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Kneser-Type Oscillation Criteria for Half-Linear Delay Differential Equations of Third Order

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Abstract: This paper delves into the analysis of oscillation characteristics within third-order quasi-linear delay equations, focusing on the canonical case. Novel sufficient conditions are introduced, aimed at discerning the nature of solutions—whether they exhibit oscillatory behavior or converge to zero. By expanding the literature, this study enriches the existing knowledge landscape within this field. One of the foundations on which we rely in proving the results is the symmetry between the positive and negative solutions, so that we can, using this feature, obtain criteria that guarantee the oscillation of all solutions. The paper enhances comprehension through the provision of illustrative examples that effectively showcase the outcomes and implications of the established findings.

Keywords: oscillatory; nonoscillatory; delay differential equation; third order; canonical

MSC: 34C10; 34K11

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1. Introduction

The focus of this article is on considering third-order delay differential equations in the form

$$\left(\beta_2(\mathrm{s})\left(\left(\beta_1(\mathrm{s})\mathcal{U}'(\mathrm{s})\right)'\right)^r\right)' + q(\mathrm{s})\mathcal{U}^r(\tau(\mathrm{s})) = 0,\ \mathrm{s} \geq \mathrm{s}_0, \tag{1}$$

where

(H_1) r is the ratio of two positive odd integers and $r > 1$;
(H_2) $q \in C([\mathrm{s}_0, \infty))$, $q(\mathrm{s}) \geq 0$;
(H_3) $\tau \in C^1([\mathrm{s}_0, \infty))$, $\tau(\mathrm{s}) \leq \mathrm{s}$, $\tau'(\mathrm{s}) > 0$, and $\lim_{\mathrm{s}\to\infty} \tau(\mathrm{s}) = \infty$;
(H_4) $\beta_2 \in C^1([\mathrm{s}_0, \infty))$, $\beta_1 \in C^2([\mathrm{s}_0, \infty))$, $\beta_1 > 0$, $\beta_2 > 0$,

$$\int_{\mathrm{s}_0}^{\infty} \frac{1}{\beta_1(\varkappa)} \mathrm{d}\varkappa = \infty \text{ and } \int_{\mathrm{s}_0}^{\infty} \frac{1}{\beta_2^{1/r}(\varkappa)} \mathrm{d}\varkappa = \infty. \tag{2}$$

Function $\mathcal{U} \in C([\mathrm{s}_{\mathcal{U}}, \infty), \mathbb{R})$, $\mathrm{s}_{\mathcal{U}} \geqslant \mathrm{s}_0$, is said to be a solution of Equation (1) if it has the property $\beta_2\left(\left(\beta_1\mathcal{U}'\right)'\right)^r \in C^1[\mathrm{s}_{\mathcal{U}}, \infty)$, and it satisfies Equation (1) for all $\mathcal{U} \in [\mathrm{s}_{\mathcal{U}}, \infty)$. We consider only those solutions $\mathcal{U}$ of Equation (1) which exist on some half-line $[\mathrm{s}_{\mathcal{U}}, \infty)$ and satisfy the condition

$$\sup\{|\mathcal{U}(\mathrm{s})| : \mathrm{s} \geqslant \mathrm{S}\} > 0,\ \text{for all } \mathrm{S} \geq \mathrm{s}_{\mathcal{U}}.$$

For any solution $\mathcal{U}$ of Equation (1), we denote by $L_i\mathcal{U}$ the ith quasiderivative of $\mathcal{U}$, that is,

$$L_0\mathcal{U} = \mathcal{U},\ L_1\mathcal{U} = \beta_1\mathcal{U}',\ L_2\mathcal{U} = \beta_2\left(\left(\beta_1\mathcal{U}'\right)'\right)^r, \text{ and } L_3\mathcal{U} = \left(\beta_2\left(\left(\beta_1\mathcal{U}'\right)'\right)^r\right)',$$

on $[s_0, \infty)$.

Delay differential equations (DDEs) are a type of ordinary differential equations (ODEs) that involve a time delay. They arise in many applications, such as control theory, population dynamics, and neuroscience, where the time delay can represent a delay in feedback, a time lag in communication, or a time delay in the response of a system. In this context, quasi-linear third-order DDEs are a subclass of DDEs that have important applications in the modeling of many physical and biological systems; see [1,2]. They are characterized by having a linear term in the derivative of the dependent variable, and a nonlinear term that depends on the product of the dependent variable and its derivative.

In recent years, there has been significant interest in the study of quasi-linear third-order DDEs and their applications. This is due in part to the fact that many real-world systems exhibit nonlinear behavior, and DDEs provide a natural framework for modeling such behavior. Moreover, the study of quasi-linear third-order DDEs has important applications in the analysis of control systems, neural networks, and biological systems, where the dynamics of the system depend on the interaction between different variables with time delays.

While even-order delay differential equations have received more attention than odd-order ones, the study of DDEs in general has gained traction in recent years. Interested readers can refer to various studies, including Parhi and Das [3], Parhi and Padhi [4,5], Baculikova et al. [6], Dzurina [7], Bohner et al [8], Chatzarakis et al. [9,10], Moaaz [11], and Almarri et al. [12,13] and the references mentioned therein.

Saker [14] investigated the oscillation behavior of nonlinear delay differential equation

$$\left(\beta_2(\mathrm{s})\left(\beta_1(\mathrm{s})\mathcal{U}'(\mathrm{s})\right)'\right)' + q(\mathrm{s})f(\mathcal{U}(\mathrm{s}-\tau)) = 0,\ \mathrm{s} \geq \mathrm{s}_0 \tag{3}$$

in the canonical case and discussed some criteria that guarantee that every solution to Equation (3) is oscillatory using Riccati transformation techniques.

Grace et al. [15] offered new criteria for the oscillation of third-order delay differential equations

$$\left(\beta_2(\mathrm{s})\left(\beta_1(\mathrm{s})\mathcal{U}'(\mathrm{s})\right)'\right)' + q(\mathrm{s})\mathcal{U}(\tau(\mathrm{s})) = 0,\ \mathrm{s} \geq \mathrm{s}_0 \tag{4}$$

in non-canonical case

$$\int_{\mathrm{s}_0}^{\infty} \frac{1}{\beta_1(\varkappa)} \mathrm{d}\varkappa < \infty \text{ and } \int_{\mathrm{s}_0}^{\infty} \frac{1}{\beta_2(\varkappa)} \mathrm{d}\varkappa < \infty.$$

Theorem 1 ([15], Theorem 3.3.)**.** *We suppose that*

$$\limsup_{\mathrm{s}\to\infty} \int_{\tau(\varkappa)}^{\mathrm{s}} q(\varkappa) \int_{\tau(\varkappa)}^{\tau(\mathrm{s})} \frac{1}{\beta_1(\vartheta)} \int_{\vartheta}^{\tau(\mathrm{s})} \frac{1}{\beta_2(v)} \mathrm{d}v\mathrm{d}\vartheta\mathrm{d}\varkappa > 1,$$

$$\int_{\mathrm{s}_0}^{\mathrm{s}} \frac{1}{\beta_1(v)} \int_{\mathrm{s}_0}^{v} \frac{1}{\beta_2(\vartheta)} \int_{\mathrm{s}_0}^{\vartheta} q(\varkappa)\mathrm{d}\varkappa\mathrm{d}\vartheta\mathrm{d}v = \infty,$$

and

$$\liminf_{\mathrm{s}\to\infty} \frac{\beta_1(\mathrm{s})\pi_1^2(\mathrm{s})}{\beta_2(\mathrm{s})} \int_{\mathrm{s}_0}^{\mathrm{s}} q(\varkappa)\mathrm{d}\varkappa = K > 0 \textit{ and } \liminf_{\mathrm{s}\to\infty} \frac{\pi_1(\tau(\mathrm{s}))}{\pi_1(\mathrm{s})} = k > 1$$

hold. We let $\{K_n\}$ be the sequence given by

$$K_n = \frac{k^{K_{n-1}} K}{1 - K_{n-1}},\ K_0 = K,\ n \in \mathbb{N},$$

and $K_i < 1$, for some $n \in \mathbb{N}$, $i = 0, 1, \ldots, n-1$. If either one of the conditions

$$K_n > \frac{1}{2} \textit{ or } \frac{K \ln k}{1 - K_n} > \frac{1}{\mathrm{e}} \textit{ or } K k^{K_n} > \frac{1}{4}$$

is fulfilled, then (4) is oscillatory, where $\beta_1(s) = \int_{s_0}^{\infty} \frac{1}{\beta_1(\varkappa)} d\varkappa$.

Saker and Dzurina [16] established that some necessary conditions guarantee that

$$\left(\beta(s)\left(\mathcal{U}''(s)\right)^r\right)' + q(s)\mathcal{U}^r(\tau(s)) = 0 \tag{5}$$

is oscillatory or that the solutions converge to zero in canonical case

$$\int_{s_0}^{\infty} \frac{1}{\beta(\varkappa)} d\varkappa = \infty.$$

Theorem 2 ([16], Theorem 2.). *We let* $\mathcal{U}$ *be a solution of (5) and* $\beta'(s) \geq 0$. *We suppose that*

$$\int_{s_0}^{\infty} \int_{v}^{\infty} \left(\frac{1}{\beta(\vartheta)} \int_{\vartheta}^{\infty} q(\varkappa) d\varkappa\right)^{1/r} d\vartheta dv = \infty \tag{6}$$

holds. If

$$\liminf_{s\to\infty} \frac{l^r s^r}{\beta(s)} \int_{s}^{\infty} q(\varkappa) \left(\frac{\tau(\varkappa)}{\varkappa}\right)^r \left(\frac{\tau(\varkappa) - s_l}{2}\right)^r d\varkappa > \frac{r^r}{(r+1)^{r+1}},$$

then $\mathcal{U}$ *is oscillatory or tends to zero as* $s \to \infty$, *where* s_l *is large enough and* $l \in (0,1)$ *is arbitrarily chosen.*

Baculikova and Dzurina [17] provided a general classification of oscillatory and asymptotic behaviors of the third-order functional differential equations of the form

$$\left(\beta(s)\left(\mathcal{U}''(s)\right)^r\right)' + q(s)f(\mathcal{U}(\tau(s))) = 0 \tag{7}$$

in the canonical case, where $f(\mathcal{U}) \in C(-\infty,\infty)$, $f' \geq 0$, $\mathcal{U}f(\mathcal{U}) > 0$ for $\mathcal{U} \neq 0$ and $-\mathcal{U}f(-\mathcal{U}y) \geq f(\mathcal{U}y) \geq f(\mathcal{U})f(y)$ for $\mathcal{U}y > 0$.

