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A New Tangent-Generated Probabilistic Approach with Symmetrical and Asymmetrical Natures: Monte Carlo Simulation with Reliability Applications

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Abstract: It is proven evidently that probability distributions have a significant role in data modeling for decision-making. Due to the indispensable role of probability distributions for data modeling in applied fields, a series of probability distributions have been introduced and implemented. However, most newly developed probability distributions involve between one and eight additional parameters. Sometimes the additional parameters lead to re-parametrization problems. Therefore, the development of new probability distributions without additional parameters is an interesting research topic. In this paper, we study a new probabilistic method without incorporating any additional parameters. The proposed approach is based on a tangent function and may be called a new tangent-G (NT-G) family of distributions. Certain properties of the NT-G distributions are derived. Based on the NT-G method, a new flexible probability distribution called a new tangent flexible Weibull (NTF-Weibull) distribution is studied. The parameters of the NTF-Weibull distribution are estimated using seven different estimation methods. Based on these eight estimations, a brief simulation of the NTF-Weibull distribution is also provided. Finally, we prove the applicability of the NTF-Weibull distribution by analyzing two waiting-time data sets taken from the reliability sector. We consider three statistical tests with a p-value to evaluate the performance and goodness of fit of the NTF-Weibull distribution.

Keywords: tangent function; flexible Weibull extension; properties; simulation; reliability data; waiting time; statistical modeling

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1. Introduction

Probability distributions play an important role in decision-making in any field of applied sciences, for instance, hydrology, the medical sector, metrology, reliability fields, air pollution, and the financial sectors [1–3]. However, it is also crystal clear and practice has proven that no particular probability distribution is capable of providing satisfactory results in all situations. Therefore, we always need new probability distributions to best describe the phenomenon under consideration. This need has led researchers to develop new probability distributions to best describe the phenomena under study [4–6].

Corresponding to the existing studies, it has practically been proved that the Weibull distribution occupies a key position in the class of continuous probability distributions. It has been many researchers' first choice of implementation when it comes to data modeling and offers the properties of the exponential and Rayleigh distributions as well [7].

Suppose $T(\in \mathbb{R}^+)$ follows the Weibull distribution with parameters $(\tau, \beta \in \mathbb{R}^+)$. Then, the cumulative distribution function (CDF) of T, expressed by $G(t)$, has the form

$$G(t) = 1 - e^{-\beta t^{\tau}}, \quad t \geq 0, \tag{1}$$

with a hazard function (HF), say $h(t)$, is given by

$$h(t) = \tau \beta t^{\tau - 1}. \tag{2}$$

The HF of the Weibull distribution has three possible shapes depending on the parameter value τ. Keeping the parameter β constant (i.e., $\beta = 1$), the shape of $h(t)$ of the Weibull distribution can be:

- Constant, when $\tau = 1$;
- Increasing, when $\tau > 1$;
- Decreasing, when $\tau < 1$.

The Weibull distribution has proven to be the best-suited model when failure phenomena occur in a single mode. However, most situations in reliability theory have failure phenomena in mixed states. In such situations, the Weibull distribution does not provide a satisfactory fitting to the data. To address this issue, numerous modifications of the Weibull distribution have been introduced to handle the failure phenomena that occur in mixed states [8–13].

Among the available literature on the modifications of the Weibull distribution, the flexible Weibull extension has received considerable attention [14]. Let T follow the flexible Weibull (F-Weibull) distribution, and then, its CDF is

$$G(t) = 1 - e^{-e^{\left(\beta t - \frac{\sigma}{t}\right)}}, \quad t \geq 0, \beta, \sigma \in \mathbb{R}^{+}. \tag{3}$$

The F-Weibull distribution is further extended by many authors to improve its data fitting capability [15–18]. The majority of the modifications are developed using additional parameters. Some of them are constructed by introducing seven or eight additional parameters [19]. Thanks to the addition of extra parameters, it significantly improves the flexibility and fitting power of the probability distributions.

However, according to some existing studies, the estimates of the parameters of such distributions become difficult to obtain and more computational works are required [20]. On the other hand, sometimes the introduction of extra parameters leads to re-parametrization problems. Therefore, it is highly desirable to develop new models without introducing extra parameters and increase distributional flexibility. In this regard, there are very small efforts to develop a new probability model without adding new parameters.

In this paper, we introduce a new probabilistic method without incorporating new parameters. The proposed probabilistic method is based on the tangent function and may be called a new tangent-G (NT-G) family of distributions.

Definition 1. *Let T follow the NT-G method if its CDF, say $F(t)$, is expressed by*

$$F(t) = 1 - \left(\frac{1 - \tan\left[\frac{\pi}{4} G(t)\right]}{1 + \tan\left[\frac{\pi}{4} G(t)\right]} \right)^{2}, \quad t \in \mathbb{R}. \tag{4}$$

In Propositions 1 and 2, we provide the mathematical description to prove that the expression provided in Equation (4) is a valid CDF.

Proposition 1. *For $F(t)$ defined in Equation (4), we have to show that*

$$\lim_{t \to -\infty} F(t) = 0,$$

and

$$\lim_{t \to \infty} F(t) = 1.$$

Proof. Using Equation (4), we have

$$F(t) = 1 - \left(\frac{1 - \tan\left[\frac{\pi}{4}G(t)\right]}{1 + \tan\left[\frac{\pi}{4}G(t)\right]}\right)^2. \tag{5}$$

Applying the limit to both sides of Equation (5), we obtain

$$\lim_{t\to-\infty} F(t) = \lim_{t\to-\infty}\left(1 - \left(\frac{1 - \tan\left[\frac{\pi}{4}G(t)\right]}{1 + \tan\left[\frac{\pi}{4}G(t)\right]}\right)^2\right),$$

$$\lim_{t\to-\infty} F(t) = 1 - \left(\frac{1 - \tan\left[\frac{\pi}{4}G(-\infty)\right]}{1 + \tan\left[\frac{\pi}{4}G(-\infty)\right]}\right)^2. \tag{6}$$

Since $G(t)$ is a valid CDF, therefore,

$$\lim_{t\to-\infty} G(t) = G(-\infty) = 0.$$

Thus, from Equation (6), we have

$$\lim_{t\to-\infty} F(t) = 1 - \left(\frac{1 - \tan(0)}{1 + \tan(0)}\right)^2,$$

$$\lim_{t\to-\infty} F(t) = 0.$$

From Equation (4), we also obtain

$$\lim_{t\to\infty} F(t) = \lim_{t\to\infty}\left(1 - \left(\frac{1 - \tan\left[\frac{\pi}{4}G(t)\right]}{1 + \tan\left[\frac{\pi}{4}G(t)\right]}\right)^2\right),$$

$$\lim_{t\to\infty} F(t) = 1 - \left(\frac{1 - \tan\left[\frac{\pi}{4}G(\infty)\right]}{1 + \tan\left[\frac{\pi}{4}G(\infty)\right]}\right)^2. \tag{7}$$

As $G(t)$ is a valid CDF, we have

$$\lim_{t\to\infty} G(t) = G(\infty) = 1.$$

Thus, from Equation (7), we have

$$\lim_{t\to\infty} F(t) = 1 - \left(\frac{1 - \tan\left(\frac{\pi}{4}\right)}{1 + \tan\left(\frac{\pi}{4}\right)}\right)^2,$$

$$\lim_{t\to\infty} F(t) = 1 - \left(\frac{1 - 1}{1 + 1}\right)^2,$$

$$\lim_{t\to\infty} F(t) = 1 - \left(\frac{0}{1 + 1}\right)^2,$$

$$\lim_{t\to\infty} F(t) = 1$$

□

Proposition 2. *The CDF $F(t)$ is differentiable.*

Proof.
$$\frac{d}{dt}F(t) = f(t).$$
The proof is straightforward and the derivative of $F(t)$ is provided in Equation (8). Corresponding to Equation (4), the probability density function (PDF) $f(t)$, given by

$$f(t) = \frac{\pi g(t) \sec^2\left[\frac{\pi}{4}G(t)\right]\left(1 - \tan\left[\frac{\pi}{4}G(t)\right]\right)}{\left(1 + \tan\left[\frac{\pi}{4}G(t)\right]\right)^3}, \qquad t \in \mathbb{R}. \tag{8}$$

The survival function (SF) of the NT-G family, say $S(t)$, is given by

$$S(t) = \left(\frac{1 - \tan\left[\frac{\pi}{4}G(t)\right]}{1 + \tan\left[\frac{\pi}{4}G(t)\right]}\right)^2, \qquad t \in \mathbb{R},$$

The HF of the NT-G family, say $h(t)$, is expressed by

$$h(t) = \frac{\pi g(t) \sec^2\left[\frac{\pi}{4}G(t)\right]\left(1 - \tan\left[\frac{\pi}{4}G(t)\right]\right)}{\left(1 - \tan\left[\frac{\pi}{4}G(t)\right]\right)\left(1 + \tan\left[\frac{\pi}{4}G(t)\right]\right)}, \qquad t \in \mathbb{R}.$$

The cumulative HF (CHF) of the NT-G family, say $H(t)$, is provided by

$$H(t) = -2\log\left(\frac{1 - \tan\left[\frac{\pi}{4}G(t)\right]}{1 + \tan\left[\frac{\pi}{4}G(t)\right]}\right), \qquad t \in \mathbb{R}.$$

□

In Section 2, we discuss a special member of the NT-G family called a new tangent flexible Weibull (NTF-Weibull) distribution. Some distributional functions of the NTF-Weibull distribution, along with their visual display, are also provided in Section. Section 3 is devoted to providing some mathematical properties of the NTF-Weibull distribution. The estimation and simulation of the NTF-Weibull distribution are provided in Section 4. Two practical data sets are considered to illustrate the NTF-Weibull distribution in Section 5. Finally, Section 6 presents some concluding remarks.

2. The NTF-Weibull Distribution

An interesting and straightforward modification of the F-Weibull distribution with CDF is given by

$$G(t) = 1 - e^{-e^{\left(\beta t - \frac{\sigma}{t^\theta}\right)}}, \qquad t \geq 0, \beta, \sigma, \theta \in \mathbb{R}^+, \tag{9}$$

and PDF

$$g(t) = \left(\beta + \frac{\theta\sigma}{t^{\theta+1}}\right)e^{\left(\beta t - \frac{\sigma}{t^\theta}\right)}e^{-e^{\left(\beta t - \frac{\sigma}{t^\theta}\right)}}, \qquad t > 0. \tag{10}$$

Clearly, for $\theta = 1$, Equation (9) reduces to Equation (3). Here, we consider the CDF in Equation (9) as a special case of the NT-G family. Using Equation (9) in Equation (4), we obtain the CDF of the proposed NTF-Weibull distribution with CDF

$$F(t) = 1 - \left(\frac{1 - \tan\left[\frac{\pi}{4}\left(1 - e^{-A(t)}\right)\right]}{1 + \tan\left[\frac{\pi}{4}\left(1 - e^{-A(t)}\right)\right]}\right)^2, \tag{11}$$

and PDF

$$f(t) = \frac{\pi\left(\beta + \frac{\theta\sigma}{t^{\theta+1}}\right)A(t)e^{-A(t)}\sec^2\left[\frac{\pi}{4}\left(1 - e^{-A(t)}\right)\right]}{\left(1 + \tan\left[\frac{\pi}{4}\left(1 - e^{-A(t)}\right)\right]\right)^3}\left(1 - \tan\left[\frac{\pi}{4}\left(1 - e^{-A(t)}\right)\right]\right), \tag{12}$$

where $A(t) = e^{\left(\beta t - \frac{\sigma}{t^{\theta}}\right)}$.

Furthermore, the SF, HF, and CHF of the NTF-Weibull distribution are given by

$$S(t) = \left(\frac{1 - \tan\left[\frac{\pi}{4}\left(1 - e^{-A(t)}\right)\right]}{1 + \tan\left[\frac{\pi}{4}\left(1 - e^{-A(t)}\right)\right]} \right)^2,$$

$$h(t) = \frac{\pi\left(\beta + \frac{\theta\sigma}{t^{\theta+1}}\right)A(t)e^{-A(t)}\sec^2\left[\frac{\pi}{4}\left(1 - e^{-A(t)}\right)\right]}{\left(1 - \tan\left[\frac{\pi}{4}\left(1 - e^{-A(t)}\right)\right]\right)\left(1 + \tan\left[\frac{\pi}{4}\left(1 - e^{-A(t)}\right)\right]\right)} \times \left(1 - \tan\left[\frac{\pi}{4}\left(1 - e^{-A(t)}\right)\right]\right),$$

and

$$H(t) = -2\log\left(\frac{1 - \tan\left[\frac{\pi}{4}\left(1 - e^{-A(t)}\right)\right]}{1 + \tan\left[\frac{\pi}{4}\left(1 - e^{-A(t)}\right)\right]} \right),$$

respectively.

Some plots of $F(t)$ and $S(t)$ of the NTF-Weibull distribution are presented in Figure 1. These plots are obtained for different vales of β, σ, and θ.

Figure 2 shows that the NTF-Weibull distribution has five different behaviors of its PDF, namely (i) decreasing (or also called reverse-J shaped) (red curve), (ii) right-skewed (green curve), (iii) symmetric shape (grey curve), (iv) left-skewed (blue curve), and (v) decreasing-increasing-decreasing (gold curve).

Furthermore, the plots of $h(t)$ of the NTF-Weibull distribution are obtained in Figure 3, which confirms five different shapes.

Figure 3 validates that $h(t)$ of the NTF-Weibull distribution can capture (i) two monotonic shapes (decreasing and increasing), and (ii) three non-monotonic shapes (bathtub, unimodal, and modified unimodal). Thus, the NTF-Weibull distribution is able to fit data sets that have single-state or mixed-state failure rates.

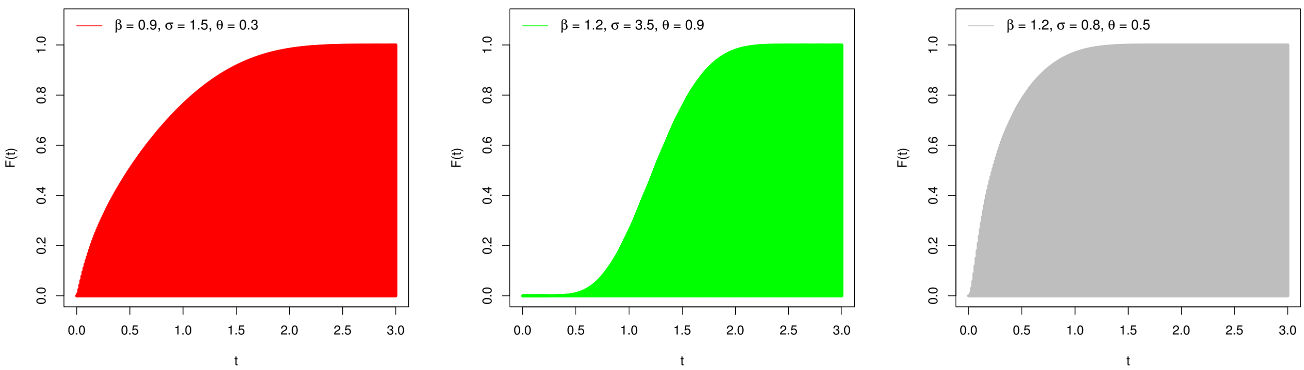

Figure 1. *Cont.*

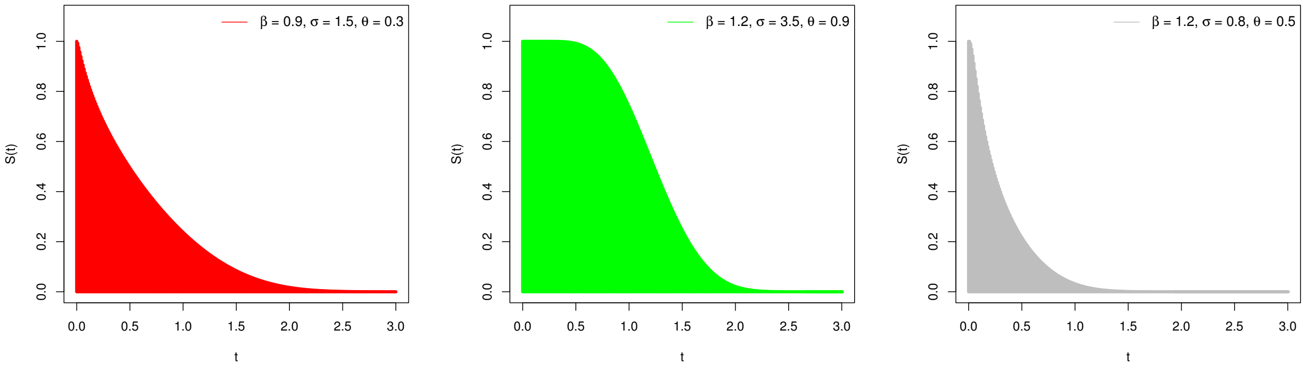

Figure 1. The visual illustrations of $F(t)$ and $S(t)$ of the NTF-Weibull distribution for different values of β, σ, and θ.

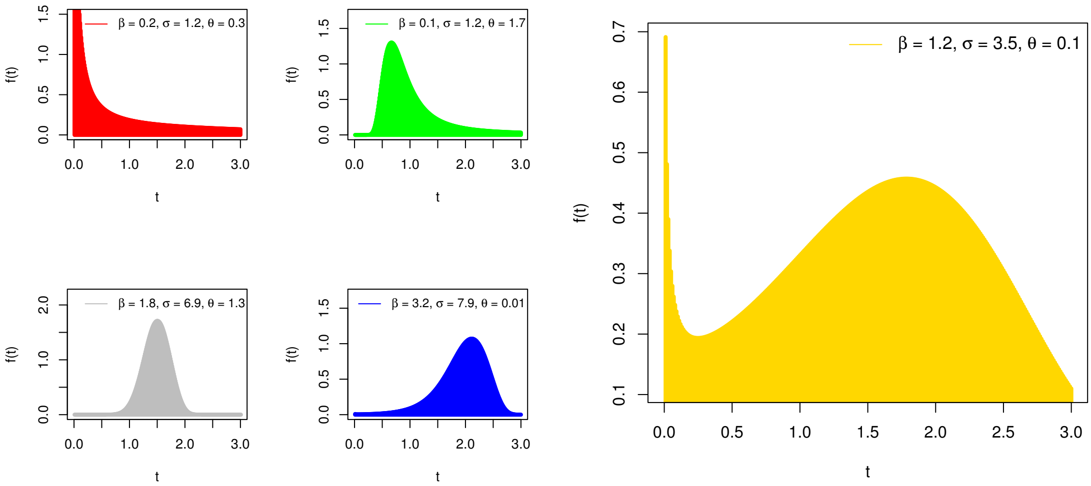

Figure 2. The visual illustrations of $f(t)$ of the NTF-Weibull distribution for different values of β, σ, and θ.

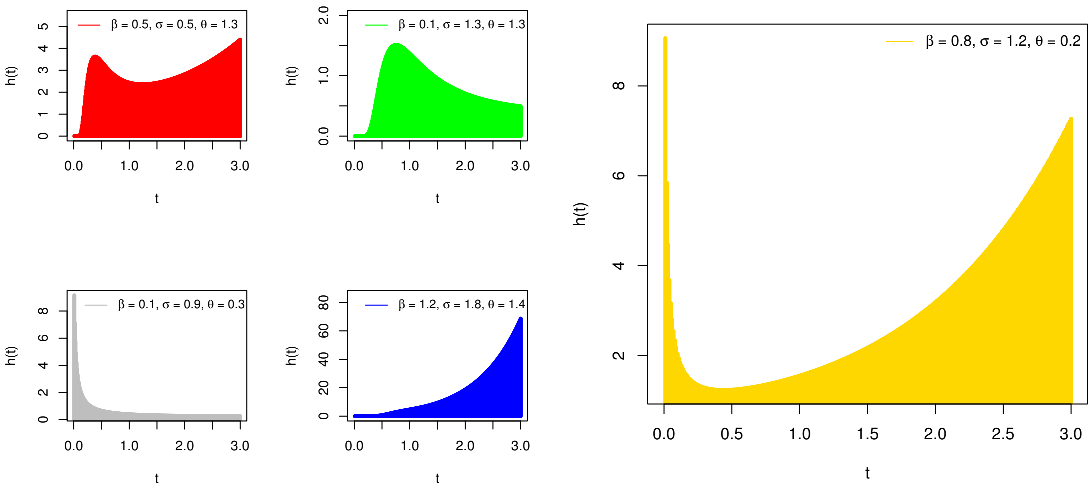

Figure 3. The visual illustrations of $h(t)$ of the NTF-Weibull distribution for different values of β, σ, and θ.

3. Distributional Properties of the NT-*G* Family

Some distributional properties of the NT-*G* distributions are studied in this section. These properties include the quantile function (QF), median, and quartiles of the NT-*G* distributions. Furthermore, the kurtosis, skewness, and r^{th} moment of the NT-*G* distributions are also derived.

In this work, we only present the mathematical derivation of the distributional properties of the NT-*G* family. The numerical results of these characteristics can be easily obtained through statistical software.

3.1. The QF of the NT-G Family

Suppose T follows the NT-*G* family with CDF $G(t)$, and its QF, say t_q, is given by

$$t_q = G^{-1}\left(\frac{4}{\pi}\tan^{-1}\left[\frac{1-\sqrt{1-q}}{1+\sqrt{1-q}}\right]\right). \tag{13}$$

From Equation (13), it is obvious that the QF of the NT-*G* class of distributions has a closed form. This is one of the interesting features of the NT-*G* distributions. Furthermore, using $q = \frac{1}{4}$, $q = \frac{1}{2}$, and $q = \frac{3}{4}$ in Equation (13), we obtain the first, second, and third quartiles of the NT-*G* distributions, respectively.

3.2. The Skewness and Kurtosis of the NT-G Family

This subsection provides the expressions of the skewness (Moor's skewness) and kurtosis (Gulton's kurtosis) of the NT-*G* family. Let T follow the NT-*G* family; then, it is

- Moor's skewness of the NT-*G* family is given by

$$\frac{Q_{2/8} - 2Q_{4/8} + Q_{6/8}}{Q_{6/8} - Q_{2/8}}. \tag{14}$$

  The terms $Q_{2/8}, Q_{4/8}$, and $Q_{6/8}$ in Equation (14), can easily be obtained by using $q = \frac{2}{8}$, $q = \frac{4}{8}$, and $q = \frac{6}{8}$ in Equation (13), respectively.
- Gulton's kurtosis of the NT-*G* family is given by

$$\frac{Q_{7/8} - Q_{5/8} - Q_{1/8} + Q_{3/8}}{Q_{6/8} - Q_{2/8}}. \tag{15}$$

  The terms $Q_{1/8}, Q_{3/8}, Q_{5/8}$, and $Q_{7/8}$ in Equation (15) can easily be obtained by using $q = \frac{1}{8}$, $q = \frac{3}{8}$, $q = \frac{5}{8}$, and $q = \frac{7}{8}$ in Equation (13), respectively.

3.3. The r^{th} Moment of the NT-G Family

In this subsection, we derive the r^{th} moment of the NT-*G* family.

Proposition 3. *Suppose $T \in \mathbb{R}$ is a NT-G distributed random variable having PDF $f(t)$. Then, the r^{th} moment of T, denoted by μ_r', is given by*

$$\mu_r' = 2\sum_{m=1}^{M}\sum_{n=0}^{\infty}\frac{(-1)^m(m+1)(m+2)}{(m+n+1)}A_r(t).$$

Proof. By definition, we know that

$$\mu_r' = \int_{\Omega} t^r f(t)dt. \tag{16}$$

Using Equation (8) in Equation (16), we obtain

$$\mu'_r = \int_\Omega t^r \frac{\pi g(t)\sec^2\left[\frac{\pi}{4}G(t)\right]\left(1-\tan\left[\frac{\pi}{4}G(t)\right]\right)}{\left(1+\tan\left[\frac{\pi}{4}G(t)\right]\right)^3} dt. \tag{17}$$

Using the series

$$\frac{1}{(1+x)^3} = \frac{1}{2}\sum_{m=1}^{M}(-1)^m(m+1)(m+2)x^m. \tag{18}$$

Using $x = \tan\left[\frac{\pi}{4}G(t)\right]$ in Equation (18), we obtain

$$\frac{1}{\left(1+\tan\left[\frac{\pi}{4}G(t)\right]\right)^3} = \frac{1}{2}\sum_{m=1}^{M}(-1)^m(m+1)(m+2)\left(\tan\left[\frac{\pi}{4}G(t)\right]\right)^m. \tag{19}$$

Using Equation (19) in Equation (17), we obtain

$$\mu'_r = \frac{\pi}{2}\sum_{m=1}^{M}(-1)^m(m+1)(m+2)\int_\Omega t^r g(t)\sec^2\left[\frac{\pi}{4}G(t)\right]\left(1-\tan\left[\frac{\pi}{4}G(t)\right]\right) \times \left(\tan\left[\frac{\pi}{4}G(t)\right]\right)^m dt,$$

$$\mu'_r = \frac{\pi}{2}\sum_{m=1}^{M}\sum_{n=0}^{\infty}(-1)^m(m+1)(m+2)\int_\Omega t^r g(t)\sec^2\left[\frac{\pi}{4}G(t)\right]\left(\tan\left[\frac{\pi}{4}G(t)\right]\right)^{m+n} dt,$$

$$\mu'_r = 2\sum_{m=1}^{M}\sum_{n=0}^{\infty}\frac{(-1)^m(m+1)(m+2)}{(m+n+1)}\int_\Omega t^r \frac{\pi}{4}(m+n+1)g(t)\sec^2\left[\frac{\pi}{4}G(t)\right] \times \left(\tan\left[\frac{\pi}{4}G(t)\right]\right)^{(m+n+1)-1} dt,$$

$$\mu'_r = 2\sum_{m=1}^{M}\sum_{n=0}^{\infty}\frac{(-1)^m(m+1)(m+2)}{(m+n+1)}A_r(t),$$

where

$$\int_\Omega t^r w(t)dt,$$

and

$$w(t) = \frac{\pi}{4}(m+n+1)g(t)\sec^2\left[\frac{\pi}{4}G(t)\right]\left(\tan\left[\frac{\pi}{4}G(t)\right]\right)^{(m+n+1)-1}. \tag{20}$$

The expression in Equation (20) represents the exponentiated PDF with exponentiated parameter $(m+n+1)$. Thus, Equation (20) is the exponentiated version of the Tangent-G (Tan-G) class [21]. Corresponding to Equation (20), the exponentiated CDF, say $W(t)$, is given by

$$W(t) = \left(\tan\left[\frac{\pi}{4}G(t)\right]\right)^{(m+n+1)}.$$

□

4. Estimation and Simulation for the NTF-Weibull Model

In this section, the most famous seven classical estimation methods are discussed and applied to estimate the NTF-Weibull parameters; each method of these seven estimation methods has its particular characteristic features and benefits.

Additionally, via the detailed simulation results of these estimation methods, the methods are numerically investigated. Here $\{t_i; i = 1, 2, \ldots, n\}$ and $\{t_{i:n}; i = 1, 2, \ldots, n\}$ refer to the random sample and associated order statistics from the NTF-Weibull distribution with PDF $f(t)$ in Equation (9), respectively.

4.1. The Estimation Methods

This subsection explores the derivation of the estimators (using different estimation methods) of the NTF-Weibull distribution, such as the maximum-likelihood estimators (MLEs), ordinary least squares estimators (OLSEs), weighted least squares estimators (WLSEs), Cramér–Von Mises estimators (CVMEs), maximum product of spacing estimators (MPSEs), Anderson–Darling estimators (ADEs), and right-tailed Anderson–Darling estimators (RADEs).

