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Uniform Convexity in Variable Exponent Sobolev Spaces

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Abstract: We prove the modular convexity of the mixed norm $L^p(\ell^2)$ on the Sobolev space $W^{1,p}(\Omega)$ in a domain $\Omega \subset \mathbb{R}^n$ under the sole assumption that the exponent $p(x)$ is bounded away from 1, i.e., we include the case $\sup\limits_{x\in\Omega} p(x) = \infty$. In particular, the mixed Sobolev norm is uniformly convex if $1 < \inf\limits_{x\in\Omega} p(x) \leq \sup\limits_{x\in\Omega} p(x) < \infty$ and $W_0^{1,p}(\Omega)$ is uniformly convex.

Keywords: fixed point; Fredholm equations; modular function spaces; variable exponent spaces

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1. Introduction

This work is devoted to the study of a uniform-convexity-like property of the modular $\rho_p : W^{1,p(x)}(\Omega) \to [0,\infty)$, defined as

$$\rho_p(u) = \int\limits_{\Omega} |\nabla u(x)|^{p(x)} dx, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $p : \Omega \to (1,\infty)$ is a measurable function and $|\mathbf{v}|$ stands for the Euclidean norm of a vector $\mathbf{v} \in \mathbb{R}^n$. We underline the fact that our work clarifies two distinct points that have so far not been covered in the literature: the consideration of the Euclidean norm $|\nabla u|$ in the Sobolev integral above for the full range $1 < \inf\limits_{\Omega} p$ and the inclusion of the case $\sup\limits_{\Omega} p = \infty$. As a by-product, we obtain the uniform convexity of the Sobolev–Luxemburg norm in the case where p is bounded away from 1 and ∞. To the best of the authors' knowledge, the uniform convexity of the Sobolev norm is new for $1 < \inf\limits_{\Omega} < p < 2$ ([1]). Introduced for the first time in the early 1930s, spaces of variable exponents acquired a new central role in mathematics after their emergence as the natural solution space for differential equations with non-standard growth. We refer the reader to [2,3] for a general treatment of variable exponent spaces and their basic properties. The consideration of the Euclidean norm in (1) is of particular importance since the corresponding Dirichlet integral is Fréchet differentiable and its derivative is precisely the p-Laplacian variable exponent $\Delta_{p(x)}$.

More specifically, let $\Omega \subset \mathbb{R}^n$ be a domain and p be a measurable function on Ω, where $1 \leq p < \infty$. It is well known that the Luxemburg norm $\|\cdot\|_p$ on the variable exponent space $L^p(\Omega)$ is uniformly convex if and only if $1 < p_- = \inf\limits_{x\in\Omega} p(x) \leq \sup\limits_{x\in\Omega} p(x) = p_+ < \infty$. Denote by $W^{m,p}(\Omega)$ the usual variable exponent Sobolev space. There are certainly infinitely many equivalent norms that can be defined on $W^{m,p}(\Omega)$. For example, if p is constant [4],

$$\|u\|_{1,p} = \left(\sum_{|\alpha|=0}^{m} \|D^{\alpha}u\|_p^p \right)^{\frac{1}{p}} \tag{2}$$

or [2,5]

$$\|u\|_{1,p} = \sum_{|\alpha|=0}^{m} \|D^{\alpha}u\|_p, \tag{3}$$

where $\|\cdot\|_p$ stands for the L^p norm. The core ideas can be reduced to the case $m = 1$; thus, in the following only the space $W^{1,p}(\Omega)$ will be considered. In general, if $|\cdot| : \mathbb{R}^n \to [0, \infty)$ is a norm on $\mathbb{R}^n$, the functional $\rho_{1,p} : W^{1,p}(\Omega) \to [0, \infty]$ defined by

$$\rho_{1,p}(u) = \int_{\Omega} |u|^p dx + \int_{\Omega} |\nabla u|^p dx \tag{4}$$

is a left-continuous convex modular, and the corresponding Luxemburg norm

$$\|u\|_{1,p} = \inf\left\{ \lambda > 0 : \rho_{1,p}\left(\frac{u}{\lambda}\right) \leq 1 \right\} \tag{5}$$

is a norm on $W^{1,p}(\Omega)$. Moreover, all such norms are equivalent. In particular, if p is constant on Ω, the functional $\|\cdot\| : W^{1,p}(\Omega) \to [0, \infty]$ defined by

$$\|u\| = \|u\|_p + \||\nabla u|\|_p \tag{6}$$

is a norm and all such norms on $W^{1,p}(\Omega)$ are topologically equivalent. The question of uniform convexity is considerably more delicate. It is easy to prove that if the variable exponent p is bounded away from 1 and ∞ on Ω, then (2) and (3) are uniformly convex. This can be seen, for example, by displaying a specific isometry between $W^{1,p}(\Omega)$ (furnished with either of those norms) and $L^p(\tilde{\Omega})$ for a suitable domain $\tilde{\Omega}$ ([2,4,5]). For a general norm of the type (5) (in particular (6)), however, the issue is more subtle. The case $|v| = \left(\sum_{j=1}^{n} v_j^2 \right)^{\frac{1}{2}}$ is of particular interest due to its applications in partial differential equations. In [1], it is shown that the Luxemburg norm on $W_0^{1,p}(\Omega)$ (defined as the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p}(\Omega)$) corresponding to the modular

