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Controller Design and Stability Analysis for a Class of Leader-Type Stochastic Nonlinear Systems

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Abstract: In this paper, the non-scaling backstepping approach is used to examine the controller design process and stability analysis of a class of leader-type stochastic nonlinear systems. By utilizing the non-scaling backstepping design method and Lyapunov method, the controller of the leader-type stochastic nonlinear system is derived. Different from the previous literature on controller design, we develop a more computationally efficient way for designing controllers because the scaling function in the coordinate transformation is not included. Meanwhile, the prescribed-time mean-square stabilization on the equilibrium and two important estimates are derived by combining the Lyapunov method with the matrix norm. Compared to the finite-time stabilization in other studies, the prescribed-time stabilization can determine the convergence time without relying on the initial value and has more real-world applicability. To illustrate the effectiveness of the controller derived in this paper, numerical examples are provided finally.

Keywords: stochastic nonlinear system; leader-type; prescribed-time mean-square stabilization; non-scaling backstepping method; Lyapunov method

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1. Introduction

During the past two decades, a lot of attention has been payed to the stability of stochastic control systems [1–5]. In contrast to deterministic systems [6–8], perturbations and unmodeled dynamical behavior in real problems are always described in the model through noise. So, stochastic control systems are also widely studied, and have yielded important results in econometrics, biology, environmental science, and other fields [9–11]. In [12], the predefined-time stability problem of nonlinear systems was discussed by using a nonlinear control strategy. A predefined time control for nonlinear polytopic systems is discussed in [13]. The bulk of the first control systems, such as steam and wattage regulators and liquid level regulators, were thought to be linear. In the actual device, certain nonlinearities were disregarded, while others were substituted by individuals with linear connections. The linear system paradigm is no longer relevant as science and technology advance, due to increasing in the variety of controlled objects and the complexity of the controls, as well as varied greater standards for control precision [6,14]. The majority of systems in practical engineering problems are nonlinear, such as electric power systems, robot control systems, multibody systems, etc. [15–18]. There are more widespread applications, even if the characteristics of nonlinear systems make the task more difficult and provide certain difficulties for our research.

In recent years, attempts have been made to study finite-time control of stochastic nonlinear systems, and Lyapunov theoretical criteria for stochastic finite-time stability have been developed [14,19,20]. Arbi et al. [21] studied the synchronization of a competitive neural network using a feedback control gain matrix based on Lyapunov stability theory. The finite-time state feedback stability of a class of continuous time nonlinear systems with conical nonlinear bounded feedback control gain perturbations and additive perturbations is proposed [6]. The mean pulse interval approach and the construction of a controller with an adequate Lyapunov function are used to investigate the finite time stability of a nonlinear pulse sampled data system [7]. The finite time random input state stability problem for a class of pulse-switched stochastic nonlinear systems is considered [22]. All the above studies require the system to be stable under the stochastic settling time, which generally depends on the choice of initial value. However, it is challenging to know the initial value, and finite-time stability is challenging to apply in practical applications. In contrasted to finite-time control, prescribed-time control enables the specification of a certain convergence time without carefully considering the initiation value, which makes the method more useful [8,23,24]. The suggested approach takes advantage of a scaling of states that grows infinitely towards terminal time through a time function, and then builds a controller that stabilizes the system in a scaling original state representation, producing an adjustment for the within a given finite time [8]. The stochastic zero controllability problem of strictly feedback nonlinear systems with random perturbations is solved by the prescribed-time mean-square stabilization problem, which offers the first feedback solution [23]. The prescribed-time mean-square stabilization problem of stochastic nonlinear systems without sensorless uncertainty is discussed via a novel non-scaling output feedback control method [24]. The study of prescribed-time control is driven by the fact that stabilization is necessary in a number of real-world applications in order to achieve the control objectives within a specified finite time [25]. Through the above literature, we can easily see that it is important to investigate the specified time control of stochastic nonlinear systems from both a theoretical and practical perspective.

Motivated by the above view, we investigate the prescribed-time mean-square stabilization of the leader-type stochastic nonlinear systems. In fact, the paper focuses on the controller design and stability analysis of a class of leader-type stochastic nonlinear systems using the non-scaling backstepping approach. By combining the Lyapunov method with the matrix norm and using the new approach to controller design, the paper derives the prescribed-time mean-square stabilization on the equilibrium and two important estimates. This approach offers advantages over finite-time stabilization as it allows for determining convergence time without relying on the initial value, making it more applicable in real-world scenarios. The effectiveness of the derived controller and the practical demonstration of the proposed approach is projected by the numerical simulation. What follows are the main contributions of this paper:

1. For stability analysis of a leader-type system model, we choose the control of the prescribed time. By the non-scalar design approach for stochastic nonlinear systems, the controller is designed in this paper. This approach allows for designing of a simpler controller because it does not use a scalar function in the coordinate transformation, which can significantly reduce the computational burden of the time-varying scalar function.
2. We considered the stability of the systems with the controller, which is obtained by the above method. It can be proved that the controller can make the system achieve stability for the prescribed time in this paper. Then, we obtain two important mean-square estimates.

The remainder of this article is organized as follows. Preliminaries and the model description are introduced in Section 2. In Section 3, we focus on the non-scaling controller design and stability analysis. To illustrate the effectiveness of the controller derived in this paper, we give two examples to show the theoretical results in Section 4. In Section 5, the conclusion is drawn.