4.1.1. The MLEs of the NTF-Weibull Distribution

In this subsection, the MLEs of the NTF-Weibull distribution parameters, β, σ and θ, are derived. The likelihood function of the NTF-Weibull distribution, for β, σ, and θ, is

$$
\begin{aligned}
L(\boldsymbol{t}; \beta, \sigma, \theta) = \pi^n \prod_{i=1}^{n} \left(\beta + \frac{\theta\sigma}{t_i^{\theta+1}}\right) e^{\sum_{i=1}^{n}\left(\beta t_i - \frac{\sigma}{t_i^{\theta}}\right)} e^{-\sum_{i=1}^{n} e^{\beta t_i - \frac{\sigma}{t_i^{\theta}}}} \\
\prod_{i=1}^{n} \sec^2\left(\frac{1}{4}\pi\left(1 - e^{-e^{\beta t_i - \frac{\sigma}{t_i^{\theta}}}}\right)\right)\left(1 - \tan\left(\frac{1}{4}\pi\left(1 - e^{-e^{\beta t_i - \frac{\sigma}{t_i^{\theta}}}}\right)\right)\right) \\
\prod_{i=1}^{n} \left(\tan\left(\frac{1}{4}\pi\left(1 - e^{-e^{\beta t_i - \frac{\sigma}{t_i^{\theta}}}}\right)\right) + 1\right)^3.
\end{aligned}
$$

The corresponding log-likelihood function $\ell(\beta, \sigma, \theta; \boldsymbol{t}) = \log L(\beta, \sigma, \theta; \boldsymbol{t})$, takes the form

$$
\begin{aligned}
\ell(\boldsymbol{t}; \beta, \sigma, \theta) = n \log(\pi) + e^{-\sum_{i=1}^{n} e^{\beta t_i - \frac{\sigma}{t_i^{\theta}}}} + \sum_{i=1}^{n}\left(\beta t_i - \frac{\sigma}{t_i^{\theta}}\right) + \sum_{i=1}^{n} \log\left(\beta + \frac{\theta\sigma}{t_i^{\theta+1}}\right) \\
+ \sum_{i=1}^{n} \log\left(1 - \tan\left(\frac{1}{4}\pi\left(1 - e^{-e^{\beta t_i - \frac{\sigma}{t_i^{\theta}}}}\right)\right)\right) \\
- 3\sum_{i=1}^{n} \log\left(\tan\left(\frac{1}{4}\pi\left(1 - e^{-e^{\beta t_i - \frac{\sigma}{t_i^{\theta}}}}\right)\right) + 1\right) \\
+ 2\sum_{i=1}^{n} \log\left(\sec\left(\frac{1}{4}\pi\left(1 - e^{-e^{\beta t_i - \frac{\sigma}{t_i^{\theta}}}}\right)\right)\right).
\end{aligned}
$$

The MLEs of the NTF-Weibull parameters, $\widehat{\beta}_{MLE}$, $\widehat{\sigma}_{MLE}$, and $\widehat{\theta}_{MLE}$, can be determined by maximizing $\ell(\boldsymbol{t}; \beta, \sigma, \theta)$ with respect to β, σ and θ or by solving the following nonlinear equations

$$
\begin{aligned}
\frac{\partial}{\partial\beta}\ell(\boldsymbol{t};\beta,\sigma,\theta) =& \frac{\pi}{2}\sum_{i=1}^{n} t_i\, e^{-e^{\beta t_i-\sigma t_i^{-\theta}}+\beta t_i-\sigma t_i^{-\theta}} \tan\left(\frac{\pi}{4}\left(1-e^{-e^{\beta t_i-\sigma t_i^{-\theta}}}\right)\right)\\
&-\frac{\pi}{4}\sum_{i=1}^{n}\frac{t_i\, e^{-e^{\beta t_i-\sigma t_i^{-\theta}}+\beta t_i-\sigma t_i^{-\theta}} \sec^2\left(\frac{\pi}{4}\left(1-e^{-e^{\beta t_i-\sigma t_i^{-\theta}}}\right)\right)}{\left(1-\tan\left(\frac{\pi}{4}\left(1-e^{-e^{\beta t_i-\sigma t_i^{-\theta}}}\right)\right)\right)}\\
&-\frac{3\pi}{4}\sum_{i=1}^{n}\frac{t_i\, e^{-e^{\beta t_i-\sigma t_i^{-\theta}}+\beta t_i-\sigma t_i^{-\theta}} \sec^2\left(\frac{\pi}{4}\left(1-e^{-e^{\beta t_i-\sigma t_i^{-\theta}}}\right)\right)}{\left(\tan\left(\frac{\pi}{4}\left(1-e^{-e^{\beta t_i-\sigma t_i^{-\theta}}}\right)\right)+1\right)}\\
&-e^{-\sum_{i=1}^{n} e^{\beta t_i-\sigma t_i^{-\theta}}}\sum_{i=1}^{n} t_i\, e^{\beta t_i-\sigma t_i^{-\theta}}+\sum_{i=1}^{n}\frac{1}{\beta+\theta\sigma t_i^{-\theta-1}}+\sum_{i=1}^{n} t_i,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial\sigma}\ell(\boldsymbol{t};\beta,\sigma,\theta) =& -\frac{\pi}{2}\sum_{i=1}^{n} t_i^{-\theta} e^{-e^{\beta t_i-\sigma t_i^{-\theta}}+\beta t_i-\sigma t_i^{-\theta}} \tan\left(\frac{\pi}{4}\left(1-e^{-e^{\beta t_i-\sigma t_i^{-\theta}}}\right)\right)\\
&+\frac{\pi}{4}\sum_{i=1}^{n}\frac{t_i^{-\theta} e^{-e^{\beta t_i-\sigma t_i^{-\theta}}+\beta t_i-\sigma t_i^{-\theta}} \sec^2\left(\frac{\pi}{4}\left(1-e^{-e^{\beta t_i-\sigma t_i^{-\theta}}}\right)\right)}{\left(1-\tan\left(\frac{\pi}{4}\left(1-e^{-e^{\beta t_i-\sigma t_i^{-\theta}}}\right)\right)\right)}\\
&+\frac{3\pi}{4}\sum_{i=1}^{n}\frac{t_i^{-\theta} e^{-e^{\beta t_i-\sigma t_i^{-\theta}}+\beta t_i-\sigma t_i^{-\theta}} \sec^2\left(\frac{\pi}{4}\left(1-e^{-e^{\beta t_i-\sigma t_i^{-\theta}}}\right)\right)}{\left(\tan\left(\frac{\pi}{4}\left(1-e^{-e^{\beta t_i-\sigma t_i^{-\theta}}}\right)\right)+1\right)}\\
&+e^{-\sum_{i=1}^{n} e^{\beta t_i-\sigma t_i^{-\theta}}}\sum_{i=1}^{n} t_i^{-\theta} e^{\beta t_i-\sigma t_i^{-\theta}}+\sum_{i=1}^{n}\frac{\theta t_i^{-\theta-1}}{\beta+\theta\sigma t_i^{-\theta-1}}-\sum_{i=1}^{n} t_i^{-\theta},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial\theta}\ell(\boldsymbol{t};\beta,\sigma,\theta) =& \frac{\pi\sigma}{2}\sum_{i=1}^{n} t_i^{-\theta}\log(t_i) e^{-e^{\beta t_i-\sigma t_i^{-\theta}}+\beta t_i-\sigma t_i^{-\theta}} \tan\left(\frac{\pi}{4}\left(1-e^{-e^{\beta t_i-\sigma t_i^{-\theta}}}\right)\right)\\
&-\frac{\pi\sigma}{4}\sum_{i=1}^{n}\frac{\log(t_i) e^{-e^{\beta t_i-\sigma t_i^{-\theta}}+\beta t_i-\sigma t_i^{-\theta}} \sec^2\left(\frac{\pi}{4}\left(1-e^{-e^{\beta t_i-\sigma t_i^{-\theta}}}\right)\right)}{t_i^{\theta}\left(1-\tan\left(\frac{\pi}{4}\left(1-e^{-e^{\beta t_i-\sigma t_i^{-\theta}}}\right)\right)\right)}\\
&-\frac{3\pi\sigma}{4}\sum_{i=1}^{n}\frac{\log(t_i) e^{-e^{\beta t_i-\sigma t_i^{-\theta}}+\beta t_i-\sigma t_i^{-\theta}} \sec^2\left(\frac{\pi}{4}\left(1-e^{-e^{\beta t_i-\sigma t_i^{-\theta}}}\right)\right)}{t_i^{\theta}\left(\tan\left(\frac{\pi}{4}\left(1-e^{-e^{\beta t_i-\sigma t_i^{-\theta}}}\right)\right)+1\right)}\\
&-\sigma e^{-\sum_{i=1}^{n} e^{\beta t_i-\sigma t_i^{-\theta}}}\sum_{i=1}^{n} t_i^{-\theta}\log(t_i) e^{\beta t_i-\sigma t_i^{-\theta}}\\
&+\sigma\sum_{i=1}^{n}\frac{t_i^{-\theta-1}-\theta t_i^{-\theta-1}\log(t_i)}{\beta+\theta\sigma t_i^{-\theta-1}}+\sigma\sum_{i=1}^{n} t_i^{-\theta}\log(t_i).
\end{aligned}
$$

The elements of the observed Fisher information matrix $H(\beta,\sigma,\theta)$ are also obtained using several programs such as SAS (PROC NLMIXED) or the R (optim) function.

4.1.2. The OLSEs and WLSEs of the NTF-Weibull Distribution

The $\widehat{\beta}_{OLSEs}$, $\widehat{\sigma}_{OLSE}$, and $\widehat{\theta}_{OLSE}$ of the NTF-Weibull parameters are obtained by minimizing the following function

$$S_{\text{OLSE}}(\beta,\sigma,\theta)=\sum_{i=1}^{n}\left(1-\left(\frac{1-\tan\left[\frac{\pi}{4}\left(1-e^{-e^{\beta t_{(i)}-\frac{\sigma}{t_{(i)}^{\theta}}}}\right)\right]}{1+\tan\left[\frac{\pi}{4}\left(1-e^{-e^{\beta t_{(i)}-\frac{\sigma}{t_{(i)}^{\theta}}}}\right)\right]}\right)^{2}-\frac{i}{n+1}\right)^{2}$$

with respect to β, σ, and θ. Similarly, these estimators are also obtained by solving the following equations:

$$\sum_{i=1}^{n}\left(1-\left(\frac{1-\tan\left[\frac{\pi}{4}\left(1-e^{-e^{\beta t_{(i)}-\frac{\sigma}{t_{(i)}^{\theta}}}}\right)\right]}{1+\tan\left[\frac{\pi}{4}\left(1-e^{-e^{\beta t_{(i)}-\frac{\sigma}{t_{(i)}^{\theta}}}}\right)\right]}\right)^{2}-\frac{i}{n+1}\right)\Omega_k\left(t_{(i)}|\beta,\sigma,\theta\right)=0,$$

for $k=1,2,3$, where

$$\begin{aligned}\Omega_1\left(t_{(i)}|\beta,\sigma,\theta\right)&=\frac{\partial}{\partial\beta}F\left(t_{(i)}|\beta,\sigma,\theta\right)\\&=-\pi\frac{t_{(i)}e^{-e^{\beta t_{(i)}-\sigma t_{(i)}^{-\theta}}+\beta t_{(i)}-\sigma t_{(i)}^{-\theta}}\left(\cot\left(\frac{\pi}{4}\left(e^{-e^{\beta t_{(i)}-\sigma t_{(i)}^{-\theta}}}+1\right)\right)-1\right)}{\left(\cot\left(\frac{1}{4}\pi\left(e^{-e^{\beta t_{(i)}-\sigma t_{(i)}^{-\theta}}}+1\right)\right)+1\right)^{3}}\\&\quad\times\csc^{2}\left(\frac{\pi}{4}\left(e^{-e^{\beta t_{(i)}-\sigma t_{(i)}^{-\theta}}}+1\right)\right),\end{aligned}\tag{21}$$

$$\begin{aligned}\Omega_2\left(x_{(i)}|\beta,\sigma,\theta\right)&=\frac{\partial}{\partial\sigma}F\left(x_{(i)}|\beta,\sigma,\theta\right)\\&=\pi\frac{t_{(i)}^{-\theta}e^{-e^{\beta t_{(i)}-\sigma t_{(i)}^{-\theta}}+\beta t_{(i)}-\sigma t_{(i)}^{-\theta}}\left(\cot\left(\frac{\pi}{4}\left(e^{-e^{\beta t_{(i)}-\sigma t_{(i)}^{-\theta}}}+1\right)\right)-1\right)}{\left(\cot\left(\frac{\pi}{4}\left(e^{-e^{\beta t_{(i)}-\sigma t_{(i)}^{-\theta}}}+1\right)\right)+1\right)^{3}}\\&\quad\times\csc^{2}\left(\frac{\pi}{4}\left(e^{-e^{\beta t_{(i)}-\sigma t_{(i)}^{-\theta}}}+1\right)\right),\end{aligned}\tag{22}$$

and

$$\Omega_3\left(x_{(i)}|\beta,\sigma,\theta\right) = \frac{\partial}{\partial\theta}F\left(x_{(i)}|\beta,\sigma,\theta\right) \tag{23}$$
$$= -\pi\sigma\frac{e^{-e^{\beta t_{(i)}-\sigma t_{(i)}^{-\theta}}+\beta t_{(i)}-\sigma t_{(i)}^{-\theta}}\left(\cot\left(\frac{\pi}{4}\left(e^{-e^{\beta t_{(i)}-\sigma t_{(i)}^{-\theta}}}+1\right)\right)-1\right)}{t_{(i)}^{\theta}\left(\cot\left(\frac{\pi}{4}\left(e^{-e^{\beta t_{(i)}-\sigma t_{(i)}^{-\theta}}}+1\right)\right)+1\right)^3}$$
$$\times \log(t_{(i)})\csc^2\left(\frac{\pi}{4}\left(e^{-e^{\beta t_{(i)}-\sigma t_{(i)}^{-\theta}}}+1\right)\right).$$

The WLSEs, $\widehat{\beta}_{WLSE}$, $\widehat{\sigma}_{WLSE}$, and $\widehat{\theta}_{WLSE}$ of the NTF-Weibull parameters can be determined by minimizing the function

$$S_{\mathrm{WLSE}}(\beta,\sigma,\theta) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\left[\frac{1-\tan\left[\frac{\pi}{4}\left(1-e^{-A\left(t_{(i)}\right)}\right)\right]}{1+\tan\left[\frac{\pi}{4}\left(1-e^{-A\left(t_{(i)}\right)}\right)\right]}\right]^2 - \frac{n+1-i}{n+1}\right]^2,$$

where $A\left(t_{(i)}\right) = e^{\beta t_{(i)}-\frac{\sigma}{t_{(i)}^{\theta}}}$. The WLSEs are also derived by solving the following equations:

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\left[\frac{1-\tan\left[\frac{\pi}{4}\left(1-e^{-A\left(t_{(i)}\right)}\right)\right]}{1+\tan\left[\frac{\pi}{4}\left(1-e^{-A\left(t_{(i)}\right)}\right)\right]}\right]^2 - \frac{n+1-i}{n+1}\right]^2\Omega_k\left(x_{(i)}|\beta,\sigma,\theta\right) = 0,$$

where Ω_k are provided in Equations (21)–(23), respectively, for $k = 1,2,3$.

4.1.3. The CVMEs of the NTF-Weibull Distribution

The CVMEs of the NTF-Weibull parameters are obtained by minimizing the following function:

$$S_{\mathrm{CVME}}(\beta,\sigma,\theta) = \frac{1}{12n} + \sum_{i=1}^{n}\left(1-\left(\frac{1-\tan\left[\frac{\pi}{4}\left(1-e^{-e^{\beta t_{(i)}-\frac{\sigma}{t_{(i)}^{\theta}}}}\right)\right]}{1+\tan\left[\frac{\pi}{4}\left(1-e^{-e^{\beta t_{(i)}-\frac{\sigma}{t_{(i)}^{\theta}}}}\right)\right]}\right)^2 - \frac{2i-1}{2n}\right)^2.$$

Moreover, the CVMEs are also determined by solving the following equations:

$$\sum_{i=1}^{n}\left(1-\left(\frac{1-\tan\left[\frac{\pi}{4}\left(1-e^{-e^{\beta t_{(i)}-\frac{\sigma}{t_{(i)}^{\theta}}}}\right)\right]}{1+\tan\left[\frac{\pi}{4}\left(1-e^{-e^{\beta t_{(i)}-\frac{\sigma}{t_{(i)}^{\theta}}}}\right)\right]}\right)^2 - \frac{2i-1}{2n}\right)\Omega_k\left(x_{(i)}|\beta,\sigma,\theta\right) = 0,$$

where Ω_k are provided in Equations (21)–(23) for $k = 1,2,3$.

4.1.4. The MPSEs of the NTF-Weibull Distribution

The $\widehat{\beta}_{MPSE}$, $\widehat{\sigma}_{MPSE}$, and $\widehat{\theta}_{MPSE}$ of the NTF-Weibull parameters are obtained by maximizing the following function:

$$G(\beta,\sigma,\theta) = \left[\prod_{i=1}^{n+1} D_i(\beta,\sigma,\theta)\right]^{\frac{1}{n+1}},$$

where $D_i(\beta,\sigma,\theta) = F(x_{i:n};\beta,\sigma,\theta) - F(x_{i-1:n};\beta,\sigma,\theta),\ \ i = 1,\ldots,n,$ where $F(x_{0:n};\beta,\sigma,\theta) = 0$ and $F(x_{n+1:n};\beta,\sigma,\theta) = 1$. The MPSEs of the NTF-Weibull parameters are also obtained by solving the following equation:

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\beta,\sigma,\theta)}\Big[\Omega_k(x_{(i)}|\boldsymbol{\zeta}) - \Omega_k(x_{(i-1)}|\beta,\sigma,\theta)\Big] = 0, \quad k = 1,2,3,$$

where Ω_k are provided in Equations (21)–(23) for $k = 1,2,3$.

4.1.5. The ADEs and RADEs of the NTF-Weibull Distribution

The ADEs of the NTF-Weibull parameters are obtained by minimizing the following function:

$$S_{\text{ADE}}(\beta,\sigma,\theta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\{\log F(x_i;\beta,\sigma,\theta) + \log \overline{F}(x_{n+1-i};\beta,\sigma,\theta)\}$$

with respect to β, σ, and θ. The ADEs are also determined by solving the following equations

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Omega_k\Big(x_{(i)}|\beta,\sigma,\theta\Big)}{F\Big(x_{(i)}|\beta,\sigma,\theta\Big)} - \frac{\Omega_j\Big(x_{(n+1-i)}|\beta,\sigma,\theta\Big)}{S\Big(x_{(n+1-i)}|\beta,\sigma,\theta\Big)}\right] = 0,\ \ k,i = 1,2,3.$$

Furthermore, the $\widehat{\beta}_{RADE}$, $\widehat{\sigma}_{RADE}$, and $\widehat{\theta}_{RADE}$ of the NTF-Weibull parameters can be determined by minimizing the function

$$S_{\text{RADE}}(\beta,\sigma,\theta) = \frac{n}{2} - 2\sum_{i=1}^{n} F(x_{(i)};\beta,\sigma,\theta) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log \overline{F}(x_{(n+1-i)};\beta,\sigma,\theta),$$

where $\overline{F}(\cdot) = S(\cdot) = 1 - F(\cdot)$ with respect to the NTF-Weibull parameters. The RADEs are also determined by solving the following equation:

$$-2\sum_{i=1}^{n}\Omega_k\Big(x_{(i)}|\beta,\sigma,\theta\Big) + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\Omega_k\Big(x_{(n+1-i)}|\beta,\sigma,\theta\Big)}{S\Big(x_{(n+1-i)}|\beta,\sigma,\theta\Big)} = 0,\ \ k = 1,2,3.$$

where Ω_k are provided in Equations (21)–(23) for $k = 1,2,3$.

4.2. Simulation Study for the NTF-Weibull Distribution

In this subsection, the behaviors of the average of the absolute value of biases ($|Bias|$), the mean square error of the estimates (MSE), and the mean relative estimates (MRE) are investigated in the case of the seven estimation methods mentioned in the previous subsection. The simulation study is performed for different combination values of β, σ, and θ. It is important to note that there are no hard and fast rules for choosing the initial values of the parameters for conducting the simulation study. Within the given range of the parameters, we can choose any values. In this paper, we use the following parameter combinations: $\beta = \{0.70, 1.20, 6.00\}, \sigma = \{0.80, 1.45, 2.50\}$, and $\theta = \{0.50, 1.50, 5.00\}$.

Furthermore, we also use the following two combination values of the parameters: $(\beta = 0.3139, \sigma = 28.8384, \theta = 2.2965)$ and $(\beta = 0.1536, \sigma = 1.2998, \theta = 0.44742)$. These two sets are the values of the estimated parameters which are calculated from the two real-life data sets in Section 5.

To test the performances of these seven parameter estimation methods for the NTF-Weibull distribution with respect to sample size n, a simulation study is conducted based on the following steps:

Step 1. Generate one thousand samples of size n from the NTF-Weibull distribution with CDF $G(t)$ given by Equation (8). This work is performed simply by quantile function and generated data from the standard uniform distribution.

Step 2. Compute the estimates for the one thousand samples, say $(\hat{\beta}_i, \hat{\sigma}_i, \hat{\theta}_i)$ for $i = 1, 2, \ldots, N$, where $N = 5000$.

Step 3. Compute the biases, mean squared errors, and mean relative estimates, respectively, by

$$|Bias_{\boldsymbol{\tau}}| = \frac{1}{N}\sum_{i=1}^{N} |\hat{\boldsymbol{\tau}}_i - \boldsymbol{\tau}|,$$

$$MSE_{\boldsymbol{\tau}} = \frac{1}{N}\sum_{i=1}^{N} (\hat{\boldsymbol{\tau}}_i - \boldsymbol{\tau})^2,$$

and

$$MRE_{\boldsymbol{\tau}} = \frac{1}{N}\sum_{i=1}^{N} \frac{|\hat{\boldsymbol{\tau}}_i - \boldsymbol{\tau}|}{\boldsymbol{\tau}},$$

where $\boldsymbol{\tau} = (\beta, \sigma, \theta)^{\intercal}$. These steps have been repeated for $n = \{10, 20, 50, 80, 120, 200, 350\}$ with mentioned special cases of parameter combinations $\boldsymbol{\tau} = (\beta, \sigma, \theta)^{\intercal}$ and for each sample size n and each parameter combination: $Bias_{\boldsymbol{\tau}}$, $MSE_{\boldsymbol{\tau}}$, and $MRE_{\boldsymbol{\tau}}$ are computed. To compute the value of the estimators, we have used the nlminb() function available with R software (version 4.3.1).

To save more space, six out of twenty-nine simulated outcomes are listed in Tables 1–6; one can easily observe that when the sample size increases, the $|Bias_{\boldsymbol{\tau}}|$, $MSE_{\boldsymbol{\tau}}$ and $MRE_{\boldsymbol{\tau}}$ decrease. Thus, the seven estimation methods of the NTF-Weibull parameters show the property of consistency for all parameter combinations.

Table 1. Simulation results for $\boldsymbol{\tau} = (\beta = 0.7, \sigma = 0.8, \theta = 0.5)^{\intercal}$.

n	**Est.**	**Est. Par.**	**WLSE**	**OLSE**	**MLE**	**MPSE**	**CVME**	**ADE**	**RADE**
10	$\|BIAS\|$	$\hat{\beta}$	$0.60887^{\{4\}}$	$0.73829^{\{6\}}$	$0.57646^{\{2\}}$	$0.53558^{\{1\}}$	$0.79201^{\{7\}}$	$0.58047^{\{3\}}$	$0.64732^{\{5\}}$
		$\hat{\sigma}$	$0.42352^{\{4\}}$	$0.47162^{\{5\}}$	$0.39492^{\{2\}}$	$0.38956^{\{1\}}$	$0.48241^{\{7\}}$	$0.40263^{\{3\}}$	$0.47738^{\{6\}}$
		$\hat{\theta}$	$0.21778^{\{4\}}$	$0.23374^{\{5\}}$	$0.20309^{\{3\}}$	$0.18770^{\{1\}}$	$0.26236^{\{6\}}$	$0.19670^{\{2\}}$	$0.28011^{\{7\}}$
	MSE	$\hat{\beta}$	$1.37076^{\{4\}}$	$2.12349^{\{6\}}$	$1.19862^{\{3\}}$	$0.86294^{\{1\}}$	$2.25726^{\{7\}}$	$1.17328^{\{2\}}$	$1.46240^{\{5\}}$
		$\hat{\sigma}$	$0.37856^{\{4\}}$	$0.46967^{\{5\}}$	$0.27776^{\{1\}}$	$0.28624^{\{2\}}$	$0.47776^{\{6.5\}}$	$0.32266^{\{3\}}$	$0.47776^{\{6.5\}}$
		$\hat{\theta}$	$0.10425^{\{3\}}$	$0.10910^{\{4\}}$	$0.11378^{\{5\}}$	$0.06548^{\{1\}}$	$0.16486^{\{6\}}$	$0.07616^{\{2\}}$	$0.18315^{\{7\}}$
	MRE	$\hat{\beta}$	$0.86981^{\{4\}}$	$1.05471^{\{6\}}$	$0.82351^{\{2\}}$	$0.76512^{\{1\}}$	$1.13145^{\{7\}}$	$0.82924^{\{3\}}$	$0.92475^{\{5\}}$
		$\hat{\sigma}$	$0.52940^{\{4\}}$	$0.58953^{\{5\}}$	$0.49365^{\{2\}}$	$0.48696^{\{1\}}$	$0.60301^{\{7\}}$	$0.50329^{\{3\}}$	$0.59673^{\{6\}}$
		$\hat{\theta}$	$0.43556^{\{4\}}$	$0.46747^{\{5\}}$	$0.40617^{\{3\}}$	$0.37540^{\{1\}}$	$0.52473^{\{6\}}$	$0.39339^{\{2\}}$	$0.56022^{\{7\}}$
	$\sum Ranks$		$35^{\{4\}}$	$47^{\{5\}}$	$23^{\{2.5\}}$	$10^{\{1\}}$	$59.5^{\{7\}}$	$23^{\{2.5\}}$	$54.5^{\{6\}}$

