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An Application of Touchard Polynomials on Subclasses of Analytic Functions

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Abstract: The aim of this work is to discuss some conditions for Touchard polynomials to be in the classes $\mathfrak{T}\mathfrak{B}_b(\rho,\sigma)$ and $\mathfrak{T}\mathfrak{K}_b(\rho,\sigma)$. Also, we obtain some connection between $\mathfrak{R}_\eta(D,E)$ and $\mathfrak{T}\mathfrak{K}_b(\rho,\sigma)$. Also, we investigate several mapping properties involving these subclasses. Further, we discuss the geometric properties of an integral operator related to the Touchard polynomial. Additionally, briefly mentioned are specific instances of our primary results. Also, several particular examples are presented.

Keywords: star-like functions of complex order b; analytic functions; convex functions of complex order b; differential equation; convolution; Touchard polynomials

MSC: 30C45; 30C80

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1. Introduction

We denote by $\mathcal{A}$ the class of functions of the following form:

$$f(z) = z + \sum_{\imath=2}^{\infty} a_\imath z^\imath \qquad (z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}). \tag{1}$$

Additionally, we let T denote the subclass of $\mathcal{A}$, which consists of functions f with the following power series expansion:

$$f(z) = z - \sum_{\imath=2}^{\infty} a_\imath z^\imath \qquad (z \in \mathbb{D};\ a_\imath > 0). \tag{2}$$

For the function $f(z)$ involved in (1) and the function $g(z)$ given as follows:

$$g(z) = z + \sum_{\imath=2}^{\infty} b_\imath z^\imath \quad (z \in \mathbb{D}),$$

the convolution (or, equivalently, the Hadamard product) of the functions $f(z)$ and $g(z)$ is defined below:

$$(f * g)(z) = z + \sum_{\imath=2}^{\infty} a_\imath b_\imath z^\imath, \qquad (z \in \mathbb{D})$$

and the integral convolution is (see [1])

$$(f \circledast g)(z) = z + \sum_{\imath=2}^{\infty} \frac{a_\imath b_\imath}{\imath} z^\imath = (g \circledast f)(z), \qquad (z \in \mathbb{D}).$$

Definition 1 ([2,3])**.** *A function $f \in \mathcal{S}_b$ if it satisfies:*

$$Re\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0 \qquad (z \in \mathbb{D};\, b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\};\, f(z) \in \mathcal{A}),$$

is called star-like of complex order b.

Definition 2 ([4,5])**.** *A function $f \in \mathcal{C}_b$ if it satisfies:*

$$Re\left\{1 + \frac{1}{b}\frac{zf''(z)}{f'(z)}\right\} > 0 \qquad (z \in \mathbb{D};\, b \in \mathbb{C}^*;\, f(z) \in \mathcal{A}),$$

is called convex of complex order b.

Definition 3 ([6,7])**.** *A function $f \in \mathcal{K}_b$ if it satisfies:*

$$Re\left\{1 + \frac{1}{b}(f'(z) - 1)\right\} > 0 \qquad (z \in \mathbb{D};\, b \in \mathbb{C}^*;\, f(z) \in \mathcal{A}),$$

is called close-to-convex of complex order b.

Definition 4 (with $n = 1$ [8])**.** *A function $f \in \mathfrak{B}_b(\rho, \sigma)$ if it satisfies:*

$$\left|\frac{1}{b}\left(\frac{zf'^2 f''(z)}{(1-\rho)f(z) + \rho f'(z)} - 1\right)\right| < \sigma \qquad (z \in \mathbb{D};\, b \in \mathbb{C}^*;\, 0 \le \rho \le 1;\, 0 < \sigma \le 1;\, f(z) \in \mathcal{A}).$$

Definition 5 (with $n = 1$ [8])**.** *A function $f \in \mathfrak{K}_b(\rho, \sigma)$ if it satisfies:*

$$\left|\frac{1}{b}(f'(z) + \rho z f''(z) - 1)\right| < \sigma \qquad (z \in \mathbb{D};\, b \in \mathbb{C}^*;\, 0 \le \rho \le 1;\, 0 < \sigma \le 1;\, f(z) \in \mathcal{A}).$$

Further, we denote by

$$\mathfrak{T}\mathfrak{B}_b(\rho, \sigma) \equiv \mathfrak{B}_b(\rho, \sigma) \cap T \qquad \mathfrak{T}\mathfrak{K}_b(\rho, \sigma) \equiv \mathfrak{K}_b(\rho, \sigma) \cap T.$$

Note that:

(1) $\mathfrak{T}\mathfrak{B}_{1-\alpha}(\rho, 1) = \mathcal{T}(\alpha, \rho)$ $(0 \le \alpha < 1)$, was investigated by Altintas [9] with $n = 1$] (see also [10] [with $n = 1$]);

(2) $\mathfrak{B}_b(0, 1) \subset \mathcal{S}_b$, $\mathfrak{K}_b(0, 1) \subset \mathcal{K}_b$ and $\mathfrak{B}_b(1, 1) \subset \mathcal{C}_b$ (see [2,4,6]);