Theorem 3 ([17], Theorem 2.). *We suppose that (6) holds. If*

$$y'(s) + q(s)f\left(\int_{s_0}^{\tau(s)} \frac{\tau(s) - \varkappa}{\beta^{1/r}(\varkappa)} d\varkappa\right) f\left(y^{1/r}(\tau(s))\right) = 0 \tag{8}$$

is oscillatory, then every solution of (7) is oscillatory or tends to zero as $s \to \infty$.

The purpose of this research is to establish new criteria that ensure all solutions to Equation (1) are oscillatory or tend to zero. The results in this paper are different from those in [18]. Our results are an extension of the results in [19] as known in the literature in the case $r = 1$. That is, our results are in the case $r > 1$.

2. Preliminary Results

This section introduces a collection of definitions and assumptions that are crucial for our paper and aids in simplifying mathematical operations. Let us define the following notations for convenience in our calculations:

$$\pi_1(s) := \int_{s_0}^{s} \frac{1}{\beta_1(\varkappa)} d\varkappa, \ \pi_2(s) := \int_{s_0}^{s} \frac{1}{\beta_2^{1/r}(\varkappa)} d\varkappa, \ \pi_{12}(s) := \int_{s_0}^{s} \frac{\pi_2(\varkappa)}{\beta_1(\varkappa)} d\varkappa,$$

$$\lambda_* := \liminf_{s\to\infty} \frac{\pi_{12}(s)}{\pi_{12}(\tau(s))},$$

$$\varrho_* := \liminf_{s\to\infty} \frac{1}{r} \beta_2^{1/r}(s) \pi_{12}^r(\tau(s)) \pi_2(s) q(s),$$

and

$$k_* := \liminf_{\mathrm{s}\to\infty} \frac{\pi_2^{\varrho_*}(\mathrm{s})}{\pi_{12}(\mathrm{s})} \int_{\mathrm{s}_0}^{\mathrm{s}} \frac{\pi_2^{1-\varrho_*}(\varkappa)}{\beta_1(\varkappa)} \mathrm{d}\varkappa, \text{ for } \varrho_* \in (0,1).$$

Remark 1. *We refer to the supremum and infimum functions as* sup *and* inf*, respectively.*

Remark 2. *All our results require that ϱ_* is positive either explicitly or implicitly. For any fixed but arbitrary $\varrho \in (0, \varrho_*)$ and $\lambda = \lambda_*$ for $\lambda_* = 1$, and $\lambda \in (1, \lambda_*)$ for $\lambda_* > 1$, there exists an $\mathrm{s}_1 \geq \mathrm{s}_0$ large enough to satisfy the following inequalities:*

$$\frac{\pi_{12}(\mathrm{s})}{\pi_{12}(\tau(\mathrm{s}))} \geq \lambda, \tag{9}$$

$$\frac{1}{r}\beta_2^{1/r}(\mathrm{s})\pi_{12}^r(\tau(\mathrm{s}))\pi_2(\mathrm{s})q(\mathrm{s}) \geq \varrho, \tag{10}$$

and

$$\frac{\pi_2^{\varrho}(\mathrm{s})}{\pi_{12}(\mathrm{s})} \int_{\mathrm{s}_0}^{\mathrm{s}} \frac{\pi_2^{1-\varrho}(\varkappa)}{\beta_1(\varkappa)} \mathrm{d}\varkappa \geq k. \tag{11}$$

Lemma 1 ([20])**.** *We assume that A and B are real numbers, $A > 0$. Then,*

$$BV - AV^{(r+1)/r} \leq \frac{r^r}{(r+1)^{r+1}} \frac{B^{r+1}}{A^r}. \tag{12}$$

Lemma 2 ([21])**.** *We let $V \in \mathbf{C}^m([\mathrm{s}_0, \infty), (0, \infty))$, $V^{(i)}(\mathrm{s}) > 0$ for $i = 1, 2, \ldots, m$, and $V^{(m+1)}(\mathrm{s}) \leq 0$, eventually. Then, eventually,*

$$\frac{V(\mathrm{s})}{V'(\mathrm{s})} \geq \frac{\epsilon}{m}\mathrm{s},$$

for every $\epsilon \in (0,1)$.

Lemma 3. *We assume that $\mathcal{U}$ is an eventually positive solution of (1). Then, $\mathcal{U}$ eventually satisfies the following cases:*

$$\begin{aligned} N_1 &: \quad \mathcal{U} > 0,\ L_1\mathcal{U} < 0,\ L_2\mathcal{U} > 0, \text{ and } L_3\mathcal{U} \leq 0, \\ N_2 &: \quad \mathcal{U} > 0,\ L_1\mathcal{U} > 0,\ L_2\mathcal{U} > 0. \end{aligned}$$

Here, we define Ω as the category of all positive solutions of (1) with x satisfying N_2.

Remark 3. *The Kneser solutions are the solutions that belong to the class N_1.*

Definition 1 ([22])**.** *If $N_2 = \varnothing$ and any Kneser solution of Equation (1) tends to zero asymptotically, then, we say Equation (1) has property A.*

3. Nonexistence of N_2-Type Solutions

This section contains several lemmas that describe the asymptotic properties of solutions belonging to the class N_2. These lemmas are instrumental in illustrating our main results.

Lemma 4. *We suppose that $\varrho_* > 0$ and $\mathcal{U} \in \Omega$. Then, for a sufficiently large* s

$(A_{1,1})$ $\lim_{\mathrm{s}\to\infty} L_2\mathcal{U}(\mathrm{s}) = \lim_{\mathrm{s}\to\infty} L_1\mathcal{U}(\mathrm{s})/\pi_2(\mathrm{s}) = \lim_{\mathrm{s}\to\infty} \mathcal{U}(\mathrm{s})/\pi_{12}(\mathrm{s}) = 0;$

$(A_{1,2})$ *$L_1\mathcal{U}/\pi_2$ is decreasing and $L_1\mathcal{U} \geq \pi_2(L_2\mathcal{U})^{1/r}$;*

$(A_{1,3})$ *$\mathcal{U}/\pi_{12}$ is decreasing and $\mathcal{U} > (\pi_{12}/\pi_2)L_1\mathcal{U}$.*

Proof. We let $\mathcal{U} \in \Omega$ and choose $s_1 \geq s_0$ such that $\mathcal{U}(\tau(s)) > 0$ and ϱ satisfies (10) for $s \geq s_1$.
$(A_{1,1})$: Since $L_2\mathcal{U}$ is a positive decreasing function, obviously

$$\lim_{s \to \infty} L_2\mathcal{U} = l \geq 0.$$

If $l > 0$, then $L_2\mathcal{U} \geq l > 0$, and so for any $\varepsilon \in (0,1)$, we have

$$\mathcal{U}(s) \geq l \int_{s_1}^{s} \frac{1}{\beta_1(\vartheta)} \int_{s_1}^{\vartheta} \frac{1}{\beta_2^{1/r}(\varkappa)} \mathrm{d}\varkappa \mathrm{d}\vartheta \geq \tilde{l}\pi_{12}(s),\ \tilde{l} = \varepsilon l.$$

Using this in (1), we obtain

$$-L_3\mathcal{U}(s) = q(s)\mathcal{U}^r(\tau(s)) \geq \tilde{l}^r q(s)\pi_{12}^r(\tau(s)).$$

Integrating from s_1 to s, we have

$$\begin{aligned}
L_2\mathcal{U}(s_1) &\geq \tilde{l}^r \int_{s_1}^{s} q(\varkappa)\pi_{12}^r(\tau(\varkappa))\mathrm{d}\varkappa \\
&\geq r\varrho\tilde{l}^r \int_{s_1}^{s} \frac{1}{\beta_2^{1/r}(\varkappa)\pi_2(\varkappa)}\mathrm{d}\varkappa \\
&= r\varrho\tilde{l}^r \ln\frac{\pi_2(s)}{\pi_2(s_1)} \to \infty \text{ as } s \to \infty,
\end{aligned}$$

which is a contradiction. Thus, $l = 0$. By using l'Hôspital's rule, we can see that $(A_{1,1})$ holds.
$(A_{1,2})$: Since $L_2\mathcal{U}$ is positive and decreasing,

$$\begin{aligned}
L_1\mathcal{U}(s) &= L_1\mathcal{U}(s_1) + \int_{s_1}^{s} \frac{1}{\beta_2^{1/r}(\varkappa)} L_2^{1/r}\mathcal{U}(\varkappa)\mathrm{d}\varkappa \\
&\geq L_1\mathcal{U}(s_1) + L_2^{1/r}\mathcal{U}(s) \int_{s_1}^{s} \frac{1}{\beta_2^{1/r}(\varkappa)}\mathrm{d}\varkappa \\
&= L_1\mathcal{U}(s_1) + (L_2\mathcal{U}(s))^{1/r}\pi_2(s) - (L_2\mathcal{U}(s))^{1/r} \int_{s_0}^{s_1} \frac{1}{\beta_2^{1/r}(\varkappa)}\mathrm{d}\varkappa.
\end{aligned}$$