2. Preliminaries and Model Description

2.1. Preliminaries

Without loss of generality in this paper, $\mathbf{R}^+ = [0, +\infty)$ denotes the set of nonnegative real numbers, $\mathbf{R}^n$ denotes the n-dimensional Euclidean space with Euclidean norm $|\cdot|$, and $\mathbf{Z}$ and $\mathbf{Z}^+$ denote the set of integers and the set of positive integers, respectively. For a given vector or matrix $\mathbf{X}$, $\mathbf{X}^T$ denotes its transpose, $Tr\{\mathbf{X}\}$ denotes its trace when $\mathbf{X}$ is square, and $|\mathbf{X}|$ is the Euclidean norm of the vector $\mathbf{X}$. Define $|\mathbf{A}| = (\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}^2)^{\frac{1}{2}}$ for a matrix $\mathbf{A}_{n\times m}$, where a_{ij} are the elements of $\mathbf{A}$. A complete probability space is presented as $(\Omega, \mathcal{F}, \mathrm{P})$ with the filtration $\{\mathcal{F}_t\}$ satisfying usual conditions. $W(t)$ is an m-dimensional Wiener process defined on $(\Omega, \mathcal{F}, \mathrm{P})$. Let $\mathcal{C}^{1,2}(\mathbf{R}^+ \times \mathbf{R}^n; \mathbf{R}^+)$ denote a class of nonnegative functions $V(t,x)$ on $\mathbf{R}^+ \times \mathbf{R}^n$ which are continuously once differentiable in t and twice in x.

We introduce the following scaling functions:

$$\mu(t) = (\frac{T}{t_0 + T - t})^m, \quad \forall t \in [t_0, t_0 + T), \tag{1}$$

where $m \geq 2$ is a integer and $T > 0$ is the freely prescribed time. Obviously, $\mu(t)$ is a monotonically increasing function on $[t_0, t_0 + T)$ with $\mu(t_0) = 1$ and $\lim_{t\to t_0+T} \mu(t) = \infty$.

Consider the following stochastic nonlinear system:

$$\mathrm{d}x = f(t,x)\mathrm{d}t + g^T(t,x)\mathrm{d}W(t), \quad \forall x_0 \in R^n, \tag{2}$$

where $x \in \mathbf{R}^n$ is the system state. The functions $f : \mathbf{R}^+ \times \mathbf{R}^n \to \mathbf{R}^n$ and $g : \mathbf{R}^+ \times \mathbf{R}^n \to \mathbf{R}^{m\times n}$ are continuous and are locally Lipschitz in x.

For any given $U(t,x) \in \mathcal{C}^{1,2}$ associated with the Itô stochastic system (2), the differential operator $\mathcal{L}$ is defined as $\mathcal{L}U \triangleq \frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} f(t,x) + \frac{1}{2} Tr\{g(t,x)\frac{\partial^2 U}{\partial x^2} g^T(t,x)\}$.

2.2. Model Description

We consider a class leader-type of stochastic nonlinear systems described by

$$\begin{cases} \mathrm{d}x_i = (x_n + f_i(t,x))\mathrm{d}t + g_i^T(t,x)\mathrm{d}W(t), & i = 1,2,\cdots,n-1, \\ \mathrm{d}x_n = (u + f_n(t,x))\mathrm{d}t + g_n^T(t,x)\mathrm{d}W(t), \end{cases} \tag{3}$$

where $x = (x_1, x_2, \cdots, x_n)^T \in \mathbf{R}^n$ and $u \in \mathbf{R}$ are the system state and control input. The function $f_i : \mathbf{R}^+ \times \mathbf{R}^n \to \mathbf{R}$ and $g_i : \mathbf{R}^+ \times \mathbf{R}^n \to \mathbf{R}^m$ are continuous and are locally Lipschitz in x, and $f_i(t,0) = 0, g_i(t,0) = 0, i = 1,2,\cdots,n$. $W(t)$ is an m-dimensional independent standard Wiener process. For system (3), we need the following assumption.

Assumption 1. *There exist positive constants c_{i1}, c_{i2} $(i = 1,2,\cdots,n-1)$ and c_{n1}, c_{n2} such that*

$$|f_i(t,x)| \leq c_{i1}|x_i|, \quad |g_i(t,x)| \leq c_{i2}|x_i|, \tag{4}$$

and

$$|f_n(t,x)| \leq c_{n1}(|x_1| + |x_2| + \cdots + |x_n|), \ |g_n(t,x)| \leq c_{n2}(|x_1| + |x_2| + \cdots + |x_n|). \tag{5}$$

Remark 1. *Assumption 1 is obtained by deforming the linear growth condition. The functions satisfying Assumption 1 exist, which is found by the simulation arithmetic in Section 4.*

Remark 2. *Different from existing works [3,8,15], etc., the proposal for a leader-type stochastic nonlinear system, which can be seen from Equation (3) and Assumption 1, is novel. This system embodies the one-to-many leadership characteristics, such as the food chain model in the ecosystem.*

With Assumption 1, the objective of this paper is to design a prescribed-time controller for the system (3), such that the system has an almost surely unique solution on $[t_0, t_0 + T)$ and the equilibrium at the origin is prescribed-time mean-square stabilization. A definition and two lemmas employed throughout this paper are demonstrated.

Definition 1 ([5]). *For the stochastic system (2) with $f(t,0) = 0$ and $g(t,0) = 0$, the equilibrium $x(t) = 0$ is a prescribed-time mean-square stabilization if there exist positive constants $k_i (1 \le i \le 4)$ such that*

$$E|x(t)|^2 \le k_1|x(t_0)|^2(1 + \mu_1^{k_2}(t))e^{-k_3\mu_1^{k_4}(t)}, \forall t \in [t_0, t_0 + T). \tag{6}$$

Lemma 1 ([5]). *Consider system (2). If there exist a nonnegative function $U(t,x) \in \mathcal{C}^{1,2}([t_0, t_0 + T) \times \mathbf{R}^n; \mathbf{R}^+)$ and positive constants c_0 and M_0 such that*

$$\lim_{|x|\to+\infty} \inf_{t\in[t_0,T_1]} U(t,x) = +\infty, \forall T_1 \in (t_0, t_0 + T), \tag{7}$$

$$\mathcal{L}U(t,x) \le -c_0\mu U + \mu M_0, \forall t \in (t_0, t_0 + T), \tag{8}$$

then the function $U(t,x)$ satisfies

$$EU(t,x(t)) \le e^{-c_0\int_{t_0}^{t}\mu(s)\mathrm{d}s}U(t_0,x_0) + \frac{M_0}{c_0}, \forall t \in [t_0, t_0 + T). \tag{9}$$

Lemma 2 ([23]). *There exist real variables x and y, for any positive real numbers a, b, m, and n, such that*

$$ax^m y^n \le b|x|^{m+n} + \frac{n}{m+n}(\frac{m+n}{m})^{-\frac{m}{n}} a^{\frac{m+n}{n}} b^{-\frac{m}{n}} |y|^{m+n}. \tag{10}$$

Applying the definition and lemmas as above, we will develop the controller design and stability analysis in the next section.