Table 1. *Cont.*

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE
20	$\lvert BIAS \rvert$	$\hat{\beta}$	$0.37392^{\{5\}}$	$0.44759^{\{6\}}$	$0.30429^{\{2\}}$	$0.28212^{\{1\}}$	$0.45567^{\{7\}}$	$0.30432^{\{3\}}$	$0.31776^{\{4\}}$
		$\hat{\sigma}$	$0.27642^{\{4\}}$	$0.30391^{\{6\}}$	$0.24755^{\{1\}}$	$0.25273^{\{2\}}$	$0.31331^{\{7\}}$	$0.25777^{\{3\}}$	$0.29891^{\{5\}}$
		$\hat{\theta}$	$0.13809^{\{4\}}$	$0.15035^{\{5\}}$	$0.11849^{\{2\}}$	$0.11785^{\{1\}}$	$0.15616^{\{6\}}$	$0.12483^{\{3\}}$	$0.17066^{\{7\}}$
	MSE	$\hat{\beta}$	$0.39700^{\{5\}}$	$0.53743^{\{6\}}$	$0.20344^{\{2\}}$	$0.15395^{\{1\}}$	$0.60555^{\{7\}}$	$0.21416^{\{3\}}$	$0.23811^{\{4\}}$
		$\hat{\sigma}$	$0.13539^{\{4\}}$	$0.17305^{\{6\}}$	$0.09948^{\{1\}}$	$0.10747^{\{2\}}$	$0.17907^{\{7\}}$	$0.11571^{\{3\}}$	$0.16040^{\{5\}}$
		$\hat{\theta}$	$0.03362^{\{4\}}$	$0.03996^{\{5\}}$	$0.02465^{\{2\}}$	$0.02241^{\{1\}}$	$0.04520^{\{6\}}$	$0.02682^{\{3\}}$	$0.05544^{\{7\}}$
	MRE	$\hat{\beta}$	$0.53417^{\{5\}}$	$0.63941^{\{6\}}$	$0.43470^{\{2\}}$	$0.40302^{\{1\}}$	$0.65095^{\{7\}}$	$0.43474^{\{3\}}$	$0.45394^{\{4\}}$
		$\hat{\sigma}$	$0.34553^{\{4\}}$	$0.37988^{\{6\}}$	$0.30943^{\{1\}}$	$0.31592^{\{2\}}$	$0.39164^{\{7\}}$	$0.32221^{\{3\}}$	$0.37364^{\{5\}}$
		$\hat{\theta}$	$0.27618^{\{4\}}$	$0.30071^{\{5\}}$	$0.23698^{\{2\}}$	$0.23570^{\{1\}}$	$0.31232^{\{6\}}$	$0.24966^{\{3\}}$	$0.34132^{\{7\}}$
	$\sum Ranks$		$39^{\{4\}}$	$51^{\{6\}}$	$15^{\{2\}}$	$12^{\{1\}}$	$60^{\{7\}}$	$27^{\{3\}}$	$48^{\{5\}}$
50	$\lvert BIAS \rvert$	$\hat{\beta}$	$0.19045^{\{5\}}$	$0.23455^{\{6\}}$	$0.15784^{\{2\}}$	$0.15710^{\{1\}}$	$0.23849^{\{7\}}$	$0.16755^{\{3\}}$	$0.16996^{\{4\}}$
		$\hat{\sigma}$	$0.16338^{\{4\}}$	$0.18079^{\{7\}}$	$0.14912^{\{1\}}$	$0.15130^{\{2\}}$	$0.17930^{\{6\}}$	$0.15390^{\{3\}}$	$0.17669^{\{5\}}$
		$\hat{\theta}$	$0.07844^{\{4\}}$	$0.08922^{\{5\}}$	$0.06872^{\{1\}}$	$0.07004^{\{2\}}$	$0.09043^{\{6\}}$	$0.07301^{\{3\}}$	$0.09648^{\{7\}}$
	MSE	$\hat{\beta}$	$0.06702^{\{5\}}$	$0.09895^{\{6\}}$	$0.04569^{\{2\}}$	$0.04084^{\{1\}}$	$0.11250^{\{7\}}$	$0.04864^{\{3\}}$	$0.05065^{\{4\}}$
		$\hat{\sigma}$	$0.04286^{\{4\}}$	$0.05384^{\{7\}}$	$0.03594^{\{1\}}$	$0.03673^{\{2\}}$	$0.05283^{\{6\}}$	$0.03861^{\{3\}}$	$0.05114^{\{5\}}$
		$\hat{\theta}$	$0.00987^{\{4\}}$	$0.01292^{\{5\}}$	$0.00781^{\{2\}}$	$0.00756^{\{1\}}$	$0.01397^{\{6\}}$	$0.00863^{\{3\}}$	$0.01564^{\{7\}}$
	MRE	$\hat{\beta}$	$0.27207^{\{5\}}$	$0.33507^{\{6\}}$	$0.22549^{\{2\}}$	$0.22443^{\{1\}}$	$0.34070^{\{7\}}$	$0.23935^{\{3\}}$	$0.24279^{\{4\}}$
		$\hat{\sigma}$	$0.20422^{\{4\}}$	$0.22599^{\{7\}}$	$0.18640^{\{1\}}$	$0.18912^{\{2\}}$	$0.22413^{\{6\}}$	$0.19238^{\{3\}}$	$0.22087^{\{5\}}$
		$\hat{\theta}$	$0.15687^{\{4\}}$	$0.17844^{\{5\}}$	$0.13744^{\{1\}}$	$0.14007^{\{2\}}$	$0.18087^{\{6\}}$	$0.14602^{\{3\}}$	$0.19296^{\{7\}}$
	$\sum Ranks$		$39^{\{4\}}$	$54^{\{6\}}$	$13^{\{1\}}$	$14^{\{2\}}$	$57^{\{7\}}$	$27^{\{3\}}$	$48^{\{5\}}$
80	$\lvert BIAS \rvert$	$\hat{\beta}$	$0.14222^{\{5\}}$	$0.17364^{\{6\}}$	$0.11881^{\{2\}}$	$0.11780^{\{1\}}$	$0.17725^{\{7\}}$	$0.13198^{\{3\}}$	$0.13358^{\{4\}}$
		$\hat{\sigma}$	$0.12372^{\{3\}}$	$0.13813^{\{6\}}$	$0.11380^{\{1\}}$	$0.11538^{\{2\}}$	$0.14086^{\{7\}}$	$0.12426^{\{4\}}$	$0.13727^{\{5\}}$
		$\hat{\theta}$	$0.05837^{\{4\}}$	$0.06886^{\{5\}}$	$0.05261^{\{1\}}$	$0.05404^{\{2\}}$	$0.07083^{\{6\}}$	$0.05820^{\{3\}}$	$0.07344^{\{7\}}$
	MSE	$\hat{\beta}$	$0.03570^{\{5\}}$	$0.05234^{\{6\}}$	$0.02461^{\{2\}}$	$0.02218^{\{1\}}$	$0.05587^{\{7\}}$	$0.02914^{\{3\}}$	$0.03064^{\{4\}}$
		$\hat{\sigma}$	$0.02473^{\{4\}}$	$0.03080^{\{6\}}$	$0.02077^{\{1\}}$	$0.02133^{\{2\}}$	$0.03170^{\{7\}}$	$0.02436^{\{3\}}$	$0.03040^{\{5\}}$
		$\hat{\theta}$	$0.00553^{\{4\}}$	$0.00761^{\{5\}}$	$0.00450^{\{1\}}$	$0.00461^{\{2\}}$	$0.00807^{\{6\}}$	$0.00541^{\{3\}}$	$0.00881^{\{7\}}$
	MRE	$\hat{\beta}$	$0.20316^{\{5\}}$	$0.24805^{\{6\}}$	$0.16973^{\{2\}}$	$0.16828^{\{1\}}$	$0.25322^{\{7\}}$	$0.18854^{\{3\}}$	$0.19083^{\{4\}}$
		$\hat{\sigma}$	$0.15465^{\{3\}}$	$0.17266^{\{6\}}$	$0.14226^{\{1\}}$	$0.14422^{\{2\}}$	$0.17607^{\{7\}}$	$0.15532^{\{4\}}$	$0.17158^{\{5\}}$
		$\hat{\theta}$	$0.11673^{\{4\}}$	$0.13773^{\{5\}}$	$0.10521^{\{1\}}$	$0.10807^{\{2\}}$	$0.14165^{\{6\}}$	$0.11640^{\{3\}}$	$0.14687^{\{7\}}$
	$\sum Ranks$		$37^{\{4\}}$	$51^{\{6\}}$	$12^{\{1\}}$	$15^{\{2\}}$	$60^{\{7\}}$	$29^{\{3\}}$	$48^{\{5\}}$
120	$\lvert BIAS \rvert$	$\hat{\beta}$	$0.11310^{\{5\}}$	$0.14189^{\{6\}}$	$0.09316^{\{1\}}$	$0.09318^{\{2\}}$	$0.14509^{\{7\}}$	$0.10891^{\{4\}}$	$0.10607^{\{3\}}$
		$\hat{\sigma}$	$0.10121^{\{4\}}$	$0.11067^{\{5\}}$	$0.09334^{\{1\}}$	$0.09406^{\{2\}}$	$0.11482^{\{7\}}$	$0.09910^{\{3\}}$	$0.11205^{\{6\}}$
		$\hat{\theta}$	$0.04837^{\{4\}}$	$0.05496^{\{5\}}$	$0.04304^{\{1\}}$	$0.04339^{\{2\}}$	$0.05711^{\{6\}}$	$0.04670^{\{3\}}$	$0.06015^{\{7\}}$
	MSE	$\hat{\beta}$	$0.02205^{\{5\}}$	$0.03416^{\{6\}}$	$0.01470^{\{2\}}$	$0.01373^{\{1\}}$	$0.03572^{\{7\}}$	$0.01990^{\{4\}}$	$0.01873^{\{3\}}$
		$\hat{\sigma}$	$0.01636^{\{4\}}$	$0.01952^{\{5\}}$	$0.01384^{\{1\}}$	$0.01387^{\{2\}}$	$0.02068^{\{7\}}$	$0.01571^{\{3\}}$	$0.02017^{\{6\}}$
		$\hat{\theta}$	$0.00371^{\{4\}}$	$0.00478^{\{5\}}$	$0.00296^{\{2\}}$	$0.00291^{\{1\}}$	$0.00528^{\{6\}}$	$0.00348^{\{3\}}$	$0.00588^{\{7\}}$
	MRE	$\hat{\beta}$	$0.16157^{\{5\}}$	$0.20269^{\{6\}}$	$0.13308^{\{1\}}$	$0.13312^{\{2\}}$	$0.20727^{\{7\}}$	$0.15559^{\{4\}}$	$0.15153^{\{3\}}$
		$\hat{\sigma}$	$0.12652^{\{4\}}$	$0.13833^{\{5\}}$	$0.11668^{\{1\}}$	$0.11758^{\{2\}}$	$0.14352^{\{7\}}$	$0.12387^{\{3\}}$	$0.14006^{\{6\}}$
		$\hat{\theta}$	$0.09675^{\{4\}}$	$0.10992^{\{5\}}$	$0.08608^{\{1\}}$	$0.08677^{\{2\}}$	$0.11422^{\{6\}}$	$0.09341^{\{3\}}$	$0.12030^{\{7\}}$
	$\sum Ranks$		$39^{\{4\}}$	$48^{\{5.5\}}$	$11^{\{1\}}$	$16^{\{2\}}$	$60^{\{7\}}$	$30^{\{3\}}$	$48^{\{5.5\}}$
200	$\lvert BIAS \rvert$	$\hat{\beta}$	$0.08601^{\{5\}}$	$0.10779^{\{7\}}$	$0.07069^{\{1\}}$	$0.07079^{\{2\}}$	$0.10675^{\{6\}}$	$0.08090^{\{4\}}$	$0.07980^{\{3\}}$
		$\hat{\sigma}$	$0.07802^{\{4\}}$	$0.08614^{\{6\}}$	$0.07242^{\{2\}}$	$0.07081^{\{1\}}$	$0.08612^{\{5\}}$	$0.07667^{\{3\}}$	$0.08650^{\{7\}}$
		$\hat{\theta}$	$0.03686^{\{4\}}$	$0.04271^{\{5\}}$	$0.03288^{\{2\}}$	$0.03240^{\{1\}}$	$0.04329^{\{6\}}$	$0.03575^{\{3\}}$	$0.04604^{\{7\}}$
	MSE	$\hat{\beta}$	$0.01212^{\{5\}}$	$0.01901^{\{7\}}$	$0.00830^{\{2\}}$	$0.00788^{\{1\}}$	$0.01850^{\{6\}}$	$0.01050^{\{4\}}$	$0.01027^{\{3\}}$
		$\hat{\sigma}$	$0.00962^{\{4\}}$	$0.01175^{\{7\}}$	$0.00819^{\{2\}}$	$0.00799^{\{1\}}$	$0.01161^{\{5\}}$	$0.00928^{\{3\}}$	$0.01168^{\{6\}}$
		$\hat{\theta}$	$0.00217^{\{4\}}$	$0.00287^{\{5\}}$	$0.00172^{\{2\}}$	$0.00163^{\{1\}}$	$0.00295^{\{6\}}$	$0.00202^{\{3\}}$	$0.00336^{\{7\}}$

Table 1. *Cont.*

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE
	MRE	$\hat{\beta}$	$0.12287^{\{5\}}$	$0.15399^{\{7\}}$	$0.10098^{\{1\}}$	$0.10113^{\{2\}}$	$0.15250^{\{6\}}$	$0.11558^{\{4\}}$	$0.11400^{\{3\}}$
		$\hat{\sigma}$	$0.09752^{\{4\}}$	$0.10768^{\{6\}}$	$0.09053^{\{2\}}$	$0.08851^{\{1\}}$	$0.10765^{\{5\}}$	$0.09583^{\{3\}}$	$0.10812^{\{7\}}$
		$\hat{\theta}$	$0.07372^{\{4\}}$	$0.08543^{\{5\}}$	$0.06577^{\{2\}}$	$0.06480^{\{1\}}$	$0.08658^{\{6\}}$	$0.07149^{\{3\}}$	$0.09209^{\{7\}}$
	$\sum Ranks$		$39^{\{4\}}$	$55^{\{7\}}$	$16^{\{2\}}$	$11^{\{1\}}$	$51^{\{6\}}$	$30^{\{3\}}$	$50^{\{5\}}$
350	$\lvert BIAS\rvert$	$\hat{\beta}$	$0.06192^{\{4\}}$	$0.08047^{\{6\}}$	$0.05199^{\{1\}}$	$0.05342^{\{2\}}$	$0.08278^{\{7\}}$	$0.06194^{\{5\}}$	$0.06084^{\{3\}}$
		$\hat{\sigma}$	$0.05775^{\{3\}}$	$0.06477^{\{5\}}$	$0.05409^{\{2\}}$	$0.05366^{\{1\}}$	$0.06535^{\{7\}}$	$0.05782^{\{4\}}$	$0.06534^{\{6\}}$
		$\hat{\theta}$	$0.02698^{\{3\}}$	$0.03219^{\{5\}}$	$0.02471^{\{1\}}$	$0.02473^{\{2\}}$	$0.03274^{\{6\}}$	$0.02736^{\{4\}}$	$0.03476^{\{7\}}$
	MSE	$\hat{\beta}$	$0.00633^{\{5\}}$	$0.01048^{\{6\}}$	$0.00430^{\{1\}}$	$0.00441^{\{2\}}$	$0.01103^{\{7\}}$	$0.00605^{\{3.5\}}$	$0.00605^{\{3.5\}}$
		$\hat{\sigma}$	$0.00527^{\{3\}}$	$0.00670^{\{6\}}$	$0.00460^{\{2\}}$	$0.00452^{\{1\}}$	$0.00669^{\{5\}}$	$0.00536^{\{4\}}$	$0.00678^{\{7\}}$
		$\hat{\theta}$	$0.00115^{\{3\}}$	$0.00166^{\{5\}}$	$0.00096^{\{2\}}$	$0.00095^{\{1\}}$	$0.00167^{\{6\}}$	$0.00119^{\{4\}}$	$0.00192^{\{7\}}$
	MRE	$\hat{\beta}$	$0.08845^{\{4\}}$	$0.11496^{\{6\}}$	$0.07427^{\{1\}}$	$0.07632^{\{2\}}$	$0.11826^{\{7\}}$	$0.08848^{\{5\}}$	$0.08692^{\{3\}}$
		$\hat{\sigma}$	$0.07218^{\{3\}}$	$0.08096^{\{5\}}$	$0.06762^{\{2\}}$	$0.06708^{\{1\}}$	$0.08168^{\{6.5\}}$	$0.07228^{\{4\}}$	$0.08168^{\{6.5\}}$
		$\hat{\theta}$	$0.05397^{\{3\}}$	$0.06438^{\{5\}}$	$0.04941^{\{1\}}$	$0.04946^{\{2\}}$	$0.06548^{\{6\}}$	$0.05471^{\{4\}}$	$0.06953^{\{7\}}$
	$\sum Ranks$		$31^{\{3\}}$	$49^{\{5\}}$	$13^{\{1\}}$	$14^{\{2\}}$	$57.5^{\{7\}}$	$37.5^{\{4\}}$	$50^{\{6\}}$

Table 2. Simulation results for $\boldsymbol{\tau} = (\beta = 0.7, \sigma = 0.8, \theta = 5.0)^{\intercal}$.

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE
10	$\lvert BIAS\rvert$	$\hat{\beta}$	$2.17164^{\{4\}}$	$2.56619^{\{6\}}$	$1.93575^{\{1\}}$	$1.96814^{\{3\}}$	$3.18915^{\{7\}}$	$1.93871^{\{2\}}$	$2.33047^{\{5\}}$
		$\hat{\sigma}$	$1.95727^{\{4\}}$	$2.29884^{\{6\}}$	$1.71946^{\{1\}}$	$1.77158^{\{3\}}$	$2.82275^{\{7\}}$	$1.75008^{\{2\}}$	$2.16671^{\{5\}}$
		$\hat{\theta}$	$2.73481^{\{4\}}$	$2.81627^{\{5\}}$	$2.21091^{\{1\}}$	$2.32924^{\{2\}}$	$3.23122^{\{7\}}$	$2.55447^{\{3\}}$	$3.17761^{\{6\}}$
	MSE	$\hat{\beta}$	$24.78846^{\{4\}}$	$29.82385^{\{6\}}$	$23.30774^{\{3\}}$	$19.76297^{\{1\}}$	$45.17990^{\{7\}}$	$20.42784^{\{2\}}$	$27.59690^{\{5\}}$
		$\hat{\sigma}$	$20.45463^{\{4\}}$	$24.38247^{\{6\}}$	$18.27920^{\{3\}}$	$16.30388^{\{1\}}$	$35.81267^{\{7\}}$	$16.74123^{\{2\}}$	$23.09970^{\{5\}}$
		$\hat{\theta}$	$13.79277^{\{5\}}$	$13.77081^{\{4\}}$	$8.78375^{\{1\}}$	$9.24588^{\{2\}}$	$18.93916^{\{6\}}$	$11.74485^{\{3\}}$	$22.43674^{\{7\}}$
	MRE	$\hat{\beta}$	$3.10235^{\{4\}}$	$3.66599^{\{6\}}$	$2.76536^{\{1\}}$	$2.81163^{\{3\}}$	$4.55592^{\{7\}}$	$2.76958^{\{2\}}$	$3.32925^{\{5\}}$
		$\hat{\sigma}$	$2.44659^{\{4\}}$	$2.87355^{\{6\}}$	$2.14932^{\{1\}}$	$2.21448^{\{3\}}$	$3.52844^{\{7\}}$	$2.18760^{\{2\}}$	$2.70838^{\{5\}}$
		$\hat{\theta}$	$0.54696^{\{4\}}$	$0.56325^{\{5\}}$	$0.44218^{\{1\}}$	$0.46585^{\{2\}}$	$0.64624^{\{7\}}$	$0.51089^{\{3\}}$	$0.63552^{\{6\}}$
	$\sum Ranks$		$37^{\{4\}}$	$50^{\{6\}}$	$13^{\{1\}}$	$20^{\{2\}}$	$62^{\{7\}}$	$21^{\{3\}}$	$49^{\{5\}}$
20	$\lvert BIAS\rvert$	$\hat{\beta}$	$1.14926^{\{5\}}$	$1.49304^{\{6\}}$	$0.70723^{\{3\}}$	$0.65958^{\{1\}}$	$1.59961^{\{7\}}$	$0.69017^{\{2\}}$	$0.96277^{\{4\}}$
		$\hat{\sigma}$	$1.01874^{\{5\}}$	$1.32991^{\{6\}}$	$0.61773^{\{3\}}$	$0.58230^{\{1\}}$	$1.42085^{\{7\}}$	$0.60984^{\{2\}}$	$0.90071^{\{4\}}$
		$\hat{\theta}$	$1.81125^{\{4\}}$	$2.09685^{\{6\}}$	$1.35363^{\{1\}}$	$1.38793^{\{2\}}$	$2.14612^{\{7\}}$	$1.51295^{\{3\}}$	$1.86586^{\{5\}}$
	MSE	$\hat{\beta}$	$8.68237^{\{5\}}$	$12.25666^{\{6\}}$	$2.65888^{\{2\}}$	$2.31122^{\{1\}}$	$15.33160^{\{7\}}$	$2.68761^{\{3\}}$	$6.25961^{\{4\}}$
		$\hat{\sigma}$	$7.14481^{\{5\}}$	$10.24052^{\{6\}}$	$1.99781^{\{2\}}$	$1.89461^{\{1\}}$	$12.49996^{\{7\}}$	$2.18072^{\{3\}}$	$5.41973^{\{4\}}$
		$\hat{\theta}$	$5.52045^{\{4\}}$	$7.53558^{\{6\}}$	$2.97521^{\{1\}}$	$3.16450^{\{2\}}$	$7.71913^{\{7\}}$	$3.95299^{\{3\}}$	$6.08044^{\{5\}}$
	MRE	$\hat{\beta}$	$1.64181^{\{5\}}$	$2.13291^{\{6\}}$	$1.01033^{\{3\}}$	$0.94226^{\{1\}}$	$2.28516^{\{7\}}$	$0.98596^{\{2\}}$	$1.37539^{\{4\}}$
		$\hat{\sigma}$	$1.27342^{\{5\}}$	$1.66239^{\{6\}}$	$0.77216^{\{3\}}$	$0.72788^{\{1\}}$	$1.77606^{\{7\}}$	$0.76230^{\{2\}}$	$1.12588^{\{4\}}$
		$\hat{\theta}$	$0.36225^{\{4\}}$	$0.41937^{\{6\}}$	$0.27073^{\{1\}}$	$0.27759^{\{2\}}$	$0.42922^{\{7\}}$	$0.30259^{\{3\}}$	$0.37317^{\{5\}}$
	$\sum Ranks$		$42^{\{5\}}$	$54^{\{6\}}$	$19^{\{2\}}$	$12^{\{1\}}$	$63^{\{7\}}$	$23^{\{3\}}$	$39^{\{4\}}$
50	$\lvert BIAS\rvert$	$\hat{\beta}$	$0.36249^{\{5\}}$	$0.54190^{\{6\}}$	$0.27586^{\{2\}}$	$0.26808^{\{1\}}$	$0.55944^{\{7\}}$	$0.29507^{\{3\}}$	$0.31411^{\{4\}}$
		$\hat{\sigma}$	$0.31211^{\{5\}}$	$0.46086^{\{6\}}$	$0.24421^{\{2\}}$	$0.23225^{\{1\}}$	$0.47568^{\{7\}}$	$0.25775^{\{3\}}$	$0.29339^{\{4\}}$
		$\hat{\theta}$	$0.96790^{\{4\}}$	$1.22062^{\{6\}}$	$0.77098^{\{1\}}$	$0.77462^{\{2\}}$	$1.22975^{\{7\}}$	$0.87013^{\{3\}}$	$1.02642^{\{5\}}$
	MSE	$\hat{\beta}$	$0.39313^{\{5\}}$	$1.39578^{\{6\}}$	$0.21039^{\{3\}}$	$0.16493^{\{1\}}$	$1.45171^{\{7\}}$	$0.20315^{\{2\}}$	$0.32531^{\{4\}}$
		$\hat{\sigma}$	$0.29204^{\{5\}}$	$1.11751^{\{6\}}$	$0.14534^{\{2\}}$	$0.12377^{\{1\}}$	$1.13071^{\{7\}}$	$0.14829^{\{3\}}$	$0.27520^{\{4\}}$
		$\hat{\theta}$	$1.53528^{\{4\}}$	$2.44308^{\{6\}}$	$0.95630^{\{1\}}$	$0.98226^{\{2\}}$	$2.47728^{\{7\}}$	$1.22966^{\{3\}}$	$1.76622^{\{5\}}$
	MRE	$\hat{\beta}$	$0.51785^{\{5\}}$	$0.77414^{\{6\}}$	$0.39408^{\{2\}}$	$0.38297^{\{1\}}$	$0.79921^{\{7\}}$	$0.42153^{\{3\}}$	$0.44873^{\{4\}}$
		$\hat{\sigma}$	$0.39013^{\{5\}}$	$0.57607^{\{6\}}$	$0.30526^{\{2\}}$	$0.29031^{\{1\}}$	$0.59460^{\{7\}}$	$0.32219^{\{3\}}$	$0.36674^{\{4\}}$
		$\hat{\theta}$	$0.19358^{\{4\}}$	$0.24412^{\{6\}}$	$0.15420^{\{1\}}$	$0.15492^{\{2\}}$	$0.24595^{\{7\}}$	$0.17403^{\{3\}}$	$0.20528^{\{5\}}$
	$\sum Ranks$		$42^{\{5\}}$	$54^{\{6\}}$	$16^{\{2\}}$	$12^{\{1\}}$	$63^{\{7\}}$	$26^{\{3\}}$	$39^{\{4\}}$

