\,, \\ &y - y_0 = t(x - x_0)\,. \end{aligned} \tag{27}$$

Now, the first and the last equation form a system of two linear equations for two variables x, y, which can be solved in the standard way. As a result, we obtain:

$$\begin{aligned} x &= \frac{x_0t^2 - 2y_0t + ax_0 + b}{t^2 - a}\,, \\ y &= \frac{-y_0t^2 + (2ax_0 + b)t - ay_0}{t^2 - a}\,, \end{aligned} \tag{28}$$

which means that we expressed x and y as rational functions of the parameter t.

Corollary 1. *There are many Euler-like substitutions. Each of them is determined by the choice of x_0, provided that $ax_0^2 + bx_0 + c \geqslant 0$. Then the point $P_0 \equiv (x_0, y_0)$ is given by:*

$$P_0 = \left(x_0, \pm\sqrt{ax_0^2 + bx_0 + c} \right), \tag{29}$$

and other points $P = (x, y) \in Q_{abc}$ *are parameterized by* (28).

In particular, the second Euler substitution corresponds to $x_0 = 0$ (provided that the graph of the quadric Q_{abc} intersects the axis y), see Figures 1 and 2. The third Euler substitution corresponds to x_0 being a root of the polynomial $ax^2 + bx + c$ (provided that the graph of Q_{abc} intersects the axis x), see Figures 3 and 4.

The first Euler substitution apparently does not fit this picture. However, its geometric interpretation is even simpler and more evident. The Formula (3) describes the family of lines parallel to asymptotes of the corresponding hyperbola, see Figure 5. We may treat it as a special case of (28) when the point (x_0, y_0) lies at a very large number. Note that points $(x_0, \pm x_0\sqrt{a})$ belong to the conic (15) in the limit for $x_0 \to \infty$.

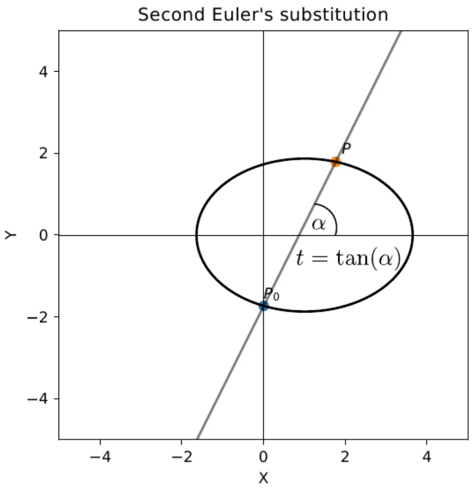

Figure 1. Geometric interpretation of the second Euler substitution in the case $a < 0$ and $c > 0$. The point P is parameterized by the slope t of the line P_0P.

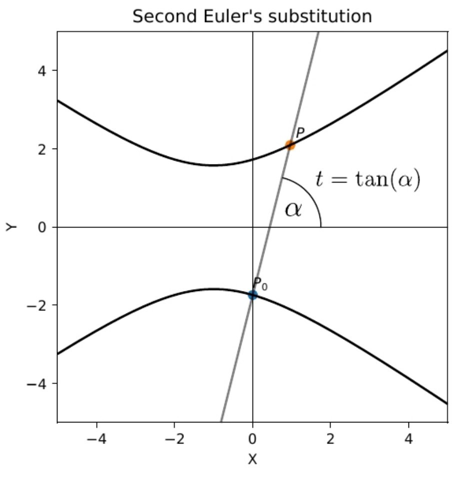

Figure 2. Geometric interpretation of the second Euler substitution in the case $a > 0$ and $c > 0$. The point P is parameterized by the slope t of the line P_0P.

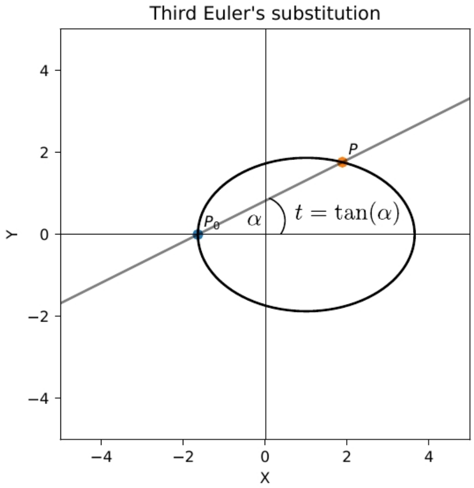

Figure 3. Geometric interpretation of the third Euler substitution in the case $a < 0$ and $\Delta > 0$. The point P is parameterized by the slope t of the line P_0P.