3. Main Results

In this section, a new controller that enables system (3) to reach stability is designed by a non-scaling backstepping design method. Based on the Lyapunov method and the matrix norm, in Theorem 1, we proved that system (3) reaches the prescribed-time mean-square stabilization. Further, two important mean-square estimates hold, which implies the effectiveness of system (3).

3.1. Controller Design

In this subsection, the controller for system (3) is designed as follows:

Step 1: For $i = 1, 2, \cdots, n-1$, we define

$$V_i = \frac{1}{4}\xi_i^4, \xi_i = x_i, \tag{11}$$

and from (3) and (4), and the definition of $\mathcal{L}$, we have

$$\begin{aligned} l\mathcal{L}V_i(\xi_i) &= \xi_i^3(x_n + f_i(t,x)) + \frac{3}{2}\xi_i^2|g_i(t,x)|^2 \\ &\le \xi_i^3 x_n + c_{i1}\xi_i^4 + \frac{3}{2}c_{i2}^2\xi_i^4 \\ &= \xi_i^3(x_n - x_i^*) + \xi_i^3 x_i^* + \xi_i^4(c_{i1} + \frac{3}{2}c_{i2}^2). \end{aligned} \tag{12}$$

Choosing

$$x_i^* = -\mu(c_i + c_{i1} + \frac{3}{2}c_{i2}^2)\xi_i \triangleq -\mu\alpha_i\xi_i, \tag{13}$$

which substitutes into (12), yields

$$\mathcal{L}V_i(\xi_i) \le -c_i\mu\xi_i^4 + \xi_i^3(x_n - x_i^*), \tag{14}$$

where $c_i > 0, i = 1, 2, \cdots, n-1$ are design parameters.

Step 2: Define

$$\xi_n = x_n + \sum_{i=1}^{n-1} \alpha_i \mu \xi_i. \tag{15}$$

By (3) and the Itô formula, we obtain

$$\begin{aligned} l\mathrm{d}\xi_n = {} & (u + f_n(t,x) + \sum_{i=1}^{n-1} (\alpha_i \mu x_n + \alpha_i \mu f_i(t,x)) \\ & + \frac{m}{T}\mu^{1+1/m} \sum_{i=1}^{n-1} \alpha_i \xi_i)\mathrm{d}t + (g_n^T(t,x) + \sum_{i=1}^{n-1} \alpha_i \mu g_i^T(t,x))\mathrm{d}W(t). \end{aligned} \tag{16}$$

We choose the Lyapunov function

$$V_n(\bar{\xi}_n) = \frac{1}{4}\sum_{i=1}^{n} \xi_i^4, \tag{17}$$

where $\bar{\xi}_n = (\xi_1, \xi_2, \cdots, \xi_n)^T$. It follows from (16), (17), and the definition of $\mathcal{L}$ that

$$\begin{aligned} \mathcal{L}V_n \leq {} & \sum_{i=1}^{n-1} \xi_i^3 (x_n + \alpha_i \mu \xi_i) - \sum_{i=1}^{n-1} c_i \mu \xi_i^4 + \xi_n^3 (u + f_n(t,x) + \sum_{i=1}^{n-1} (\alpha_i \mu x_n + \alpha_i \mu f_i(t,x)) \\ & + \frac{m}{T}\mu^{1+1/m} \sum_{i=1}^{n-1} \alpha_i \xi_i) + \frac{3}{2}\xi_n^2 |g_n^T(t,x) + \sum_{i=1}^{n-1} \alpha_i \mu g_i^T(t,x)|^2. \end{aligned} \tag{18}$$

By (5), we have

$$|f_n(t,x)| \leq \hat{c}_{n1}(\mu|\xi_1| + \mu|\xi_2| + \cdots + |\xi_n|), \tag{19}$$

and

$$|g_n(t,x)| \leq \hat{c}_{n2}(\mu|\xi_1| + \mu|\xi_2| + \cdots + |\xi_n|), \tag{20}$$

where $\hat{c}_{n1}$ and $\hat{c}_{n2}$ are positive constants. In the following, the notations $a_{i,j}, a_{i,n,r}, i,j = 1,\cdots,n-1, j \neq i; r = 1,\cdots,6$ are all the arbitrary positive constants. Using Lemma 2, we yield

$$\begin{aligned} |\sum_{i=1}^{n-1} \xi_i^3 (x_n + \alpha_i \mu \xi_i)| = {} & |\sum_{i=1}^{n-1} \xi_i^3 (\alpha_i \mu \xi_i + \xi_n - \sum_{i=1}^{n-1} \alpha_i \mu \xi_i)| \\ \leq {} & |\sum_{i=1}^{n-1} \xi_i^3 (\xi_n + \sum_{\substack{j=1 \\ j\neq i}}^{n-1} \alpha_j \mu \xi_j)| \\ \leq {} & \sum_{i=1}^{n-1} (a_{i,n,1} + \sum_{\substack{j=1 \\ j\neq i}}^{n-1} a_{i,j} + \frac{1}{4}(\frac{4}{3}a_{j,i})^{-3} \alpha_i^4)\mu \xi_i^4 + (\sum_{i=1}^{n-1} \frac{1}{4}(\frac{4}{3}a_{i,n,1})^{-3})\xi_n^4. \end{aligned} \tag{21}$$

From (19) and Lemma 2, we obtain

$$\begin{aligned} |\xi_n^3 f_n(t,x)| \leq {} & |\xi_n|^3 \hat{c}_{n1}(\mu|\xi_1| + \mu|\xi_2| + \cdots + |\xi_n|) \\ \leq {} & \hat{c}_{n1}\xi_n^4 + \sum_{i=1}^{n-1} (a_{i,n,2}\mu \xi_i^4 + \frac{3}{4}(4a_{i,n,2})^{-1/3} \hat{c}_{n1}^{4/3} \mu \xi_n^4) \\ = {} & \sum_{i=1}^{n-1} a_{i,n,2}\mu \xi_i^4 + (\hat{c}_{n1} + \sum_{i=1}^{n-1} \frac{3}{4}(4a_{i,n,2})^{-1/3} \hat{c}_{n1}^{4/3} \mu)\xi_n^4. \end{aligned} \tag{22}$$