$$\rho(u) = \int_{\Omega} \left(\sum_{j=1}^{n} |D^j u|^2 \right)^{\frac{p}{2}} dx$$

is uniformly convex on $W_0^{1,p}(\Omega)$ provided $2 \leq p \leq \sup_{\Omega} p(x) < \infty$ in Ω. On the other hand, it has been shown in [6] that under the sole condition $1 < p_-$, the *modular* $\rho_p : L^p(\Omega) \to [0, \infty]$ defined by

$$\rho_p(u) = \int_{\Omega} |u|^p \, dx$$

possesses a uniform-convexity-like property. This work aims at obtaining uniform convexity results in the case of an unbounded exponent, i.e., $p_+ = \infty$, for the Sobolev modular (4), in the particular case where the norm $|\cdot|$ is the Euclidean norm. As a by-product, we show that the Sobolev space $W_0^{1,p}(\Omega)$ is uniformly convex when furnished with the norm

$$\|u\|_{1,p} = \inf\left\{ \lambda > 0, \int_{\Omega} \left| \sum_{1}^{n} \left(\frac{D^j u}{\lambda} \right)^2 \right|^{\frac{p}{2}} dx \leq 1 \right\}.$$

2. Inequalities

In this section, vector Clarkson-type inequalities are proven. These inequalities will be of utmost importance in the following. First, the scalar case is discussed, for whose proof references are given.

Lemma 1. *For $a, b \in \mathbb{R}$, $|a| + |b| \neq 0$, $1 \leq p \leq 2$ ([7]):*

$$\left|\frac{a+b}{2}\right|^p + \frac{p(p-1)}{2^{p+1}} \frac{|a-b|^2}{(|a|+|b|)^{2-p}} \leq \frac{1}{2}(|a|^p + |b|p). \tag{7}$$

In addition, if $p \geq 2$ it holds ([8]):

$$\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \leq \frac{1}{2}(|a|^p + |b|^p). \tag{8}$$

Inequalities (7) and (8) in turn, imply their own validity in the complex case.

Lemma 2. *For $1 < p \leq 2$, $z_1 \in \mathbb{C}$, $z_2 \in \mathbb{C}$, $|z_1|^2 + |z_2|^2 \neq 0$, it holds*

$$\left|\frac{z_1+z_2}{2}\right|^p + \frac{p(p-1)}{2^{p+1}} \frac{|z_1-z_2|^2}{(|z_1|^2+|z_2|^2)^{\frac{2-p}{2}}} \leq \frac{1}{2}(|z_1|^p + |z_2|^p). \tag{9}$$

In addition, if $p \geq 2$, one has, for any two complex numbers z_1 and z_2,

$$\left|\frac{z_1+z_2}{2}\right|^p + \left|\frac{z_1-z_2}{2}\right|^p \leq \frac{1}{2}(|z_1|^p + |z_2|^p). \tag{10}$$

Proof. Let us first focus on the case $1 < p \leq 2$. Before we prove the inequality (9), we will need the following estimate

$$0 \leq (p-1)^{\frac{2}{2-p}} < e^{-2}, \tag{11}$$

for $1 \leq p < 2$. Set $g(p) = (p-1)^{\frac{2}{2-p}}$. It is easy to show that $g(1) = 0$ and that $g(p) \to e^{-2}$ when $p \to 2^-$. Furthermore,

$$g'(p) = 2(p-1)^{\frac{p}{2-p}}(2-p)^{-2}(2-p+(p-1)\ln(p-1)).$$

Writing $h(p) = 2 - p + (p-1)\ln(p-1)$, it follows that $h(1) = 1$, $h(2) = 0$ and $h'(p) = \ln(p-1) < 0$. Thus, $h(p) > 0$ on $[1, 2)$, and hence $g'(p) > 0$, which gives the estimate. By setting $w = z_2 z_1^{-1} = re^{i\theta}$, $-\pi < \theta \leq \pi$, $r > 0$, it is easy to rewrite the target inequality (9) as

$$\left|\frac{1+re^{i\theta}}{2}\right|^p + \frac{p(p-1)}{2^{p+1}} \frac{|1-re^{i\theta}|^2}{(1+r^2)^{\frac{2-p}{2}}} \leq \frac{1}{2}(1+r^p). \tag{12}$$

Fix r and denote the left-hand side by $F(\theta)$, i.e.,

$$F(\theta) = \frac{1}{2^p}\left(1 + 2r\cos\theta + r^2\right)^{\frac{p}{2}} + \frac{p(p-1)}{2^{p+1}(1+r^2)^{\frac{2-p}{2}}}\left(1 - 2r\cos\theta + r^2\right).$$