In view of $(A_{1,1})$, we see that

$$L_1\mathcal{U}(s_1) - (L_2\mathcal{U}(s))^{1/r} \int_{s_0}^{s_1} \frac{1}{\beta_2^{1/r}(\varkappa)}\mathrm{d}\varkappa > 0.$$

Thus,

$$L_1\mathcal{U}(s) > (L_2\mathcal{U}(s))^{1/r}\pi_2(s),$$

and, consequently,

$$\left(\frac{L_1\mathcal{U}}{\pi_2}\right)'(s) = \frac{(L_2\mathcal{U}(s))_2^{1/r}\pi(s) - L_1\mathcal{U}(s)}{\beta_2^{1/r}(s)\pi_2^2(s)} < 0,\ s \geq s_2.$$

$(A_{1,3})$: Since $L_1\mathcal{U}/\pi_2$ is a decreasing function tending to zero,

$$\begin{aligned} \mathcal{U}(\mathrm{s}) &= \mathcal{U}(\mathrm{s}_2) + \int_{\mathrm{s}_2}^{\mathrm{s}} \frac{L_1\mathcal{U}(\varkappa)}{\pi_2(\varkappa)} \frac{\pi_2(\varkappa)}{\beta_1(\varkappa)} \mathrm{d}\varkappa \\ &\geq \mathcal{U}(\mathrm{s}_2) + \frac{L_1\mathcal{U}(\mathrm{s})}{\pi_2(\mathrm{s})} \int_{\mathrm{s}_2}^{\mathrm{s}} \frac{\pi_2(\varkappa)}{\beta_1(\varkappa)} \mathrm{d}\varkappa \\ &\geq \mathcal{U}(\mathrm{s}_2) + \frac{L_1\mathcal{U}(\mathrm{s})}{\pi_2(\mathrm{s})} \pi_{12}(\mathrm{s}) + \frac{L_1\mathcal{U}(\mathrm{s})}{\pi_2(\mathrm{s})} \int_{\mathrm{s}_0}^{\mathrm{s}_2} \frac{\pi_2(\varkappa)}{\beta_1(\varkappa)} \mathrm{d}\varkappa \\ &> \frac{L_1\mathcal{U}(\mathrm{s})}{\pi_2(\mathrm{s})} \pi_{12}(\mathrm{s}). \end{aligned}$$

Therefore,

$$\left(\frac{\mathcal{U}}{\pi_{12}}\right)'(\mathrm{s}) = \frac{L_1\mathcal{U}(\mathrm{s})\pi_{12}(\mathrm{s}) - \mathcal{U}(\mathrm{s})\pi_2(\mathrm{s})}{\beta_1(\mathrm{s})\pi_{12}^2(\mathrm{s})} < 0.$$

□

The following lemma provides further properties of solutions that are classified under the category N_2.

Lemma 5. *We assume that $\varrho_* > 0$ and $\mathcal{U} \in \Omega$. Then, for $\varrho \in (0, \varrho_*)$ and a sufficiently large* s

$(A_{2,1})$ $L_1\mathcal{U}/\pi_2^{1-\varrho_*}$ *is decreasing, and* $(1-\varrho_*)L_1\mathcal{U} > \pi_2(L_2\mathcal{U})^{1/r}$;

$(A_{2,2})$ $\lim_{\mathrm{s}\to\infty} L_1\mathcal{U}(\mathrm{s})/\pi_2^{1-\varrho_*}(\mathrm{s}) = 0$;

$(A_{2,3})$ $\mathcal{U}/\pi_{12}^{1/k}$ *is decreasing and* $\mathcal{U} > k(\pi_{12}/\pi_2)L_1\mathcal{U}$.

Proof. We let $\mathcal{U} \in \Omega$ and choose $\mathrm{s}_1 \geq \mathrm{s}_0$ such that $\mathcal{U}(\tau(\mathrm{s})) > 0$ and parts $(A_{1,1})$-$(A_{1,3})$ in Lemma 4 hold for $\mathrm{s} \geq \mathrm{s}_1 \geq \mathrm{s}_0$ and choose $\varrho \in (\varrho_*/(1+\varrho_*), \varrho_*)$ and $k \leq k_*$ satisfying (10) and (11), respectively, for $\mathrm{s} \geq \mathrm{s}_1$.
Since

$$\frac{\varrho}{1-\varrho} > \varrho_*,$$

there exist constants $c_1 \in (0,1)$ and $c_2 > 0$ such that

$$\frac{c_1\varrho}{1-\varrho} > \varrho_* + c_2. \tag{13}$$

$(A_{2,1})$: We define

$$z(\mathrm{s}) = L_1\mathcal{U}(\mathrm{s}) - \pi_2(\mathrm{s})(L_2\mathcal{U}(\mathrm{s}))^{1/r}. \tag{14}$$

This, according to $(A_{1,2})$, is obviously positive. Differentiating z and employing (1) and (10), we have

$$\begin{aligned} z'(\mathrm{s}) &= \left(L_1\mathcal{U}(\mathrm{s}) - (L_2\mathcal{U}(\mathrm{s}))^{1/r}\pi_2(\mathrm{s})\right)' \\ &= -\frac{1}{r}\pi_2(\mathrm{s})(L_2\mathcal{U}(\mathrm{s}))^{1/r-1}L_3\mathcal{U}(\mathrm{s}) \\ &= \frac{1}{r}\pi_2(\mathrm{s})q(\mathrm{s})\mathcal{U}^r(\tau(\mathrm{s}))(L_2\mathcal{U}(\mathrm{s}))^{1/r-1} \\ &\geq \varrho\frac{\mathcal{U}^r(\tau(\mathrm{s}))}{\beta_2^{1/r}(\mathrm{s})\pi_{12}^r(\tau(\mathrm{s}))}(L_2\mathcal{U}(\mathrm{s}))^{1/r-1}. \end{aligned} \tag{15}$$

By virtue of $(A_{1,3})$, we have

$$z'(\mathrm{s}) \geq \varrho\frac{\mathcal{U}^r(\mathrm{s})}{\beta_2^{1/r}(\mathrm{s})\pi_{12}^r(\mathrm{s})}(L_2\mathcal{U}(\mathrm{s}))^{1/r-1}. \tag{16}$$

From $(A_{1,2})$ and $(A_{1,3})$, we see that

$$\frac{\mathcal{U}(\mathrm{s})}{\pi_{12}(\mathrm{s})} > \frac{L_1\mathcal{U}(\mathrm{s})}{\pi_2(\mathrm{s})} > (L_2\mathcal{U}(\mathrm{s}))^{1/r}.$$

Since $r > 1$, then

$$\left(\frac{\mathcal{U}(\mathrm{s})}{\pi_{12}(\mathrm{s})}\right)^{1-r} < \left(\frac{L_1\mathcal{U}(\mathrm{s})}{\pi_2(\mathrm{s})}\right)^{1-r} < (L_2\mathcal{U}(\mathrm{s}))^{(1-r)/r}. \tag{17}$$

Substituting previous inequality in (16), we obtain

$$z'(\mathrm{s}) \geq \varrho \frac{\mathcal{U}^r(\mathrm{s})}{\beta_2^{1/r}(\mathrm{s})\pi_{12}^r(\mathrm{s})}\left(\frac{\mathcal{U}(\mathrm{s})}{\pi_{12}(\mathrm{s})}\right)^{1-r} = \varrho \frac{\mathcal{U}(\mathrm{s})}{\beta_2^{1/r}(\mathrm{s})\pi_{12}(\mathrm{s})} \geq \varrho \frac{L_1\mathcal{U}(\mathrm{s})}{\beta_2^{1/r}(\mathrm{s})\pi_2(\mathrm{s})}. \tag{18}$$

Integrating from s_2 to s and using the fact that $L_1\mathcal{U}/\pi_2$ is decreasing and tends to zero asymptotically, we have

$$\begin{aligned} z(\mathrm{s}) &\geq z(\mathrm{s}_2) + \varrho \int_{\mathrm{s}_2}^{\mathrm{s}} \frac{L_1\mathcal{U}(\varkappa)}{\beta_2^{1/r}(\varkappa)\pi_2(\varkappa)}\mathrm{d}\varkappa \geq z(\mathrm{s}_2) + \varrho \frac{L_1\mathcal{U}(\mathrm{s})}{\pi_2(\mathrm{s})}\int_{\mathrm{s}_2}^{\mathrm{s}} \frac{1}{\beta_2^{1/r}(\varkappa)}\mathrm{d}\varkappa \\ &= z(\mathrm{s}_2) + \varrho \frac{L_1\mathcal{U}(\mathrm{s})}{\pi_2(\mathrm{s})}\pi_2(\mathrm{s}) - \varrho \frac{L_1\mathcal{U}(\mathrm{s})}{\pi_2(\mathrm{s})}\int_{\mathrm{s}_0}^{\mathrm{s}_2} \frac{1}{\beta_2^{1/r}(\varkappa)}\mathrm{d}\varkappa > \varrho L_1\mathcal{U}(\mathrm{s}). \end{aligned} \tag{19}$$

Then,

$$(1-\varrho)L_1\mathcal{U}(\mathrm{s}) > \pi_2(\mathrm{s})(L_2\mathcal{U}(\mathrm{s}))^{1/r},$$

and

$$\left(\frac{L_1\mathcal{U}(\mathrm{s})}{\pi_2^{1-\varrho}(\mathrm{s})}\right)' = \frac{(L_2\mathcal{U}(\mathrm{s}))^{1/r}\pi_2(\mathrm{s}) - (1-\varrho)L_1\mathcal{U}(\mathrm{s})}{\beta_2^{1/r}(\mathrm{s})\pi_2^{2-\varrho}(\mathrm{s})} < 0. \tag{20}$$