It follows from (15) and Lemma 2 that

$$\begin{aligned}|\xi_n^3\sum_{i=1}^{n-1}\alpha_i\mu x_n| &\le |\sum_{i=1}^{n-1}\alpha_i\mu\xi_n^4+\sum_{i=1}^{n-1}(\sum_{j=1}^{n-1}\alpha_j)\alpha_i\mu^2\xi_i\xi_n^3| \\ &\le \sum_{i=1}^{n-1}a_{i,n,3}\mu\xi_i^4+(\sum_{i=1}^{n-1}\alpha_i\mu+\sum_{i=1}^{n-1}\frac{3}{4}(4a_{i,n,3})^{-1/3}(\alpha_i\sum_{j=1}^{n-1}\alpha_j)^{4/3}\mu^{7/3})\xi_n^4.\end{aligned} \tag{23}$$

According to (5), it can be deduced that

$$|\xi_n^3\sum_{i=1}^{n-1}\alpha_i\mu f_i(t,x)| \le |\sum_{i=1}^{n-1}\alpha_i c_{i1}\mu\xi_i\xi_n^3| \le \sum_{i=1}^{n-1}a_{i,n,4}\mu\xi_i^4+\sum_{i=1}^{n-1}\frac{3}{4}(4a_{i,n,4})^{-1/3}(c_{i1}\alpha_i)^{4/3}\mu\xi_n^4. \tag{24}$$

Using Lemma 2, it is easy to obtain

$$\begin{aligned}|\xi_n^3\frac{m}{T}\mu^{1+1/m}\sum_{i=1}^{n-1}\alpha_i\xi_i| &= |\sum_{i=1}^{n-1}\frac{m}{T}\alpha_i\mu^{1+1/m}\xi_i\xi_n^3| \\ &\le \sum_{i=1}^{n-1}a_{i,n,5}\mu\xi_i^4+\sum_{i=1}^{n-1}\frac{3}{4}(4a_{i,n,5})^{-1/3}(\frac{m}{T}\alpha_i)^{4/3}\mu^{5/3}\xi_n^4.\end{aligned} \tag{25}$$

Through (5), (20) and Lemma 2, we have

$$\begin{aligned}&\frac{3}{2}\xi_n^2|g_n^T(t,x)+\sum_{i=1}^{n-1}\alpha_i\mu g_i^T(t,x)|^2 \\ &\le \sum_{i=1}^{n-1}a_{i,n,6}\mu\xi_i^4+(\sum_{i=1}^{n-1}\frac{3}{2}n\hat{c}_{n2}+\frac{1}{4}(a_{i,n,6})^{-1}(\frac{3}{2}n\hat{c}_{n2}+\frac{3}{2}nc_{i2}\alpha_i)^2\mu^3)\xi_n^4.\end{aligned} \tag{26}$$

Substituting (21)–(26) into (18), it yields

$$\begin{aligned}\mathcal{L}V_n \le &-\sum_{i=1}^{n-1}(c_i-a_i)\mu\xi_i^4+\xi_n^3u+\xi_n^4(\sum_{i=1}^{n-1}\frac{1}{4}(\frac{4}{3}a_{i,n,1})^{-3}+\hat{c}_{n1}+\sum_{i=1}^{n-1}\frac{3}{4}(4a_{i,n,2})^{-1/3}\hat{c}_{n1}^{4/3}\mu \\ &+\sum_{i=1}^{n-1}\alpha_i\mu+\sum_{i=1}^{n-1}\frac{3}{4}(4a_{i,n,3})^{-1/3}(\alpha_i\sum_{j=1}^{n-1}\alpha_j)^{4/3}\mu^{7/3}+\sum_{i=1}^{n-1}\frac{3}{4}(4a_{i,n,4})^{-1/3}(c_{i1}\alpha_i)^{4/3}\mu \\ &+\sum_{i=1}^{n-1}\frac{3}{4}(4a_{i,n,5})^{-1/3}(\frac{m}{T}\alpha_i)^{4/3}\mu^{5/3}+\sum_{i=1}^{n-1}\frac{1}{4}(a_{i,n,6})^{-1}(\frac{3}{2}n\hat{c}_{n2}+\frac{3}{2}nc_{i2}\alpha_i)^2\mu^3+\frac{3}{2}nc_{n2}^2),\end{aligned} \tag{27}$$

where

$$a_i=\sum_{r=1}^{6}a_{i,n,r}+\sum_{\substack{j=1\\ j\neq i}}^{n-1}a_{i,j}+\frac{1}{4}(\frac{4}{3}a_{j,i})^{-1/3}\alpha_i^4,\ i=1,\cdots,n-1.$$

Choosing the virtual controller

$$\begin{aligned}u = &-\mu^3\xi_n(\sum_{i=1}^{n-1}\frac{1}{4}(\frac{4}{3}a_{i,n,1})^{-3}+\hat{c}_{n1}+\sum_{i=1}^{n-1}\frac{3}{4}(4a_{i,n,2})^{-1/3}\hat{c}_{n1}^{4/3} \\ &+\sum_{i=1}^{n-1}\alpha_i+\sum_{i=1}^{n-1}\frac{3}{4}(4a_{i,n,3})^{-1/3}(\alpha_i\sum_{j=1}^{n-1}\alpha_j)^{4/3}+\sum_{i=1}^{n-1}\frac{3}{4}(4a_{i,n,4})^{-1/3}(c_{i1}\alpha_i)^{4/3} \\ &+\sum_{i=1}^{n-1}\frac{3}{4}(4a_{i,n,5})^{-1/3}(\frac{m}{T}\alpha_i)^{4/3}+\sum_{i=1}^{n-1}\frac{1}{4}(a_{i,n,6})^{-1}(\frac{3}{2}n\hat{c}_{n2}+\frac{3}{2}nc_{i2}\alpha_i^2)+\frac{3}{2}nc_{n2}^2) \\ \triangleq &-\mu^3\alpha_n\xi_n.\end{aligned} \tag{28}$$