We have

$$F'(\theta) = \frac{2pr\sin\theta}{2^{p+1}}\left(-(1+2r\cos\theta+r^2)^{\frac{p}{2}-1} + \frac{p-1}{(1+r^2)^{\frac{2-p}{2}}}\right).$$

It is readily seen that for $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$, it holds

$$-(1+r^2)^{\frac{2-p}{2}} + (p-1)(1+2r\cos\theta + r^2)^{\frac{2-p}{2}} \leq (p-1)(1+r)^{2-p} - (1+r^2)^{\frac{2-p}{2}}.$$

We claim that

$$(p-1)(1+r)^{2-p} - (1+r^2)^{\frac{2-p}{2}} \leq 0.$$

Indeed, it is enough to show that

$$(p-1)^{\frac{2}{p-2}} \leq \frac{1+r^2}{(1+r)^2}.$$

This follows directly from estimate (11) and the fact that $e^2 > 2$. Therefore, F increases on $\left(-\dfrac{\pi}{2}, 0\right)$ and decreases on $\left(0, \dfrac{\pi}{2}\right)$, i.e., on $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ one has

$$F(\theta) \leq F(0) = \left|\frac{1+r}{2}\right|^p + \frac{p(p-1)}{2^{p+1}} \frac{|1-r|^2}{(1+r^2)^{\frac{2-p}{2}}}. \tag{13}$$

On the other hand, on $\left(-\pi, -\dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}\pi\right]$, one has $1 + 2r\cos\theta + r^2 < 1 + r^2$. Consequently,

$$(1+2r\cos\theta + r^2)^{\frac{p-2}{2}} > (1+r^2)^{\frac{p-2}{2}} \geq (p-1)(1+r^2)^{\frac{p-2}{2}}.$$

Thus, $F(\theta)$ increases on $\left(-\pi, -\dfrac{\pi}{2}\right)$ and decreases on $\left(\dfrac{\pi}{2}, \pi\right)$ and the bound in (13) holds on $(-\pi, \pi]$. On account of inequality (7), $F(0)$ is bounded above by the right-hand side of inequality (12), and this observation proves the desired inequality.

The proof of (10), for $p > 2$, follows by the same arguments and will be omitted. □

Using the above lemma, we are ready to state and prove the vector version of the fundamental inequalities of Lemma 1 in any Hilbert space.

Theorem 1. *Let* $\mathbf{u}$, $\mathbf{v}$ *be vectors in a Hilbert space* $(\mathbb{H}, \|\cdot\|)$. *If* $1 \leq p \leq 2$, *it holds that*

$$\left\|\frac{\mathbf{u}+\mathbf{v}}{2}\right\|^p + \frac{p(p-1)}{2^{p+1}} \frac{\|\mathbf{u}-\mathbf{v}\|^2}{(\|\mathbf{u}\| + \|\mathbf{v}\|)^{2-p}} \leq \frac{1}{2}(\|\mathbf{u}\|^p + \|\mathbf{v}\|^p), \tag{14}$$

provided $\|\mathbf{u}\| + \|\mathbf{v}\| \neq 0$. *In addition, if* $p \geq 2$, *it holds that*

$$\left\|\frac{\mathbf{u}+\mathbf{v}}{2}\right\|^p + \left\|\frac{\mathbf{u}-\mathbf{v}}{2}\right\|^p \leq \frac{1}{2}(\|\mathbf{u}\|^p + \|\mathbf{v}\|^p). \tag{15}$$

Proof. If the vectors $\mathbf{u}$, $\mathbf{v}$ are linearly dependent, the two inequalities reduce to the scalar case. Assume that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Set W as the subspace of $\mathbb{H}$ spanned by these two vectors. Using Gram–Schmidt, there exists an orthonormal basis $\{\mathbf{I}, \mathbf{J}\}$ of W. We have

$$\mathbf{u} = x\mathbf{I} + y\mathbf{J} \quad \textit{and} \quad \mathbf{v} = a\mathbf{I} + b\mathbf{J},$$

for $(x, y) \in \mathbb{R}^2$ and $(a, b) \in \mathbb{R}^2$. Set $z_1 = x + iy$ and $z_2 = a + ib$ in $\mathbb{C}$. Clearly, the following hold

$$\begin{cases} \|\mathbf{u}\|^2 &= |z_1|^2 = x^2 + y^2, \\ \|\mathbf{v}\|^2 &= |z_2|^2 = a^2 + b^2, \\ \|\mathbf{u}+\mathbf{v}\|^2 &= |z_1 + z_2|^2 = (x+a)^2 + (y+b)^2, \\ \|\mathbf{u}-\mathbf{v}\|^2 &= |z_1 - z_2|^2 = (x-a)^2 + (y-b)^2. \end{cases}$$