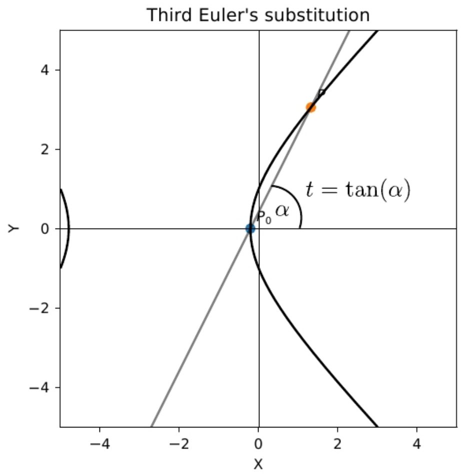

Figure 4. Geometric interpretation of the third Euler substitution in the case $a > 0$. The point P is parameterized by the slope t of the line P_0P.

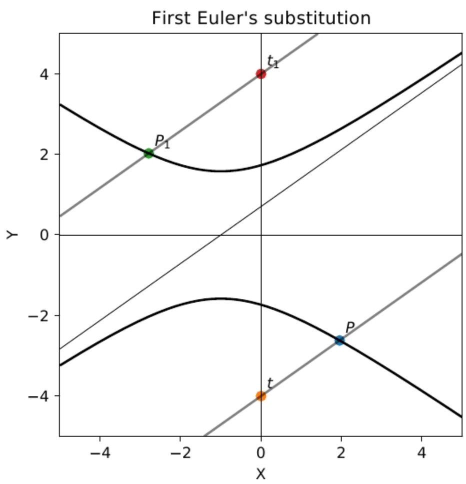

Figure 5. Geometric interpretation of the first Euler substitution. The points P and P_1 are parameterized by intersections t and t_1, respectively, of the y-axis with the line parallel to one of the asymptotes of the hyperbola $y^2 = ax^2 + bx + c$.

4. New Insights from the Geometric Interpretation

The description given in the previous section is more or less known (see, e.g., [2,8]), although we are not aware of any reference containing all these details. We are going to derive from this geometric picture more quite interesting consequences.

First of all, we identify characteristic points on the graph of a quadratic curve which can be chosen as P_0 in the most natural way: vertices (M_1, M_2, R_1, R_2) and intersections with coordinate axes (R_1, R_2, V_1, V_2); see Figures 6 and 7.

In particular, in the case of the second Euler substitution, $P_0 = V_2$ (see Figures 1 and 2) or $P_0 = V_1$, while in the case of the third Euler substitution $P_0 = R_1$ (see Figure 3) or $P_0 = R_2$ (see Figure 4). The first Euler substitution is related to P_0.

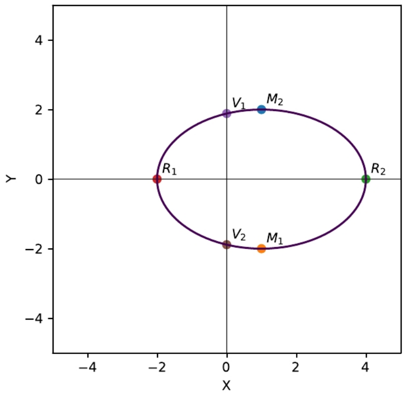

Figure 6. Characteristic points on the graph of an ellipse: intersections with the coordinate axes (provided that they exist) and extremes (minimum M_1 and maximum M_2).