The following important inequation is obtained

$$\mathcal{L}V_n \leq -\sum_{i=1}^{n}(c_i - a_i)\mu^{\delta_i}\xi_i^4, \tag{29}$$

where $c_n > 0$ is a design parameter, $a_n = 0$, $\delta_1 = \cdots = \delta_{n-1} = 1$, $\delta_n = 3$. Considering the design parameters as

$$c_i > a_i, i = 1, 2, \cdots, n, \tag{30}$$

from (29) and (30), we have

$$\mathcal{L}V_n \leq -\sum_{i=1}^{n}(c_i - a_i)\mu^{\delta_i}\xi_i^4 \leq -c\mu V_n < 0, \tag{31}$$

where $c = 4\min_{1\leq i\leq n}\{c_i - a_i\}$.

Remark 3. *In this subsection, we propose a non-scaling backstepping design scheme for a class of leader-type stochastic nonlinear system (3) to achieve mean-square stability at a prescribed time. The advantage of this design is that it does not use time-varying μ for the coordinate transformation $\xi_n = x_n - \sum_{i=1}^{n-1} x_i^*$. Fundamentally different from the scaling approach developed by [24], each step of the controller design of our method is performed using the scaling transformation containing μ, which is more time efficient in the computational process.*

3.2. Mean-Square Stability Analysis

In the following theorem, we state the main stability results on the system (3).

Theorem 1. *Consider the plant consisting of (3) and (28). The following conclusion hold under Assumption 1: the equilibrium at the origin of the plant is the prescribed-time mean-square stabilization with $\lim_{t\to t_0+T} E|x|^2 = \lim_{t\to t_0+T} Eu^2 = 0$. Moreover, for $\forall t \in [t_0, t_0+T)$, the following important estimates hold:*

$$\begin{aligned} E|x|^2 \leq& \sqrt{n}(n + (\sum_{i=1}^{n-1}\alpha_i^2)\mu^2) \\ &\cdot(\sum_{i=1}^{n-1} x_i^4(t_0) + (x_n(t_0) + \mu\sum_{i=1}^{n-1}\alpha_i x_i(t_0))^4)^{1/2} e^{-\frac{cT^m}{2(m-1)}(\frac{1}{(t_0+T-t)^{m-1}} - \frac{1}{T^{m-1}})}, \forall t \in [t_0, t_0+T), \end{aligned} \tag{32}$$

and

$$\begin{aligned} Eu^2 \leq& \sqrt{n}\alpha_n^2\mu^{2\delta_n} \\ &\cdot(\sum_{i=1}^{n-1} x_i^4(t_0) + (x_n(t_0) + \mu\sum_{i=1}^{n-1}\alpha_i x_i(t_0))^4)^{1/2} e^{-\frac{cT^m}{2(m-1)}(\frac{1}{(t_0+T-t)^{m-1}} - \frac{1}{T^{m-1}})}, \forall t \in [t_0, t_0+T). \end{aligned} \tag{33}$$

Proof. Let $\xi = \{\xi_1, \xi_2, \cdots, \xi_n\}^T$. From (28), the n dimension system state x is satisfied

$$x = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ -\alpha_1\mu & -\alpha_2\mu & \cdots & -\alpha_{n-1}\mu & 1 \end{bmatrix}\xi. \tag{34}$$

By (34), we obtain

$$|x| \leq (n + (\sum_{i=1}^{n-1} \alpha_i^2)\mu^2)^{1/2}|\xi|, \tag{35}$$

which means that

$$|\xi| \geq (n + (\sum_{i=1}^{n-1} \alpha_i^2)\mu^2)^{-1/2}|x|. \tag{36}$$

Noting (17) and (36), the conditions (7) and (8) in Lemma 1 hold. Therefore, by Lemma 1, we have

$$EV_n \leq e^{-c\int_{t_0}^{t} \mu(s)\mathrm{d}\mu} V_n(t_0, x_0), \quad \forall t \in [t_0, t_0 + T). \tag{37}$$

Utilising Cauchy–Schwarz inequality, one has

$$\begin{aligned} lE|\xi|^2 &\leq 2\sqrt{n}(EV_n)^{1/2} \leq 2\sqrt{n}e^{-\frac{c}{2}\int_{t_0}^{t} \mu(s)\mathrm{d}\mu} V_n^{1/2}(t_0, x_0) \\ &\leq 2\sqrt{n}e^{-\frac{cT^m}{2(m-1)}(\frac{1}{(t_0+T-t)^{m-1}} - \frac{1}{T^{m-1}})} V_n^{1/2}(t_0, x_0), \quad \forall t \in [t_0, t_0 + T). \end{aligned} \tag{38}$$

Combining (35) and (38), we can obtain

$$\begin{aligned} lE|x|^2 \leq& \sqrt{n}(n + (\sum_{i=1}^{n-1} \alpha_i^2)\mu^2) \\ &\cdot (\sum_{i=1}^{n-1} x_i^4(t_0) + (x_n(t_0) + \mu \sum_{i=1}^{n-1} \alpha_i x_i(t_0))^4)^{1/2} \\ &\cdot e^{-\frac{cT^m}{2(m-1)}(\frac{1}{(t_0+T-t)^{m-1}} - \frac{1}{T^{m-1}})}, \quad \forall t \in [t_0, t_0 + T). \end{aligned} \tag{39}$$