Lemma 2 implies

$$\left|\frac{z_1+z_2}{2}\right|^p + \frac{p(p-1)}{2^{p+1}}\frac{|z_1-z_2|^2}{(|z_1|^2+|z_2|^2)^{\frac{2-p}{2}}} \le \frac{1}{2}(|z_1|^p+|z_2|^p),$$

for $1 < p \le 2$, and for $p \ge 2$, we have

$$\left|\frac{z_1+z_2}{2}\right|^p + \left|\frac{z_1-z_2}{2}\right|^p \le \frac{1}{2}(|z_1|^p+|z_2|^p),$$

which obviously implies

$$\left\|\frac{\mathbf{u}+\mathbf{v}}{2}\right\|^p + \frac{p(p-1)}{2^{p+1}}\frac{\|\mathbf{u}-\mathbf{v}\|^2}{(\|\mathbf{u}\|+\|\mathbf{v}\|)^{2-p}} \le \frac{1}{2}(\|\mathbf{u}\|^p+\|\mathbf{v}\|^p),$$

for $1 < p \le 2$, provided $\|\mathbf{u}\| + \|\mathbf{v}\| \neq 0$. Additionally, if $p \ge 2$, it holds that

$$\left\|\frac{\mathbf{u}+\mathbf{v}}{2}\right\|^p + \left\|\frac{\mathbf{u}-\mathbf{v}}{2}\right\|^p \le \frac{1}{2}(\|\mathbf{u}\|^p+\|\mathbf{v}\|^p).$$

The proof of Theorem 1 is complete. □

3. Variable Exponent Spaces

It is by today's standards abundantly clear that the normed space structure is much too stringent to completely capture certain mathematical subtleties that are only visible under a more flexible lens. To name an example (in fact, it may be the most important to understand the aim of this work), the variable exponent p-Laplacian is modular in nature. With this in mind, we set out to a present brief summary of definitions and known results. The reader is referred to [2,3,9,10] for a more detailed discussion of the topics briefly outlined in this section.

Definition 1. *[10–13] A convex modular on a real vector space X is a function $\varrho : X \to [0,\infty]$ satisfying the following conditions:*

(1) $\varrho(x) = 0$ *if and only if* $x = 0$;
(2) $\varrho(\alpha x) = \varrho(x)$, *if* $|\alpha| = 1$;
(3) $\varrho(\alpha x + (1-\alpha)y) \le \alpha\varrho(x) + (1-\alpha)\varrho(y)$, *for any* $\alpha \in [0,1]$ *and any* $x, y \in X$.

Furthermore, it is considered that ϱ exhibits left-continuity when, for all $x \in X$,

$$\lim_{r\to 1^-} \varrho(rx) = \varrho(x).$$

A modular function defined on a vector space X naturally gives rise to a modular space.

Definition 2. *When a convex modular function ϱ is defined on the vector space X, the resulting modular space consists of the following set:*

$$X_\varrho = \{x \in X;\ \lim_{\alpha\to 0} \varrho(\alpha x) = 0\}.$$

The Luxemburg norm, denoted as $\|\cdot\|_\varrho$ and defined on the vector space X_ϱ, is given by the following expression:

$$\|x\|_\varrho := \inf\left\{\alpha > 0;\ \varrho\left(\frac{x}{\alpha}\right) \le 1\right\}.$$

In preparation for the next section, a concept related to the geometry of modular spaces is introduced [14]. Specifically, for $x \in X$ and $r > 0$, it is a natural and, as shall be seen in Section 5, relevant question, whether the modular ball $\{y \in X : \rho(x-y) < r\}$ is uniformly convex in the modular sense. Though an exhaustive discussion of this subject is beyond the scope of this work [2,10], the following type of uniform convexity introduced in [14] will have far reaching consequences in the applications to be discussed in Section 5. Notice that one can routinely verify that Definition 3 generalizes the idea of the norm-uniform convexity of a ball in a normed space.

Definition 3 ([14])**.** *Given a modular ϱ on a vector space X, we introduce the following uniform convexity-type properties of ϱ:*

(a) *Let $r > 0$ and $\varepsilon > 0$ be given. Define*

$$D_2(r,\varepsilon) := \left\{(x,y);\ x,y \in X_\varrho, \varrho(x) \le r, \varrho(y) \le r, \varrho\left(\frac{x-y}{2}\right) \ge \varepsilon r\right\}.$$

Set

$$\delta_2(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\,\varrho\left(\frac{x+y}{2}\right);\ (x,y) \in D_2(r,\varepsilon)\right\}$$

if $D_2(r,\varepsilon) \neq \emptyset$ and write $\delta_2(r,\varepsilon) = 1$ otherwise. ϱ is said to satisfy $(UC2)$ if for every $r > 0$ and $\varepsilon > 0$, one has $\delta_2(r,\varepsilon) > 0$. Observe that given $r > 0$, $\varepsilon > 0$ can be chosen small enough so that $D_2(r,\varepsilon) \neq \emptyset$.