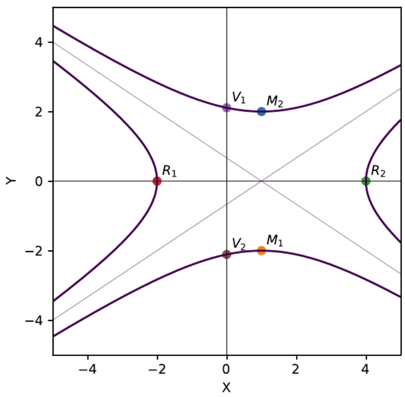

Figure 7. Characteristic points on the graphs of hyperbolas (two hyperbolas with the same $|a|$ are presented): intersections with the coordinate axes (V_1, V_2, R_1, R_2) and extremes (M_1, M_2).

4.1. Fourth Euler's Substitution

The geometric approach presented above includes all three classical Euler's substitutions, but it is still missing vertices M_1 and M_2. Therefore, it is natural to introduce another (fourth) Euler's substitution, geometrically related to missing vertices: $P_0 = M_1$ (see Figures 8 and 9) or $P_0 = M_2$.

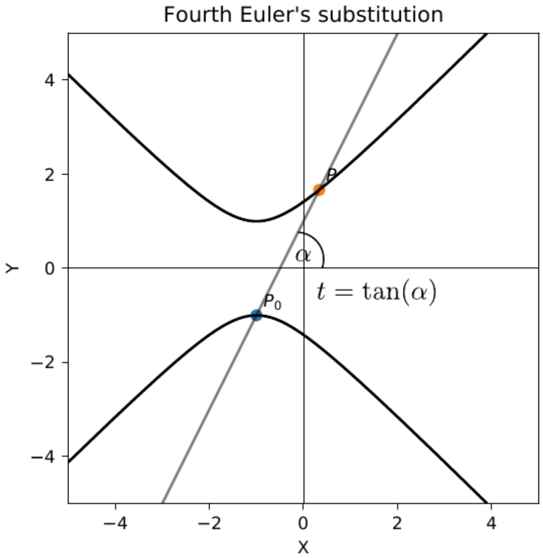

Figure 8. Geometric interpretation of the fourth Euler substitution in the case $a > 0$. The point P is parameterized by the slope t of the line P_0P, where $P_0 = M_1$.

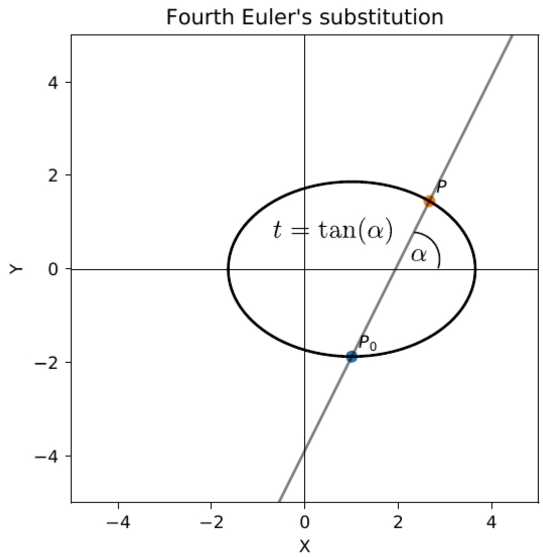

Figure 9. Geometric interpretation of the fourth Euler substitution in the case $a < 0$. The point P is parameterized by the slope t of the line P_0P, where $P_0 = M_1$.

The algebraic description of the fourth Euler substitution is based on the canonical form of the quadratic polynomial:

$$y = \sqrt{a(x-p)^2 + q} \tag{30}$$

where

$$p = -\frac{b}{2a}\,, \qquad q = -\frac{\Delta}{4a}\,. \tag{31}$$

The fourth Euler substitution is defined by:

$$y = \sqrt{q} + (x-p)t\,. \tag{32}$$

Squaring both sides we get:

$$a(x-p)^2 + q = q + 2(x-p)t\sqrt{q} + (x-p)^2t^2. \tag{33}$$

The constant q cancels out and dividing both sides by $x - p$, we obtain

$$a(x-p) = 2t\sqrt{q} + (x-p)t^2, \tag{34}$$

which is linear in x. Hence

$$x - p = \frac{2t\sqrt{q}}{a - t^2}, \tag{35}$$

and using (32) we get

$$y = \frac{a+t^2}{a-t^2}\sqrt{q}\,. \tag{36}$$

Thus we have a rational dependence of x and y on the parameter t. Moreover,

$$\frac{dx}{dt} = \frac{2(a+t^2)\sqrt{q}}{(a-t^2)^2}\,, \tag{37}$$

and we can easily transform the irrational integral function (1) into an integral function rational with respect to t.