We deduce directly from (20) and from property $L_1\mathcal{U}$ that $\varrho < 1$ is increasing. Using this in (18), additionally, taking into account (13), we obtain

$$\begin{aligned} z(\mathrm{s}) &\geq z(\mathrm{s}_3) + \varrho \int_{\mathrm{s}_3}^{\mathrm{s}} \frac{L_1\mathcal{U}(\varkappa)}{\beta_2^{1/r}(\varkappa)\pi_2(\varkappa)}\mathrm{d}\varkappa \\ &\geq z(\mathrm{s}_3) + \varrho \frac{L_1\mathcal{U}(\mathrm{s})}{\pi_2^{1-\varrho}(\mathrm{s})}\int_{\mathrm{s}_3}^{\mathrm{s}} \frac{1}{\beta_2^{1/r}(\varkappa)\pi_2^{\varrho}(\varkappa)}\mathrm{d}\varkappa \\ &\geq \frac{\varrho}{1-\varrho}\frac{L_1\mathcal{U}(\mathrm{s})}{\pi_2^{1-\varrho}(\mathrm{s})}\left(\pi_2^{1-\varrho}(\mathrm{s}) - \pi_2^{1-\varrho}(\mathrm{s}_3)\right) \\ &\geq \frac{c_1\varrho}{1-\varrho}L_1\mathcal{U}(\mathrm{s}) \\ &\geq (\varrho_* + c_2)L_1\mathcal{U}(\mathrm{s}), \end{aligned}$$

which implies

$$(1-\varrho_*)L_1\mathcal{U}(\mathrm{s}) > (1-\varrho_*-c_2)L_1\mathcal{U}(\mathrm{s}) > (L_2\mathcal{U}(\mathrm{s}))^{1/r}\pi_2(\mathrm{s}),$$

and

$$\left(\frac{L_1\mathcal{U}(\mathrm{s})}{\pi_2^{1-\varrho_*-c_2}(\mathrm{s})}\right)' < 0. \tag{21}$$

The conclusion then immediately follows.
$(A_{2,2})$: Obviously, (21) also implies that $L_1\mathcal{U}/\pi_2^{1-\varrho_*} \to 0$ as $\mathrm{s} \to \infty$, since otherwise

$$\frac{L_1\mathcal{U}(\mathrm{s})}{\pi_2^{1-\varrho_*-c_2}(\mathrm{s})} = \frac{L_1\mathcal{U}(\mathrm{s})}{\pi_2^{1-\varrho_*}(\mathrm{s})}\pi_2^{c_2}(\mathrm{s}) \to \infty \text{ as } \mathrm{s} \to \infty, \tag{22}$$

which is a contradiction.
$(A_{2,3})$: Using that by $(A_{2,1})$ and $(A_{2,2})$, $L_1\mathcal{U}/\pi_2^{1-\varrho_*}$ is a decreasing, we have

$$\begin{aligned}
\mathcal{U}(\mathrm{s}) &= \mathcal{U}(\mathrm{s}_4) + \int_{\mathrm{s}_4}^{\mathrm{s}} \frac{L_1\mathcal{U}(\varkappa)}{\pi_2^{1-\varrho_*}(\varkappa)} \frac{\pi_2^{1-\varrho_*}(\varkappa)}{\beta_1(\varkappa)} \mathrm{d}\varkappa \\
&\geq \mathcal{U}(\mathrm{s}_4) + \frac{L_1\mathcal{U}(\mathrm{s})}{\pi_2^{1-\varrho_*}(\mathrm{s})} \int_{\mathrm{s}_4}^{\mathrm{s}} \frac{\pi_2^{1-\varrho_*}(\varkappa)}{\beta_1(\varkappa)} \mathrm{d}\varkappa \\
&= \mathcal{U}(\mathrm{s}_4) + \frac{L_1\mathcal{U}(\mathrm{s})}{\pi_2^{1-\varrho_*}(\mathrm{s})} \int_{\mathrm{s}_0}^{\mathrm{s}} \frac{\pi_2^{1-\varrho_*}(\varkappa)}{\beta_1(\varkappa)} \mathrm{d}\varkappa - \frac{L_1\mathcal{U}(\mathrm{s})}{\pi_2^{1-\varrho_*}(\mathrm{s})} \int_{\mathrm{s}_0}^{\mathrm{s}_4} \frac{\pi_2^{1-\varrho_*}(\varkappa)}{\beta_1(\varkappa)} \mathrm{d}\varkappa \\
&> \frac{L_1\mathcal{U}(\mathrm{s})}{\pi_2^{1-\varrho_*}(\mathrm{s})} \int_{\mathrm{s}_0}^{\mathrm{s}} \frac{\pi_2^{1-\varrho_*}(\varkappa)}{\beta_1(\varkappa)} \mathrm{d}\varkappa \\
&\geq k\frac{\pi_{12}(\mathrm{s})}{\pi_2(\mathrm{s})} L_1\mathcal{U}(\mathrm{s}).
\end{aligned}$$

Therefore,

$$\left(\frac{\mathcal{U}(\mathrm{s})}{\pi_{12}^{1/k}(\mathrm{s})}\right)' = \frac{k\pi_{12}(\mathrm{s})L_1\mathcal{U}(\mathrm{s}) - \pi_2(\mathrm{s})\mathcal{U}(\mathrm{s})}{k\beta_1(\mathrm{s})\pi_2^{1/k+1}(\mathrm{s})} < 0.$$

The proof of Lemma is complete. □

Corollary 1. *We suppose that* $\varrho_* \geq 1$. *Then,* $N_2 = \varnothing$.

Proof. This follows from

$$(1 - \varrho_*)L_1\mathcal{U}(\mathrm{s}) > (L_2\mathcal{U}(\mathrm{s}))^{1/r}\pi_2(\mathrm{s}),$$

and the property that L_2 is positive. □

Corollary 2. *We suppose that* $\varrho_* > 0$ *and* $\lambda_* = \infty$. *Then,* $\Omega = \varnothing$.

Proof. We let $\mathcal{U} \in \Omega$ and choose $\mathrm{s}_1 \geq \mathrm{s}_0$ such that $\mathcal{U}(\tau(\mathrm{s})) > 0$ and parts $(A_{2,1})$-$(A_{2,3})$ in Lemma 4 hold for $\mathrm{s} \geq \mathrm{s}_1 \geq \mathrm{s}_0$ and choose fixed but arbitrarily large $\lambda \leq \lambda_*$, $\varrho \leq \varrho_*$, and $k \leq k_*$ satisfying (9), (10) and (11), respectively, for $\mathrm{s} \geq \mathrm{s}_1$. Using (15) and the decreasing of $\mathcal{U}/\beta_{12}^{1/k}$, we have

$$\begin{aligned}
z'(\mathrm{s}) &\geq \varrho \frac{\mathcal{U}^r(\tau(\mathrm{s}))}{\beta_2^{1/r}(\mathrm{s})\pi_{12}^{r/k}(\tau(\mathrm{s}))\pi_{12}^{r(1-1/k)}(\tau(\mathrm{s}))} (L_2\mathcal{U}(\mathrm{s}))^{1/r-1} \\
&\geq \varrho \frac{\mathcal{U}^r(\mathrm{s})}{\pi_{12}^{r/k}(\mathrm{s})} \frac{1}{\beta_2^{1/r}(\mathrm{s})\pi_{12}^{r(1-1/k)}(\tau(\mathrm{s}))} (L_2\mathcal{U}(\mathrm{s}))^{1/r-1}.
\end{aligned}$$

Using $(A_{2,3})$, (17) and (9), we obtain

$$\begin{aligned} z'(\mathrm{s}) &\geq \varrho \frac{\mathcal{U}^r(\mathrm{s})}{\pi_{12}^{r/k}(\mathrm{s})} \frac{1}{\beta_2^{1/r}(\mathrm{s})\pi_{12}^{r(1-1/k)}(\tau(\mathrm{s}))}\left(\frac{\mathcal{U}(\mathrm{s})}{\pi_{12}(\mathrm{s})}\right)^{1-r} \\ &= \varrho \frac{\pi_{12}^{r(1-1/k)}(\mathrm{s})}{\beta_2^{1/r}(\mathrm{s})\pi_{12}^{r(1-1/k)}(\tau(\mathrm{s}))} \frac{\mathcal{U}(\mathrm{s})}{\pi_{12}(\mathrm{s})} \\ &\geq \varrho \frac{\lambda^{r(1-1/k)}}{\beta_2^{1/r}(\mathrm{s})} \frac{\mathcal{U}(\mathrm{s})}{\pi_{12}(\mathrm{s})} \geq \varrho k \lambda^{r(1-1/k)} \frac{L_1\mathcal{U}(\mathrm{s})}{\beta_2^{1/r}(\mathrm{s})\pi_2(\mathrm{s})}. \end{aligned}$$

Integrating the last inequality from s_2 to s and using that $L_1\mathcal{U}/\pi_2$ is a decreasing function tending to zero, we obtain

$$z(\mathrm{s}) \geq k\varrho\lambda^{r(1-1/k)} L_1\mathcal{U}(\mathrm{s}). \tag{23}$$

Consequently,

$$\left(1 - k\varrho\lambda^{r(1-1/k)}\right) L_1\mathcal{U}(\mathrm{s}) \geq (L_2\mathcal{U}(\mathrm{s}))^{1/r}\pi_2(\mathrm{s}).$$

We can choose $\lambda > (1/k\varrho)^{k/r(k-1)}$, since λ can be arbitrarily large, which is contrary to the fact that $L_2\mathcal{U}$ is positive.