Table 2. *Cont.*

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE
80	$\lvert BIAS \rvert$	$\hat{\beta}$	$0.25287^{\{5\}}$	$0.35366^{\{7\}}$	$0.19880^{\{2\}}$	$0.19171^{\{1\}}$	$0.34624^{\{6\}}$	$0.22248^{\{4\}}$	$0.22049^{\{3\}}$
		$\hat{\sigma}$	$0.21571^{\{5\}}$	$0.29647^{\{7\}}$	$0.17403^{\{2\}}$	$0.16553^{\{1\}}$	$0.29076^{\{6\}}$	$0.19544^{\{3\}}$	$0.20620^{\{4\}}$
		$\hat{\theta}$	$0.72061^{\{4\}}$	$0.93340^{\{7\}}$	$0.58485^{\{1\}}$	$0.58630^{\{2\}}$	$0.91626^{\{6\}}$	$0.68297^{\{3\}}$	$0.78025^{\{5\}}$
	MSE	$\hat{\beta}$	$0.14989^{\{5\}}$	$0.32826^{\{7\}}$	$0.08796^{\{2\}}$	$0.07046^{\{1\}}$	$0.29585^{\{6\}}$	$0.10073^{\{3\}}$	$0.11252^{\{4\}}$
		$\hat{\sigma}$	$0.10586^{\{5\}}$	$0.23463^{\{7\}}$	$0.06140^{\{2\}}$	$0.05414^{\{1\}}$	$0.20715^{\{6\}}$	$0.07578^{\{3\}}$	$0.09145^{\{4\}}$
		$\hat{\theta}$	$0.85234^{\{4\}}$	$1.39407^{\{7\}}$	$0.54755^{\{1\}}$	$0.56244^{\{2\}}$	$1.35161^{\{6\}}$	$0.75592^{\{3\}}$	$0.98275^{\{5\}}$
	MRE	$\hat{\beta}$	$0.36125^{\{5\}}$	$0.50523^{\{7\}}$	$0.28400^{\{2\}}$	$0.27387^{\{1\}}$	$0.49463^{\{6\}}$	$0.31782^{\{4\}}$	$0.31499^{\{3\}}$
		$\hat{\sigma}$	$0.26964^{\{5\}}$	$0.37059^{\{7\}}$	$0.21754^{\{2\}}$	$0.20691^{\{1\}}$	$0.36345^{\{6\}}$	$0.24430^{\{3\}}$	$0.25775^{\{4\}}$
		$\hat{\theta}$	$0.14412^{\{4\}}$	$0.18668^{\{7\}}$	$0.11697^{\{1\}}$	$0.11726^{\{2\}}$	$0.18325^{\{6\}}$	$0.13659^{\{3\}}$	$0.15605^{\{5\}}$
	$\sum Ranks$		$42^{\{5\}}$	$63^{\{7\}}$	$15^{\{2\}}$	$12^{\{1\}}$	$54^{\{6\}}$	$29^{\{3\}}$	$37^{\{4\}}$
120	$\lvert BIAS \rvert$	$\hat{\beta}$	$0.19298^{\{5\}}$	$0.27368^{\{6\}}$	$0.14708^{\{2\}}$	$0.14695^{\{1\}}$	$0.27818^{\{7\}}$	$0.17285^{\{3\}}$	$0.17353^{\{4\}}$
		$\hat{\sigma}$	$0.16665^{\{5\}}$	$0.22724^{\{6\}}$	$0.13386^{\{2\}}$	$0.12979^{\{1\}}$	$0.22976^{\{7\}}$	$0.15024^{\{3\}}$	$0.16197^{\{4\}}$
		$\hat{\theta}$	$0.58067^{\{4\}}$	$0.75365^{\{6\}}$	$0.46817^{\{1\}}$	$0.47256^{\{2\}}$	$0.76043^{\{7\}}$	$0.54272^{\{3\}}$	$0.62279^{\{5\}}$
	MSE	$\hat{\beta}$	$0.07372^{\{5\}}$	$0.16209^{\{6\}}$	$0.04237^{\{2\}}$	$0.03746^{\{1\}}$	$0.17445^{\{7\}}$	$0.05439^{\{3\}}$	$0.05820^{\{4\}}$
		$\hat{\sigma}$	$0.05397^{\{5\}}$	$0.11253^{\{6\}}$	$0.03253^{\{2\}}$	$0.03014^{\{1\}}$	$0.11658^{\{7\}}$	$0.04006^{\{3\}}$	$0.04916^{\{4\}}$
		$\hat{\theta}$	$0.54563^{\{4\}}$	$0.90960^{\{6\}}$	$0.34813^{\{1\}}$	$0.36579^{\{2\}}$	$0.93278^{\{7\}}$	$0.48227^{\{3\}}$	$0.61896^{\{5\}}$
	MRE	$\hat{\beta}$	$0.27568^{\{5\}}$	$0.39097^{\{6\}}$	$0.21011^{\{2\}}$	$0.20993^{\{1\}}$	$0.39739^{\{7\}}$	$0.24693^{\{3\}}$	$0.24790^{\{4\}}$
		$\hat{\sigma}$	$0.20831^{\{5\}}$	$0.28405^{\{6\}}$	$0.16733^{\{2\}}$	$0.16224^{\{1\}}$	$0.28720^{\{7\}}$	$0.18781^{\{3\}}$	$0.20246^{\{4\}}$
		$\hat{\theta}$	$0.11613^{\{4\}}$	$0.15073^{\{6\}}$	$0.09363^{\{1\}}$	$0.09451^{\{2\}}$	$0.15209^{\{7\}}$	$0.10854^{\{3\}}$	$0.12456^{\{5\}}$
	$\sum Ranks$		$42^{\{5\}}$	$54^{\{6\}}$	$15^{\{2\}}$	$12^{\{1\}}$	$63^{\{7\}}$	$27^{\{3\}}$	$39^{\{4\}}$
200	$\lvert BIAS \rvert$	$\hat{\beta}$	$0.13638^{\{5\}}$	$0.20096^{\{6\}}$	$0.11052^{\{2\}}$	$0.10880^{\{1\}}$	$0.20321^{\{7\}}$	$0.12913^{\{4\}}$	$0.12693^{\{3\}}$
		$\hat{\sigma}$	$0.11814^{\{4\}}$	$0.16603^{\{6\}}$	$0.09998^{\{2\}}$	$0.09463^{\{1\}}$	$0.16928^{\{7\}}$	$0.11213^{\{3\}}$	$0.11860^{\{5\}}$
		$\hat{\theta}$	$0.42701^{\{4\}}$	$0.57573^{\{6\}}$	$0.36266^{\{2\}}$	$0.34321^{\{1\}}$	$0.58932^{\{7\}}$	$0.40964^{\{3\}}$	$0.47361^{\{5\}}$
	MSE	$\hat{\beta}$	$0.03333^{\{5\}}$	$0.07435^{\{6\}}$	$0.02205^{\{2\}}$	$0.01951^{\{1\}}$	$0.07483^{\{7\}}$	$0.02855^{\{4\}}$	$0.02767^{\{3\}}$
		$\hat{\sigma}$	$0.02413^{\{5\}}$	$0.05030^{\{6\}}$	$0.01705^{\{2\}}$	$0.01539^{\{1\}}$	$0.05136^{\{7\}}$	$0.02088^{\{3\}}$	$0.02347^{\{4\}}$
		$\hat{\theta}$	$0.29226^{\{4\}}$	$0.52373^{\{6\}}$	$0.20872^{\{2\}}$	$0.19797^{\{1\}}$	$0.54385^{\{7\}}$	$0.26377^{\{3\}}$	$0.35237^{\{5\}}$
	MRE	$\hat{\beta}$	$0.19483^{\{5\}}$	$0.28709^{\{6\}}$	$0.15789^{\{2\}}$	$0.15543^{\{1\}}$	$0.29031^{\{7\}}$	$0.18448^{\{4\}}$	$0.18133^{\{3\}}$
		$\hat{\sigma}$	$0.14767^{\{4\}}$	$0.20754^{\{6\}}$	$0.12498^{\{2\}}$	$0.11829^{\{1\}}$	$0.21160^{\{7\}}$	$0.14016^{\{3\}}$	$0.14824^{\{5\}}$
		$\hat{\theta}$	$0.08540^{\{4\}}$	$0.11515^{\{6\}}$	$0.07253^{\{2\}}$	$0.06864^{\{1\}}$	$0.11786^{\{7\}}$	$0.08193^{\{3\}}$	$0.09472^{\{5\}}$
	$\sum Ranks$		$40^{\{5\}}$	$54^{\{6\}}$	$18^{\{2\}}$	$9^{\{1\}}$	$63^{\{7\}}$	$30^{\{3\}}$	$38^{\{4\}}$
350	$\lvert BIAS \rvert$	$\hat{\beta}$	$0.10073^{\{5\}}$	$0.14860^{\{7\}}$	$0.07789^{\{1\}}$	$0.08008^{\{2\}}$	$0.14750^{\{6\}}$	$0.09917^{\{4\}}$	$0.09646^{\{3\}}$
		$\hat{\sigma}$	$0.08681^{\{4\}}$	$0.12280^{\{6\}}$	$0.07106^{\{2\}}$	$0.07069^{\{1\}}$	$0.12354^{\{7\}}$	$0.08627^{\{3\}}$	$0.09069^{\{5\}}$
		$\hat{\theta}$	$0.31930^{\{4\}}$	$0.43579^{\{6\}}$	$0.26175^{\{2\}}$	$0.25552^{\{1\}}$	$0.43963^{\{7\}}$	$0.31641^{\{3\}}$	$0.35959^{\{5\}}$
	MSE	$\hat{\beta}$	$0.01719^{\{5\}}$	$0.03789^{\{6\}}$	$0.01014^{\{2\}}$	$0.01012^{\{1\}}$	$0.03880^{\{7\}}$	$0.01646^{\{4\}}$	$0.01536^{\{3\}}$
		$\hat{\sigma}$	$0.01252^{\{4\}}$	$0.02553^{\{6\}}$	$0.00820^{\{1\}}$	$0.00821^{\{2\}}$	$0.02658^{\{7\}}$	$0.01210^{\{3\}}$	$0.01334^{\{5\}}$
		$\hat{\theta}$	$0.16346^{\{4\}}$	$0.30303^{\{6\}}$	$0.10690^{\{1\}}$	$0.11313^{\{2\}}$	$0.30610^{\{7\}}$	$0.15805^{\{3\}}$	$0.20438^{\{5\}}$
	MRE	$\hat{\beta}$	$0.14390^{\{5\}}$	$0.21228^{\{7\}}$	$0.11127^{\{1\}}$	$0.11440^{\{2\}}$	$0.21071^{\{6\}}$	$0.14166^{\{4\}}$	$0.13780^{\{3\}}$
		$\hat{\sigma}$	$0.10851^{\{4\}}$	$0.15350^{\{6\}}$	$0.08883^{\{2\}}$	$0.08837^{\{1\}}$	$0.15443^{\{7\}}$	$0.10784^{\{3\}}$	$0.11336^{\{5\}}$
		$\hat{\theta}$	$0.06386^{\{4\}}$	$0.08716^{\{6\}}$	$0.05235^{\{2\}}$	$0.05110^{\{1\}}$	$0.08793^{\{7\}}$	$0.06328^{\{3\}}$	$0.07192^{\{5\}}$
	$\sum Ranks$		$39^{\{4.5\}}$	$56^{\{6\}}$	$14^{\{2\}}$	$13^{\{1\}}$	$61^{\{7\}}$	$30^{\{3\}}$	$39^{\{4.5\}}$

Table 3. Simulation results for $\boldsymbol{\tau} = (\beta = 1.2, \sigma = 0.8, \theta = 1.5)^{\intercal}$.

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE
10	$\lvert BIAS \rvert$	$\hat{\beta}$	$1.61883^{\{4\}}$	$1.83450^{\{6\}}$	$1.39432^{\{2\}}$	$1.38861^{\{1\}}$	$2.13812^{\{7\}}$	$1.51719^{\{3\}}$	$1.67556^{\{5\}}$
		$\hat{\sigma}$	$1.13490^{\{4\}}$	$1.27787^{\{6\}}$	$0.91308^{\{1\}}$	$0.98440^{\{2\}}$	$1.44333^{\{7\}}$	$1.06127^{\{3\}}$	$1.22443^{\{5\}}$
		$\hat{\theta}$	$0.89425^{\{4\}}$	$0.96267^{\{5\}}$	$0.70268^{\{1\}}$	$0.77148^{\{2\}}$	$1.09051^{\{6\}}$	$0.86847^{\{3\}}$	$1.10177^{\{7\}}$
	MSE	$\hat{\beta}$	$8.15443^{\{4\}}$	$9.79201^{\{6\}}$	$7.24438^{\{3\}}$	$5.95280^{\{1\}}$	$14.44607^{\{7\}}$	$7.19181^{\{2\}}$	$9.17721^{\{5\}}$
		$\hat{\sigma}$	$3.96615^{\{4\}}$	$4.80821^{\{6\}}$	$2.90587^{\{1\}}$	$2.98428^{\{2\}}$	$6.40603^{\{7\}}$	$3.54416^{\{3\}}$	$4.65791^{\{5\}}$
		$\hat{\theta}$	$1.48436^{\{4\}}$	$1.63046^{\{5\}}$	$0.92649^{\{1\}}$	$1.05179^{\{2\}}$	$2.35471^{\{6\}}$	$1.39582^{\{3\}}$	$2.51402^{\{7\}}$
	MRE	$\hat{\beta}$	$1.34902^{\{4\}}$	$1.52875^{\{6\}}$	$1.16193^{\{2\}}$	$1.15717^{\{1\}}$	$1.78177^{\{7\}}$	$1.26432^{\{3\}}$	$1.39630^{\{5\}}$
		$\hat{\sigma}$	$1.41863^{\{4\}}$	$1.59734^{\{6\}}$	$1.14135^{\{1\}}$	$1.23050^{\{2\}}$	$1.80417^{\{7\}}$	$1.32658^{\{3\}}$	$1.53054^{\{5\}}$
		$\hat{\theta}$	$0.59617^{\{4\}}$	$0.64178^{\{5\}}$	$0.46846^{\{1\}}$	$0.51432^{\{2\}}$	$0.72701^{\{6\}}$	$0.57898^{\{3\}}$	$0.73451^{\{7\}}$
	$\sum Ranks$		$36^{\{4\}}$	$51^{\{5.5\}}$	$13^{\{1\}}$	$15^{\{2\}}$	$60^{\{7\}}$	$26^{\{3\}}$	$51^{\{5.5\}}$
20	$\lvert BIAS \rvert$	$\hat{\beta}$	$0.96420^{\{5\}}$	$1.15693^{\{6\}}$	$0.69439^{\{2\}}$	$0.69069^{\{1\}}$	$1.23453^{\{7\}}$	$0.77471^{\{3\}}$	$0.89192^{\{4\}}$
		$\hat{\sigma}$	$0.66038^{\{4\}}$	$0.78981^{\{6\}}$	$0.46434^{\{1\}}$	$0.48123^{\{2\}}$	$0.83497^{\{7\}}$	$0.53293^{\{3\}}$	$0.66486^{\{5\}}$
		$\hat{\theta}$	$0.59285^{\{4\}}$	$0.66792^{\{5\}}$	$0.44481^{\{1\}}$	$0.47026^{\{2\}}$	$0.69063^{\{7\}}$	$0.51403^{\{3\}}$	$0.67399^{\{6\}}$
	MSE	$\hat{\beta}$	$2.82518^{\{5\}}$	$3.96010^{\{6\}}$	$1.34225^{\{2\}}$	$1.20840^{\{1\}}$	$4.79176^{\{7\}}$	$1.75067^{\{3\}}$	$2.37220^{\{4\}}$
		$\hat{\sigma}$	$1.38806^{\{5\}}$	$1.96056^{\{6\}}$	$0.56525^{\{1\}}$	$0.61674^{\{2\}}$	$2.26838^{\{7\}}$	$0.82708^{\{3\}}$	$1.32803^{\{4\}}$
		$\hat{\theta}$	$0.59536^{\{4\}}$	$0.75293^{\{5\}}$	$0.33084^{\{1\}}$	$0.35196^{\{2\}}$	$0.82265^{\{7\}}$	$0.43311^{\{3\}}$	$0.80082^{\{6\}}$
	MRE	$\hat{\beta}$	$0.80350^{\{5\}}$	$0.96411^{\{6\}}$	$0.57866^{\{2\}}$	$0.57558^{\{1\}}$	$1.02877^{\{7\}}$	$0.64559^{\{3\}}$	$0.74327^{\{4\}}$
		$\hat{\sigma}$	$0.82547^{\{4\}}$	$0.98726^{\{6\}}$	$0.58043^{\{1\}}$	$0.60154^{\{2\}}$	$1.04371^{\{7\}}$	$0.66617^{\{3\}}$	$0.83108^{\{5\}}$
		$\hat{\theta}$	$0.39523^{\{4\}}$	$0.44528^{\{5\}}$	$0.29654^{\{1\}}$	$0.31350^{\{2\}}$	$0.46042^{\{7\}}$	$0.34269^{\{3\}}$	$0.44932^{\{6\}}$
	$\sum Ranks$		$40^{\{4\}}$	$51^{\{6\}}$	$12^{\{1\}}$	$15^{\{2\}}$	$63^{\{7\}}$	$27^{\{3\}}$	$44^{\{5\}}$
50	$\lvert BIAS \rvert$	$\hat{\beta}$	$0.45317^{\{5\}}$	$0.59738^{\{7\}}$	$0.34779^{\{1\}}$	$0.34881^{\{2\}}$	$0.58612^{\{6\}}$	$0.39572^{\{3\}}$	$0.41576^{\{4\}}$
		$\hat{\sigma}$	$0.30495^{\{4\}}$	$0.39471^{\{7\}}$	$0.23577^{\{1\}}$	$0.24333^{\{2\}}$	$0.38298^{\{6\}}$	$0.27081^{\{3\}}$	$0.31210^{\{5\}}$
		$\hat{\theta}$	$0.33318^{\{4\}}$	$0.40962^{\{7\}}$	$0.25819^{\{1\}}$	$0.27437^{\{2\}}$	$0.39546^{\{6\}}$	$0.30821^{\{3\}}$	$0.37884^{\{5\}}$
	MSE	$\hat{\beta}$	$0.42413^{\{5\}}$	$0.81322^{\{6\}}$	$0.25045^{\{2\}}$	$0.22840^{\{1\}}$	$0.82657^{\{7\}}$	$0.30969^{\{3\}}$	$0.36453^{\{4\}}$
		$\hat{\sigma}$	$0.19329^{\{4\}}$	$0.37481^{\{7\}}$	$0.10797^{\{1\}}$	$0.11525^{\{2\}}$	$0.36491^{\{6\}}$	$0.14657^{\{3\}}$	$0.20330^{\{5\}}$
		$\hat{\theta}$	$0.17961^{\{4\}}$	$0.27307^{\{7\}}$	$0.10883^{\{1\}}$	$0.12018^{\{2\}}$	$0.25580^{\{6\}}$	$0.15093^{\{3\}}$	$0.23399^{\{5\}}$
	MRE	$\hat{\beta}$	$0.37764^{\{5\}}$	$0.49782^{\{7\}}$	$0.28983^{\{1\}}$	$0.29068^{\{2\}}$	$0.48843^{\{6\}}$	$0.32976^{\{3\}}$	$0.34646^{\{4\}}$
		$\hat{\sigma}$	$0.38118^{\{4\}}$	$0.49338^{\{7\}}$	$0.29471^{\{1\}}$	$0.30416^{\{2\}}$	$0.47872^{\{6\}}$	$0.33852^{\{3\}}$	$0.39013^{\{5\}}$
		$\hat{\theta}$	$0.22212^{\{4\}}$	$0.27308^{\{7\}}$	$0.17212^{\{1\}}$	$0.18291^{\{2\}}$	$0.26364^{\{6\}}$	$0.20547^{\{3\}}$	$0.25256^{\{5\}}$
	$\sum Ranks$		$39^{\{4\}}$	$62^{\{7\}}$	$10^{\{1\}}$	$17^{\{2\}}$	$55^{\{6\}}$	$27^{\{3\}}$	$42^{\{5\}}$
80	$\lvert BIAS \rvert$	$\hat{\beta}$	$0.33661^{\{5\}}$	$0.42561^{\{6\}}$	$0.25646^{\{1\}}$	$0.26434^{\{2\}}$	$0.43411^{\{7\}}$	$0.29472^{\{3\}}$	$0.31970^{\{4\}}$
		$\hat{\sigma}$	$0.22721^{\{4\}}$	$0.27580^{\{6\}}$	$0.17916^{\{1\}}$	$0.18473^{\{2\}}$	$0.28176^{\{7\}}$	$0.20255^{\{3\}}$	$0.24062^{\{5\}}$
		$\hat{\theta}$	$0.25761^{\{4\}}$	$0.30494^{\{6\}}$	$0.20352^{\{1\}}$	$0.21336^{\{2\}}$	$0.31146^{\{7\}}$	$0.23385^{\{3\}}$	$0.29776^{\{5\}}$
	MSE	$\hat{\beta}$	$0.21754^{\{5\}}$	$0.35924^{\{6\}}$	$0.12137^{\{2\}}$	$0.11984^{\{1\}}$	$0.38380^{\{7\}}$	$0.15754^{\{3\}}$	$0.19094^{\{4\}}$
		$\hat{\sigma}$	$0.09701^{\{4\}}$	$0.15514^{\{6\}}$	$0.05580^{\{1\}}$	$0.05827^{\{2\}}$	$0.16399^{\{7\}}$	$0.07254^{\{3\}}$	$0.10567^{\{5\}}$
		$\hat{\theta}$	$0.10631^{\{4\}}$	$0.14912^{\{6\}}$	$0.06563^{\{1\}}$	$0.07174^{\{2\}}$	$0.15442^{\{7\}}$	$0.08690^{\{3\}}$	$0.14301^{\{5\}}$
	MRE	$\hat{\beta}$	$0.28051^{\{5\}}$	$0.35468^{\{6\}}$	$0.21372^{\{1\}}$	$0.22028^{\{2\}}$	$0.36176^{\{7\}}$	$0.24560^{\{3\}}$	$0.26641^{\{4\}}$
		$\hat{\sigma}$	$0.28401^{\{4\}}$	$0.34476^{\{6\}}$	$0.22395^{\{1\}}$	$0.23092^{\{2\}}$	$0.35220^{\{7\}}$	$0.25319^{\{3\}}$	$0.30077^{\{5\}}$
		$\hat{\theta}$	$0.17174^{\{4\}}$	$0.20330^{\{6\}}$	$0.13568^{\{1\}}$	$0.14224^{\{2\}}$	$0.20764^{\{7\}}$	$0.15590^{\{3\}}$	$0.19851^{\{5\}}$
	$\sum Ranks$		$39^{\{4\}}$	$54^{\{6\}}$	$10^{\{1\}}$	$17^{\{2\}}$	$63^{\{7\}}$	$27^{\{3\}}$	$42^{\{5\}}$

Table 3. *Cont.*

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE
120	$\|BIAS\|$	$\hat{\beta}$	$0.25959^{\{5\}}$	$0.33466^{\{6\}}$	$0.20339^{\{1\}}$	$0.20805^{\{2\}}$	$0.33981^{\{7\}}$	$0.24027^{\{3\}}$	$0.25812^{\{4\}}$
		$\hat{\sigma}$	$0.17301^{\{4\}}$	$0.21765^{\{6\}}$	$0.14289^{\{1\}}$	$0.14375^{\{2\}}$	$0.22049^{\{7\}}$	$0.16324^{\{3\}}$	$0.19026^{\{5\}}$
		$\hat{\theta}$	$0.19755^{\{4\}}$	$0.24953^{\{6\}}$	$0.16393^{\{1\}}$	$0.16768^{\{2\}}$	$0.25378^{\{7\}}$	$0.18975^{\{3\}}$	$0.23965^{\{5\}}$
	MSE	$\hat{\beta}$	$0.11982^{\{5\}}$	$0.20054^{\{6\}}$	$0.07374^{\{2\}}$	$0.07102^{\{1\}}$	$0.21202^{\{7\}}$	$0.10020^{\{3\}}$	$0.11823^{\{4\}}$
		$\hat{\sigma}$	$0.05177^{\{4\}}$	$0.08547^{\{6\}}$	$0.03505^{\{2\}}$	$0.03455^{\{1\}}$	$0.09034^{\{7\}}$	$0.04512^{\{3\}}$	$0.06332^{\{5\}}$
		$\hat{\theta}$	$0.06128^{\{4\}}$	$0.09946^{\{6\}}$	$0.04328^{\{1\}}$	$0.04497^{\{2\}}$	$0.10299^{\{7\}}$	$0.05713^{\{3\}}$	$0.09200^{\{5\}}$
	MRE	$\hat{\beta}$	$0.21633^{\{5\}}$	$0.27888^{\{6\}}$	$0.16949^{\{1\}}$	$0.17337^{\{2\}}$	$0.28318^{\{7\}}$	$0.20023^{\{3\}}$	$0.21510^{\{4\}}$
		$\hat{\sigma}$	$0.21626^{\{4\}}$	$0.27206^{\{6\}}$	$0.17861^{\{1\}}$	$0.17969^{\{2\}}$	$0.27561^{\{7\}}$	$0.20405^{\{3\}}$	$0.23783^{\{5\}}$
		$\hat{\theta}$	$0.13170^{\{4\}}$	$0.16636^{\{6\}}$	$0.10929^{\{1\}}$	$0.11179^{\{2\}}$	$0.16918^{\{7\}}$	$0.12650^{\{3\}}$	$0.15976^{\{5\}}$
	$\sum Ranks$		$39^{\{4\}}$	$54^{\{6\}}$	$11^{\{1\}}$	$16^{\{2\}}$	$63^{\{7\}}$	$27^{\{3\}}$	$42^{\{5\}}$
200	$\|BIAS\|$	$\hat{\beta}$	$0.19199^{\{5\}}$	$0.25400^{\{6\}}$	$0.15630^{\{1\}}$	$0.16013^{\{2\}}$	$0.25480^{\{7\}}$	$0.18021^{\{3\}}$	$0.19174^{\{4\}}$
		$\hat{\sigma}$	$0.12957^{\{4\}}$	$0.16411^{\{6\}}$	$0.11104^{\{1\}}$	$0.11183^{\{2\}}$	$0.16526^{\{7\}}$	$0.12218^{\{3\}}$	$0.14346^{\{5\}}$
		$\hat{\theta}$	$0.15188^{\{4\}}$	$0.19055^{\{6\}}$	$0.12897^{\{1\}}$	$0.13181^{\{2\}}$	$0.19104^{\{7\}}$	$0.14468^{\{3\}}$	$0.18334^{\{5\}}$
	MSE	$\hat{\beta}$	$0.06215^{\{4\}}$	$0.11099^{\{6\}}$	$0.04102^{\{2\}}$	$0.04053^{\{1\}}$	$0.11239^{\{7\}}$	$0.05300^{\{3\}}$	$0.06255^{\{5\}}$
		$\hat{\sigma}$	$0.02790^{\{4\}}$	$0.04625^{\{6\}}$	$0.02015^{\{1\}}$	$0.02024^{\{2\}}$	$0.04731^{\{7\}}$	$0.02428^{\{3\}}$	$0.03457^{\{5\}}$
		$\hat{\theta}$	$0.03626^{\{4\}}$	$0.05731^{\{6\}}$	$0.02639^{\{1\}}$	$0.02710^{\{2\}}$	$0.05774^{\{7\}}$	$0.03301^{\{3\}}$	$0.05386^{\{5\}}$
	MRE	$\hat{\beta}$	$0.15999^{\{5\}}$	$0.21166^{\{6\}}$	$0.13025^{\{1\}}$	$0.13345^{\{2\}}$	$0.21233^{\{7\}}$	$0.15017^{\{3\}}$	$0.15978^{\{4\}}$
		$\hat{\sigma}$	$0.16196^{\{4\}}$	$0.20514^{\{6\}}$	$0.13880^{\{1\}}$	$0.13979^{\{2\}}$	$0.20657^{\{7\}}$	$0.15272^{\{3\}}$	$0.17933^{\{5\}}$
		$\hat{\theta}$	$0.10126^{\{4\}}$	$0.12703^{\{6\}}$	$0.08598^{\{1\}}$	$0.08787^{\{2\}}$	$0.12736^{\{7\}}$	$0.09645^{\{3\}}$	$0.12223^{\{5\}}$
	$\sum Ranks$		$38^{\{4\}}$	$54^{\{6\}}$	$10^{\{1\}}$	$17^{\{2\}}$	$63^{\{7\}}$	$27^{\{3\}}$	$43^{\{5\}}$
350	$\|BIAS\|$	$\hat{\beta}$	$0.14174^{\{4\}}$	$0.18793^{\{6\}}$	$0.11379^{\{1\}}$	$0.11672^{\{2\}}$	$0.18795^{\{7\}}$	$0.13730^{\{3\}}$	$0.14399^{\{5\}}$
		$\hat{\sigma}$	$0.09575^{\{4\}}$	$0.12052^{\{6\}}$	$0.08087^{\{1\}}$	$0.08091^{\{2\}}$	$0.12126^{\{7\}}$	$0.09388^{\{3\}}$	$0.10728^{\{5\}}$
		$\hat{\theta}$	$0.11198^{\{4\}}$	$0.14185^{\{6\}}$	$0.09574^{\{1\}}$	$0.09639^{\{2\}}$	$0.14362^{\{7\}}$	$0.11123^{\{3\}}$	$0.13661^{\{5\}}$
	MSE	$\hat{\beta}$	$0.03302^{\{4\}}$	$0.05861^{\{7\}}$	$0.02106^{\{1\}}$	$0.02153^{\{2\}}$	$0.05815^{\{6\}}$	$0.03039^{\{3\}}$	$0.03358^{\{5\}}$
		$\hat{\sigma}$	$0.01489^{\{4\}}$	$0.02430^{\{7\}}$	$0.01039^{\{1\}}$	$0.01054^{\{2\}}$	$0.02403^{\{6\}}$	$0.01425^{\{3\}}$	$0.01868^{\{5\}}$
		$\hat{\theta}$	$0.02001^{\{4\}}$	$0.03185^{\{6\}}$	$0.01439^{\{1\}}$	$0.01465^{\{2\}}$	$0.03242^{\{7\}}$	$0.01949^{\{3\}}$	$0.02944^{\{5\}}$
	MRE	$\hat{\beta}$	$0.11812^{\{4\}}$	$0.15661^{\{6\}}$	$0.09483^{\{1\}}$	$0.09727^{\{2\}}$	$0.15662^{\{7\}}$	$0.11442^{\{3\}}$	$0.11999^{\{5\}}$
		$\hat{\sigma}$	$0.11969^{\{4\}}$	$0.15066^{\{6\}}$	$0.10108^{\{1\}}$	$0.10114^{\{2\}}$	$0.15158^{\{7\}}$	$0.11735^{\{3\}}$	$0.13410^{\{5\}}$
		$\hat{\theta}$	$0.07466^{\{4\}}$	$0.09456^{\{6\}}$	$0.06383^{\{1\}}$	$0.06426^{\{2\}}$	$0.09574^{\{7\}}$	$0.07415^{\{3\}}$	$0.09108^{\{5\}}$
	$\sum Ranks$		$36^{\{4\}}$	$56^{\{6\}}$	$9^{\{1\}}$	$18^{\{2\}}$	$61^{\{7\}}$	$27^{\{3\}}$	$45^{\{5\}}$

Table 4. Simulation results for $\boldsymbol{\tau} = (\beta = 1.2, \sigma = 2.5, \theta = 5.0)^{\intercal}$.