4.2. Simplifying Euler's First Substitution

A geometric approach suggests some modifications or new variants of the existing rational parameterizations. Introducing a new parameter τ

$$\tau = b \mp 2t\sqrt{a} \quad \Longrightarrow \quad t = \mp\frac{(\tau - b)}{2\sqrt{a}}\,, \tag{38}$$

and substituting it into (4) and (5), we obtain the following simplification of the first Euler substitution:

$$x = \frac{1}{4a}\left(\tau + \frac{\Delta}{\tau} - 2b\right), \qquad y = \mp\frac{1}{4\sqrt{a}}\left(\tau - \frac{\Delta}{\tau}\right). \tag{39}$$

Geometrically, the parameter t is related to intersections with the y axis (compare Figure 5), while the parameter τ is related to intersections with the vertical symmetry axis (i.e., the line $x = p$). Indeed, the parameter $\tau = 0$ corresponds to the line passing through the point $(p, 0)$ and this is one of two asymptotes (that is why $x \to \infty$ and $y \to \infty$ for $\tau \to 0$).

4.3. Euler's First Substitution as a Limit of the Generic Case

We are going to show that the first Euler substitution can be derived from the generic case (28) by taking a suitable limit $x_0 \to \infty$ and $y_0 \to \infty$. We consider the pencil of lines $y - y_0 = t(x - x_0)$ (compare (24)) but as a parameter we take the ordinate τ of the intersection of the line P_0P with the y-axis (i.e., $y = \tau$ for $x = 0$). Hence

$$t = \frac{y_0 - \tau}{x_0}\,. \tag{40}$$

This change of variable, replacing t with τ, works for any $x_0 \neq 0$. Therefore, the second Euler substitution (related to the case $x_0 = 0$) is excluded.

Substituting (40) into (28) we obtain

$$x = \frac{(\tau^2 - c)x_0}{bx_0 + c + \tau^2 - 2\tau y_0}\,, \qquad y = \frac{\tau(bx_0 + 2c) - (c + \tau^2)y_0}{bx_0 + c + \tau^2 - 2\tau y_0}\,. \tag{41}$$

Assuming $a > 0$ and taking into account $y_0^2 = ax_0^2 + bx_0 + c$ we take the limit $|x_0| \to \infty$. Hence $y_0 \to \pm x_0\sqrt{a}$ and

$$x \underset{|x_0|\to\infty}{\longrightarrow} \frac{\tau^2 - c}{b \mp 2\tau\sqrt{a}}\,, \qquad y \underset{|x_0|\to\infty}{\longrightarrow} \frac{\tau b \mp (c + \tau^2)\sqrt{a}}{b \mp 2\tau\sqrt{a}}\,. \tag{42}$$

Comparing (42) with (4) and (5) we easily see that both solutions are identical, provided that we identify τ with t. Note, of course, that the t parameter given by (40) is different from the t parameter used in Section 2.1.

5. Euler's Substitutions versus Trigonometric Substitutions

Another popular method for computing irrational integrals (1) consists of making a suitable trigonometric or hyperbolic substitution. We use the canonical form of the quadratic curve (compare (30)):

$$y^2 = a(x-p)^2 + q \, . \tag{43}$$

Assuming $q \neq 0$ (otherwise y depends linearly on x) we introduce new variables ξ, η as folows:

$$\eta = \frac{y}{\sqrt{|q|}} \, , \qquad \xi = \frac{(x-p)\sqrt{|a|}}{\sqrt{|q|}} \, . \tag{44}$$

Then (43) becomes

$$\eta^2 = (\operatorname{sgn} a)\, \xi^2 + \operatorname{sgn} q \, , \tag{45}$$

because $a/|a| = \operatorname{sgn} a$, etc.