(3) $\mathfrak{B}_1(0, 1) = \mathcal{S}^*$ (see [1]);

(4) $\mathfrak{B}_{e^{-i\theta}\cos\theta}(0, 1) = \mathcal{S}^\theta$, where $\mathcal{S}^\theta$ is the class of spiral-like functions ($|\theta| < \frac{\pi}{2}$; θ is the real value) (see [11]);

(5) $\mathfrak{B}_{1-\alpha}(0, 1) = \mathcal{S}^*(\alpha)$ $(0 \le \alpha < 1)$ (see [12]);

(6) $\mathfrak{B}_{(1-\alpha)e^{-i\theta}\cos\theta}(0, 1) = \mathcal{S}^\theta(\alpha)$ $(0 \le \alpha < 1)$, where $\mathcal{S}^\theta(\alpha)$ is the class of spiral-like functions of order α ($|\theta| < \frac{\pi}{2}$; θ is the real value) (see [13]);

(7) $\mathfrak{B}_1(1, 1) = \mathcal{C}$ (see [1]);

(8) $\mathfrak{B}_{e^{-i\theta}\cos\theta}(1, 1) = \mathcal{C}^\theta$, where $\mathcal{C}^\theta$ is the class of $\theta-$Robertson-type functions ($|\theta| < \frac{\pi}{2}$; θ is the real value) (see [14]);

(9) $\mathfrak{B}_{1-\alpha}(1, 1) = \mathcal{C}(\alpha)$, $(0 \le \alpha < 1)$ (see [12]);

(10) $\mathfrak{T}\mathfrak{B}_b(0, \sigma) = \mathcal{T}\mathcal{S}_b(\sigma) = \left\{f \in T : \left|\frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right| < \sigma\right\}$ (see [6]);

(11) $\mathfrak{T}\mathfrak{B}_b(1, \sigma) = \mathcal{T}\mathcal{C}_b(\sigma) = \left\{f \in T : \left|\frac{1}{b}\left(\frac{zf''(z)}{f'(z)} - 1\right)\right| < \sigma\right\}$ (see [6]);

(12) $\mathfrak{TR}_b(0,\sigma) = \mathcal{TK}_b(\sigma) = \left\{ f \in T : \left| \frac{1}{b}(f'(z)-1) \right| < \sigma \right\}$ (see [6]);
(13) $\mathfrak{TR}_{1-\alpha}(0,1) = \mathcal{TK}(\alpha) = \{ f \in T : |f'(z)-1| < 1-\alpha \}$ (see [15]);
(14) $\mathfrak{TR}_1(0,1) = \mathcal{TK} = \{ f \in T : |f'(z)-1| < 1 \}$ (see [16]).

Definition 6 ([17]). *A function* $f \in \mathfrak{R}_\eta(D,E)$ *if it satisfies:*

$$\left| \frac{f'(z)-1}{(D-E)\eta - E(f'(z)-1)} \right| < 1 \qquad (z \in \mathbb{D};\ \eta \in \mathbb{C}^*;\ -1 \le E < D \le 1).$$

Note that:
(1) $\mathfrak{R}_1(\alpha,-\alpha) = \mathfrak{R}(\alpha) = \left\{ f \in \mathcal{A} : \left| \frac{f'(z)-1}{f'(z)+1} \right| < \alpha \quad (z \in \mathbb{D};\ 0 < \alpha \le 1) \right\}$ (see [18]);
(2) $\mathfrak{R}_{e^{-i\theta}\cos\theta}(1-2\alpha,-1) = \mathfrak{R}_\theta(\alpha) = \{ f \in \mathcal{A} : \mathrm{Re}(e^{i\theta}(f'(z)-\alpha)) > 0 \quad (z \in \mathbb{D};\ |\theta| < \frac{\pi}{2};\ 0 \le \alpha < 1)$ (see [19]).

In 1939, Jacques Touchard [20] studied the Touchard polynomials, also called Bell polynomials [21–23], consisting of a binomial-type polynomial sequence defined by

$$\mathcal{T}_\kappa(x) = \sum_{\imath=0}^{\kappa} \mathcal{S}(\kappa,\imath)x^\imath = \sum_{\imath=0}^{\kappa} \left(\begin{array}{c} \kappa \\ \imath \end{array} \right) x^\imath,$$

where $\mathcal{S}(\kappa,\imath)$ is a Stirling number of the second kind.