The proof of Corollary is complete. □

Corollary 3. *Suppose that $\varrho_* > 0$ and $k_* = \infty$. Then $\Omega = \varnothing$.*

Proof. The proof is omitted as it can be obtained by following the same steps as in Corollary 2, taking into account that k can take on an arbitrarily large value. □

Remark 4. *For $\varrho_* \in (0,1)$, $k_* \in [1,\infty)$ and $\lambda_* \in [1,\infty)$, we deduce that*

$$\varrho_0 = \varrho_*,$$

$$\varrho_n = \frac{\varrho_0 k_{n-1} \lambda_*^{r(1-1/k_{n-1})}}{1-\varrho_{n-1}}, \; n \in \mathbb{N}, \tag{24}$$

where k_n satisfies

$$k_n = \liminf_{\mathrm{s}\to\infty} \frac{\pi_2^{\varrho_n}(\mathrm{s})}{\pi_{12}(\mathrm{s})} \int_{\mathrm{s}_0}^{\mathrm{s}} \frac{\pi_2^{1-\varrho_n}(\varkappa)}{\beta_1(\varkappa)} \mathrm{d}\varkappa, \; n \in \mathbb{N}_0. \tag{25}$$

If $\varrho_i < 1$ and $k_i \in [1,\infty)$ for $i = 0,1,\ldots,n$, then ϱ_{n+1} exists. In this case, we obtain

$$\frac{\varrho_1}{\varrho_0} = \frac{k_0 \lambda^{r(1-1/k_0)}}{1-\varrho_0} > 1,$$

and

$$\begin{aligned} k_1 &= \liminf_{\mathrm{s}\to\infty} \frac{\pi_2^{\varrho_1}(\mathrm{s})}{\pi_{12}(\mathrm{s})} \int_{\mathrm{s}_0}^{\mathrm{s}} \frac{\pi_2^{1-\varrho_1}(\varkappa)}{\beta_1(\varkappa)} \mathrm{d}\varkappa = \liminf_{\mathrm{s}\to\infty} \frac{\pi_2^{\varrho_1}(\mathrm{s})}{\pi_{12}(\mathrm{s})} \int_{\mathrm{s}_0}^{\mathrm{s}} \frac{\pi_2^{1-\varrho_0-(\varrho_1-\varrho_0)}(\varkappa)}{\beta_1(\varkappa)} \mathrm{d}\varkappa \\ &\geq \liminf_{\mathrm{s}\to\infty} \frac{\pi_2^{\varrho_0}(\mathrm{s})}{\pi_{12}(\mathrm{s})} \int_{\mathrm{s}_0}^{\mathrm{s}} \frac{\pi_2^{1-\varrho_0}(\varkappa)}{\beta_1(\varkappa)} \mathrm{d}\varkappa = k_0. \end{aligned}$$

Therefore, we can conclude that

$$k_1 \geq k_0.$$

By using induction on n, we can also obtain

$$\frac{\varrho_{n+1}}{\varrho_n} \geq \ell_n > 1, \tag{26}$$

where

$$\ell_0 := \frac{k_0 \lambda_*^{r(1-1/k_{n-1})}}{1-\varrho_0},$$

$$\ell_n := \frac{k_n \lambda_*^{r(1/k_{n-1}-1/k_n)}(1-\varrho_{n-1})}{k_{n-1}(1-\varrho_n)},\ n \in \mathbb{N}, \tag{27}$$

with

$$k_n \geq k_{n-1}.$$

In the following, we can suppose that λ_*, k_*, ϱ_* are well defined $\lambda_* \in [1,\infty)$, $k_* \in [1,\infty)$, and $\varrho_* \in (0,1)$.

Lemma 6. *We suppose that $\delta_* > 0$ and $\mathcal{U} \in \Omega$. Then, for any $n \in \mathbb{N}_0$, ϱ_n and k_n defined by (24) and (25), respectively, and for a suficiently large t*

$(A_{n,1})$ $L_1\mathcal{U}/\pi_2^{1-\varrho_n}$ *is decreasing, and* $(1-\varrho_n)L_1\mathcal{U} > (L_2\mathcal{U})^{1/r}\pi_2$;

$(A_{n,2})$ $\lim_{s\to\infty} L_1\mathcal{U}(s)/\pi_2^{1-\varrho_n}(s) = 0$;

$(A_{n,3})$ $\mathcal{U}/\pi_{12}^{1/\varepsilon_n k_n}$ *is decreasing and* $\mathcal{U} > \varepsilon_n k_n(\pi_{12}/\pi_2)L_1\mathcal{U}$ *for any* $\varepsilon_n \in (0,1)$.

Proof. We let $\mathcal{U} \in \Omega$ with $\mathcal{U}(\tau(s)) > 0$ and parts $(A_{1,1})$-$(A_{1,3})$ in Lemma 4 hold for $s \geq s_1 \geq s_0$ and choose fixed but arbitrarily large $\varrho \leq \varrho_*$, and $k \leq k_*$ satisfying (10) and (11), respectively, for $s \geq s_1$. We proceed by induction on n. For $n = 0$, the conclusion follows from Lemma 5 with $\varepsilon_0 = k/k_*$. Next, we assume that $(A_{n,1})$–$(A_{n,3})$ hold for $n \geq 1$ for $s \geq s_n \geq s_1$. We need to show that they each hold for $n+1$.
$(A_{n+1,1})$: Using $(A_{n,3})$ in (15), we obtain

$$\begin{aligned} z'(s) &\geq \varrho \frac{\mathcal{U}^r(\tau(s))}{\beta_2^{1/r}(s)\pi_{12}^{r/\varepsilon_n k_n}(\tau(s))\pi_{12}^{r(1-1/\varepsilon_n k_n)}(\tau(s))}(L_2\mathcal{U}(s))^{1/r-1} \\ &\geq \varrho \frac{\mathcal{U}^r(s)}{\beta_2^{1/r}(s)\pi_{12}^{r/\varepsilon_n k_n}(s)\pi_{12}^{r(1-1/\varepsilon_n k_n)}(\tau(s))}\left(\frac{\mathcal{U}(s)}{\pi_{12}(s)}\right)^{1-r} \\ &= \varrho \frac{\pi_{12}^{r(1-1/\varepsilon_n k_n)}(s)}{\pi_{12}^{r(1-1/\varepsilon_n k_n)}(\tau(s))}\frac{\mathcal{U}(s)}{\beta_2^{1/r}(s)\pi_{12}(s)} \\ &\geq \varepsilon_n k_n \varrho \lambda^{r(1-1/\varepsilon_n k_n)} \frac{L_1\mathcal{U}(s)}{\beta_2^{1/r}(s)\pi_2(s)}. \end{aligned}$$

Integrating the above inequality from s_n to s and using $(A_{n,1})$ and $(A_{n,2})$, we have

$$\begin{aligned} z(s) &\geq z(s_n) + \varepsilon_n k_n \varrho \lambda^{r(1-1/\varepsilon_n k_n)} \int_{s_n}^{s} \frac{L_1\mathcal{U}(\varkappa)}{\beta_2^{1/r}(\varkappa)\pi_2(\varkappa)} \mathrm{d}\varkappa \\ &\geq z(s_n) + \varepsilon_n k_n \varrho \lambda^{r(1-1/\varepsilon_n k_n)} \frac{L_1\mathcal{U}(s)}{\pi_2^{1-\varrho_n}(s)} \int_{s_n}^{s} \frac{1}{\beta_2^{1/r}(\varkappa)\pi_2^{\varrho_n}(\varkappa)} \mathrm{d}\varkappa \\ &\geq z(s_n) + \frac{\varepsilon_n k_n \varrho \lambda^{r(1-1/\varepsilon_n k_n)}}{1-\varrho_n} \frac{L_1\mathcal{U}(s)}{\pi_2^{1-\varrho_n}(s)} \left[\pi_2^{1-\varrho_n}(s) - \pi_2^{1-\varrho_n}(s_n)\right] \\ &> \frac{\varepsilon_n k_n \varrho \lambda^{r(1-1/\varepsilon_n k_n)}}{1-\varrho_n} L_1\mathcal{U}(s) = \mu \varrho_{n+1} L_1\mathcal{U}(s), \end{aligned} \tag{28}$$

where

$$\mu = \frac{\varrho}{\varrho_*}\varepsilon_n \frac{\lambda^{r(1-1/\varepsilon_n k_n)}}{\lambda_*^{r(1-1/k_n)}} \in (0,1),$$

and

$$\lim_{\substack{\lambda \to \lambda_* \\ \varepsilon_n \to 1 \\ \varrho \to \varrho_*}} \mu = 1.$$

We choose μ such that

$$\mu > \frac{1}{1 - \varrho_n + \varrho_{n+1}} = \frac{1}{1 + \varrho_n(\ell_n - 1)}, \tag{29}$$

where ℓ_n satisfies (26). Then,

$$\frac{\mu \varrho_{n+1}}{1 - \mu \varrho_{n+1}} > \frac{\varrho_{n+1}}{(1 + \varrho_n(\ell_n - 1))\left(1 - \frac{\ell_n \varrho_n}{1 + \varrho_n(\ell_n - 1)}\right)} = \frac{\varrho_{n+1}}{1 - \varrho_n},$$

and there exist two constants $c_1 \in (0,1)$ and $c_2 > 0$ such that

$$c_1 \frac{\mu(1 - \varrho_n)\varrho_{n+1}}{1 - \mu \varrho_{n+1}} > \varrho_{n+1} + c_2.$$