Noting that $c > 0$, $T > 0$, $m \geq 2$ and the definition of u, we obtain

$$\lim_{t \to t_0 + T} E|x|^2 = 0. \tag{40}$$

From (28) and (38), we obtain

$$\begin{aligned} lEu^2 \leq& \sqrt{n}\alpha_n^2 \mu^{2\delta_n} (\sum_{i=1}^{n-1} x_i^4(t_0) + (x_n(t_0) + \mu \sum_{i=1}^{n-1} \alpha_i x_i(t_0))^4)^{1/2} \\ &\cdot e^{-\frac{cT^m}{2(m-1)}(\frac{1}{(t_0+T-t)^{m-1}} - \frac{1}{T^{m-1}})}, \quad \forall t \in [t_0, t_0 + T), \end{aligned} \tag{41}$$

and

$$\lim_{t \to t_0 + T} E|u|^2 = 0. \tag{42}$$

The proof is completed. □

Remark 4. *In this subsection, we propose a non-scaling backstepping design scheme for stochastic nonlinear system (3) to achieve prescribed-time mean-square stabilization. Theorem 1 is proven to hold by the form of the controller designed in Section 3.1 and the matrix norm. Compared to finite-time stabilization [7,14,19], etc., prescribed-time mean-square stabilization can define a known specific convergence time, regardless of the initial conditions, and has more practical applications.*

4. Simulation Example

In this section, we give two simulation examples to show the effectiveness of the prescribed-time control design schemes developed in the last section.

Example 1. *Consider the circuit system shown in Figure 1. By Ohm's law, this system is described as:*

$$\begin{cases} \dfrac{\mathrm{d}u_c}{\mathrm{d}t} = \dfrac{u - u_c}{R_c C}, \\ \dfrac{\mathrm{d}i_L}{\mathrm{d}t} = \dfrac{u - i_L R_L}{L}, \\ \dfrac{\mathrm{d}u}{\mathrm{d}t} = \dfrac{Bd}{m}(F - B(\dfrac{u - u_c}{R_c} + i_L)d), \end{cases} \tag{43}$$

where m represents the mass, d represents the distance between two parallel guides, B represents the magnetic induction intensity, F represents the external force, C denotes the capacitor, L denotes the inductor, R_c and R_L are the resistance of C and L, respectively, and u and u_c are the voltage, i_L denotes the current through L.

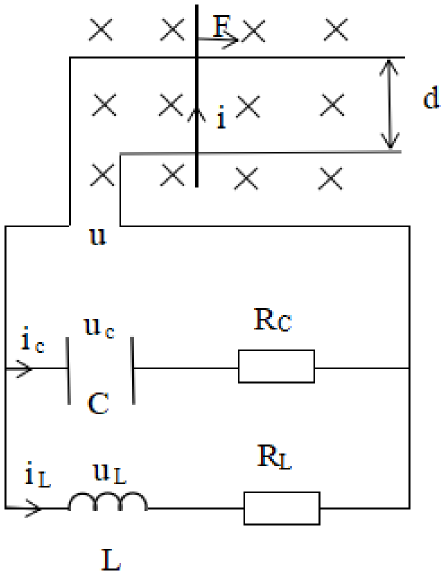


Figure 1. Circuit system.

For the above system by physical deformation with suitable parameter selection, and then organize the form of this text system, the form of the system is obtained as follows:

$$\begin{cases} \mathrm{d}x_1 = (x_3 - 10x_1)\mathrm{d}t + x_1 \mathrm{d}\omega, \\ \mathrm{d}x_2 = (x_3 - x_2)\mathrm{d}t + x_2 \mathrm{d}\omega, \\ \mathrm{d}x_3 = (u + \dfrac{x_1}{900} - \dfrac{x_2}{18} - \dfrac{x_3}{900})\mathrm{d}t + (x_1 + x_2 + x_3)\mathrm{d}\omega. \end{cases} \tag{44}$$

Step 1: Define

$$V_1 = \frac{1}{4}\xi_1^4, \quad \xi_1 = x_1.$$

From (44), we have

$$\mathrm{d}\xi_1 = \mathrm{d}x_1 = (x_3 - 10\xi_1)\mathrm{d}t + \xi_1 \mathrm{d}\omega.$$

By the definition of $\mathcal{L}$, we have

$$\mathcal{L}V_1 \le \xi_1^3(x_3 + 10\xi_1) + \frac{3}{2}\xi_1^2|\xi_1|^2. \tag{45}$$

Choosing

$$x_1^* = -(c_1 + 10 + \frac{3}{2})\mu\xi_1.$$

which substitutes into (45), it yields

$$\mathcal{L}V_1 \leq -c_1\mu\xi_1^4 + \xi_1^3(x_3 - x_1^*). \tag{46}$$

Next, we define

$$V_2 = \frac{1}{4}\xi_2^4, \quad \xi_2 = x_2.$$

From (44), we have

$$\mathrm{d}\xi_2 = \mathrm{d}x_2 = (x_3 - \xi_2)\mathrm{d}t + \xi_2\mathrm{d}\omega.$$

By the definition of $\mathcal{L}$, we have

$$\mathcal{L}V_2 \leq \xi_2^3(x_3 + \xi_2) + \frac{3}{2}\xi_2^2|\xi_2|^2. \tag{47}$$

Choosing

$$x_2^* = -(c_2 + 1 + \frac{3}{2})\mu\xi_2,$$

which substitutes into (47), and yields

$$\mathcal{L}V_2 \leq -c_2\mu\xi_2^4 + \xi_2^3(x_3 - x_2^*). \tag{48}$$

Step 2: Define

$$V_3 = \frac{1}{4}(\xi_1^4 + \xi_2^4 + \xi_3^4), \quad \xi_3 = x_3 + (c_1 + \frac{23}{2})\mu\xi_1 + (c_2 + \frac{5}{2})\mu\xi_2.$$

From (43), we have

$$\begin{aligned}\mathrm{d}\xi_3 = {} & [u + \frac{\xi_1}{900} - \frac{\xi_2}{18} - \frac{\xi_3}{900} + (c_1 + \frac{23}{2})\mu(x_3 + 10\xi_1) + (c_2 + \frac{5}{2})\mu(x_3 + \xi_2) + 2((c_1 + \frac{23}{2})\xi_1 \\ & + (c_2 + \frac{5}{2})\xi_2)\mu^{3/2}]\mathrm{d}t + [\xi_1 + \xi_2 + x_3 + (c_1 + \frac{23}{2})\mu\xi_1 + (c_2 + \frac{5}{2})\mu\xi_2]\mathrm{d}\omega.\end{aligned}$$