(b) *ϱ is said to satisfy $(UUC2)$ if for every $s \ge 0$ and $\varepsilon > 0$, there exists $\eta_2(s,\varepsilon) > 0$ depending on s and ε such that*

$$\delta_2(r,\varepsilon) > \eta_2(s,\varepsilon) > 0 \ for\ r > s.$$

Definition 4 ([2])**.** *A convex modular ρ on a vector space V is said to be uniformly convex (in short (UC^*)) if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every $u \in V$ and $v \in V$:*

$$\rho\left(\frac{u-v}{2}\right) > \varepsilon\,\frac{\rho(u)+\rho(v)}{2} \ \ implies \ \ \rho\left(\frac{u+v}{2}\right) \le (1-\delta)\,\frac{\rho(u)+\rho(v)}{2}.$$

Notice that if ρ happens to be a norm, then the preceding definition is the usual uniform convexity for norms.

4. Modular Uniform Convexity in Variable Exponent Lebesgue–Sobolev Spaces

The class of variable exponent Lebesgue spaces was first introduced in 1931 [15]. The interested reader can consult [2,3,9] for an exhaustive treatment of these spaces. This section will focus on the modular uniform convexity properties of such spaces. We open the section with standard definitions.

Definition 5. *Let $\Omega \subset \mathbb{R}^n$ be a domain. The notation $\mathcal{M}(\Omega)$ will be used for the vector space of all real-valued, Borel-measurable functions defined on Ω. Let $\mathcal{P}(\Omega)$ be the subset of $\mathcal{M}$ consisting of functions $p : \Omega \longrightarrow [1,\infty]$. For each such p, define the set $\Omega_\infty := \{x \in \Omega : p(x) = \infty\}$. The function $\varrho : \mathcal{M}(\Omega) \longrightarrow [0,\infty]$, defined by*

$$\varrho(u) = \int\limits_{\Omega\setminus\Omega_\infty} |u(x)|^{p(x)} d\mu + \sup_{x\in\Omega_\infty} |u(x)|,$$

is a convex and continuous modular on $\mathcal{M}(\Omega)$. The associated modular vector space is denoted by $L^{p(\cdot)}(\Omega)$ or simply $L^p(\Omega)$ if no confusion arises.

Definition 6. *For Ω and p as in the preceding definition, $W^{1,p}(\Omega)$ will stand for the vector subspace of $L^p(\Omega)$ consisting of functions whose weak derivatives also belong to $L^p(\Omega)$. The Sobolev space $W^{1,p}(\Omega)$ will be equipped with the convex modular $\rho : W^{1,p}(\Omega) \to [0,\infty]$ defined as:*

$$\rho(u) = \varrho(u) + \varrho(|\nabla u|) = \varrho(u) + \varrho\left(\left(\sum_{j=1}^{n}\left(\frac{\partial u}{\partial x_j}\right)^2\right)^{\frac{1}{2}}\right)$$

and the corresponding Luxemburg norm introduced in Definition 2 will be denoted by $\|\cdot\|_\rho$.

The following result follows easily from Theorem 1.

Corollary 1. *Set* $\Omega_1 = \{x \in \Omega : p(x) \geq 2\}$. *For* $u \in W^{1,p}(\Omega)$, $v \in W^{1,p}(\Omega)$ *it holds that*

$$\int_{\Omega_1}\left|\frac{\nabla(u+v)}{2}\right|^p dx + \int_{\Omega_1}\left|\frac{\nabla(u-v)}{2}\right|^p dx \leq \frac{1}{2}\left(\int_{\Omega_1}|\nabla u|^p\,dx + \int_{\Omega_1}|\nabla v|^p\,dx\right)$$

and

$$\int_{\Omega\setminus\Omega_1}\left|\frac{\nabla(u+v)}{2}\right|^p dx + \int_{\Omega\setminus\Omega_1}\frac{p(p-1)|\nabla(u-v)|^2}{2^{p+1}(|\nabla u|^2+|\nabla v|^2)^{\frac{p-2}{p}}}\,dx \leq \frac{1}{2}\int_{\Omega\setminus\Omega_1}|\nabla u|^p\,dx + \frac{1}{2}\int_{\Omega\setminus\Omega_1}|\nabla v|^p\,dx.$$

The next result will be crucial to establish the main result of this work.

Theorem 2. *Let* $\Omega \subseteq \mathbb{R}^n$ *be a domain and* $p \in \mathcal{P}(\Omega)$ *finite a.e. The functional* $\varrho : W^{1,p}(\Omega) \to [0,\infty]$ *defined by*

$$\varrho(\mathbf{u}) = \int_{\Omega}\left(\sum_{1}^{n} u_j^2\right)^{\frac{p(x)}{2}} dx$$

(here u_j *denotes the* j^{th} *partial derivative of* u*) is a convex pseudomodular (i.e., it has all the properties exhibited in Definition 1, except (1)) and is* (UC^*) *provided* $p_- > 1$. *Moreover, when restricted to* $W_0^{1,p}(\Omega)$, ϱ *is* (UC^*) *when* $p_- > 1$.