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE
10	$\|BIAS\|$	$\hat{\beta}$	$3.50751^{\{4\}}$	$3.99253^{\{5\}}$	$3.12284^{\{2\}}$	$3.01694^{\{1\}}$	$4.63888^{\{7\}}$	$3.28017^{\{3\}}$	$4.02401^{\{6\}}$
		$\hat{\sigma}$	$3.62898^{\{4\}}$	$4.09498^{\{5\}}$	$3.28330^{\{2\}}$	$3.13956^{\{1\}}$	$4.79897^{\{7\}}$	$3.44073^{\{3\}}$	$4.21679^{\{6\}}$
		$\hat{\theta}$	$3.68240^{\{4\}}$	$3.87953^{\{5\}}$	$2.80447^{\{1\}}$	$3.13008^{\{2\}}$	$4.29365^{\{6\}}$	$3.43073^{\{3\}}$	$4.37125^{\{7\}}$
	MSE	$\hat{\beta}$	$45.51368^{\{4\}}$	$54.48658^{\{5\}}$	$44.48986^{\{3\}}$	$33.29330^{\{1\}}$	$78.14396^{\{7\}}$	$41.74686^{\{2\}}$	$59.06149^{\{6\}}$
		$\hat{\sigma}$	$46.37063^{\{4\}}$	$55.13927^{\{5\}}$	$45.49430^{\{3\}}$	$34.26872^{\{1\}}$	$79.73699^{\{7\}}$	$43.01654^{\{2\}}$	$62.04876^{\{6\}}$
		$\hat{\theta}$	$24.06897^{\{4\}}$	$26.26159^{\{5\}}$	$13.46685^{\{1\}}$	$16.25558^{\{2\}}$	$34.65714^{\{6\}}$	$21.35817^{\{3\}}$	$38.77643^{\{7\}}$
	MRE	$\hat{\beta}$	$2.92293^{\{4\}}$	$3.32710^{\{5\}}$	$2.60236^{\{2\}}$	$2.51412^{\{1\}}$	$3.86574^{\{7\}}$	$2.73347^{\{3\}}$	$3.35334^{\{6\}}$
		$\hat{\sigma}$	$1.45159^{\{4\}}$	$1.63799^{\{5\}}$	$1.31332^{\{2\}}$	$1.25582^{\{1\}}$	$1.91959^{\{7\}}$	$1.37629^{\{3\}}$	$1.68672^{\{6\}}$
		$\hat{\theta}$	$0.73648^{\{4\}}$	$0.77591^{\{5\}}$	$0.56089^{\{1\}}$	$0.62602^{\{2\}}$	$0.85873^{\{6\}}$	$0.68615^{\{3\}}$	$0.87425^{\{7\}}$
	$\sum Ranks$		$36^{\{4\}}$	$45^{\{5\}}$	$17^{\{2\}}$	$12^{\{1\}}$	$60^{\{7\}}$	$25^{\{3\}}$	$57^{\{6\}}$

Table 4. *Cont.*

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE
20	$\lvert BIAS \rvert$	$\hat{\beta}$	$2.17785^{\{5\}}$	$2.66017^{\{6\}}$	$1.32239^{\{1\}}$	$1.40936^{\{2\}}$	$2.69450^{\{7\}}$	$1.49833^{\{3\}}$	$1.88819^{\{4\}}$
		$\hat{\sigma}$	$2.27496^{\{5\}}$	$2.75612^{\{6\}}$	$1.43459^{\{1\}}$	$1.50799^{\{2\}}$	$2.81820^{\{7\}}$	$1.61584^{\{3\}}$	$2.00427^{\{4\}}$
		$\hat{\theta}$	$2.48506^{\{4\}}$	$2.88259^{\{6\}}$	$1.83594^{\{1\}}$	$1.99198^{\{2\}}$	$3.01822^{\{7\}}$	$2.13959^{\{3\}}$	$2.71653^{\{5\}}$
	MSE	$\hat{\beta}$	$21.39154^{\{5\}}$	$28.54696^{\{6\}}$	$8.95163^{\{1\}}$	$9.56936^{\{2\}}$	$30.38177^{\{7\}}$	$11.12534^{\{3\}}$	$16.36185^{\{4\}}$
		$\hat{\sigma}$	$21.99405^{\{5\}}$	$29.18711^{\{6\}}$	$9.43874^{\{1\}}$	$10.03672^{\{2\}}$	$31.34826^{\{7\}}$	$11.66575^{\{3\}}$	$17.11858^{\{4\}}$
		$\hat{\theta}$	$10.17036^{\{4\}}$	$13.40818^{\{6\}}$	$5.47715^{\{1\}}$	$6.33525^{\{2\}}$	$15.25726^{\{7\}}$	$7.60348^{\{3\}}$	$12.64608^{\{5\}}$
	MRE	$\hat{\beta}$	$1.81488^{\{5\}}$	$2.21680^{\{6\}}$	$1.10199^{\{1\}}$	$1.17447^{\{2\}}$	$2.24541^{\{7\}}$	$1.24861^{\{3\}}$	$1.57349^{\{4\}}$
		$\hat{\sigma}$	$0.90999^{\{5\}}$	$1.10245^{\{6\}}$	$0.57384^{\{1\}}$	$0.60320^{\{2\}}$	$1.12728^{\{7\}}$	$0.64634^{\{3\}}$	$0.80171^{\{4\}}$
		$\hat{\theta}$	$0.49701^{\{4\}}$	$0.57652^{\{6\}}$	$0.36719^{\{1\}}$	$0.39840^{\{2\}}$	$0.60364^{\{7\}}$	$0.42792^{\{3\}}$	$0.54331^{\{5\}}$
	$\sum Ranks$		$42^{\{5\}}$	$54^{\{6\}}$	$9^{\{1\}}$	$18^{\{2\}}$	$63^{\{7\}}$	$27^{\{3\}}$	$39^{\{4\}}$
50	$\lvert BIAS \rvert$	$\hat{\beta}$	$0.73410^{\{5\}}$	$1.09430^{\{7\}}$	$0.50741^{\{2\}}$	$0.50636^{\{1\}}$	$1.08212^{\{6\}}$	$0.58144^{\{3\}}$	$0.68869^{\{4\}}$
		$\hat{\sigma}$	$0.79831^{\{5\}}$	$1.16655^{\{7\}}$	$0.58062^{\{2\}}$	$0.57060^{\{1\}}$	$1.15555^{\{6\}}$	$0.65111^{\{3\}}$	$0.76066^{\{4\}}$
		$\hat{\theta}$	$1.40746^{\{4\}}$	$1.74112^{\{7\}}$	$1.05840^{\{1\}}$	$1.12932^{\{2\}}$	$1.73203^{\{6\}}$	$1.25684^{\{3\}}$	$1.54111^{\{5\}}$
	MSE	$\hat{\beta}$	$2.21893^{\{5\}}$	$5.58442^{\{7\}}$	$0.78408^{\{1\}}$	$0.79888^{\{2\}}$	$5.48416^{\{6\}}$	$1.06751^{\{3\}}$	$1.87645^{\{4\}}$
		$\hat{\sigma}$	$2.39674^{\{5\}}$	$5.85385^{\{7\}}$	$0.91319^{\{2\}}$	$0.90965^{\{1\}}$	$5.76114^{\{6\}}$	$1.20283^{\{3\}}$	$2.05866^{\{4\}}$
		$\hat{\theta}$	$3.20665^{\{4\}}$	$4.83747^{\{7\}}$	$1.80768^{\{1\}}$	$2.04796^{\{2\}}$	$4.80309^{\{6\}}$	$2.53262^{\{3\}}$	$3.85672^{\{5\}}$
	MRE	$\hat{\beta}$	$0.61175^{\{5\}}$	$0.91192^{\{7\}}$	$0.42285^{\{2\}}$	$0.42196^{\{1\}}$	$0.90176^{\{6\}}$	$0.48453^{\{3\}}$	$0.57391^{\{4\}}$
		$\hat{\sigma}$	$0.31932^{\{5\}}$	$0.46662^{\{7\}}$	$0.23225^{\{2\}}$	$0.22824^{\{1\}}$	$0.46222^{\{6\}}$	$0.26045^{\{3\}}$	$0.30427^{\{4\}}$
		$\hat{\theta}$	$0.28149^{\{4\}}$	$0.34822^{\{7\}}$	$0.21168^{\{1\}}$	$0.22586^{\{2\}}$	$0.34641^{\{6\}}$	$0.25137^{\{3\}}$	$0.30822^{\{5\}}$
	$\sum Ranks$		$42^{\{5\}}$	$63^{\{7\}}$	$14^{\{2\}}$	$13^{\{1\}}$	$54^{\{6\}}$	$27^{\{3\}}$	$39^{\{4\}}$
80	$\lvert BIAS \rvert$	$\hat{\beta}$	$0.47492^{\{5\}}$	$0.69765^{\{7\}}$	$0.34830^{\{2\}}$	$0.33970^{\{1\}}$	$0.68324^{\{6\}}$	$0.39955^{\{3\}}$	$0.44377^{\{4\}}$
		$\hat{\sigma}$	$0.52678^{\{5\}}$	$0.75261^{\{7\}}$	$0.40071^{\{2\}}$	$0.38586^{\{1\}}$	$0.74167^{\{6\}}$	$0.45346^{\{3\}}$	$0.49961^{\{4\}}$
		$\hat{\theta}$	$1.05848^{\{4\}}$	$1.34044^{\{6\}}$	$0.82536^{\{1\}}$	$0.82746^{\{2\}}$	$1.34523^{\{7\}}$	$0.96776^{\{3\}}$	$1.15003^{\{5\}}$
	MSE	$\hat{\beta}$	$0.56717^{\{4\}}$	$1.83577^{\{6\}}$	$0.25717^{\{2\}}$	$0.22868^{\{1\}}$	$1.85520^{\{7\}}$	$0.35684^{\{3\}}$	$0.59312^{\{5\}}$
		$\hat{\sigma}$	$0.64953^{\{4\}}$	$1.97624^{\{6\}}$	$0.32144^{\{2\}}$	$0.28523^{\{1\}}$	$1.99629^{\{7\}}$	$0.43169^{\{3\}}$	$0.68577^{\{5\}}$
		$\hat{\theta}$	$1.76694^{\{4\}}$	$2.89548^{\{6\}}$	$1.06751^{\{1\}}$	$1.11300^{\{2\}}$	$2.89687^{\{7\}}$	$1.50023^{\{3\}}$	$2.11458^{\{5\}}$
	MRE	$\hat{\beta}$	$0.39577^{\{5\}}$	$0.58137^{\{7\}}$	$0.29025^{\{2\}}$	$0.28308^{\{1\}}$	$0.56937^{\{6\}}$	$0.33295^{\{3\}}$	$0.36981^{\{4\}}$
		$\hat{\sigma}$	$0.21071^{\{5\}}$	$0.30104^{\{7\}}$	$0.16028^{\{2\}}$	$0.15435^{\{1\}}$	$0.29667^{\{6\}}$	$0.18138^{\{3\}}$	$0.19984^{\{4\}}$
		$\hat{\theta}$	$0.21170^{\{4\}}$	$0.26809^{\{6\}}$	$0.16507^{\{1\}}$	$0.16549^{\{2\}}$	$0.26905^{\{7\}}$	$0.19355^{\{3\}}$	$0.23001^{\{5\}}$
	$\sum Ranks$		$40^{\{4\}}$	$58^{\{6\}}$	$15^{\{2\}}$	$12^{\{1\}}$	$59^{\{7\}}$	$27^{\{3\}}$	$41^{\{5\}}$
120	$\lvert BIAS \rvert$	$\hat{\beta}$	$0.35372^{\{5\}}$	$0.48890^{\{6\}}$	$0.26330^{\{1\}}$	$0.26424^{\{2\}}$	$0.50546^{\{7\}}$	$0.31334^{\{3\}}$	$0.35335^{\{4\}}$
		$\hat{\sigma}$	$0.39911^{\{4\}}$	$0.53564^{\{6\}}$	$0.30424^{\{1\}}$	$0.30735^{\{2\}}$	$0.55207^{\{7\}}$	$0.35673^{\{3\}}$	$0.40003^{\{5\}}$
		$\hat{\theta}$	$0.83974^{\{4\}}$	$1.05534^{\{6\}}$	$0.64770^{\{1\}}$	$0.66126^{\{2\}}$	$1.07889^{\{7\}}$	$0.78561^{\{3\}}$	$0.94446^{\{5\}}$
	MSE	$\hat{\beta}$	$0.25984^{\{4\}}$	$0.65605^{\{6\}}$	$0.13687^{\{2\}}$	$0.12532^{\{1\}}$	$0.67761^{\{7\}}$	$0.18779^{\{3\}}$	$0.32276^{\{5\}}$
		$\hat{\sigma}$	$0.31303^{\{4\}}$	$0.73514^{\{6\}}$	$0.17489^{\{2\}}$	$0.16464^{\{1\}}$	$0.75981^{\{7\}}$	$0.23494^{\{3\}}$	$0.38221^{\{5\}}$
		$\hat{\theta}$	$1.12103^{\{4\}}$	$1.79847^{\{6\}}$	$0.66117^{\{1\}}$	$0.72344^{\{2\}}$	$1.85886^{\{7\}}$	$0.97410^{\{3\}}$	$1.42223^{\{5\}}$
	MRE	$\hat{\beta}$	$0.29476^{\{5\}}$	$0.40741^{\{6\}}$	$0.21942^{\{1\}}$	$0.22020^{\{2\}}$	$0.42121^{\{7\}}$	$0.26111^{\{3\}}$	$0.29446^{\{4\}}$
		$\hat{\sigma}$	$0.15964^{\{4\}}$	$0.21425^{\{6\}}$	$0.12169^{\{1\}}$	$0.12294^{\{2\}}$	$0.22083^{\{7\}}$	$0.14269^{\{3\}}$	$0.16001^{\{5\}}$
		$\hat{\theta}$	$0.16795^{\{4\}}$	$0.21107^{\{6\}}$	$0.12954^{\{1\}}$	$0.13225^{\{2\}}$	$0.21578^{\{7\}}$	$0.15712^{\{3\}}$	$0.18889^{\{5\}}$
	$\sum Ranks$		$38^{\{4\}}$	$54^{\{6\}}$	$11^{\{1\}}$	$16^{\{2\}}$	$63^{\{7\}}$	$27^{\{3\}}$	$43^{\{5\}}$
200	$\lvert BIAS \rvert$	$\hat{\beta}$	$0.25351^{\{4\}}$	$0.35854^{\{7\}}$	$0.19644^{\{2\}}$	$0.19211^{\{1\}}$	$0.35646^{\{6\}}$	$0.23886^{\{3\}}$	$0.25630^{\{5\}}$
		$\hat{\sigma}$	$0.29012^{\{4\}}$	$0.39653^{\{7\}}$	$0.22833^{\{2\}}$	$0.22525^{\{1\}}$	$0.39254^{\{6\}}$	$0.27257^{\{3\}}$	$0.29145^{\{5\}}$
		$\hat{\theta}$	$0.63371^{\{4\}}$	$0.83782^{\{7\}}$	$0.49993^{\{2\}}$	$0.48641^{\{1\}}$	$0.83212^{\{6\}}$	$0.61183^{\{3\}}$	$0.71477^{\{5\}}$
	MSE	$\hat{\beta}$	$0.11661^{\{4\}}$	$0.25278^{\{7\}}$	$0.06900^{\{2\}}$	$0.06405^{\{1\}}$	$0.24926^{\{6\}}$	$0.10051^{\{3\}}$	$0.12544^{\{5\}}$
		$\hat{\sigma}$	$0.14784^{\{4\}}$	$0.29755^{\{7\}}$	$0.09003^{\{2\}}$	$0.08566^{\{1\}}$	$0.29332^{\{6\}}$	$0.12828^{\{3\}}$	$0.15678^{\{5\}}$
		$\hat{\theta}$	$0.63520^{\{4\}}$	$1.09743^{\{7\}}$	$0.39475^{\{1\}}$	$0.40549^{\{2\}}$	$1.09576^{\{6\}}$	$0.58497^{\{3\}}$	$0.80534^{\{5\}}$

Table 4. *Cont.*

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE
	MRE	$\hat{\beta}$	$0.21126^{\{4\}}$	$0.29878^{\{7\}}$	$0.16370^{\{2\}}$	$0.16010^{\{1\}}$	$0.29705^{\{6\}}$	$0.19905^{\{3\}}$	$0.21358^{\{5\}}$
		$\hat{\sigma}$	$0.11605^{\{4\}}$	$0.15861^{\{7\}}$	$0.09133^{\{2\}}$	$0.09010^{\{1\}}$	$0.15702^{\{6\}}$	$0.10903^{\{3\}}$	$0.11658^{\{5\}}$
		$\hat{\theta}$	$0.12674^{\{4\}}$	$0.16756^{\{7\}}$	$0.09999^{\{2\}}$	$0.09728^{\{1\}}$	$0.16642^{\{6\}}$	$0.12237^{\{3\}}$	$0.14295^{\{5\}}$
	$\sum Ranks$		$36^{\{4\}}$	$63^{\{7\}}$	$17^{\{2\}}$	$10^{\{1\}}$	$54^{\{6\}}$	$27^{\{3\}}$	$45^{\{5\}}$
350	$\|BIAS\|$	$\hat{\beta}$	$0.18224^{\{4\}}$	$0.25505^{\{7\}}$	$0.14489^{\{2\}}$	$0.13861^{\{1\}}$	$0.24962^{\{6\}}$	$0.17629^{\{3\}}$	$0.18852^{\{5\}}$
		$\hat{\sigma}$	$0.20725^{\{4\}}$	$0.28091^{\{7\}}$	$0.17018^{\{2\}}$	$0.16220^{\{1\}}$	$0.27691^{\{6\}}$	$0.20320^{\{3\}}$	$0.21407^{\{5\}}$
		$\hat{\theta}$	$0.46624^{\{4\}}$	$0.61826^{\{7\}}$	$0.37805^{\{2\}}$	$0.34783^{\{1\}}$	$0.61273^{\{6\}}$	$0.45505^{\{3\}}$	$0.54059^{\{5\}}$
	MSE	$\hat{\beta}$	$0.05658^{\{4\}}$	$0.11685^{\{7\}}$	$0.03706^{\{2\}}$	$0.03296^{\{1\}}$	$0.10875^{\{6\}}$	$0.05123^{\{3\}}$	$0.06176^{\{5\}}$
		$\hat{\sigma}$	$0.07240^{\{4\}}$	$0.13801^{\{7\}}$	$0.04977^{\{2\}}$	$0.04476^{\{1\}}$	$0.13173^{\{6\}}$	$0.06713^{\{3\}}$	$0.07774^{\{5\}}$
		$\hat{\theta}$	$0.34228^{\{4\}}$	$0.60986^{\{7\}}$	$0.22954^{\{2\}}$	$0.22412^{\{1\}}$	$0.58870^{\{6\}}$	$0.32805^{\{3\}}$	$0.46326^{\{5\}}$
	MRE	$\hat{\beta}$	$0.15186^{\{4\}}$	$0.21254^{\{7\}}$	$0.12074^{\{2\}}$	$0.11551^{\{1\}}$	$0.20801^{\{6\}}$	$0.14691^{\{3\}}$	$0.15710^{\{5\}}$
		$\hat{\sigma}$	$0.08290^{\{4\}}$	$0.11237^{\{7\}}$	$0.06807^{\{2\}}$	$0.06488^{\{1\}}$	$0.11076^{\{6\}}$	$0.08128^{\{3\}}$	$0.08563^{\{5\}}$
		$\hat{\theta}$	$0.09325^{\{4\}}$	$0.12365^{\{7\}}$	$0.07561^{\{2\}}$	$0.06957^{\{1\}}$	$0.12255^{\{6\}}$	$0.09101^{\{3\}}$	$0.10812^{\{5\}}$
	$\sum Ranks$		$36^{\{4\}}$	$63^{\{7\}}$	$18^{\{2\}}$	$9^{\{1\}}$	$54^{\{6\}}$	$27^{\{3\}}$	$45^{\{5\}}$

Table 5. Simulation results for $\boldsymbol{\tau} = (\beta = 0.31390, \sigma = 28.83840, \theta = 2.29650)^{\intercal}$.

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE
10	$\|BIAS\|$	$\hat{\beta}$	$0.16571^{\{1\}}$	$0.17572^{\{3\}}$	$0.20433^{\{6\}}$	$0.16820^{\{2\}}$	$0.21024^{\{7\}}$	$0.17771^{\{4\}}$	$0.19267^{\{5\}}$
		$\hat{\sigma}$	$2.68115^{\{5\}}$	$2.48849^{\{3\}}$	$5.19391^{\{7\}}$	$2.15650^{\{2\}}$	$2.71885^{\{6\}}$	$2.57780^{\{4\}}$	$2.05666^{\{1\}}$
		$\hat{\theta}$	$0.21904^{\{1\}}$	$0.24825^{\{2\}}$	$0.31150^{\{7\}}$	$0.27100^{\{6\}}$	$0.25246^{\{3\}}$	$0.25917^{\{5\}}$	$0.25796^{\{4\}}$
	MSE	$\hat{\beta}$	$0.05658^{\{2\}}$	$0.05687^{\{3\}}$	$0.10054^{\{7\}}$	$0.05523^{\{1\}}$	$0.09424^{\{6\}}$	$0.06962^{\{4\}}$	$0.08575^{\{5\}}$
		$\hat{\sigma}$	$15.14475^{\{6\}}$	$13.01903^{\{2\}}$	$56.64649^{\{7\}}$	$13.69833^{\{3\}}$	$15.07967^{\{5\}}$	$14.41006^{\{4\}}$	$7.97328^{\{1\}}$
		$\hat{\theta}$	$0.08314^{\{1\}}$	$0.10079^{\{2\}}$	$0.17183^{\{7\}}$	$0.12111^{\{6\}}$	$0.11061^{\{5\}}$	$0.11004^{\{4\}}$	$0.10778^{\{3\}}$
	MRE	$\hat{\beta}$	$0.52791^{\{1\}}$	$0.55979^{\{3\}}$	$0.65093^{\{6\}}$	$0.53585^{\{2\}}$	$0.66977^{\{7\}}$	$0.56613^{\{4\}}$	$0.61380^{\{5\}}$
		$\hat{\sigma}$	$0.09297^{\{5\}}$	$0.08629^{\{3\}}$	$0.18010^{\{7\}}$	$0.07478^{\{2\}}$	$0.09428^{\{6\}}$	$0.08939^{\{4\}}$	$0.07132^{\{1\}}$
		$\hat{\theta}$	$0.09538^{\{1\}}$	$0.10810^{\{2\}}$	$0.13564^{\{7\}}$	$0.11801^{\{6\}}$	$0.10993^{\{3\}}$	$0.11286^{\{5\}}$	$0.11233^{\{4\}}$
	$\sum Ranks$		$23^{\{1.5\}}$	$23^{\{1.5\}}$	$61^{\{7\}}$	$30^{\{4\}}$	$48^{\{6\}}$	$38^{\{5\}}$	$29^{\{3\}}$
20	$\|BIAS\|$	$\hat{\beta}$	$0.11143^{\{3\}}$	$0.12516^{\{6\}}$	$0.11945^{\{5\}}$	$0.10973^{\{2\}}$	$0.13811^{\{7\}}$	$0.10920^{\{1\}}$	$0.11253^{\{4\}}$
		$\hat{\sigma}$	$2.41605^{\{5\}}$	$2.10634^{\{3\}}$	$4.24938^{\{7\}}$	$1.94027^{\{1\}}$	$2.27587^{\{4\}}$	$2.43885^{\{6\}}$	$1.99898^{\{2\}}$
		$\hat{\theta}$	$0.15628^{\{1\}}$	$0.18049^{\{3\}}$	$0.23428^{\{7\}}$	$0.19247^{\{6\}}$	$0.18046^{\{2\}}$	$0.18148^{\{4\}}$	$0.18228^{\{5\}}$
	MSE	$\hat{\beta}$	$0.02338^{\{3\}}$	$0.02792^{\{5\}}$	$0.03048^{\{6\}}$	$0.01975^{\{1\}}$	$0.03897^{\{7\}}$	$0.02174^{\{2\}}$	$0.02607^{\{4\}}$
		$\hat{\sigma}$	$11.66546^{\{4\}}$	$8.61562^{\{2\}}$	$39.20109^{\{7\}}$	$11.75748^{\{5\}}$	$10.34974^{\{3\}}$	$12.61878^{\{6\}}$	$7.34804^{\{1\}}$
		$\hat{\theta}$	$0.04311^{\{1\}}$	$0.05583^{\{4\}}$	$0.10275^{\{7\}}$	$0.06444^{\{6\}}$	$0.06029^{\{5\}}$	$0.05482^{\{3\}}$	$0.05226^{\{2\}}$
	MRE	$\hat{\beta}$	$0.35498^{\{3\}}$	$0.39873^{\{6\}}$	$0.38052^{\{5\}}$	$0.34957^{\{2\}}$	$0.43999^{\{7\}}$	$0.34789^{\{1\}}$	$0.35848^{\{4\}}$
		$\hat{\sigma}$	$0.08378^{\{5\}}$	$0.07304^{\{3\}}$	$0.14735^{\{7\}}$	$0.06728^{\{1\}}$	$0.07892^{\{4\}}$	$0.08457^{\{6\}}$	$0.06932^{\{2\}}$
		$\hat{\theta}$	$0.06805^{\{1\}}$	$0.07859^{\{3\}}$	$0.10201^{\{7\}}$	$0.08381^{\{6\}}$	$0.07858^{\{2\}}$	$0.07902^{\{4\}}$	$0.07938^{\{5\}}$
	$\sum Ranks$		$26^{\{1\}}$	$35^{\{5\}}$	$58^{\{7\}}$	$30^{\{3\}}$	$41^{\{6\}}$	$33^{\{4\}}$	$29^{\{2\}}$
50	$\|BIAS\|$	$\hat{\beta}$	$0.06451^{\{1\}}$	$0.07832^{\{6\}}$	$0.06899^{\{5\}}$	$0.06709^{\{4\}}$	$0.07984^{\{7\}}$	$0.06576^{\{3\}}$	$0.06534^{\{2\}}$
		$\hat{\sigma}$	$1.98238^{\{6\}}$	$1.69960^{\{3\}}$	$3.23910^{\{7\}}$	$1.51781^{\{1\}}$	$1.69502^{\{2\}}$	$1.97686^{\{5\}}$	$1.79749^{\{4\}}$
		$\hat{\theta}$	$0.09433^{\{1\}}$	$0.11326^{\{3\}}$	$0.16874^{\{7\}}$	$0.12981^{\{6\}}$	$0.11039^{\{2\}}$	$0.12099^{\{4\}}$	$0.12174^{\{5\}}$
	MSE	$\hat{\beta}$	$0.00684^{\{1\}}$	$0.01045^{\{6\}}$	$0.00871^{\{5\}}$	$0.00698^{\{2\}}$	$0.01149^{\{7\}}$	$0.00700^{\{3\}}$	$0.00713^{\{4\}}$
		$\hat{\sigma}$	$7.47900^{\{5\}}$	$5.33016^{\{1\}}$	$25.94710^{\{7\}}$	$7.23531^{\{4\}}$	$5.43109^{\{2\}}$	$7.93882^{\{6\}}$	$5.67923^{\{3\}}$
		$\hat{\theta}$	$0.01583^{\{1\}}$	$0.02295^{\{3\}}$	$0.05457^{\{7\}}$	$0.03018^{\{6\}}$	$0.02289^{\{2\}}$	$0.02494^{\{5\}}$	$0.02409^{\{4\}}$