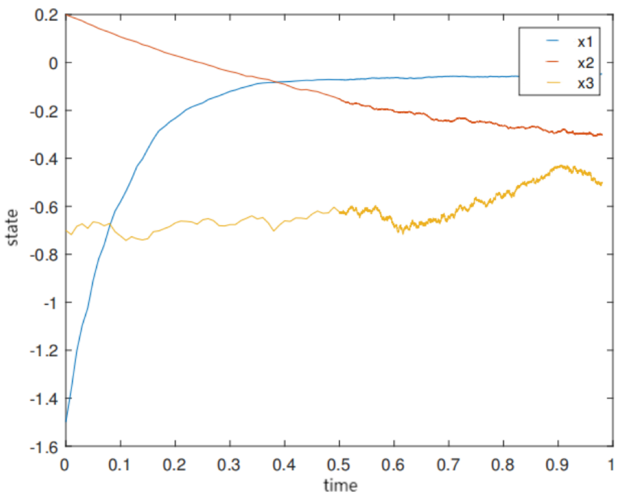
Considering the definition of $\mathcal{L}$, we have

$$\begin{aligned}\mathcal{L}V_3 \leq {} & -c_1\mu\xi_1^4 + \xi_1^3(x_3 - x_1^*) - c_2\mu\xi_2^4 + \xi_2^3(x_3 - x_2^*) + \xi_3^3[u + \frac{\xi_1}{900} - \frac{\xi_2}{18} - \frac{\xi_3}{900} \\ & + (c_1 + \frac{23}{2})\mu(x_3 + 10\xi_1) + (c_2 + \frac{5}{2})\mu(x_3 + \xi_2) + 2((c_1 + \frac{23}{2})\xi_1 + (c_2 + \frac{5}{2})\xi_2)\mu^{3/2}] \\ & + \frac{3}{2}\xi_3^2|\xi_1 + \xi_2 + \xi_3|^2.\end{aligned} \tag{49}$$

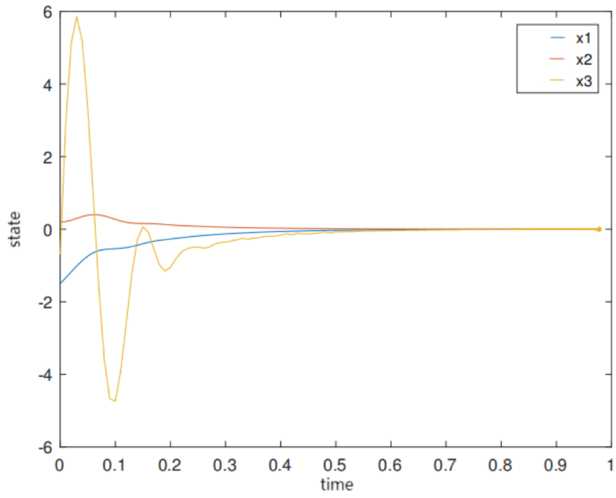
According to the design procedure developed in Section 3, the controller is designed as

$$\begin{aligned}u = {} & -\mu^3(\frac{1}{2}(\frac{4}{3})^{-3} + \frac{1}{900}(1 + \alpha_1)\frac{3}{4}4^{-1/3} + (\frac{1}{900}\alpha_2 + \frac{1}{18})\frac{3}{4}4^{-1/3} \\ & + \frac{1}{900} + (\alpha_1 + \alpha_2)(1 + \frac{3}{4}4^{-1/3}\alpha_1) + (\alpha_1 + \alpha_2)(\frac{3}{4}4^{-1/3}\alpha_2) \\ & + 10\frac{3}{4}4^{-1/3}\alpha_1 + \frac{3}{4}4^{-1/3}\alpha_2 + 2\frac{3}{4}4^{-1/3}(\alpha_1 + \alpha_2) + 3 + \frac{3}{2})\xi_n\end{aligned} \tag{50}$$

For simulation, we select $t_0 = 0, T = 1, m = 2$, randomly set the initial conditions as $x_1(0) = -1.5, x_2(0) = 0.2, x_3(0) = -0.7$, and the parameters $c_1 = \frac{7}{2}, c_2 = \frac{5}{2}$. Figure 2a shows the state of system (44) without control u applied. Figure 2b shows the response of system (44) and (50). Figure 3 illustrates the effectiveness of controller u. According to Figure 4, we find that $\lim_{t\to 1} E|x|^2 = \lim_{t\to 1} Eu^2 = 0$, which means that the prescribed-time mean-square stabilization is achieved. Therefore, the effectiveness of the controller design developed in Section 3 is demonstrated.



(**a**) No control applied



(**b**) Control applied

Figure 2. Comparison of no control applied and control applied for Example 1.

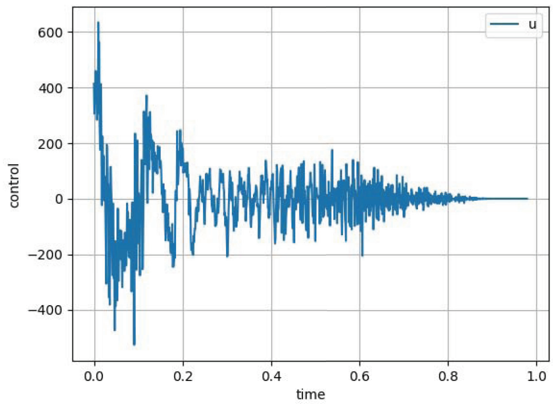
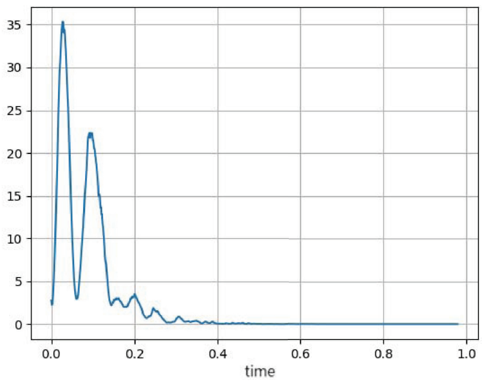
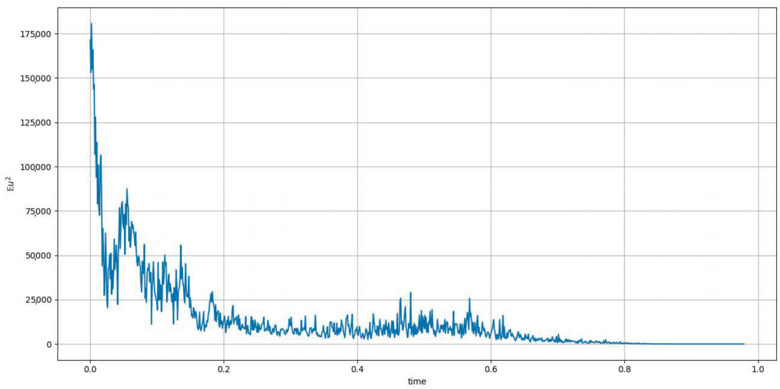