According to Definition (14) of z, we deduce that

$$(1 - \mu \varrho_{n+1}) L_1 \mathcal{U}(\mathrm{s}) > (L_2 \mathcal{U}(\mathrm{s}))^{1/r} \pi_2(\mathrm{s}),$$

and

$$\left(\frac{L_1 \mathcal{U}(\mathrm{s})}{\pi_2^{1 - \mu \varrho_{n+1}}(\mathrm{s})} \right)' < 0.$$

Using the above monotonicity in (28), we see that

$$\begin{aligned}
z(\mathrm{s}) &\geq z(\mathrm{s}_n) + \varepsilon_n k_n \varrho \lambda^{r(1 - 1/\varepsilon_n k_n)} \int_{\mathrm{s}_n}^{\mathrm{s}} \frac{L_1 \mathcal{U}(\varkappa)}{\beta_2^{1/r}(\varkappa) \pi_2(\varkappa)} \mathrm{d}\varkappa \\
&\geq z(\mathrm{s}_n) + \frac{\varepsilon_n k_n \varrho \lambda^{r(1 - 1/\varepsilon_n k_n)}}{1 - \mu \varrho_{n+1}} \frac{L_1 \mathcal{U}(\mathrm{s})}{\pi_2^{1 - \mu \varrho_{n+1}}(\mathrm{s})} \left(\pi_2^{1 - \mu \varrho_{n+1}}(\mathrm{s}) - \pi_2^{1 - \mu \varrho_{n+1}}(\mathrm{s}_n) \right) \\
&\geq \frac{c_1 \varepsilon_n k_n \varrho \lambda^{r(1 - 1/\varepsilon_n k_n)}}{1 - \mu \varrho_{n+1}} L_1 \mathcal{U}(\mathrm{s}) \\
&= c_1 \mu \varrho_{n+1} \frac{1 - \varrho_n}{1 - \mu \varrho_{n+1}} L_1 \mathcal{U}(\mathrm{s}) \\
&> (\varrho_{n+1} + c_2) L_1 \mathcal{U}(\mathrm{s}).
\end{aligned}$$

Then,

$$(1 - \varrho_{n+1} - c_2) L_1 \mathcal{U}(\mathrm{s}) > (L_2 \mathcal{U}(\mathrm{s}))^{1/r} \pi_2(\mathrm{s}), \tag{30}$$

and

$$\left(\frac{L_1 \mathcal{U}(\mathrm{s})}{\pi_2^{1 - \varrho_{n+1} - c_2}(\mathrm{s})} \right)' < 0. \tag{31}$$

This leads to the conclusion.
$(A_{n+1,2})$: Clearly, (31) also implies that $L_1 \mathcal{U} / \pi_2^{1 - \varrho_{n+1}} \to 0$ as $\mathrm{s} \to \infty$, since otherwise

$$\frac{L_1 \mathcal{U}(\mathrm{s})}{\pi_2^{1 - \varrho_{n+1} - c_2}(\mathrm{s})} = \frac{L_1 \mathcal{U}(\mathrm{s})}{\pi_2^{1 - \varrho_{n+1}}(\mathrm{s})} \pi_2^{c_2}(\mathrm{s}) \to \infty \text{ as } \mathrm{s} \to \infty, \tag{32}$$

which is a contradiction.
$(A_{n+1,3})$: Using that by $(A_{n+1,1})$ and $(A_{n+1,2})$, $L_1\mathcal{U}/\pi_2^{1-\varrho_{n+1}}$ is decreasing, we obtain, for any $\varepsilon_n \in (0,1)$,

$$\begin{aligned}
\mathcal{U}(\mathrm{s}) &= \mathcal{U}(\mathrm{s}_n'') + \int_{\mathrm{s}_n''}^{\mathrm{s}} \frac{L_1\mathcal{U}(\varkappa)}{\pi_2^{1-\varrho_{n+1}}(\varkappa)} \frac{\pi_2^{1-\varrho_{n+1}}(\varkappa)}{\beta_1(\varkappa)}\mathrm{d}\varkappa \\
&\geq \mathcal{U}(\mathrm{s}_n'') + \frac{L_1\mathcal{U}(\mathrm{s})}{\pi_2^{1-\varrho_{n+1}}(\mathrm{s})} \int_{\mathrm{s}_n''}^{\mathrm{s}} \frac{\pi_2^{1-\varrho_{n+1}}(\varkappa)}{\beta_1(\varkappa)}\mathrm{d}\varkappa \\
&= \mathcal{U}(\mathrm{s}_n'') + \frac{L_1\mathcal{U}(\mathrm{s})}{\pi_2^{1-\varrho_{n+1}}(\mathrm{s})} \int_{\mathrm{s}_0}^{\mathrm{s}} \frac{\pi_2^{1-\varrho_{n+1}}(\varkappa)}{\beta_1(\varkappa)}\mathrm{d}\varkappa - \frac{L_1\mathcal{U}(\mathrm{s})}{\pi_2^{1-\varrho_{n+1}}(\mathrm{s})} \int_{\mathrm{s}_0}^{\mathrm{s}_n''} \frac{\pi_2^{1-\varrho_{n+1}}(\varkappa)}{\beta_1(\varkappa)}\mathrm{d}\varkappa \\
&> \frac{L_1\mathcal{U}(\mathrm{s})}{\pi_2^{1-\varrho_{n+1}}(\mathrm{s})} \int_{\mathrm{s}_0}^{\mathrm{s}} \frac{\pi_2^{1-\varrho_{n+1}}(\varkappa)}{\beta_1(\varkappa)}\mathrm{d}\varkappa \\
&\geq \varepsilon_{n+1}k_{n+1}\frac{\pi_{12}(\mathrm{s})}{\pi_2(\mathrm{s})}L_1\mathcal{U}(\mathrm{s}),
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{\mathcal{U}(\mathrm{s})}{\pi_{12}^{1/\varepsilon_{n+1}k_{n+1}}(\mathrm{s})}\right)' &= \frac{\varepsilon_{n+1}k_{n+1}\pi_{12}^{1/\varepsilon_{n+1}k_{n+1}}(\mathrm{s})L_1\mathcal{U}(\mathrm{s}) - \pi_{12}^{1/\varepsilon_{n+1}k_{n+1}-1}(\mathrm{s})\pi_2(\mathrm{s})\mathcal{U}(\mathrm{s})}{\varepsilon_{n+1}k_{n+1}\beta_1(\mathrm{s})\pi_{12}^{2/\varepsilon_{n+1}k_{n+1}}(\mathrm{s})} \\
&= \frac{\varepsilon_{n+1}k_{n+1}\pi_{12}(\mathrm{s})L_1\mathcal{U}(\mathrm{s}) - \pi_2(\mathrm{s})\mathcal{U}(\mathrm{s})}{\varepsilon_{n+1}k_{n+1}\beta_1(\mathrm{s})\pi_{12}^{1/\varepsilon_{n+1}k_{n+1}+1}(\mathrm{s})} < 0.
\end{aligned}$$

The proof of Lemma is complete. □

Corollary 4. *We assume that $\varrho_i < 1$, $i = 0,1,2,\ldots,n-1$ and $\varrho_n \geq 1$. Then, $N_2 = \varnothing$.*

Proof. This follows directly from

$$(1-\varrho_n)L_1\mathcal{U}(\mathrm{s}) > (L_2\mathcal{U}(\mathrm{s}))^{1/r}\pi_2(\mathrm{s}),$$

and the fact that L_2 is positive. □

By applying the previous corollary and Equation (26), we can see that the sequence $\{\varrho_n\}$ defined in (24) is both increasing and bounded from above. Hence, there exists

$$\lim_{n\to\infty}\varrho_n = \varrho_j \in (0,1).$$

Moreover, we can satisfy the equation

$$\varrho_j = \frac{\varrho_* k_j \lambda_*^{r(1-1/k_j)}}{1-\varrho_j}, \tag{33}$$

where

$$k_j = \liminf_{\mathrm{s}\to\infty}\frac{\pi_2^{\varrho_j}(\mathrm{s})}{\pi_{12}(\mathrm{s})}\int_{\mathrm{s}_0}^{\mathrm{s}}\frac{\pi_2^{1-\varrho_j}(\varkappa)}{\beta_1(\mathrm{s})}\mathrm{d}\varkappa.$$

This allows us collection of important results that directly imply the nonexistence of N_2 type solutions.

Lemma 7. *We assume that $\lambda_* < \infty$ and that (33) does not possess a root on $(0,1)$. Then, $\Omega = \varnothing$.*

Corollary 5. *We assume that $\lambda_* < \infty$. If*

$$\varrho_* > \max\left\{ \frac{\varrho_j (1-\varrho_j) \lambda_*^{r\left(1/k_j - 1\right)}}{k_j} : 0 < \varrho_j < 1 \right\}, \tag{34}$$

then $\Omega = \varnothing$.

Lemma 8. *We assume that (2) hold. Furthermore, we assume that there exists $\rho \in C^1([s_0,\infty),(0,\infty))$ such that*

$$\limsup_{s\to\infty} \int_{s_0}^{s} \left(\rho(\varkappa) q(\varkappa) \left(\frac{\tau(\varkappa)}{\varkappa} \right)^{2/\epsilon} - \frac{\beta_1^r(\varkappa)(\rho'(\varkappa))_+^{r+1}}{(r+1)^{r+1} \pi_2^r(\varkappa) \rho^r(\varkappa)} \right) d\varkappa = \infty, \tag{35}$$

where $(\rho'(s))_+ = \max\{0, \rho'(s)\}$. Then, $N_2 = \varnothing$.