Table 5. *Cont.*

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE
	MRE	$\hat{\beta}$	$0.20551^{\{1\}}$	$0.24950^{\{6\}}$	$0.21979^{\{5\}}$	$0.21372^{\{4\}}$	$0.25436^{\{7\}}$	$0.20949^{\{3\}}$	$0.20817^{\{2\}}$
		$\hat{\sigma}$	$0.06874^{\{6\}}$	$0.05894^{\{3\}}$	$0.11232^{\{7\}}$	$0.05263^{\{1\}}$	$0.05878^{\{2\}}$	$0.06855^{\{5\}}$	$0.06233^{\{4\}}$
		$\hat{\theta}$	$0.04108^{\{1\}}$	$0.04932^{\{3\}}$	$0.07348^{\{7\}}$	$0.05652^{\{6\}}$	$0.04807^{\{2\}}$	$0.05268^{\{4\}}$	$0.05301^{\{5\}}$
	$\sum Ranks$		$23^{\{1\}}$	$34^{\{4.5\}}$	$57^{\{7\}}$	$34^{\{4.5\}}$	$33^{\{2.5\}}$	$38^{\{6\}}$	$33^{\{2.5\}}$
80	$\lvert BIAS \rvert$	$\hat{\beta}$	$0.05051^{\{1\}}$	$0.06078^{\{7\}}$	$0.05608^{\{5\}}$	$0.05295^{\{3\}}$	$0.06036^{\{6\}}$	$0.05330^{\{4\}}$	$0.05123^{\{2\}}$
		$\hat{\sigma}$	$1.77552^{\{5\}}$	$1.45832^{\{3\}}$	$2.76014^{\{7\}}$	$1.35821^{\{1\}}$	$1.43350^{\{2\}}$	$1.79124^{\{6\}}$	$1.64098^{\{4\}}$
		$\hat{\theta}$	$0.07935^{\{1\}}$	$0.08676^{\{3\}}$	$0.14378^{\{7\}}$	$0.10607^{\{6\}}$	$0.08128^{\{2\}}$	$0.10348^{\{5\}}$	$0.10049^{\{4\}}$
	MSE	$\hat{\beta}$	$0.00412^{\{1\}}$	$0.00605^{\{6\}}$	$0.00551^{\{5\}}$	$0.00423^{\{3\}}$	$0.00632^{\{7\}}$	$0.00454^{\{4\}}$	$0.00422^{\{2\}}$
		$\hat{\sigma}$	$5.89034^{\{4\}}$	$3.84061^{\{2\}}$	$19.00348^{\{7\}}$	$6.11006^{\{5\}}$	$3.60788^{\{1\}}$	$6.66675^{\{6\}}$	$4.69830^{\{3\}}$
		$\hat{\theta}$	$0.01170^{\{1\}}$	$0.01385^{\{3\}}$	$0.04060^{\{7\}}$	$0.02094^{\{6\}}$	$0.01241^{\{2\}}$	$0.01856^{\{5\}}$	$0.01665^{\{4\}}$
	MRE	$\hat{\beta}$	$0.16090^{\{1\}}$	$0.19362^{\{7\}}$	$0.17866^{\{5\}}$	$0.16869^{\{3\}}$	$0.19230^{\{6\}}$	$0.16981^{\{4\}}$	$0.16321^{\{2\}}$
		$\hat{\sigma}$	$0.06157^{\{5\}}$	$0.05057^{\{3\}}$	$0.09571^{\{7\}}$	$0.04710^{\{1\}}$	$0.04971^{\{2\}}$	$0.06211^{\{6\}}$	$0.05690^{\{4\}}$
		$\hat{\theta}$	$0.03455^{\{1\}}$	$0.03778^{\{3\}}$	$0.06261^{\{7\}}$	$0.04619^{\{6\}}$	$0.03540^{\{2\}}$	$0.04506^{\{5\}}$	$0.04376^{\{4\}}$
	$\sum Ranks$		$20^{\{1\}}$	$37^{\{5\}}$	$57^{\{7\}}$	$34^{\{4\}}$	$30^{\{3\}}$	$45^{\{6\}}$	$29^{\{2\}}$
120	$\lvert BIAS \rvert$	$\hat{\beta}$	$0.03958^{\{1\}}$	$0.04869^{\{6\}}$	$0.04269^{\{5\}}$	$0.04216^{\{3\}}$	$0.04885^{\{7\}}$	$0.04250^{\{4\}}$	$0.04135^{\{2\}}$
		$\hat{\sigma}$	$1.52231^{\{5\}}$	$1.22959^{\{2\}}$	$2.40680^{\{7\}}$	$1.11146^{\{1\}}$	$1.25655^{\{3\}}$	$1.67741^{\{6\}}$	$1.49157^{\{4\}}$
		$\hat{\theta}$	$0.06394^{\{1\}}$	$0.06720^{\{3\}}$	$0.11862^{\{7\}}$	$0.08364^{\{4\}}$	$0.06611^{\{2\}}$	$0.08630^{\{6\}}$	$0.08402^{\{5\}}$
	MSE	$\hat{\beta}$	$0.00252^{\{1\}}$	$0.00384^{\{6\}}$	$0.00304^{\{5\}}$	$0.00264^{\{2\}}$	$0.00395^{\{7\}}$	$0.00284^{\{4\}}$	$0.00268^{\{3\}}$
		$\hat{\sigma}$	$4.17387^{\{5\}}$	$2.70456^{\{1\}}$	$14.31864^{\{7\}}$	$3.73477^{\{3\}}$	$2.74010^{\{2\}}$	$5.48047^{\{6\}}$	$3.75980^{\{4\}}$
		$\hat{\theta}$	$0.00768^{\{1\}}$	$0.00816^{\{2.5\}}$	$0.02749^{\{7\}}$	$0.01255^{\{5\}}$	$0.00816^{\{2.5\}}$	$0.01319^{\{6\}}$	$0.01169^{\{4\}}$
	MRE	$\hat{\beta}$	$0.12609^{\{1\}}$	$0.15512^{\{6\}}$	$0.13599^{\{5\}}$	$0.13431^{\{3\}}$	$0.15563^{\{7\}}$	$0.13540^{\{4\}}$	$0.13173^{\{2\}}$
		$\hat{\sigma}$	$0.05279^{\{5\}}$	$0.04264^{\{2\}}$	$0.08346^{\{7\}}$	$0.03854^{\{1\}}$	$0.04357^{\{3\}}$	$0.05817^{\{6\}}$	$0.05172^{\{4\}}$
		$\hat{\theta}$	$0.02784^{\{1\}}$	$0.02926^{\{3\}}$	$0.05165^{\{7\}}$	$0.03642^{\{4\}}$	$0.02879^{\{2\}}$	$0.03758^{\{6\}}$	$0.03659^{\{5\}}$
	$\sum Ranks$		$21^{\{1\}}$	$31.5^{\{3\}}$	$57^{\{7\}}$	$26^{\{2\}}$	$35.5^{\{5\}}$	$48^{\{6\}}$	$33^{\{4\}}$
200	$\lvert BIAS \rvert$	$\hat{\beta}$	$0.03092^{\{1\}}$	$0.03648^{\{7\}}$	$0.03369^{\{5\}}$	$0.03191^{\{2\}}$	$0.03613^{\{6\}}$	$0.03301^{\{4\}}$	$0.03233^{\{3\}}$
		$\hat{\sigma}$	$1.31130^{\{4\}}$	$1.01936^{\{2\}}$	$2.08576^{\{7\}}$	$0.90749^{\{1\}}$	$1.02520^{\{3\}}$	$1.43113^{\{6\}}$	$1.34179^{\{5\}}$
		$\hat{\theta}$	$0.05366^{\{3\}}$	$0.04802^{\{2\}}$	$0.09795^{\{7\}}$	$0.06543^{\{4\}}$	$0.04574^{\{1\}}$	$0.06815^{\{5\}}$	$0.06921^{\{6\}}$
	MSE	$\hat{\beta}$	$0.00150^{\{1\}}$	$0.00212^{\{6\}}$	$0.00183^{\{5\}}$	$0.00156^{\{2\}}$	$0.00215^{\{7\}}$	$0.00169^{\{4\}}$	$0.00160^{\{3\}}$
		$\hat{\sigma}$	$3.23318^{\{5\}}$	$1.79603^{\{1\}}$	$10.58787^{\{7\}}$	$2.64350^{\{3\}}$	$1.84750^{\{2\}}$	$3.77993^{\{6\}}$	$3.09525^{\{4\}}$
		$\hat{\theta}$	$0.00568^{\{3\}}$	$0.00428^{\{2\}}$	$0.01892^{\{7\}}$	$0.00845^{\{6\}}$	$0.00403^{\{1\}}$	$0.00834^{\{5\}}$	$0.00806^{\{4\}}$
	MRE	$\hat{\beta}$	$0.09851^{\{1\}}$	$0.11622^{\{7\}}$	$0.10732^{\{5\}}$	$0.10167^{\{2\}}$	$0.11510^{\{6\}}$	$0.10515^{\{4\}}$	$0.10298^{\{3\}}$
		$\hat{\sigma}$	$0.04547^{\{4\}}$	$0.03535^{\{2\}}$	$0.07233^{\{7\}}$	$0.03147^{\{1\}}$	$0.03555^{\{3\}}$	$0.04963^{\{6\}}$	$0.04653^{\{5\}}$
		$\hat{\theta}$	$0.02336^{\{3\}}$	$0.02091^{\{2\}}$	$0.04265^{\{7\}}$	$0.02849^{\{4\}}$	$0.01992^{\{1\}}$	$0.02968^{\{5\}}$	$0.03014^{\{6\}}$
	$\sum Ranks$		$25^{\{1.5\}}$	$31^{\{4\}}$	$57^{\{7\}}$	$25^{\{1.5\}}$	$30^{\{3\}}$	$45^{\{6\}}$	$39^{\{5\}}$
350	$\lvert BIAS \rvert$	$\hat{\beta}$	$0.02368^{\{1\}}$	$0.02750^{\{6.5\}}$	$0.02519^{\{5\}}$	$0.02390^{\{2\}}$	$0.02750^{\{6.5\}}$	$0.02487^{\{4\}}$	$0.02428^{\{3\}}$
		$\hat{\sigma}$	$1.09755^{\{4\}}$	$0.83161^{\{2\}}$	$1.69409^{\{7\}}$	$0.66836^{\{1\}}$	$0.83484^{\{3\}}$	$1.27575^{\{6\}}$	$1.16324^{\{5\}}$
		$\hat{\theta}$	$0.04552^{\{3\}}$	$0.03451^{\{2\}}$	$0.07705^{\{7\}}$	$0.04819^{\{4\}}$	$0.03407^{\{1\}}$	$0.05611^{\{6\}}$	$0.05464^{\{5\}}$
	MSE	$\hat{\beta}$	$0.00090^{\{2\}}$	$0.00119^{\{6\}}$	$0.00102^{\{5\}}$	$0.00086^{\{1\}}$	$0.00122^{\{7\}}$	$0.00097^{\{4\}}$	$0.00091^{\{3\}}$
		$\hat{\sigma}$	$2.21457^{\{4\}}$	$1.18727^{\{1\}}$	$6.62988^{\{7\}}$	$1.44502^{\{3\}}$	$1.19911^{\{2\}}$	$2.90414^{\{6\}}$	$2.22198^{\{5\}}$
		$\hat{\theta}$	$0.00405^{\{3\}}$	$0.00219^{\{2\}}$	$0.01196^{\{7\}}$	$0.00435^{\{4\}}$	$0.00218^{\{1\}}$	$0.00565^{\{6\}}$	$0.00514^{\{5\}}$
	MRE	$\hat{\beta}$	$0.07543^{\{1\}}$	$0.08760^{\{6\}}$	$0.08024^{\{5\}}$	$0.07613^{\{2\}}$	$0.08761^{\{7\}}$	$0.07923^{\{4\}}$	$0.07736^{\{3\}}$
		$\hat{\sigma}$	$0.03806^{\{4\}}$	$0.02884^{\{2\}}$	$0.05874^{\{7\}}$	$0.02318^{\{1\}}$	$0.02895^{\{3\}}$	$0.04424^{\{6\}}$	$0.04034^{\{5\}}$
		$\hat{\theta}$	$0.01982^{\{3\}}$	$0.01503^{\{2\}}$	$0.03355^{\{7\}}$	$0.02098^{\{4\}}$	$0.01484^{\{1\}}$	$0.02443^{\{6\}}$	$0.02379^{\{5\}}$
	$\sum Ranks$		$25^{\{2\}}$	$29.5^{\{3\}}$	$57^{\{7\}}$	$22^{\{1\}}$	$31.5^{\{4\}}$	$48^{\{6\}}$	$39^{\{5\}}$

Table 6. Simulation results for $\boldsymbol{\tau} = (\beta = 0.15360, \sigma = 1.29980, \theta = 0.44742)^\intercal$.

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE
10	$\lvert BIAS\rvert$	$\hat{\beta}$	$0.13197^{\{3\}}$	$0.14831^{\{6\}}$	$0.14214^{\{5\}}$	$0.12465^{\{1\}}$	$0.17463^{\{7\}}$	$0.13066^{\{2\}}$	$0.13317^{\{4\}}$
		$\hat{\sigma}$	$0.35106^{\{1\}}$	$0.36696^{\{3\}}$	$0.40754^{\{5\}}$	$0.35831^{\{2\}}$	$0.42224^{\{7\}}$	$0.37654^{\{4\}}$	$0.41925^{\{6\}}$
		$\hat{\theta}$	$0.15913^{\{3\}}$	$0.17276^{\{5\}}$	$0.16159^{\{4\}}$	$0.15403^{\{1\}}$	$0.19460^{\{6\}}$	$0.15528^{\{2\}}$	$0.20672^{\{7\}}$
	MSE	$\hat{\beta}$	$0.05112^{\{1\}}$	$0.06113^{\{5\}}$	$0.06513^{\{6\}}$	$0.05521^{\{4\}}$	$0.08703^{\{7\}}$	$0.05386^{\{2\}}$	$0.05486^{\{3\}}$
		$\hat{\sigma}$	$0.21120^{\{1\}}$	$0.22975^{\{3\}}$	$0.27755^{\{5\}}$	$0.22106^{\{2\}}$	$0.30049^{\{7\}}$	$0.23600^{\{4\}}$	$0.29068^{\{6\}}$
		$\hat{\theta}$	$0.04926^{\{3\}}$	$0.05830^{\{5\}}$	$0.05641^{\{4\}}$	$0.04095^{\{1\}}$	$0.07984^{\{6\}}$	$0.04536^{\{2\}}$	$0.09235^{\{7\}}$
	MRE	$\hat{\beta}$	$0.85918^{\{3\}}$	$0.96555^{\{6\}}$	$0.92536^{\{5\}}$	$0.81150^{\{1\}}$	$1.13689^{\{7\}}$	$0.85068^{\{2\}}$	$0.86702^{\{4\}}$
		$\hat{\sigma}$	$0.27009^{\{1\}}$	$0.28232^{\{3\}}$	$0.31354^{\{5\}}$	$0.27567^{\{2\}}$	$0.32485^{\{7\}}$	$0.28969^{\{4\}}$	$0.32255^{\{6\}}$
		$\hat{\theta}$	$0.35566^{\{3\}}$	$0.38612^{\{5\}}$	$0.36116^{\{4\}}$	$0.34427^{\{1\}}$	$0.43494^{\{6\}}$	$0.34705^{\{2\}}$	$0.46202^{\{7\}}$
	$\sum Ranks$		$19^{\{2\}}$	$41^{\{4\}}$	$43^{\{5\}}$	$15^{\{1\}}$	$60^{\{7\}}$	$24^{\{3\}}$	$50^{\{6\}}$
20	$\lvert BIAS\rvert$	$\hat{\beta}$	$0.08489^{\{5\}}$	$0.09620^{\{6\}}$	$0.07059^{\{3\}}$	$0.06460^{\{1\}}$	$0.10381^{\{7\}}$	$0.06802^{\{2\}}$	$0.07242^{\{4\}}$
		$\hat{\sigma}$	$0.25444^{\{2\}}$	$0.26663^{\{4\}}$	$0.26676^{\{5\}}$	$0.24266^{\{1\}}$	$0.28615^{\{7\}}$	$0.25668^{\{3\}}$	$0.27901^{\{6\}}$
		$\hat{\theta}$	$0.10714^{\{4\}}$	$0.11758^{\{5\}}$	$0.10151^{\{3\}}$	$0.09597^{\{1\}}$	$0.12346^{\{6\}}$	$0.09970^{\{2\}}$	$0.13355^{\{7\}}$
	MSE	$\hat{\beta}$	$0.01899^{\{5\}}$	$0.02365^{\{6\}}$	$0.01197^{\{3\}}$	$0.00926^{\{1\}}$	$0.02844^{\{7\}}$	$0.01100^{\{2\}}$	$0.01255^{\{4\}}$
		$\hat{\sigma}$	$0.10671^{\{2\}}$	$0.11829^{\{5\}}$	$0.11704^{\{4\}}$	$0.09653^{\{1\}}$	$0.13369^{\{7\}}$	$0.10730^{\{3\}}$	$0.12714^{\{6\}}$
		$\hat{\theta}$	$0.02014^{\{4\}}$	$0.02378^{\{5\}}$	$0.01935^{\{3\}}$	$0.01501^{\{1\}}$	$0.02759^{\{6\}}$	$0.01702^{\{2\}}$	$0.03235^{\{7\}}$
	MRE	$\hat{\beta}$	$0.55267^{\{5\}}$	$0.62629^{\{6\}}$	$0.45959^{\{3\}}$	$0.42055^{\{1\}}$	$0.67585^{\{7\}}$	$0.44284^{\{2\}}$	$0.47148^{\{4\}}$
		$\hat{\sigma}$	$0.19575^{\{2\}}$	$0.20513^{\{4\}}$	$0.20523^{\{5\}}$	$0.18669^{\{1\}}$	$0.22015^{\{7\}}$	$0.19747^{\{3\}}$	$0.21466^{\{6\}}$
		$\hat{\theta}$	$0.23945^{\{4\}}$	$0.26281^{\{5\}}$	$0.22689^{\{3\}}$	$0.21449^{\{1\}}$	$0.27593^{\{6\}}$	$0.22282^{\{2\}}$	$0.29848^{\{7\}}$
	$\sum Ranks$		$33^{\{4\}}$	$46^{\{5\}}$	$32^{\{3\}}$	$9^{\{1\}}$	$60^{\{7\}}$	$21^{\{2\}}$	$51^{\{6\}}$
50	$\lvert BIAS\rvert$	$\hat{\beta}$	$0.04283^{\{5\}}$	$0.05116^{\{6\}}$	$0.03519^{\{2\}}$	$0.03366^{\{1\}}$	$0.05433^{\{7\}}$	$0.03803^{\{4\}}$	$0.03668^{\{3\}}$
		$\hat{\sigma}$	$0.16135^{\{3\}}$	$0.16889^{\{5\}}$	$0.16164^{\{4\}}$	$0.15022^{\{1\}}$	$0.17241^{\{7\}}$	$0.15918^{\{2\}}$	$0.16979^{\{6\}}$
		$\hat{\theta}$	$0.06402^{\{4\}}$	$0.06985^{\{5\}}$	$0.05888^{\{2\}}$	$0.05680^{\{1\}}$	$0.07179^{\{6\}}$	$0.06079^{\{3\}}$	$0.07736^{\{7\}}$
	MSE	$\hat{\beta}$	$0.00359^{\{5\}}$	$0.00524^{\{6\}}$	$0.00237^{\{2\}}$	$0.00190^{\{1\}}$	$0.00595^{\{7\}}$	$0.00266^{\{4\}}$	$0.00246^{\{3\}}$
		$\hat{\sigma}$	$0.04180^{\{4\}}$	$0.04512^{\{5\}}$	$0.04167^{\{3\}}$	$0.03629^{\{1\}}$	$0.04764^{\{7\}}$	$0.04044^{\{2\}}$	$0.04634^{\{6\}}$
		$\hat{\theta}$	$0.00663^{\{4\}}$	$0.00792^{\{5\}}$	$0.00573^{\{2\}}$	$0.00509^{\{1\}}$	$0.00865^{\{6\}}$	$0.00604^{\{3\}}$	$0.01001^{\{7\}}$
	MRE	$\hat{\beta}$	$0.27885^{\{5\}}$	$0.33304^{\{6\}}$	$0.22911^{\{2\}}$	$0.21914^{\{1\}}$	$0.35369^{\{7\}}$	$0.24759^{\{4\}}$	$0.23879^{\{3\}}$
		$\hat{\sigma}$	$0.12414^{\{3\}}$	$0.12994^{\{5\}}$	$0.12436^{\{4\}}$	$0.11557^{\{1\}}$	$0.13265^{\{7\}}$	$0.12247^{\{2\}}$	$0.13063^{\{6\}}$
		$\hat{\theta}$	$0.14308^{\{4\}}$	$0.15611^{\{5\}}$	$0.13160^{\{2\}}$	$0.12695^{\{1\}}$	$0.16044^{\{6\}}$	$0.13587^{\{3\}}$	$0.17291^{\{7\}}$
	$\sum Ranks$		$37^{\{4\}}$	$48^{\{5.5\}}$	$23^{\{2\}}$	$9^{\{1\}}$	$60^{\{7\}}$	$27^{\{3\}}$	$48^{\{5.5\}}$
80	$\lvert BIAS\rvert$	$\hat{\beta}$	$0.03184^{\{5\}}$	$0.03927^{\{6\}}$	$0.02617^{\{1\}}$	$0.02630^{\{2\}}$	$0.04030^{\{7\}}$	$0.02869^{\{4\}}$	$0.02838^{\{3\}}$
		$\hat{\sigma}$	$0.12668^{\{4\}}$	$0.13475^{\{7\}}$	$0.12457^{\{2\}}$	$0.12255^{\{1\}}$	$0.13467^{\{6\}}$	$0.12643^{\{3\}}$	$0.13342^{\{5\}}$
		$\hat{\theta}$	$0.04771^{\{4\}}$	$0.05541^{\{6\}}$	$0.04437^{\{1\}}$	$0.04546^{\{2\}}$	$0.05507^{\{5\}}$	$0.04700^{\{3\}}$	$0.06030^{\{7\}}$
	MSE	$\hat{\beta}$	$0.00190^{\{5\}}$	$0.00274^{\{6\}}$	$0.00124^{\{2\}}$	$0.00113^{\{1\}}$	$0.00301^{\{7\}}$	$0.00138^{\{4\}}$	$0.00135^{\{3\}}$
		$\hat{\sigma}$	$0.02546^{\{4\}}$	$0.02909^{\{7\}}$	$0.02447^{\{2\}}$	$0.02394^{\{1\}}$	$0.02887^{\{6\}}$	$0.02539^{\{3\}}$	$0.02799^{\{5\}}$
		$\hat{\theta}$	$0.00371^{\{4\}}$	$0.00495^{\{5.5\}}$	$0.00320^{\{1\}}$	$0.00322^{\{2\}}$	$0.00495^{\{5.5\}}$	$0.00356^{\{3\}}$	$0.00594^{\{7\}}$
	MRE	$\hat{\beta}$	$0.20729^{\{5\}}$	$0.25568^{\{6\}}$	$0.17041^{\{1\}}$	$0.17120^{\{2\}}$	$0.26237^{\{7\}}$	$0.18675^{\{4\}}$	$0.18475^{\{3\}}$
		$\hat{\sigma}$	$0.09746^{\{4\}}$	$0.10367^{\{7\}}$	$0.09583^{\{2\}}$	$0.09428^{\{1\}}$	$0.10360^{\{6\}}$	$0.09727^{\{3\}}$	$0.10264^{\{5\}}$
		$\hat{\theta}$	$0.10664^{\{4\}}$	$0.12385^{\{6\}}$	$0.09917^{\{1\}}$	$0.10161^{\{2\}}$	$0.12308^{\{5\}}$	$0.10505^{\{3\}}$	$0.13477^{\{7\}}$
	$\sum Ranks$		$39^{\{4\}}$	$56.5^{\{7\}}$	$13^{\{1\}}$	$14^{\{2\}}$	$54.5^{\{6\}}$	$30^{\{3\}}$	$45^{\{5\}}$
120	$\lvert BIAS\rvert$	$\hat{\beta}$	$0.02503^{\{5\}}$	$0.03154^{\{6\}}$	$0.02056^{\{1\}}$	$0.02088^{\{2\}}$	$0.03243^{\{7\}}$	$0.02338^{\{4\}}$	$0.02302^{\{3\}}$
		$\hat{\sigma}$	$0.10291^{\{3\}}$	$0.10731^{\{5\}}$	$0.10115^{\{2\}}$	$0.09911^{\{1\}}$	$0.11027^{\{7\}}$	$0.10570^{\{4\}}$	$0.10822^{\{6\}}$
		$\hat{\theta}$	$0.03913^{\{4\}}$	$0.04485^{\{5\}}$	$0.03615^{\{1\}}$	$0.03683^{\{2\}}$	$0.04561^{\{6\}}$	$0.03844^{\{3\}}$	$0.04944^{\{7\}}$
	MSE	$\hat{\beta}$	$0.00107^{\{5\}}$	$0.00171^{\{6\}}$	$0.00071^{\{2\}}$	$0.00069^{\{1\}}$	$0.00182^{\{7\}}$	$0.00090^{\{4\}}$	$0.00088^{\{3\}}$
		$\hat{\sigma}$	$0.01678^{\{3\}}$	$0.01845^{\{5\}}$	$0.01615^{\{2\}}$	$0.01569^{\{1\}}$	$0.01911^{\{7\}}$	$0.01764^{\{4\}}$	$0.01855^{\{6\}}$
		$\hat{\theta}$	$0.00240^{\{4\}}$	$0.00322^{\{5\}}$	$0.00209^{\{2\}}$	$0.00207^{\{1\}}$	$0.00329^{\{6\}}$	$0.00234^{\{3\}}$	$0.00393^{\{7\}}$