Figure 3. Effectiveness of controller for Example 1.



(**a**) $E|x|^2$



(**b**) Eu^2

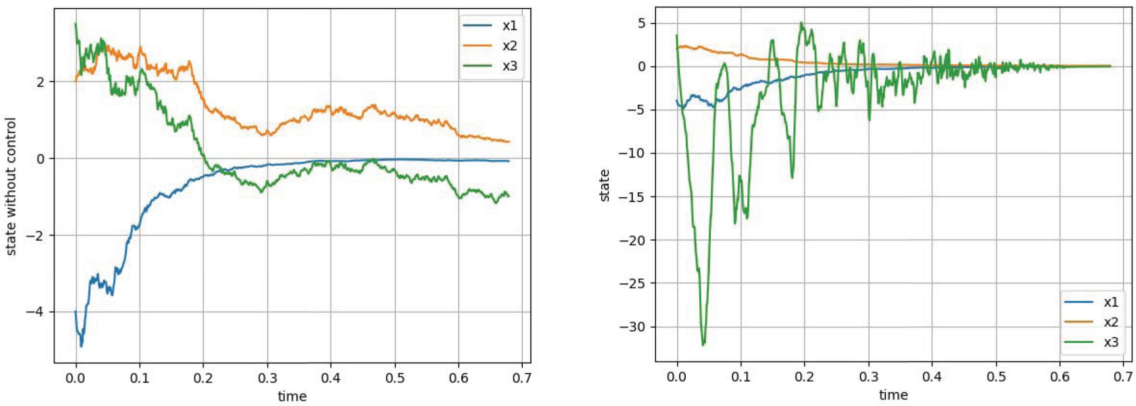
Figure 4. Stability analysis for Example 1.

In the next example, we choose a nonlinear system to verify the effectiveness of the method in this paper.

Example 2. *Consider the following systems:*

$$\begin{cases} \mathrm{d}x_1 = (x_3 - 10x_1)\mathrm{d}t + x_1\mathrm{d}\omega, \\ \mathrm{d}x_2 = (x_3 - x_2\max\{\cos x_2, 0.5\})\mathrm{d}t + x_2\mathrm{d}\omega, \\ \mathrm{d}x_3 = (u + \sin x_1 - \sin(x_2 - x_3))\mathrm{d}t + (x_1 + x_2 + x_3)\mathrm{d}\omega. \end{cases} \tag{51}$$

The derivation of this example will not be repeated. For simulation, we select $t_0 = 0$, $T = 0.7$, $m = 2$, and randomly set the initial conditions as $x_1(0) = -4, x_2(0) = 2$, $x_3(0) = 3.5$. Figure 5a shows the state of system (51) without control u applied. Figure 5b shows the response of system (51) with controller u for Example 2. Figure 6 illustrates the effectiveness of controller u. Therefore, the validity of the controller design developed in Section 3 is demonstrated.



(**a**) No control applied (**b**) Control applied

Figure 5. Comparison of no control applied and control applied for Example 2.

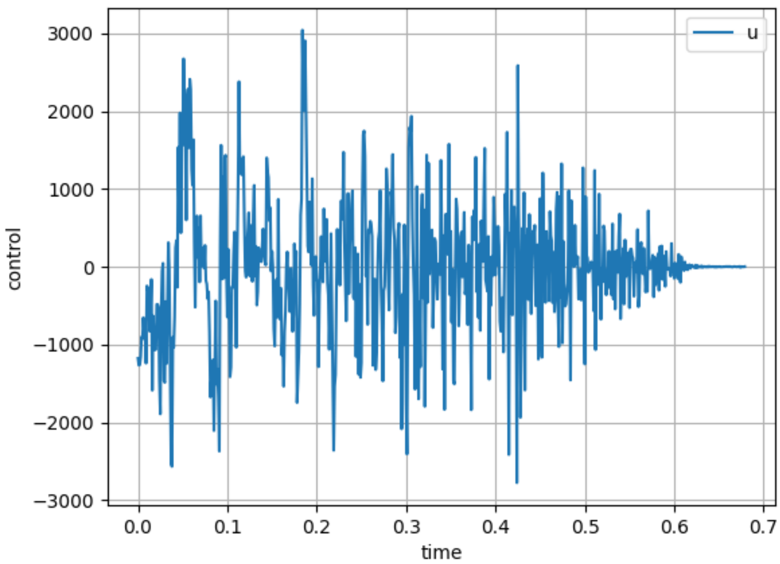


Figure 6. Effectiveness of controller for Example 2.

5. Conclusions

In this paper, we discussed the specified time mean square stability problem of a class of leader-type stochastic nonlinear system (3) with the help of a new controller. Firstly, we defined new Lyapunov functions (11) and (17) for system (3). By developing a scale-free backstepping design method, a new controller has been designed to ensure that the equilibrium of the system origin is the time mean square stability specified in Theorem 1.

At the same time, two important estimates, (32) and (33), were given. In addition, the special cases of circuit systems (43) and nonlinear systems (51) achieved the specified time mean square stability through the application of controllers. In future work, it is necessary to consider the influence of the time-delay of leader-type stochastic nonlinear systems.

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