Proof. We assume the contrary, that $\mathcal{U} \in \Omega$. Now, we define

$$w(s) = \rho(s) \frac{L_2 \mathcal{U}(s)}{\mathcal{U}^r(s)},\ s \ge s_1, \tag{36}$$

then, $w(s) > 0$ and

$$\begin{aligned} w'(s) &= \rho'(s) \frac{L_2\mathcal{U}(s)}{\mathcal{U}^r(s)} + \rho(s) \frac{L_3\mathcal{U}(s)}{\mathcal{U}^r(s)} - r\rho(s) \frac{L_2\mathcal{U}(s)}{\mathcal{U}^r(s)} \frac{\mathcal{U}'(s)}{\mathcal{U}(s)} \\ &= \rho'(s) \frac{L_2\mathcal{U}(s)}{\mathcal{U}^r(s)} + \rho(s) \frac{L_3\mathcal{U}(s)}{\mathcal{U}^r(s)} - r\rho(s) \frac{L_2\mathcal{U}(s)}{\mathcal{U}^r(s)} \frac{1}{\beta_1(s)} \frac{L_1\mathcal{U}(s)}{\mathcal{U}(s)} \\ &= -\rho(s) q(s) \frac{\mathcal{U}^r(\tau(s))}{\mathcal{U}^r(s)} + \frac{\rho'(s)}{\rho(s)} w(s) - r w(s) \frac{1}{\beta_1(s)} \frac{L_1\mathcal{U}(s)}{\mathcal{U}(s)}. \end{aligned}$$

Then, in view of (1) and from Lemma 4, in view of $(A_{1,2})$, we have

$$w'(s) \le -\rho(s) q(s) \frac{\mathcal{U}^r(\tau(s))}{\mathcal{U}^r(s)} + \frac{\rho'(s)}{\rho(s)} w(s) - r \frac{\pi_2(s)}{\beta_1(s)} w(s) \frac{(L_2\mathcal{U})^{1/r}}{\mathcal{U}(s)}.$$

Since $\mathcal{U} > 0$, $L_1\mathcal{U} > 0$, $L_2\mathcal{U} > 0$, from Lemma 2, we obtain

$$\frac{\mathcal{U}(s)}{\mathcal{U}'(s)} \ge \frac{\epsilon}{2} s.$$

Integrating the last inequality from $\tau(s)$ to s, we find

$$\frac{\mathcal{U}(\tau(s))}{\mathcal{U}(s)} \ge \left(\frac{\tau(s)}{s} \right)^{2/\epsilon},$$

which implies that

$$w'(s) \le -\rho(s) q(s) \left(\frac{\tau(s)}{s} \right)^{2r/\epsilon} + \frac{(\rho'(s))_+}{\rho(s)} w(s) - \frac{r\pi_2(s)}{\beta_1(s)\rho^{1/r}(s)} w^{1+1/r}(s). \tag{37}$$

Setting

$$B = \frac{(\rho'(s))_+}{\rho(s)} \text{ and } B = \frac{r\pi_2(s)}{\beta_1(s)\rho^{1/r}(s)},$$

and using Lemma 1, we see that

$$\frac{(\rho'(\mathrm{s}))_+}{\rho(\mathrm{s})}w(\mathrm{s}) - \frac{r\pi_2(\mathrm{s})}{\beta_1(\mathrm{s})\rho^{1/r}(\mathrm{s})}w^{1+1/r}(\mathrm{s}) \leq \frac{\beta_1^r(\mathrm{s})(\rho'(\mathrm{s}))_+^{r+1}}{(r+1)^{r+1}\pi_2^r(\mathrm{s})\rho^r(\mathrm{s})}. \tag{38}$$

Thus, from (37) and (38), we obtain

$$w'(\mathrm{s}) \leq -\left(\rho(\mathrm{s})q(\mathrm{s})\left(\frac{\tau(\mathrm{s})}{\mathrm{s}}\right)^{2r/\epsilon} - \frac{\beta_1^r(\mathrm{s})(\rho'(\mathrm{s}))_+^{r+1}}{(r+1)^{r+1}\pi_2^r(\mathrm{s})\rho^r(\mathrm{s})}\right). \tag{39}$$

Integrating (39) from s_1 to s, we obtain

$$-w(\mathrm{s}_1) < w(\mathrm{s}) - w(\mathrm{s}_1) \leq -\int_{\mathrm{s}_1}^{\mathrm{s}}\left(\rho(\varkappa)q(\varkappa)\left(\frac{\tau(\varkappa)}{\varkappa}\right)^{2r/\epsilon} - \frac{\beta_1^r(\varkappa)(\rho'(\varkappa))_+^{r+1}}{(r+1)^{r+1}\pi_2^r(\varkappa)\rho^r(\varkappa)}\right)\mathrm{d}\varkappa,$$

which yields

$$\int_{\mathrm{s}_1}^{\mathrm{s}}\left(\rho(\varkappa)q(\varkappa)\left(\frac{\tau(\varkappa)}{\varkappa}\right)^{2r/\epsilon} - \frac{\beta_1^r(\varkappa)(\rho'(\varkappa))_+^{r+1}}{(r+1)^{r+1}\pi_2^r(\varkappa)\rho^r(\varkappa)}\right)\mathrm{d}\varkappa < w(\mathrm{s}_1),$$

for all large s. This is a contradiction to (35). □

4. Convergence to Zero of Kneser Solutions

In the following part, we provide results that ensure the asymptotic convergence of any Kneser solution to zero. We start by highlighting a crucial fact that an unbounded nonoscillatory solution can exist only if

$$\int_{\mathrm{s}_0}^{\infty} q(\varkappa)\mathrm{d}\varkappa < \infty. \tag{40}$$

The proof is stated briefly for the reader's convenience.

Lemma 9. *Suppose that*

$$\int_{\mathrm{s}_0}^{\infty} q(\varkappa)\mathrm{d}\varkappa = \infty. \tag{41}$$

Then (1) has property A.

Proof. We suppose, on the contrary, that $\mathcal{U}$ is a positive solution of (1), that is, $\mathcal{U}(\mathrm{s}) \geq l > 0$ for $\mathrm{s} \geq \mathrm{s}_1$. By integrating (1) from s_2 to s, we have

$$\begin{aligned} L_2\mathcal{U}(\mathrm{s}) &= L_2\mathcal{U}(\mathrm{s}_2) - \int_{\mathrm{s}_2}^{\mathrm{s}} q(\varkappa)\mathcal{U}^r(\tau(\varkappa))\mathrm{d}\varkappa \\ &\leq L_2\mathcal{U}(\mathrm{s}_2) - l^r\int_{\mathrm{s}_2}^{\mathrm{s}} q(\varkappa)\mathrm{d}\varkappa \to -\infty \text{ as } \mathrm{s} \to \infty, \end{aligned}$$

which contradicts the positivity of $L_2\mathcal{U}$. □

Therefore, we assume the validity of Equation (40) without further explanation. We then distinguish between two cases,

$$\int_{\mathrm{s}_0}^{\infty}\left(\frac{1}{\beta_2(\vartheta)}\int_{\vartheta}^{\infty} q(\varkappa)\mathrm{d}\varkappa\right)^{1/r}\mathrm{d}\vartheta = \infty, \tag{42}$$

and

$$\int_{\mathrm{s}_0}^{\infty}\left(\frac{1}{\beta_2(\vartheta)}\int_{\vartheta}^{\infty} q(\varkappa)\mathrm{d}\varkappa\right)^{1/r}\mathrm{d}\vartheta < \infty. \tag{43}$$

Lemma 10. *We assume either (42) or*

$$\int_{s_0}^{\infty} \frac{1}{\beta_1(v)} \int_{v}^{\infty} \left(\frac{1}{\beta_2(\vartheta)} \int_{\vartheta}^{\infty} q(\varkappa) d\varkappa \right)^{1/r} d\vartheta dv = \infty. \tag{44}$$

If $\mathcal{U}$ is a Kneser solution of (1), then $\lim_{s\to\infty} \mathcal{U}(s) = 0$.

Proof. We suppose $\mathcal{U} \in N_1$ and choose $s_1 \geq s_0$ such that $\mathcal{U}(\tau(s)) > 0$ on $[s_1, \infty)$. Obviously, there is a finite number l such that $\lim_{s\to\infty} \mathcal{U}(s) = l \geq 0$. We suppose that $l > 0$. Then, there exists $s_2 \geq s_1$ such that $\mathcal{U}(\tau(s)) > l$ for $s \geq s_2$.
If (42) holds, then, by Integrating (1) from s to ∞, we obtain

$$L_2\mathcal{U}(s) \geq \int_{s}^{\infty} q(\varkappa)\mathcal{U}^r(\tau(\varkappa)) d\varkappa > l^r \int_{s}^{\infty} q(\varkappa) d\varkappa,$$

that is,

$$(L_1\mathcal{U}(s))' > \frac{l}{\beta_2^{1/r}(s)} \left(\int_{s}^{\infty} q(\varkappa) d\varkappa \right)^{1/r}. \tag{45}$$

Integrating (45) from s_2 to s, we obtain

$$-L_1\mathcal{U}(s) > -L_1\mathcal{U}(s_2) - l^r \int_{s_2}^{s} \frac{1}{\beta_2^{1/r}(\vartheta)} \left(\int_{\vartheta}^{\infty} q(\varkappa) d\varkappa \right)^{1/r} d\vartheta \to -\infty \text{ as } s \to \infty,$$

which contradicts the positivity of $-L_1\mathcal{U}$.
If (44) holds, then, by Integrating (45) from s to ∞, we have

$$-\mathcal{U}'(s) > \frac{l}{\beta_1(s)} \int_{s}^{\infty} \left(\frac{1}{\beta_2(\vartheta)} \int_{\vartheta}^{\infty} q(\varkappa) d\varkappa \right)^{1/r} d\vartheta,$$

and therefore,

$$\mathcal{U}(s) \leq \mathcal{U}(s_2) - l^r \int_{s_2}^{s} \frac{1}{\beta_1(v)} \int_{v}^{\infty} \left(\frac{1}{\beta_2(\vartheta)} \int_{\vartheta}^{\infty} q(\varkappa) d\varkappa \right)^{1/r} d\vartheta dv \to -\infty \text{ as } s \to \infty, \tag{46}$$

which contradicts the fact that $\mathcal{U}$ is positive. □

5. Property A of (1)

After combining the results from the previous two sections, we present the main results of this research as follows:

Theorem 4. *We suppose that $\varrho_* \geq 1$, and either (42) or (44) holds. Then, (1) has property A..*

Theorem 5. *We suppose that $\varrho_* > 0$, $\lambda_* = \infty$, and either (42) or (44) holds. Then, (1) has property A.*

Theorem 6. *We suppose that $\varrho_* > 0$ and $k_* = \infty$, and either (42) or (44) holds. Then, (1) has property A.*

Theorem 7. *We suppose that $\varrho_i < 1$, $i = 0, 1, 2, \ldots, n-1$, and $\varrho_n \geq 1$ and either (42) or (44) holds. Then, (1) has property A.*

Theorem 8. *We suppose that $\lambda_* < \infty$, (34), and either (42) or (44) holds. Then, (1) has property A.*

Theorem 9. *We suppose that (35) and either (42) or (44) holds. Then, (1) has property A.*

Theorem 10. *We suppose that*

$$\varrho_* > \max\left\{\frac{\varrho_j(1-\varrho_j)\left(2-\varrho_f\right)\lambda_*^{-r\varrho_j/2}}{k_j} : 0 < \varrho_j < 1\right\},$$

and either (42) or (44) holds. Then, (1) has property A.

6. Examples

We provide some examples in this section to demonstrate and validate our results.

Example 1. *We consider the third-order delay differential equation*

$$\left(\mathrm{e}^{-3\mathrm{s}}(\mathcal{U}''(\mathrm{s}))^3\right)' + q(\mathrm{s})\mathcal{U}^3\left(\frac{1}{2}\mathrm{s}\right) = 0, \tag{47}$$

where $\mathrm{s} > 1$ *and* $\tau \in (0,1)$*. It is easy to see that*

$$\beta_1(\mathrm{s}) = 1,\ \beta_2(\mathrm{s}) = \mathrm{e}^{-3\mathrm{s}},\ \pi_1(\mathrm{s}) \sim \mathrm{s},\ \pi_2(\mathrm{s}) \sim \mathrm{e}^{\mathrm{s}},\ \pi_{12}(\mathrm{s}) \sim \mathrm{e}^{\mathrm{s}}.$$

Then,

$$\lambda_* := \liminf_{\mathrm{s}\to\infty} \frac{\pi_{12}(\mathrm{s})}{\pi_{12}(\tau(\mathrm{s}))} = \liminf_{\mathrm{s}\to\infty} \mathrm{e}^{\mathrm{s}/2} = \infty,$$

and

$$\varrho_* := \liminf_{\mathrm{s}\to\infty} \frac{1}{r}\beta_2^{1/r}(\mathrm{s})\pi_{12}^r(\tau(\mathrm{s}))\pi_2(\mathrm{s})q(\mathrm{s}) = \liminf_{\mathrm{s}\to\infty} \frac{1}{3}q(\mathrm{s})\mathrm{e}^{3\mathrm{s}/2} > 0.$$

Clearly, if we let $q(\mathrm{s}) = \mathrm{e}^{-\mathrm{s}}$*, then*

$$\int_1^\infty \left(\frac{1}{\beta_2(\vartheta)}\int_\vartheta^\infty q(\varkappa)\mathrm{d}\varkappa\right)^{1/3}\mathrm{d}\vartheta = \int_1^\infty \mathrm{e}^{\vartheta}\left(\int_\vartheta^\infty \mathrm{e}^{-\varkappa}\mathrm{d}\varkappa\right)^{1/3}\mathrm{d}\vartheta = \int_1^\infty \mathrm{e}^{2\vartheta/3}\mathrm{d}\vartheta = \infty.$$

Thus, Theorem 5 is satisfied and so (47) has property A.
If we set $\rho(\mathrm{s}) = \mathrm{e}^{4\mathrm{s}}$*, then*

$$\begin{aligned} & \limsup_{\mathrm{s}\to\infty} \int_{\mathrm{s}_1}^{\mathrm{s}} \left(\rho(\varkappa)q(\varkappa)\left(\frac{\tau(\varkappa)}{\varkappa}\right)^{2r/\epsilon} - \frac{\beta_1^r(\varkappa)(\rho'(\varkappa))_+^{r+1}}{(r+1)^{r+1}\pi_2^r(\varkappa)\rho^r(\varkappa)}\right)\mathrm{d}\varkappa \\ = & \limsup_{\mathrm{s}\to\infty} \int_{\mathrm{s}_1}^{\mathrm{s}} \left(\frac{1}{2^{6/\epsilon}}\mathrm{e}^{3\varkappa} - \frac{\mathrm{e}^{16\varkappa}}{\mathrm{e}^{3\varkappa}\mathrm{e}^{12\varkappa}}\right)\mathrm{d}\varkappa \\ = & \limsup_{\mathrm{s}\to\infty} \int_{\mathrm{s}_1}^{\mathrm{s}} \left(\frac{1}{2^{6/\epsilon}}\mathrm{e}^{3\varkappa} - \mathrm{e}^{\varkappa}\right)\mathrm{d}\varkappa = \infty. \end{aligned}$$

Therefore, Theorem 9 is satisfied, and so (47) has property A.

Example 2. *We consider*

$$\left(\frac{1}{\mathrm{s}^3}\left(\left(\frac{1}{\mathrm{s}}\mathcal{U}'(\mathrm{s})\right)'\right)^3\right)' + q_0\mathcal{U}^3(\tau\mathrm{s}) = 0, \tag{48}$$

where $\mathrm{s} > 1$, $\tau \in (0,1)$ *and* $q_0 \geq 0$*. Clearly,*

$$\beta_1(\mathrm{s}) = \frac{1}{\mathrm{s}},\ \beta_2(\mathrm{s}) = \frac{1}{\mathrm{s}^3},\ \pi_1(\mathrm{s}) \sim \mathrm{s}^2/2,\ \pi_2(\mathrm{s}) \sim \mathrm{s}^2/2,\ \pi_{12}(\mathrm{s}) \sim \mathrm{s}^4/8.$$

Then,

$$\lambda_* := \liminf_{s\to\infty} \frac{\pi_{12}(s)}{\pi_{12}(\tau(s))} = \liminf_{s\to\infty} \frac{s^4/8}{s^4\tau^4/8} = \frac{1}{\tau^4},$$

and

$$\begin{aligned}\int_{s_0}^{\infty}\left(\frac{1}{\beta_2(\vartheta)}\int_{\vartheta}^{\infty} q(\varkappa)\mathrm{d}\varkappa\right)^{1/r}\mathrm{d}\vartheta &= \int_{s_0}^{\infty}\left(\frac{1}{\vartheta^3}\int_{\vartheta}^{\infty} q_0\mathrm{d}\varkappa\right)^{1/3}\mathrm{d}\vartheta \\ &= q_0^{1/3}\int_{s_0}^{\infty}\frac{1}{\vartheta^{2/3}}\mathrm{d}\vartheta = \infty.\end{aligned}$$

Thus, Theorem 10 is satisfied and so (48) has property A.
If we set $\rho(s) = s^{\gamma}$, $\gamma \geq 1$, then Theorem 9 is satisfied, and so (48) has property A.

Example 3. *We consider*

$$\left(\left(\mathcal{U}''(s)\right)^5\right)' + \frac{5}{s^6}\mathcal{U}^5\left(\frac{1}{2}s\right) = 0. \tag{49}$$

Clearly,

$$\beta_1(s) = 1,\ \beta_2(s) = 1,\ \pi_1(s) \sim s,\ \pi_2(s) \sim s,\ \pi_{12}(s) \sim s^2/2.$$

Then,

$$\lambda_* := \liminf_{s\to\infty} \frac{\pi_{12}(s)}{\pi_{12}(\tau(s))} = \liminf_{s\to\infty} \frac{s^2}{\left(\frac{1}{2}s\right)^2} = 4,$$

and

$$\begin{aligned}\int_{s_0}^{\infty}\left(\frac{1}{\beta_2(\vartheta)}\int_{\vartheta}^{\infty} q(\varkappa)\mathrm{d}\varkappa\right)^{1/r}\mathrm{d}\vartheta &= \int_{s_0}^{\infty}\left(\int_{\vartheta}^{\infty}\frac{1}{\varkappa^6}\mathrm{d}\varkappa\right)^{1/5}\mathrm{d}\vartheta \\ &= \int_{s_0}^{\infty}\frac{1}{\vartheta}\mathrm{d}\vartheta = \lim_{\vartheta\to\infty}\ln\vartheta = \infty.\end{aligned}$$

Thus, Theorem 10 is satisfied, and so (49) has property A.

7. Conclusions

This paper introduced a novel oscillation criterion tailored for third-order delay differential equations, subsequently refining it through the application of an iterative approach under specific conditions. The criteria established herein provide a robust assurance that Equation (1) adheres to property A, ensuring that all solutions of Equation (1) invariably either oscillate or asymptotically approach zero as $s \to \infty$. Our research not only enriches the existing scholarly discourse on this subject, but also lays the groundwork for future investigations. Our future investigations aim to delve into higher-order delay differential equations,

$$\left(\beta_2(s)\left(\left(\beta_1(s)\mathcal{U}'(s)\right)^{(n-2)}\right)^r\right)' + q(s)\mathcal{U}^r(\tau(s)) = 0,\ s \geq s_0.$$

Through these endeavors, we aim to further illuminate the intricate dynamics of such equations and contribute to the advancement of mathematical understanding in this domain

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