Table 6. *Cont.*

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE
	MRE	$\hat{\beta}$	$0.16294^{\{5\}}$	$0.20533^{\{6\}}$	$0.13386^{\{1\}}$	$0.13597^{\{2\}}$	$0.21114^{\{7\}}$	$0.15224^{\{4\}}$	$0.14988^{\{3\}}$
		$\hat{\sigma}$	$0.07917^{\{3\}}$	$0.08256^{\{5\}}$	$0.07782^{\{2\}}$	$0.07625^{\{1\}}$	$0.08483^{\{7\}}$	$0.08132^{\{4\}}$	$0.08326^{\{6\}}$
		$\hat{\theta}$	$0.08747^{\{4\}}$	$0.10024^{\{5\}}$	$0.08080^{\{1\}}$	$0.08232^{\{2\}}$	$0.10193^{\{6\}}$	$0.08592^{\{3\}}$	$0.11049^{\{7\}}$
	$\sum Ranks$		$36^{\{4\}}$	$48^{\{5.5\}}$	$14^{\{2\}}$	$13^{\{1\}}$	$60^{\{7\}}$	$33^{\{3\}}$	$48^{\{5.5\}}$
200	$\lvert BIAS \rvert$	$\hat{\beta}$	$0.01873^{\{5\}}$	$0.02428^{\{7\}}$	$0.01554^{\{1\}}$	$0.01606^{\{2\}}$	$0.02375^{\{6\}}$	$0.01757^{\{3\}}$	$0.01774^{\{4\}}$
		$\hat{\sigma}$	$0.08089^{\{4\}}$	$0.08354^{\{6\}}$	$0.07748^{\{2\}}$	$0.07746^{\{1\}}$	$0.08355^{\{7\}}$	$0.08014^{\{3\}}$	$0.08327^{\{5\}}$
		$\hat{\theta}$	$0.02976^{\{4\}}$	$0.03417^{\{5\}}$	$0.02734^{\{1\}}$	$0.02774^{\{2\}}$	$0.03480^{\{6\}}$	$0.02950^{\{3\}}$	$0.03697^{\{7\}}$
	MSE	$\hat{\beta}$	$0.00059^{\{5\}}$	$0.00097^{\{7\}}$	$0.00039^{\{1\}}$	$0.00040^{\{2\}}$	$0.00096^{\{6\}}$	$0.00051^{\{3.5\}}$	$0.00051^{\{3.5\}}$
		$\hat{\sigma}$	$0.01038^{\{4\}}$	$0.01117^{\{7\}}$	$0.00945^{\{1\}}$	$0.00946^{\{2\}}$	$0.01101^{\{6\}}$	$0.01011^{\{3\}}$	$0.01093^{\{5\}}$
		$\hat{\theta}$	$0.00142^{\{4\}}$	$0.00187^{\{5\}}$	$0.00119^{\{1\}}$	$0.00122^{\{2\}}$	$0.00194^{\{6\}}$	$0.00137^{\{3\}}$	$0.00223^{\{7\}}$
	MRE	$\hat{\beta}$	$0.12191^{\{5\}}$	$0.15810^{\{7\}}$	$0.10114^{\{1\}}$	$0.10452^{\{2\}}$	$0.15465^{\{6\}}$	$0.11440^{\{3\}}$	$0.11552^{\{4\}}$
		$\hat{\sigma}$	$0.06223^{\{4\}}$	$0.06427^{\{6\}}$	$0.05961^{\{2\}}$	$0.05960^{\{1\}}$	$0.06428^{\{7\}}$	$0.06166^{\{3\}}$	$0.06407^{\{5\}}$
		$\hat{\theta}$	$0.06651^{\{4\}}$	$0.07637^{\{5\}}$	$0.06110^{\{1\}}$	$0.06199^{\{2\}}$	$0.07778^{\{6\}}$	$0.06592^{\{3\}}$	$0.08263^{\{7\}}$
	$\sum Ranks$		$39^{\{4\}}$	$55^{\{6\}}$	$11^{\{1\}}$	$16^{\{2\}}$	$56^{\{7\}}$	$27.5^{\{3\}}$	$47.5^{\{5\}}$
350	$\lvert BIAS \rvert$	$\hat{\beta}$	$0.01378^{\{5\}}$	$0.01800^{\{7\}}$	$0.01154^{\{1\}}$	$0.01202^{\{2\}}$	$0.01784^{\{6\}}$	$0.01367^{\{4\}}$	$0.01330^{\{3\}}$
		$\hat{\sigma}$	$0.05947^{\{3\}}$	$0.06296^{\{6\}}$	$0.05877^{\{1\}}$	$0.05899^{\{2\}}$	$0.06361^{\{7\}}$	$0.06041^{\{4\}}$	$0.06293^{\{5\}}$
		$\hat{\theta}$	$0.02229^{\{4\}}$	$0.02521^{\{5\}}$	$0.02074^{\{1\}}$	$0.02124^{\{2\}}$	$0.02636^{\{6\}}$	$0.02191^{\{3\}}$	$0.02806^{\{7\}}$
	MSE	$\hat{\beta}$	$0.00031^{\{5\}}$	$0.00052^{\{6.5\}}$	$0.00021^{\{1\}}$	$0.00023^{\{2\}}$	$0.00052^{\{6.5\}}$	$0.00030^{\{4\}}$	$0.00028^{\{3\}}$
		$\hat{\sigma}$	$0.00565^{\{3\}}$	$0.00631^{\{6\}}$	$0.00539^{\{1\}}$	$0.00545^{\{2\}}$	$0.00642^{\{7\}}$	$0.00580^{\{4\}}$	$0.00617^{\{5\}}$
		$\hat{\theta}$	$0.00078^{\{4\}}$	$0.00101^{\{5\}}$	$0.00068^{\{1\}}$	$0.00071^{\{2\}}$	$0.00109^{\{6\}}$	$0.00076^{\{3\}}$	$0.00126^{\{7\}}$
	MRE	$\hat{\beta}$	$0.08971^{\{5\}}$	$0.11717^{\{7\}}$	$0.07512^{\{1\}}$	$0.07826^{\{2\}}$	$0.11612^{\{6\}}$	$0.08899^{\{4\}}$	$0.08660^{\{3\}}$
		$\hat{\sigma}$	$0.04576^{\{3\}}$	$0.04844^{\{6\}}$	$0.04522^{\{1\}}$	$0.04538^{\{2\}}$	$0.04894^{\{7\}}$	$0.04648^{\{4\}}$	$0.04842^{\{5\}}$
		$\hat{\theta}$	$0.04982^{\{4\}}$	$0.05636^{\{5\}}$	$0.04636^{\{1\}}$	$0.04748^{\{2\}}$	$0.05891^{\{6\}}$	$0.04898^{\{3\}}$	$0.06271^{\{7\}}$
	$\sum Ranks$		$36^{\{4\}}$	$53.5^{\{6\}}$	$9^{\{1\}}$	$18^{\{2\}}$	$57.5^{\{7\}}$	$33^{\{3\}}$	$45^{\{5\}}$

Furthermore, the partial and overall ranks of all estimators are shown in Table 7. From Table 7, we can conclude that the MLE outperforms all the other estimators with an overall score of 357. Thus, the MLE method is recommended for estimating the parameter of the NTF-Weibull distribution.

Table 7. Partial and overall ranks of the classical estimation methods for several parametric values.

τ^{T}	n	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE
	10	4	5	2.5	1	7	2.5	6
	20	4	6	2	1	7	3	5
	50	4	6	1	2	7	3	5
$(\beta = 0.70, \sigma = 0.80, \theta = 0.50)$	80	4	6	1	2	7	3	5
	120	4	5.5	1	2	7	3	5.5
	200	4	7	2	1	6	3	5
	350	3	5	1	2	7	4	6
	10	4	5	1	2	7	3	6
	20	5	6	1	2	7	3	4
	50	5	6	1	2	7	3	4
$(\beta = 0.70, \sigma = 0.80, \theta = 1.50)$	80	5	6	1	2	7	3	4
	120	4	6	1	2	7	3	5
	200	4.5	7	1	2	6	3	4.5
	350	4	6	2	1	7	3	5

Table 7. *Cont.*

$\tau^{\top}$	n	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE
$(\beta = 0.70, \sigma = 0.80, \theta = 5.00)$	10	4	6	1	2	7	3	5
	20	5	6	2	1	7	3	4
	50	5	6	2	1	7	3	4
	80	5	7	2	1	6	3	4
	120	5	6	2	1	7	3	4
	200	5	6	2	1	7	3	4
	350	4.5	6	2	1	7	3	4.5
$(\beta = 0.70, \sigma = 1.45, \theta = 0.50)$	10	4	5	1	2	6.5	3	6.5
	20	4	6	1	2	7	3	5
	50	4	7	2	1	6	3	5
	80	4	6	2	1	7	3	5
	120	4	5.5	2	1	7	3	5.5
	200	4	6	1	2	7	3	5
	350	4	7	1	2	6	3	5
$(\beta = 0.70, \sigma = 1.45, \theta = 1.50)$	10	4	5	2	1	7	3	6
	20	5	6	1	2	7	3	4
	50	5	7	1	2	6	3	4
	80	5	6	1	2	7	3	4
	120	4	7	2	1	6	3	5
	200	4	7	1	2	6	3	5
	350	4	6	2	1	7	3	5
$(\beta = 0.70, \sigma = 1.45, \theta = 5.00)$	10	4	5.5	2.5	1	7	2.5	5.5
	20	5	6	1	2	7	3	4
	50	5	7	2	1	6	3	4
	80	5	6	1	2	7	3	4
	120	5	6	2	1	7	3	4
	200	5	7	2	1	6	3	4
	350	5	6	2	1	7	3	4
$(\beta = 0.70, \sigma = 2.50, \theta = 0.50)$	10	4	5	2	1	6	3	7
	20	4	6	1	2	7	3	5
	50	4	6	1	2	7	3	5
	80	4	7	1	2	6	3	5
	120	4	5.5	1	2	7	3	5.5
	200	4	5.5	1	2	7	3	5.5
	350	4	7	1	2	6	3	5
$(\beta = 0.70, \sigma = 2.50, \theta = 1.50)$	10	4	5	1	2	7	3	6
	20	4.5	6	1	2	7	3	4.5
	50	5	6	1	2	7	3	4
	80	5	6	1	2	7	3	4
	120	5	6	1	2	7	3	4
	200	4	6	1	2	7	3	5
	350	4	7	2	1	6	3	5
$(\beta = 0.70, \sigma = 2.50, \theta = 5.00)$	10	4	5	2	1	7	3	6
	20	5	6	1	2	7	3	4
	50	5	6	2	1	7	3	4
	80	5	6	2	1	7	3	4
	120	5	7	2	1	6	3	4
	200	5	7	2	1	6	3.5	3.5
	350	5	6	2	1	7	4	3

Table 7. *Cont.*

$\tau^{\intercal}$	n	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE
$(\beta = 1.20, \sigma = 0.80, \theta = 0.50)$	10	4	5	2	1	7	3	6
	20	4	6	1	2	7	3	5
	50	4	6	1	2	7	3	5
	80	4	6	1	2	7	3	5
	120	4	5	1	2	6	3	7
	200	4	5.5	1	2	7	3	5.5
	350	4	5	1	2	6.5	3	6.5
$(\beta = 1.20, \sigma = 0.80, \theta = 1.50)$	10	4	5.5	1	2	7	3	5.5
	20	4	6	1	2	7	3	5
	50	4	7	1	2	6	3	5
	80	4	6	1	2	7	3	5
	120	4	6	1	2	7	3	5
	200	4	6	1	2	7	3	5
	350	4	6	1	2	7	3	5
$(\beta = 1.20, \sigma = 0.80, \theta = 5.00)$	10	4	5	1	2	7	3	6
	20	5	6	1	2	7	3	4
	50	4	6	1	2	7	3	5
	80	4	7	2	1	6	3	5
	120	4	6	1	2	7	3	5
	200	4	7	2	1	6	3	5
	350	4	7	2	1	6	3	5
$(\beta = 1.20, \sigma = 1.45, \theta = 0.50)$	10	4	5	1	2	7	3	6
	20	4	6	1	2	7	3	5
	50	4	7	1	2	6	3	5
	80	4	6	1	2	7	3	5
	120	4	5.5	2	1	7	3	5.5
	200	4	5.5	1	2	7	3	5.5
	350	3	5.5	1	2	7	4	5.5
$(\beta = 1.20, \sigma = 1.45, \theta = 1.50)$	10	4	5	1	2	7	3	6
	20	4.5	6	1	2	7	3	4.5
	50	4	7	1	2	6	3	5
	80	4	6	1	2	7	3	5
	120	4	6	1	2	7	3	5
	200	4	7	1	2	6	3	5
	350	4	7	2	1	6	3	5
$(\beta = 1.20, \sigma = 1.45, \theta = 5.00)$	10	4	5	2	1	7	3	6
	20	5	6	1	2	7	3	4
	50	4	7	1	2	6	3	5
	80	4	6	2	1	7	3	5
	120	4	6	2	1	7	3	5
	200	4	6	2	1	7	3	5
	350	4	6	2	1	7	3	5
$(\beta = 1.20, \sigma = 2.50, \theta = 0.50)$	10	4	5	1	2	7	3	6
	20	4	5.5	1	2	7	3	5.5
	50	4	5.5	1	2	7	3	5.5
	80	4	7	1	2	6	3	5
	120	4	5.5	1	2	7	3	5.5
	200	4	7	1	2	6	3	5
	350	4	6	1	2	7	3	5
$(\beta = 1.20, \sigma = 2.50, \theta = 1.50)$	10	4	5	1	2	7	3	6
	20	5	6	1	2	7	3	4
	50	4	6	1	2	7	3	5
	80	4	6	1	2	7	3	5
	120	4	6	2	1	7	3	5
	200	4	7	1	2	6	3	5
	350	4	7	1	2	6	3	5

Table 7. *Cont.*

$\tau^{\top}$	n	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE
$(\beta = 1.20, \sigma = 2.50, \theta = 5.00)$	10	4	5	2	1	7	3	6
	20	5	6	1	2	7	3	4
	50	5	7	2	1	6	3	4
	80	4	6	2	1	7	3	5
	120	4	6	1	2	7	3	5
	200	4	7	2	1	6	3	5
	350	4	7	2	1	6	3	5
$(\beta = 6.00, \sigma = 0.80, \theta = 0.50)$	10	4	5	1	2	6	3	7
	20	4	5	1	2	7	3	6
	50	4	7	1	2	5.5	3	5.5
	80	4	7	1	2	5	3	6
	120	4	6.5	1	2	5	3	6.5
	200	4	5	1	2	7	3	6
	350	3	5	1	2	6.5	4	6.5
$(\beta = 6.00, \sigma = 0.80, \theta = 1.50)$	10	4	3	1	2	6	5	7
	20	4	6	1	2	7	3	5
	50	4	6	1	2	7	3	5
	80	4	7	1	2	6	3	5
	120	4	6	1	2	7	3	5
	200	3	7	2	1	6	4	5
	350	3	6	2	1	7	4	5
$(\beta = 6.00, \sigma = 0.80, \theta = 5.00)$	10	4	2.5	6	7	1	5	2.5
	20	6	3	4	7	1	5	2
	50	6	4	5	7	3	2	1
	80	5	7	1	2	6	3	4
	120	4	7	2	1	6	3	5
	200	3	7	2	1	6	4	5
	350	3	6	2	1	7	4	5
$(\beta = 6.00, \sigma = 1.45, \theta = 0.50)$	10	4	5	1	2	7	3	6
	20	4	6	1	2	7	3	5
	50	4	6	1	2	7	3	5
	80	4	7	1	2	5	3	6
	120	4	6	1	2	5	3	7
	200	4	6.5	1	2	5	3	6.5
	350	3	5	2	1	6	4	7
$(\beta = 6.00, \sigma = 1.45, \theta = 1.50)$	10	4	3	2	1	6	5	7
	20	4	6	1	2	7	3	5
	50	4	7	1	2	6	3	5
	80	4	6	1	2	7	3	5
	120	4	7	1	2	6	3	5
	200	4	7	2	1	6	3	5
	350	3	7	2	1	6	4	5
$(\beta = 6.00, \sigma = 1.45, \theta = 5.00)$	10	4	2	7	6	1	5	3
	20	6	2	5	7	1	4	3
	50	7	6	1	5	4	3	2
	80	5	7	2	1	6	4	3
	120	3	6	2	1	7	4	5
	200	3	7	2	1	6	4	5
	350	3	7	2	1	6	4	5
$(\beta = 6.00, \sigma = 2.50, \theta = 0.50)$	10	4	5	2	1	6	3	7
	20	4	5.5	1	2	7	3	5.5
	50	4	5.5	1	2	7	3	5.5
	80	4	5.5	1	2	7	3	5.5
	120	4	7	1	2	5.5	3	5.5
	200	4	7	2	1	6	3	5
	350	4	7	2	1	5	3	6

Table 7. *Cont.*

$\tau^{\top}$	n	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE
$(\beta = 6.00, \sigma = 2.50, \theta = 1.50)$	10	5	3	2	1	6	4	7
	20	4	6	2	1	7	3	5
	50	4	6	1	2	7	3	5
	80	4	7	1	2	6	3	5
	120	4	7	2	1	6	3	5
	200	4	7	2	1	6	3	5
	350	4	7	2	1	6	3	5
$(\beta = 6.00, \sigma = 2.50, \theta = 5.00)$	10	3	1	6	4.5	2	7	4.5
	20	7	1	6	4	3	5	2
	50	7	6	1	2	5	3	4
	80	4.5	7	2	1	6	3	4.5
	120	4	7	2	1	6	3	5
	200	4	6	2	1	7	3	5
	350	3	7	2	1	6	4	5
β = 0.31390, σ = 28.83840, θ = 2.29650	10	1.5	1.5	7	4	6	5	3
	20	1	5	7	3	6	4	2
	50	1	4.5	7	4.5	2.5	6	2.5
	80	1	5	7	4	3	6	2
	120	1	3	7	2	5	6	4
	200	1	4	7	1.5	3	6	5
	350	1.5	3	7	1	4	6	5
β = 0.15360, σ = 1.29980, θ = 0.44742	10	2	4	5	1	7	3	6
	20	4	5	3	1	7	2	6
	50	4	5.5	2	1	7	3	5.5
	80	4	7	1	2	6	3	5
	120	4	5.5	2	1	7	3	5.5
	200	4	6	1	2	7	3	5
	350	4	6	1	2	7	3	5
$\sum$ Ranks		**828.5**	**1187.5**	**357**	**368.5**	**1283**	**657.5**	**1002**
Overall Rank		**4**	**6**	**1**	**2**	**7**	**3**	**5**

5. Data Analyses Using the Reliability Data Sets

This section demonstrates application of the NTF-Weibull distribution in the field of reliability. For the practical demonstration of the NTF-Weibull distribution, two reliability data sets are considered. The next subsection provides a brief description of both reliability data sets.

5.1. Description of the Reliability Data Sets

This subsection provides a comprehensive description of reliable data sets that are considered for illustrative purposes of the NTF-Weibull distribution.

The first data set (it may be represented as Data 1) is taken from the mechanical engineering sector; see Appendix A. This data set represents the strength of the fibers tensioned at 20 mm gauge lengths [22,23]. Some descriptive measures of the mechanical engineering data set are mean = $\bar{t} = 3.059$, variance = $var(t) = 0.3855436$, maximum = 5.020, minimum = 1.901, range = 3.119, skewness = 0.6328486, kurtosis = 3.286345, first quartile = Q_1 = 2.554, second quartile = $Q_2 = 2.996$, third quartile = $Q_3 = 3.421$.

The second data set (it may be represented as Data 2) is considered from the field of reliability engineering; see Appendix A. It represents the waiting time between the failures of the reactor pumps [24]. Some descriptive measures of the reliability engineering data set are $\bar{t} = 1.578$, $var(t) = 3.727516$, maximum = 6.560, minimum = 0.062, range = 6.498, skewness = 1.364345, kurtosis = 3.544534, $Q_1 = 0.310$, $Q_2 = 0.614$, and $Q_3 = 2.041$.

Furthermore, some descriptive plots of Data 1 and Data 2 are presented in Figure 4 and Figure 5, respectively. These plots include the kernel density, histogram, box plot, and violin plot of Data 1 and Data 2.

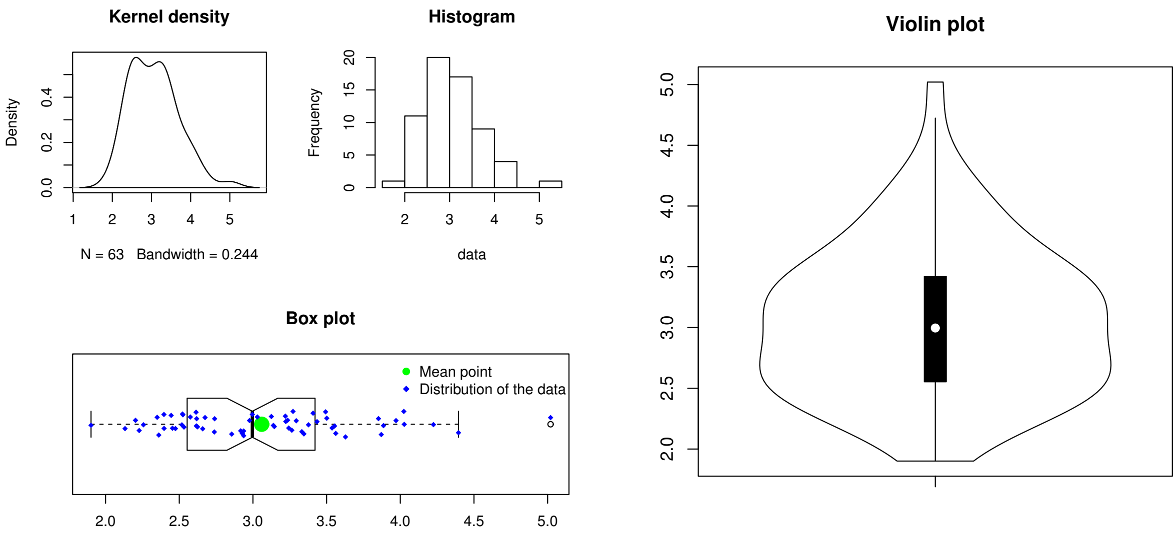

Figure 4. Some descriptive plots of Data 1.

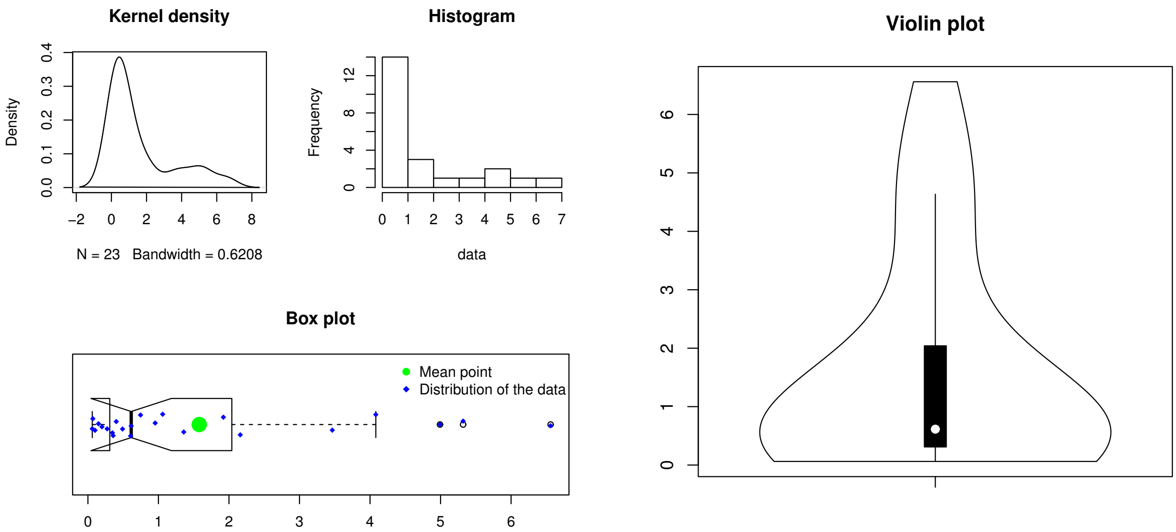

Figure 5. Some descriptive plots of Data 2.

5.2. The Rival Distributions

This subsection provides some rival distributions to verify that the NTF-Weibull distribution can outperform some existing distributions when considering reliability applications. Therefore, we apply the NTF-Weibull distribution to analyze these two mechanical and reliability engineering data sets and compare its results with the original two-parameter Weibull distribution [25], F-Weibull distribution [14], exponentiated Weibull (E-Weibull) distribution [26], new exponential Weibull (NE-Weibull) distribution [27], logarithmic Weibull (L-Weibull) distribution [28], and flexible generalized skew-normal (FGSN) distribution [29]. The PDFs of these selected probability distributions are as follows:

- Weibull distribution

$$f(t) = \tau\beta t^{\tau-1}e^{-\beta t^{\tau}}, \qquad t,\tau,\beta \in \mathbb{R}^{+}.$$

- F-Weibull distribution

$$f(t) = \left(\delta + \frac{\beta}{t^2}\right) e^{\left(\delta t - \frac{\beta}{t}\right)} e^{-e^{\left(\delta t - \frac{\beta}{t}\right)}}, \qquad t, \delta, \beta \in \mathbb{R}^+.$$

- E-Weibull distribution

$$f(t) = \alpha\tau\beta t^{\tau-1} e^{-\beta t^{\tau}} \left(1 - e^{-\beta t^{\tau}}\right)^{\alpha-1}, \qquad t, \alpha, \tau, \beta \in \mathbb{R}^+.$$

- ETX-Weibull distribution

$$f(t) = \frac{\tau\beta\lambda(\lambda-1)t^{\tau-1}e^{-\beta t^{\tau}}}{\left(\lambda - 1 + e^{-\beta t^{\tau}}\right)^2}, \qquad t, \tau, \beta \in \mathbb{R}^+, \lambda > 1.$$

- L-Weibull distribution

$$f(t) = \frac{\kappa\gamma\tau\beta t^{\tau-1}e^{-\beta t^{\tau}}\left[1 + \gamma - \log\left(1 - e^{-\beta t^{\tau}}\right)\right]}{\left(\gamma - \log\left(1 - e^{-\beta t^{\tau}}\right)\right)^2} \left(1 - \frac{\gamma\left(1 - e^{-\beta t^{\tau}}\right)}{\gamma - \log\left(1 - e^{-\beta t^{\tau}}\right)}\right)^{\kappa-1},$$

where $t, \tau, \beta, \kappa, \gamma \in \mathbb{R}^+$.

- FGSN distribution

$$f(x) = 2\phi_1\left(t; \xi, s^2\right) \Phi\left(\frac{a(t-\zeta)}{s} + \frac{b(t-\zeta)^3}{s^3}\right), \qquad t, \zeta \in \mathbb{R},\ a, b, s \in \mathbb{R}^+.$$

5.3. The Selection Criteria

To find out the best-suited probability model for the mechanical and reliability engineering data sets, we consider three statistical tests (i.e., selection and decision tools) and four information criteria. Among the fitted set of models, a probability distribution with the smallest values of these tests indicates the best-suited fit. The mathematical expressions of the selected statistical tests and information criteria are given by

- The Anderson–Darling (AD) test

$$-n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F(t_i) + \log \bar{F}(t_{n-i+1})\right].$$

- The Cramer–Von Mises (CVM) test

$$\frac{1}{12n} + \sum_{i=1}^{n}\left(\frac{2i-1}{2n} - F(t_i)\right)^2.$$

- The Kolmogorov–Smirnov (KS) test

$$sup_t|F_n(t) - F(t)|.$$

- Akaike information criterion (AIC)

$$2k - 2\ell(.).$$

- Consistent Akaike information criterion (CAIC)

$$\frac{2nk}{n-k-1} - 2\ell(.).$$

- Bayesian information criterion (BIC)

$$k\log(n) - \ell(.).$$

- Hannan–Quinn information criterion (HQIC)

$$2k\log[\log(n)] - 2\ell(.).$$

In the above expressions, n and t_i, respectively, represent the size and i^{th} observation of the data. The term k represents the number of model parameters, whereas $F_n(t)$ and sup_t represent the empirical CDF and supremum of the set of distances, respectively. In addition to these tests, the p-value for the fitted distributions is also computed. A higher p-value of a model indicates a close fit. The numerical values of the MLEs, statistical tests, and p-value of the fitted distributions are obtained using the R software with the "method = BFGS" algorithm.

5.4. Analysis of Data Set 1

Using Data 1, the values of $\hat{\tau}_{MLE}$, $\hat{\beta}_{MLE}$, $\hat{\sigma}_{MLE}$, $\hat{\theta}_{MLE}$, $\hat{\alpha}_{MLE}$, $\hat{\lambda}_{MLE}$, $\hat{\kappa}_{MLE}$, $\hat{\gamma}_{MLE}$, $\hat{a}_{MLE}$, $\hat{b}_{MLE}$, $\hat{s}_{MLE}$, and $\hat{\xi}_{MLE}$ of the fitted distributions are reported in Table 8. From Section 4, we can see that the MLEs $(\hat{\beta}_{MLE}, \hat{\sigma}_{MLE}, \hat{\theta}_{MLE})$ of the parameters (β, σ, θ) of the NTF-Weibull distribution are not explicit forms, and therefore, to ensure the uniqueness of $\hat{\beta}_{MLE}$, $\hat{\sigma}_{MLE}$, and $\hat{\theta}_{MLE}$, we plot the profiles of their LLF; see Figure 6. The profiles plot in Figure 6 shows that $\hat{\beta}_{MLE}$, $\hat{\sigma}_{MLE}$, and $\hat{\theta}_{MLE}$ have unique values.

The values of the statistical tests and information criteria of the fitted models are reported in Tables 9 and 10. From Tables 9 and 10, we can easily observe that:

- The NTF-Weibull distribution has the smallest values of the evaluation criteria.
- The NTF-Weibull distribution has a high p-value compared to the rival distributions.

These facts show that the NTF-Weibull distribution is the best model for the mechanical engineering data set considered in this paper.

In addition to the numerical evaluation, we also provide a graphical evaluation of these distributions. For the graphical evaluation, we consider the Kaplan–Meier SF, empirical CDF, PDF, and quantile–quantile (QQ) plots; see Figure 7. The fitted plots in Figure 7 show that the NTF-Weibull distribution closely fits the mechanical engineering data set.