Proof. This proof follows along the same lines as that of Theorem 3 in [6]. We provide the details in the interest of completeness. It is obvious that ϱ is a convex modular on $L^p(\Omega)$. In the course of the proof, it will be understood that $|(x_1, x_2, ..., x_n)| = \left(\sum_{j=1}^{n} x_j^2\right)^{\frac{1}{2}}$ and for a subset $A \subseteq \Omega$, we set $\varrho_A(\mathbf{u}) = \int_A |\mathbf{u}|^p dx$. Let $\varepsilon \in (0,1]$. Assume

$$\varrho\left(\frac{\mathbf{u}-\mathbf{v}}{2}\right) \geq \varepsilon\left(\frac{\varrho(\mathbf{u})+\varrho(\mathbf{v})}{2}\right).$$

Set $\Omega_1 = \{x \in \Omega : p(x) \geq 2\}$; then, necessarily, either

$$\int_{\Omega_1}\left|\frac{\nabla\mathbf{u}-\nabla\mathbf{v}}{2}\right|^p \geq \frac{\varepsilon}{2}\left(\frac{\varrho(\mathbf{u})+\varrho(\mathbf{v})}{2}\right) \tag{16}$$

or

$$\int_{\Omega\setminus\Omega_1}\left|\frac{\nabla\mathbf{u}-\nabla\mathbf{v}}{2}\right|^p \geq \frac{\varepsilon}{2}\left(\frac{\varrho(\mathbf{u})+\varrho(\mathbf{v})}{2}\right). \tag{17}$$

In the first case, by virtue of Theorem 1, it is readily concluded that

$$\int_{\Omega_1} \left| \frac{\nabla \mathbf{u} + \nabla \mathbf{v}}{2} \right|^p \leq \frac{1}{2} \left(\int_{\Omega_1} |\nabla \mathbf{u}|^p \, dx + \int_{\Omega_1} |\nabla \mathbf{v}|^p \, dx \right) - \frac{\varepsilon}{2} \left(\frac{\varrho(\mathbf{u}) + \varrho(\mathbf{v})}{2} \right)$$

which implies

$$\int_{\Omega_1} \left| \frac{\nabla \mathbf{u} + \nabla \mathbf{v}}{2} \right|^p \leq \frac{1}{2} \left(\int_{\Omega_1} |\nabla \mathbf{u}|^p \, dx + \int_{\Omega_1} |\nabla \mathbf{v}|^p \, dx \right) - \frac{\varepsilon}{2} \left(\frac{\varrho_{\Omega_1}(\mathbf{u}) + \varrho_{\Omega_1}(\mathbf{v})}{2} \right).$$

In all,

$$\begin{aligned}
\varrho\left(\frac{\mathbf{u} + \mathbf{v}}{2} \right) &= \left(\int_{\Omega_1} + \int_{\Omega \setminus \Omega_1} \right) \left| \frac{\nabla \mathbf{u} + \nabla \mathbf{v}}{2} \right|^p dx \\
&= \int_{\Omega_1} \left| \frac{\nabla \mathbf{u} + \nabla \mathbf{v}}{2} \right|^p dx + \int_{\Omega \setminus \Omega_1} \left| \frac{\nabla \mathbf{u} + \nabla \mathbf{v}}{2} \right|^p dx \\
&\leq \frac{1}{2} \left(\int_{\Omega_1} |\nabla \mathbf{u}|^p \, dx + \int_{\Omega_1} |\nabla \mathbf{v}|^p \, dx \right) - \frac{\varepsilon}{2} \left(\frac{\varrho(\mathbf{u}) + \varrho(\mathbf{v})}{2} \right) \\
&\quad + \int_{\Omega \setminus \Omega_1} \left| \frac{\nabla \mathbf{u} + \nabla \mathbf{v}}{2} \right|^p dx \\
&\leq \frac{1}{2} \left(\int_{\Omega_1} |\nabla \mathbf{u}|^p \, dx + \int_{\Omega_1} |\nabla \mathbf{v}|^p \, dx \right) - \frac{\varepsilon}{2} \left(\frac{\varrho(\mathbf{u}) + \varrho(\mathbf{v})}{2} \right) \\
&\quad + \frac{1}{2} \left(\int_{\Omega \setminus \Omega_1} |\nabla \mathbf{u}|^p \, dx + \int_{\Omega \setminus \Omega_1} |\nabla \mathbf{v}|^p \, dx \right) \\
&= \frac{1}{2} (\varrho(\mathbf{u}) + \varrho(\mathbf{v})) - \frac{\varepsilon}{2} \left(\frac{\varrho(\mathbf{u}) + \varrho(\mathbf{v})}{2} \right) \\
&= (1 - \varepsilon/2) \left(\frac{\varrho(\mathbf{u}) + \varrho(\mathbf{v})}{2} \right).
\end{aligned}$$