Thus we have three separate cases (in the fourth case –both signs negative– there are no real solutions), where trigonometric or hyperbolic substitutions are well known:

$$\begin{aligned} \eta &= \sqrt{\xi^2 - 1} \quad \Longrightarrow \quad \xi = \cosh\vartheta \, , \; \eta = \sinh\vartheta \, , \\ \eta &= \sqrt{1 - \xi^2} \quad \Longrightarrow \quad \xi = \cos\vartheta \, , \; \eta = \sin\vartheta \, , \\ \eta &= \sqrt{\xi^2 + 1} \quad \Longrightarrow \quad \xi = \sinh\vartheta \, , \; \eta = \cosh\vartheta \, . \end{aligned} \tag{46}$$

Is it better than Euler's substitutions? This is a matter of taste. Perhaps it is easier to memorize, however, one has to remember that integrals of trigonometric or hyperbolic functions have to be converted into integrals of rational functions by another substitution:

$$t = \tan\frac{\theta}{2} \quad \text{or} \quad t = \tanh\frac{\theta}{2} \, . \tag{47}$$

6. Conclusions

We presented and discussed a geometric approach to Euler substitutions. One consequence of this thorough discussion was the introduction of a fourth Euler substitution, in addition to three traditionally mentioned Euler substitutions. In fact, we can say that more (one parameter family) Euler-like substitutions exist and can be further modified or simplified by suitable linear or fractional linear transformations.

Surprisingly, the subject of constructing rational parametrization of algebraic curves (rationalizing roots) has recently become important in the context of Feynman integrals and computations in high energy particle physics [12,13]. Furthermore, Euler's substitutions were applied for reducing square roots in some mathematical finance calculations [14]. It would be interesting to appply, in those fields, some geometric ideas presented in this paper.

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References

1. Euler Substitution. *Wikipedia*. Available online: https://en.wikipedia.org/wiki/Euler_substitution (accessed on 8 August 2023).
2. Euler Substitutions. *Encyclopedia of Mathematics*. Available online: https://encyclopediaofmath.org/wiki/Euler_substitutions (accessed on 18 July 2023).
3. Radożycki, T. *Solving Problems in Mathematical Analysis*; Part I, Section 14.5; Springer: Cham, Switzerland, 2020.
4. Fikhtengol'ts, G.M. *The Fundamentals of Mathematical Analysis*; Chapter 10, Section 170; Pergamon Press: Oxford, UK, 1965; Volume 1.
5. Piskunov, N. *Differential and Integral Calculus*; Mir Publishers: Moscow, Russia, 1969.
6. Ryzhik, I.M.; Gradshteyn, I.S. *Table of Integrals, Sums, Series, and Products*; Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoj Literatury: Moscow, Russia, 1951. (In Russian)
7. Byerly W.E. *Elements of the Integral Calculus*; Ginn and Company: Boston, MA, USA, 1892.
8. Fikhtengol'ts, G.M. *Kurs Differentsial'nogo i Integral'nogo Ischisleniia*; Chapter 8, Section 282; Fizmatgiz: Moscow, Russia, 1959; Volume 2. (In Russian)
9. Lucht, P. The Euler Substitutions. Technical Report. Available online: https://www.researchgate.net/publication/309907150_The_Eu\protect\discretionary{\char\hyphenchar\font}{}{}ler_Substitutions (accessed on 18 July 2023).
10. Boyadzhiev, K.N. *Special Techniques for Solving Integrals: Examples and Problems*; World Scientific: Singapore, 2021.
11. Euler, L. *Institutiones Calculi Integralis*; Part 1, Section 1, Chapter 2; Bruce, I., Translator; Imperial Academy of Sciences: St. Petersburg, Russia, 1768; Volume 1. Available online: http://www.17centurymaths.com/ (accessed on 8 August 2023).
12. Besier, M.; van Straten, D.; Weinzierl, S. Rationalizing roots: An algorithmic approach. *Commun. Number Theory Phys.* **2019**, *13*, 253–297. [CrossRef]
13. Besier, M.; Festi, D. Rationalizability of square roots. *J. Symb. Comput.* **2021**, *106*, 48–67. [CrossRef]
14. Trinh, Y.T.; Hanzon, B. Option Pricing and CVA Calculations using the Monte Carlo-Tree (MC-Tree) Method. *arXiv* **2022**, arXiv:2202.00785.