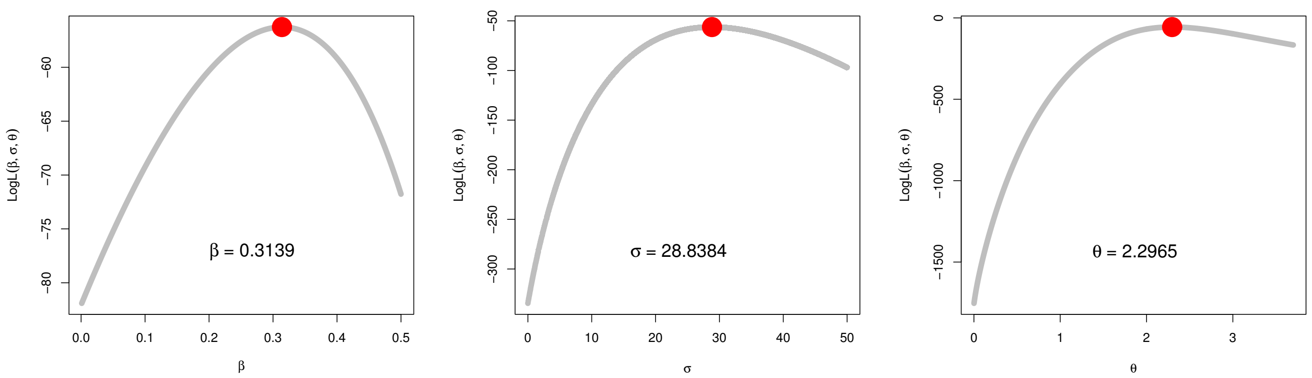

Figure 6. The profiles of the LLF of $\hat{\beta}_{MLE}$, $\hat{\sigma}_{MLE}$, and $\hat{\theta}_{MLE}$ of the NTF-Weibull distribution for Data 1.

Table 8. The values of $\hat{\tau}_{MLE}$, $\hat{\beta}_{MLE}$, $\hat{\sigma}_{MLE}$, $\hat{\theta}_{MLE}$, $\hat{\alpha}_{MLE}$, $\hat{\lambda}_{MLE}$, $\hat{\kappa}_{MLE}$, $\hat{\gamma}_{MLE}$, $\hat{a}_{MLE}$, $\hat{b}_{MLE}$, $\hat{s}_{MLE}$, and $\hat{\xi}_{MLE}$ along with the standard errors (in parenthesise) of the fitted distributions using Data 1.

Models	$\hat{\tau}_{MLE}$	$\hat{\beta}_{MLE}$	$\hat{\sigma}_{MLE}$	$\hat{\theta}_{MLE}$	$\hat{\alpha}_{MLE}$	$\hat{\lambda}_{MLE}$	$\hat{\kappa}_{MLE}$	$\hat{\gamma}_{MLE}$	$\hat{a}_{MLE}$	$\hat{b}_{MLE}$	$\hat{s}_{MLE}$	$\hat{\xi}_{MLE}$
NTF-Weibull	-	0.3139 (0.1009)	28.8384 (12.2965)	2.2965 (0.5123)	-	-	-	-	-	-	-	-
Weibull	4.9013 (0.1866)	0.0030 (0.0006)	-	-	-	-	-	-	-	-	-	-
F-Weibull	-	0.7466 (0.0639)	8.2688 (0.8418)	-	-	-	-	-	-	-	-	
E-Weibull	1.5090 (0.7061)	0.7392 (0.9308)	-	-	31.9126 (59.1279)	-	-	-	-	-	-	-
ETX-Weibull	1.8356 (1.1188)	0.3275 (0.6296)	-	-	-	12.5241 (21.8193)	-	-	-	-	-	-
L-Weibull	2.4850 (0.03271)	0.6117 (0.0401)	-	-	-	-	0.1769 (0.0328)	0.0056 (0.0022)	-	-	-	-
FGSN	-	-	-	-	-	-	-	-	4.8069 (2.6411)	0.6756 (28.9924)	2.2514 (0.1388)	1.0209 (0.1550)

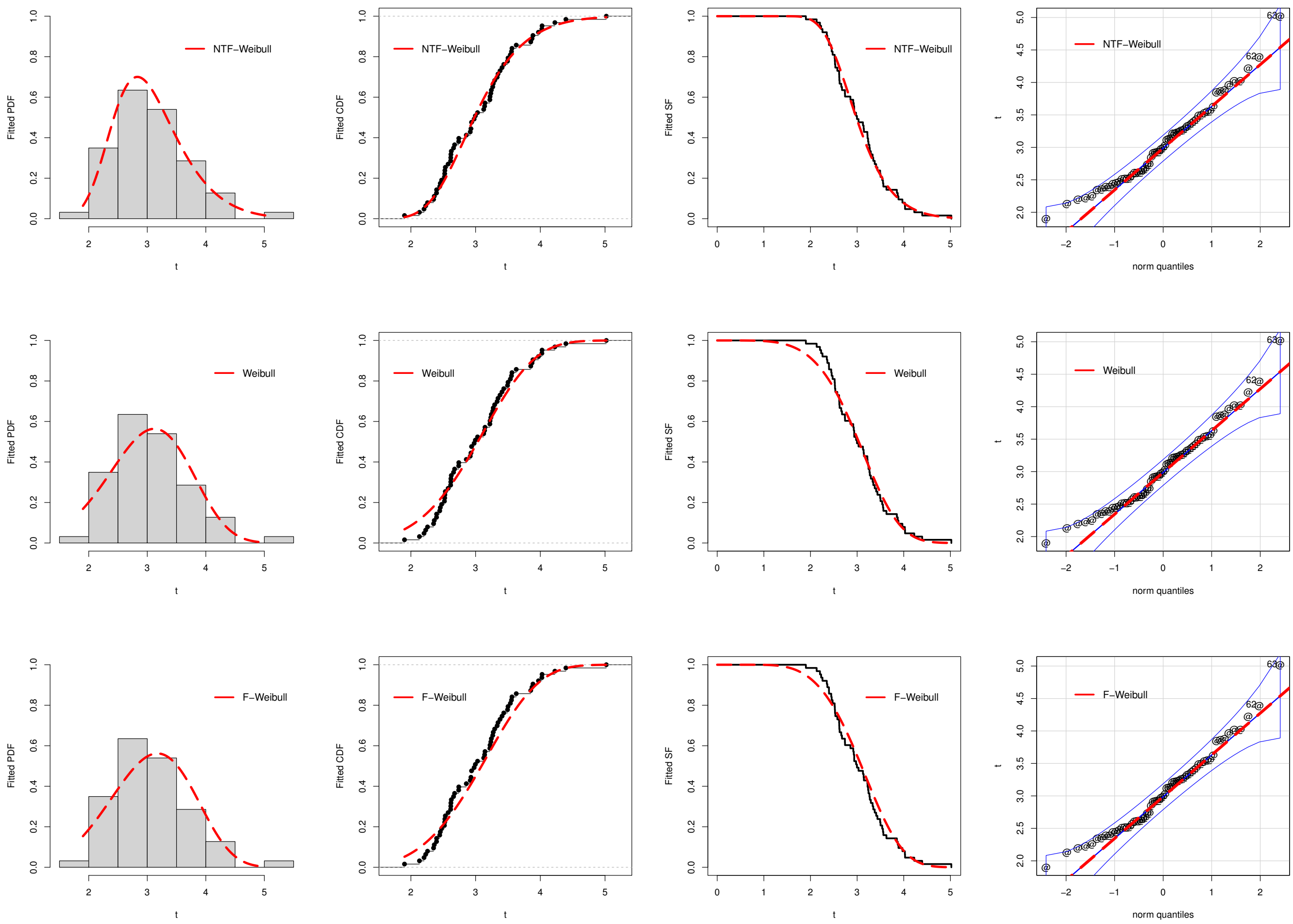

Figure 7. *Cont.*

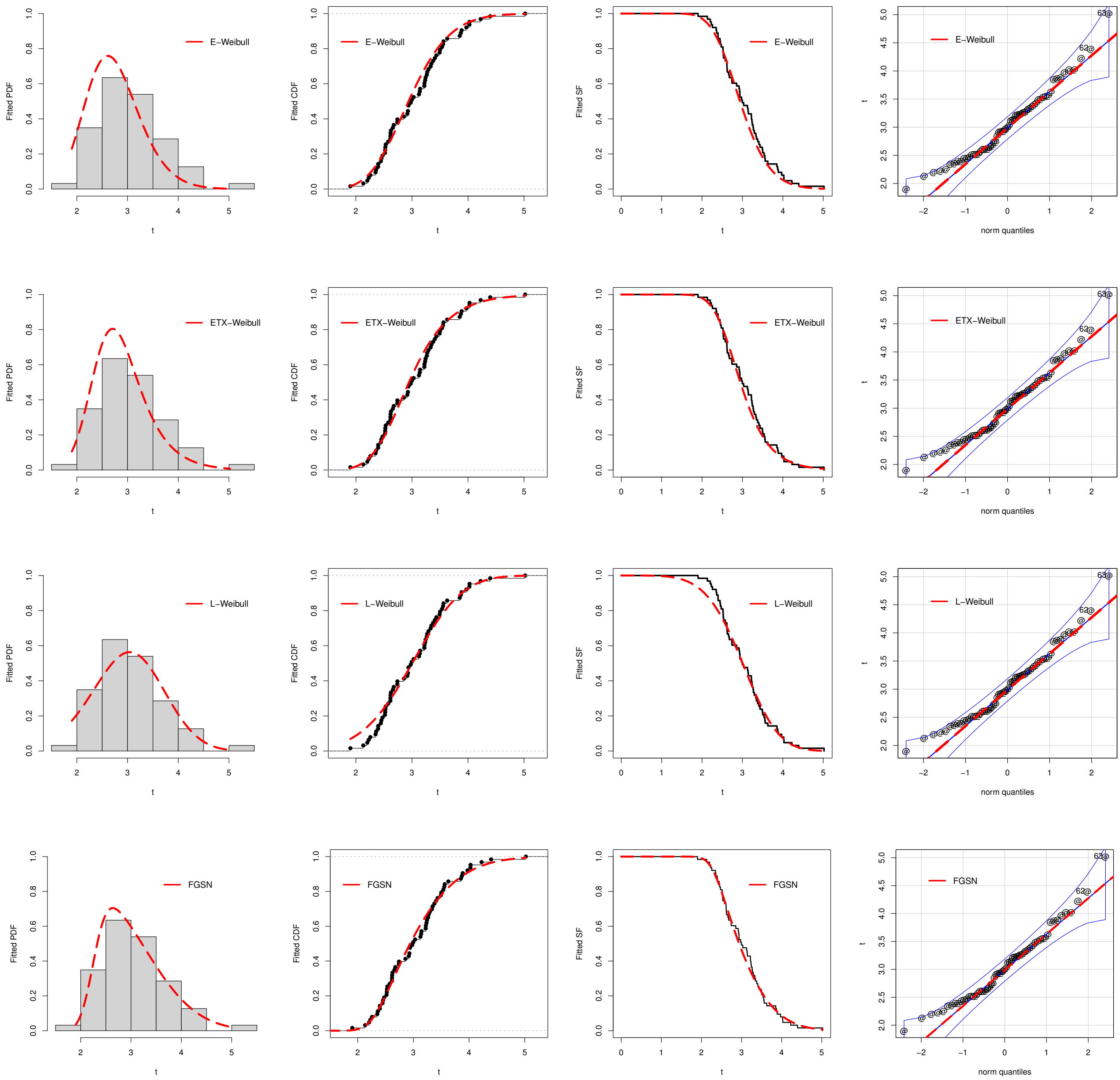

Figure 7. The graphical evaluation of the fitted models using the mechanical engineering data set.

5.5. Analysis of Data Set 2

Corresponding to Data 2, the values of $\hat{\tau}_{MLE}$, $\hat{\beta}_{MLE}$, $\hat{\sigma}_{MLE}$, $\hat{\theta}_{MLE}$, $\hat{\alpha}_{MLE}$, $\hat{\lambda}_{MLE}$, $\hat{\kappa}_{MLE}$, $\hat{\gamma}_{MLE}$, $\hat{a}_{MLE}$, $\hat{b}_{MLE}$, $\hat{s}_{MLE}$, and $\hat{\xi}_{MLE}$ are presented in Table 11. Using Data 2, the uniqueness of $\hat{\beta}_{MLE}$, $\hat{\sigma}_{MLE}$, and $\hat{\theta}_{MLE}$ of the NTF-Weibull distribution is shown in Figure 8. The profiles of the LLF in Figure 8 reveal that $\hat{\beta}_{MLE}$, $\hat{\sigma}_{MLE}$, and $\hat{\theta}_{MLE}$ have unique values.

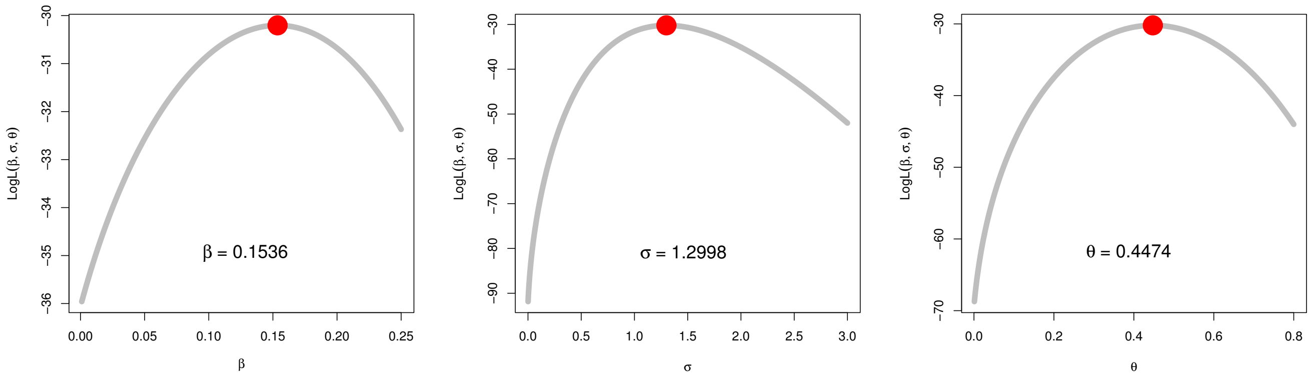

Figure 8. The profiles of the LLF of $\hat{\beta}_{MLE}, \hat{\sigma}_{MLE}$, and $\hat{\theta}_{MLE}$ of the NTF-Weibull distribution using Data 2.

After obtaining the MLEs, next we provide the numerical results of the considered statistical tests for the fitted distributions. From the numerical evaluation in Tables 12 and 13, it is obvious that the NTF-Weibull distribution also provides the best fit for the reliability engineering data set than the rival distributions. Based on the numerical description of the fitted models in Tables 12 and 13, we can easily observe that the NTF-Weibull distribution again performs better than the selected competing distributions. In other words, we can say that the NTF-Weibull distribution proved to achieve a satisfactory fit. Therefore, we have the evidence to claim the best-fitting power of the NTF-Weibull distribution for the considered reliability data set.

Table 9. For Data 1, the values of the statistical tests of the fitted distributions.

Models	CVM	AD	KS	*p*-Value
NTF-Weibull	0.05867	0.31704	0.07833	0.83420
Weibull	0.12223	0.85085	0.10017	0.55210
F-Weibull	0.12129	0.84557	0.08698	0.72710
E-Weibull	0.06120	0.32704	0.07997	0.81520
ETX-Weibull	0.07225	0.38696	0.08451	0.75910
L-Weibull	0.06916	0.37172	0.08347	0.77230
FGSN	0.05292	0.28126	0.08576	0.74300

Table 10. For Data 1, the values of the information criteria of the fitted distributions.

Models	AIC	BIC	CAIC	HQIC
NTF-Weibull	118.5254	118.9322	124.9548	121.0541
Weibull	128.1359	128.3359	132.4222	129.8217
F-Weibull	127.1997	127.3997	131.4860	128.8855
E-Weibull	120.6269	121.0337	127.0563	123.1556
ETX-Weibull	121.4615	121.8683	127.8909	123.9902
L-Weibull	121.8455	122.5351	130.4180	125.2171
FGSN	119.8780	120.5676	128.4505	123.2496

Table 11. The values of $\hat{\tau}_{MLE}$, $\hat{\beta}_{MLE}$, $\hat{\sigma}_{MLE}$, $\hat{\theta}_{MLE}$, $\hat{\alpha}_{MLE}$, $\hat{\lambda}_{MLE}$, $\hat{\kappa}_{MLE}$, $\hat{\gamma}_{MLE}$, $\hat{a}_{MLE}$, $\hat{b}_{MLE}$, $\hat{s}_{MLE}$, and $\hat{\xi}_{MLE}$ along with the standard errors (in parenthesise) of the fitted distributions using Data 2.

Models	$\hat{\tau}_{MLE}$	$\hat{\beta}_{MLE}$	$\hat{\sigma}_{MLE}$	$\hat{\theta}_{MLE}$	$\hat{\alpha}_{MLE}$	$\hat{\lambda}_{MLE}$	$\hat{\kappa}_{MLE}$	$\hat{\gamma}_{MLE}$	$\hat{a}_{MLE}$	$\hat{b}_{MLE}$	$\hat{s}_{MLE}$	$\hat{\xi}_{MLE}$
NTF-Weibull	-	0.1536 (0.0587)	1.2998 (0.3080)	0.44742 (0.1089)	-	-	-	-	-	-	-	-
Weibull	0.8091 (0.1297)	0.7642 (0.1825)	-	-	-	-	-	-	-	-	-	-
F-Weibull	-	0.2071 (0.0431)	0.2587 (0.0656)	-	-	-	-	-	-	-	-	-
E-Weibull	0.3047 (0.2709)	2.9744 (2.7846)	-	-	9.8073 (26.4870)	-	-	-	-	-	-	-
ETX-Weibull	0.4149 (0.4581)	1.6090 (2.2204)	-	-	-	4.0848 (9.6394)	-	-	-	-	-	-
L-Weibull	0.8089 (0.0270)	7.8478 (0.0622)	-	-	-	-	0.1005 (0.0676)	5.5115 (0.1195)	-	-	-	-
FGSN	-	-	-	-	-	-	-	-	5.0386 (2.1265)	5.3983 (3.6989)	1.6377 (0.1552)	1.8891 (0.2786)

Table 12. For Data 2, the values of the statistical tests of the fitted distributions.

Models	**CVM**	**AD**	**KS**	***p*-Value**
NTF-Weibull	0.01892	0.15392	0.08408	0.99220
Weibull	0.06548	0.43105	0.11838	0.86680
F-Weibull	0.04121	0.26699	0.13848	0.71910
E-Weibull	0.02350	0.21322	0.09706	0.96730
ETX-Weibull	0.02642	0.23125	0.10019	0.95730
L-Weibull	0.06173	0.40561	0.11907	0.86240
FGSN	0.13312	0.83032	0.28191	0.04138

Table 13. For Data 2, the values of the information criteria of the fitted distributions.

Models	**AIC**	**BIC**	**CAIC**	**HQIC**
NTF-Weibull	66.4045	67.6677	69.8110	67.2612
Weibull	69.0278	69.6278	71.2988	69.5989
F-Weibull	67.7658	68.3658	70.0368	68.3369
E-Weibull	69.6644	70.9275	73.0709	70.5211
ETX-Weibull	70.1203	71.3834	73.5268	70.9770
L-Weibull	71.6381	73.8603	76.1801	72.7804
FGSN	82.1866	84.4088	86.7285	83.3289

Like Data 1, again we provide a visual comparison of the fitted distributions. Based on the given estimated plots in Figure 9, we can see that the NTF-Weibull distribution again provides the best fit compared to the other distributions.

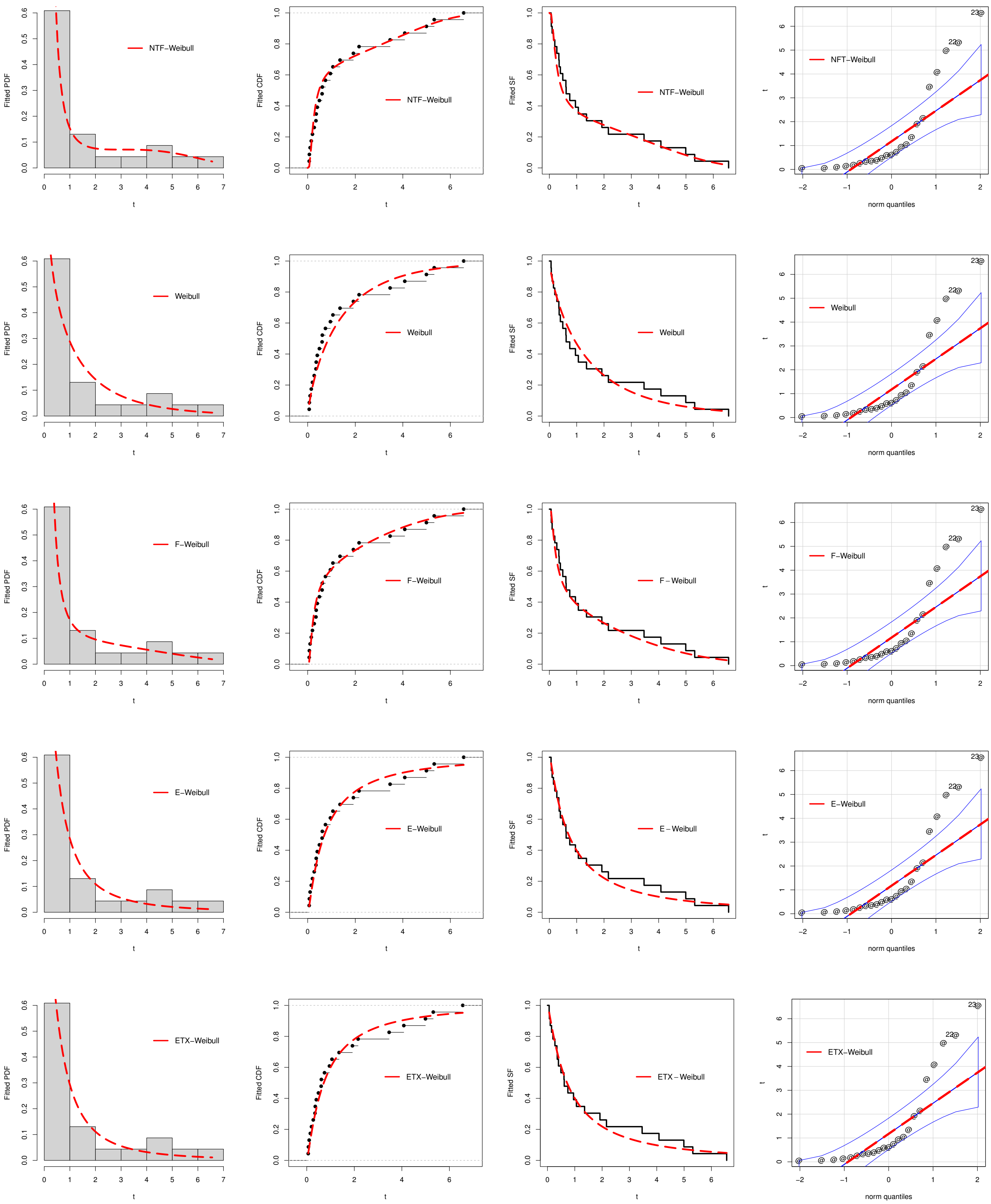

Figure 9. *Cont.*

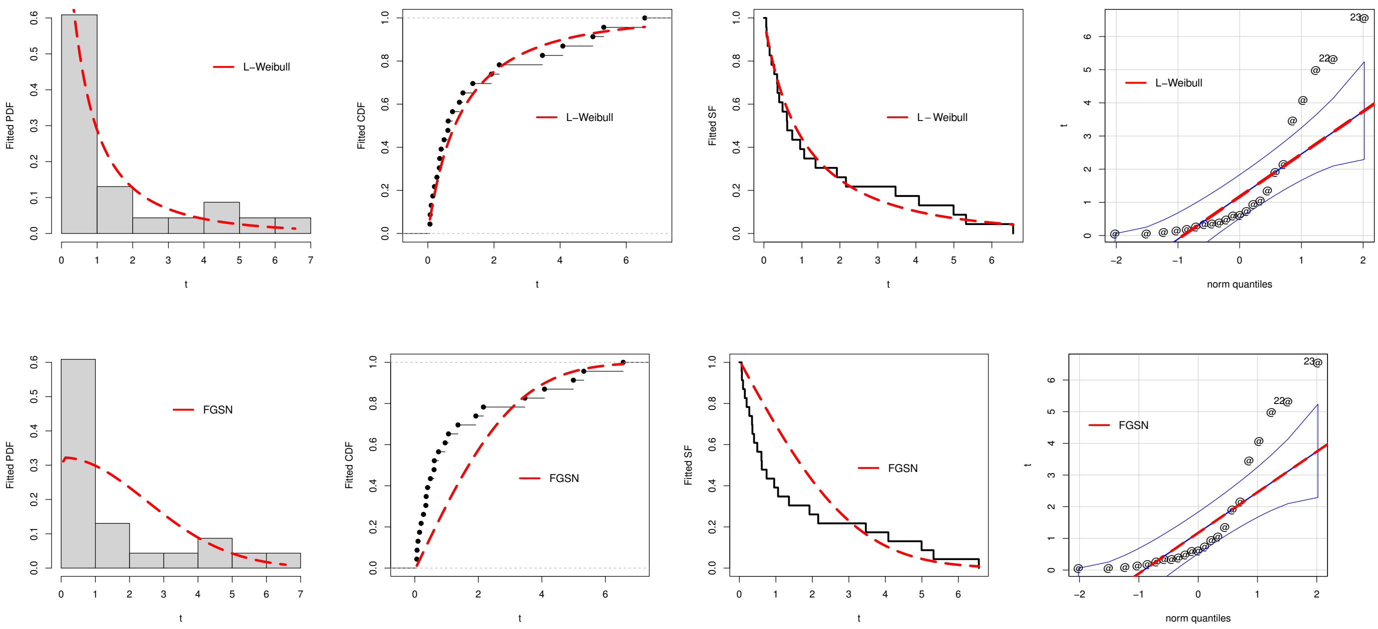

Figure 9. The graphical evaluation of the fitted models using the reliability engineering data set.

6. Concluding Remarks

This paper contributed to the literature by introducing a new useful distributional approach based on the trigonometric function. The proposed method was mainly based on the tangent function and called a new tangent-*G* family of distributions. One of the key advantages of the NT-*G* family is that it can improve the distributional flexibility of the classical model without any additional parameters. Certain mathematical properties of the NT-*G* family were derived. Based on the NT-*G* family approach, a new probability distribution, called a new tangent flexible Weibull distribution, was introduced. Seven estimation methods were incorporated to estimate the parameters of the NTF-Weibull distribution. A comprehensive simulation study was conducted to evaluate the estimators of the NTF-Weibull distribution. Based on our results using the Monte Carlo simulation study, it was observed that the maximumlikelihood method is suitable to implement for estimating the parameters of the NT-*G* family of distributions. Finally, two real-life data applications were considered to illustrate the usefulness of the NTF-Weibull distribution. Both applications were considered from the reliability engineering field. The first data set represented the strength of the fibers. The strength was tensioned at 20 mm gauge lengths. The second data set represented the waiting time until the failures' occurrence of the reactor pumps. Using three well-known comparison tools (i.e., statistical tests) with *p*-values, it was observed that the NTF-Weibull distribution closely fits the reliability data sets more than the rival distributions.

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Data Availability Statement: The data sets are available from the corresponding author upon request.

Conflicts of Interest: The authors declare no competing interest.

Appendix A

Table A1. The observations of Data 1 and Data 2.

Descriptions	Observations
Data 1	1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020
Data 2	2.160, 0.746, 0.402, 0.954, 0.491, 6.560, 4.992, 0.347, 0.150, 0.358, 0.101, 1.359, 3.465, 1.060, 0.614, 1.921, 4.082, 0.199, 0.605, 0.273, 0.070, 0.062, 5.320

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