The last statement settles the issue in case (16) holds. If instead (17) holds, define

$$\Omega_2 = \left\{ x \in \Omega \setminus \Omega_1 : |\nabla \mathbf{u} - \nabla \mathbf{v}| < \frac{\varepsilon}{4} \left(|\nabla \mathbf{u}|^2 + |\nabla \mathbf{v}|^2 \right)^{\frac{1}{2}} \right\}.$$

It follows

$$\begin{aligned}
\int_{\Omega_2} \left| \frac{\nabla \mathbf{u} - \nabla \mathbf{v}}{2} \right|^p dx &\leq \int_{\Omega_2} \left| \frac{\varepsilon}{4} \frac{1}{2} (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{v}|^2)^{\frac{1}{2}} \right|^p dx \\
&\leq \frac{\varepsilon}{4} \int_{\Omega_2} \left| \frac{1}{2} (|\nabla \mathbf{u}| + |\nabla \mathbf{v}|) \right|^p dx \\
&\leq \frac{\varepsilon}{8} (\varrho(\mathbf{u}) + \varrho(\mathbf{v})).
\end{aligned}$$

Set $\Omega^* = \Omega \setminus (\Omega_1 \bigcup \Omega_2)$. On account of inequality (17), it is readily obtained that

$$\int_{\Omega^*} \left| \frac{\nabla(\mathbf{u}-\mathbf{v})}{2} \right|^p dx = \left(\int_{\Omega\setminus\Omega_1} - \int_{\Omega_2} \right) \left| \frac{\nabla(\mathbf{u}-\mathbf{v})}{2} \right|^p dx$$
$$\geq \frac{\varepsilon}{2}\left(\frac{\varrho(\mathbf{u})+\varrho(\mathbf{v})}{2} \right) - \frac{\varepsilon}{4}\left(\frac{\varrho(\mathbf{u})+\varrho(\mathbf{v})}{2} \right)$$
$$= \frac{\varepsilon}{4}\left(\frac{\varrho(\mathbf{u})+\varrho(\mathbf{v})}{2} \right).$$

By definition of Ω^*, one has

$$(p_- - 1)\frac{\varepsilon}{8}\left| \frac{\nabla(\mathbf{u}-\mathbf{v})}{2} \right|^p \leq \frac{p(p-1)}{2}\left(\frac{\varepsilon}{4}\right)^{2-p}\left| \frac{\nabla(\mathbf{u}-\mathbf{v})}{2} \right|^p$$
$$\leq \frac{p(p-1)}{2}\left(\frac{|\nabla(\mathbf{u}-\mathbf{v}|)}{(|\nabla\mathbf{u}|^2+|\nabla\mathbf{v}|^2)^{\frac{1}{2}}} \right)^{2-p}\left| \frac{\nabla(\mathbf{u}-\mathbf{v})}{2} \right|^p,$$

which implies

$$\left| \frac{\nabla(\mathbf{u}+\mathbf{v})}{2} \right|^p + (p_- - 1)\frac{\varepsilon}{8}\left| \frac{\nabla(\mathbf{u}-\mathbf{v})}{2} \right|^p \leq \frac{1}{2}(|\nabla\mathbf{u}|^p + |\nabla\mathbf{v}|^p),$$

on account of the first part of Theorem 1. Integrating the above inequality and taking into consideration (17), it follows that

$$\int_{\Omega^*}\left| \frac{\nabla(\mathbf{u}+\mathbf{v})}{2} \right|^p dx \leq \frac{1}{2}\int_{\Omega^*} |\nabla\mathbf{u}|^p\, dx + \frac{1}{2}\int_{\Omega^*} |\nabla\mathbf{v}|^p - (p_- - 1)\frac{\varepsilon^2}{32}\left(\frac{\varrho(\mathbf{u})+\varrho(\mathbf{v})}{2} \right).$$

Finally,

$$\varrho\left(\frac{\mathbf{u}+\mathbf{v}}{2}\right) = \left(\int_{\Omega_1\bigcup\Omega_2} + \int_{\Omega^*} \right)\left| \frac{\nabla(\mathbf{u}+\mathbf{v})}{2} \right|^p dx$$
$$\leq \frac{1}{2}\left(\int_{\Omega_1\bigcup\Omega_2} |\nabla\mathbf{u}|^p\, dx + \int_{\Omega_1\bigcup\Omega_2} |\nabla\mathbf{v}|^p\, dx \right)$$
$$+ \frac{1}{2}\left(\int_{\Omega^*} |\nabla\mathbf{u}|^p\, dx + \int_{\Omega^*} |\nabla\mathbf{v}|^p \right) - (p_- - 1)\frac{\varepsilon^2}{32}\left(\frac{\varrho(\mathbf{u})+\varrho(\mathbf{v})}{2} \right)$$
$$= \frac{1}{2}(\varrho(\mathbf{u})+\varrho(\mathbf{v})) - (p_- - 1)\frac{\varepsilon^2}{32}\left(\frac{\varrho(\mathbf{u})+\varrho(\mathbf{v})}{2} \right)$$
$$\leq \left(1 - (p_- - 1)\frac{\varepsilon^2}{32} \right)\left(\frac{\varrho(\mathbf{u})+\varrho(\mathbf{v})}{2} \right).$$

If we set

$$\delta = \min\left\{ \frac{\varepsilon}{2}, \left(1 - (p_- - 1)\frac{\varepsilon^2}{32} \right) \right\},$$

then $\delta > 0$ since $p_- > 1$, and the following holds

$$\varrho\left(\frac{\mathbf{u}+\mathbf{v}}{2}\right) \leq (1-\delta)\left(\frac{\varrho(\mathbf{u})+\varrho(\mathbf{v})}{2} \right),$$

i.e., ϱ is (UC^*) as claimed. $\square$

Using Lemma 1 and along the same lines, the following theorem can be proven (see also [6]).

Theorem 3. *If $1 < p_- \leq p < \infty$ in Ω, the modular $\rho : L^p(\Omega) \to [0,\infty]$ defined by*

$$\rho(u) = \int_\Omega |u|^p \, dx$$

is (UC^*).

The following is the main result of this section.

Theorem 4. *For $1 < p_- \leq p < \infty$, the modular $\rho : W^{1,p}(\Omega) \to [0,\infty]$ defined by*

$$\rho(u) = \int_\Omega \left(|u|^p dx + |\nabla u|^p \right) dx$$

is (UC^*).

Proof. The proof follows immediately from Theorem 2, Theorem 3 and the fact that uniform convexity is preserved under sums ([2], Lemma 2.4.16). □

5. Uniform Convexity of the Luxemburg Norm on $W_0^{1,p}(\Omega)$

This section is devoted to the proof of the fundamental result that for a variable exponent p bounded away from 1 and ∞, the Luxemburg norm on the Sobolev space $W_0^{1,p}(\Omega)$ is uniformly convex. The originality in this section is the range $1 < p < 2$. A few well-known facts about the modular spaces L^p are summarized below.

Theorem 5 ([2,14])**.** *Assume $p_+ < \infty$. In the notation of Definition 5, for any $u \in L^p(\Omega)$,*

(i) $\varrho(u) = 1$ *if and only if* $\|u\|_\varrho = 1$.

(ii) $\min\{\|u\|_\varrho^{p_+}, \|u\|_\varrho^{p_-}\} \leq \varrho(u) \leq \max\{\|u\|_\varrho^{p_+}, \|u\|_\varrho^{p_-}\}$.

Let us recall the following definition.

Definition 7. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain and $p \in \mathcal{P}(\Omega)$ be an admissible exponent. Denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$ (Definition 6) by $W_0^{1,p}(\Omega)$.*

The following theorem is well known.

Theorem 6 ([3,16])**.** *For $1 < p_- \leq p_+ < \infty$, then on $W_0^{1,p}(\Omega)$ the norm $\||\nabla u|\|_\varrho$ is equivalent to the Luxemburg norm $\|u\|_\rho$. Specifically, there exists $\alpha > 0$ depending only on Ω and p such that for all $u \in W_0^{1,p}(\Omega)$,*

$$\frac{1}{\alpha} \|u\|_\rho \leq \||\nabla u|\|_\varrho \leq \alpha \|u\|_\rho \, .$$

The following result holds.

Theorem 7. *Let $\Omega \subset \mathbb{R}^n$ be a domain and p be an admissible exponent with $1 < p_- \leq p_+ < \infty$. Then, the Sobolev–Luxemburg norm $\| \cdot \|_\rho$ on $W^{1,p}(\Omega)$ is uniformly convex. Likewise, the norm $u \to \||\nabla u|\|_\varrho$ defined on $W_0^{1,p}(\Omega)$ is uniformly convex.*

Proof. Let $0 < \varepsilon < 2$ and take u and v with $\|u\|_\rho = \|v\|_\rho = 1$, $\|u - v\|_\rho \geq \varepsilon$; that is, by virtue of Theorem 5, $\rho(u) = \rho(v) = 1$ and $\rho\left(\frac{u-v}{2}\right) \geq \left(\frac{\varepsilon}{2}\right)^{p_+}$. On account of Theorem 4 for some $\eta > 0$ it holds $\rho(\frac{u+v}{2}) < 1 - \eta$. On account of (ii) in Theorem 5, it follows that $\left\|\frac{u+v}{2}\right\|_\rho < 1 - \theta$, for some $0 < \theta < 1$. $\square$

The rest of the claim follows along the same lines from a direct application of Theorem 2.

6. Conclusions

In conclusion, we proved the modular uniform convexity of the Sobolev space $W^{1,p(x)}(\Omega)$ in the case $\sup_{x\in\Omega} p(x) = p_+ = \infty$. We have also proven that the Luxemburg norm in $W^{1,p}(\Omega)$ is uniformly convex even for $1 < p_- < 2$. To the best of our knowledge, both results are new and have concrete applications in the study of the solvability of boundary value problems involving partial differential equations with non-standard growth.

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