If X is a random variable with a Poisson distribution with an expected value γ, then its κth moment is $E(X^\kappa) = \mathcal{T}_\kappa(\gamma)$, leading to the definition:

$$\mathcal{T}_\kappa(\gamma) = e^{-\gamma} \sum_{\imath=0}^{\infty} \frac{\gamma^\imath \imath^\kappa}{\imath!} \qquad (\gamma > 0;\ \kappa \ge 0).$$

In order to study the different inventory problems of the permutations when the cycles have specific features, Jacques Touchard studied these polynomials and generalized the Bell polynomials. In addition, he developed and researched a class of related polynomials. This new approach can be used to solve integral equations, both linear and nonlinear. Since it is difficult to solve integral equations analytically, we must often find approximations for the solutions. In this situation, the linear Volterra integro-differential equation is solved using the Touchard polynomial approach. Both linear and nonlinear Volterra integral equations have been solved using the Touchard polynomial method.

Lately, Murugusundaramoorthy and Porwal [24] introduced and defined a function $\mathcal{T}_\kappa(\gamma;z)$ as follows:

$$\mathcal{T}_\kappa(\gamma;z) = z + e^{-\gamma} \sum_{\imath=2}^{\infty} \frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!} z^\imath \qquad (z \in \mathbb{D};\ \gamma > 0;\ \kappa \ge 0). \tag{3}$$

The above series convergence is infinite according to the ratio test. Also, they introduced $\mathcal{F}_\kappa(\gamma;z)$ as follows:

$$\mathcal{F}_\kappa(\gamma;z) = 2z - \mathcal{T}_\kappa(\gamma;z) = z - e^{-\gamma} \sum_{\imath=2}^{\infty} \frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!} z^\imath \qquad (z \in \mathbb{D};\ \gamma > 0;\ \kappa \ge 0). \tag{4}$$

Next, we define the convolution operator $\mathcal{TF}_\kappa(\gamma;z)$ for functions f given by (2) as follows

$$\mathcal{TF}_\kappa(\gamma;z) = \mathcal{F}_\kappa(\gamma;z) * f(z) = z - e^{-\gamma} \sum_{\imath=2}^{\infty} \frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!} a_\imath z^\imath \qquad (z \in \mathbb{D};\ \gamma > 0;\ \kappa \ge 0). \tag{5}$$

Also, we define the functions

$$\begin{aligned}\Phi_\kappa(\mu,\gamma;z) &= (1-\mu)\mathcal{T}_\kappa(\gamma;z)+\mu z(\mathcal{T}_\kappa(\gamma;z))' \\ &= z+e^{-\gamma}\sum_{\imath=2}^{\infty}[1+\mu(\imath-1)]\frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!}z^\imath \qquad (\mu\geq 0),\end{aligned}$$

and

$$\begin{aligned}\Phi^*_\kappa(\mu,\gamma;z) &= 2z-\Phi_\kappa(\mu,\gamma;z) \\ &= z-e^{-\gamma}\sum_{\imath=2}^{\infty}[1+\mu(\imath-1)]\frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!}z^\imath \qquad (\mu\geq 0).\end{aligned} \tag{6}$$

Also, we define $\mathcal{P}_\kappa(\gamma;z):\mathcal{A}\to\mathcal{A}$ by the convolution as

$$\mathcal{P}_\kappa(\gamma;z)=\mathcal{F}_\kappa(\gamma;z)*f(z)=z-e^{-\gamma}\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!}a_\imath z^\imath.$$

and $\mathcal{R}_\kappa(\gamma;z):\mathcal{A}\to\mathcal{A}$ by the integral convolution as

$$\mathcal{R}_\kappa(\gamma;z)=\mathcal{F}_\kappa(\gamma;z)\circledast f(z)=z-e^{-\gamma}\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!}\frac{a_\imath}{\imath}z^\imath.$$

Also, we define $\mathcal{Q}_\kappa(\mu,\gamma;z):\mathcal{A}\to\mathcal{A}$ by the convolution as

$$\mathcal{Q}_\kappa(\mu,\gamma;z)=\Phi^*_\kappa(\mu,\gamma;z)*f(z)=z-e^{-\gamma}\sum_{\imath=2}^{\infty}[1+\mu(\imath-1)]\frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!}a_\imath z^\imath \qquad (\mu\geq 0) \tag{7}$$

and $\mathcal{G}_\kappa(\mu,\gamma;z):\mathcal{A}\to\mathcal{A}$ by the integral convolution as

$$\mathcal{G}_\kappa(\mu,\gamma;z)=\Phi^*_\kappa(\mu,\gamma;z)\circledast f(z)=z-e^{-\gamma}\sum_{\imath=2}^{\infty}[1+\mu(\imath-1)]\frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!}\frac{a_\imath}{\imath}z^\imath \qquad (\mu\geq 0). \tag{8}$$

Definition 7. *The κ^{th} moment of the Poisson distribution about the origin is*

$$\chi'_\kappa=e^{-\gamma}\sum_{\imath=0}^{\infty}\frac{\gamma^\imath\imath^\kappa}{\imath!}.$$

In our study, we will use the following lemmas.

Lemma 1 ([8] with $n=1$)**.** *A function $f\in\mathfrak{TB}_b(\rho,\sigma)$ is of the form* (2) *if and only if*

$$\sum_{\imath=2}^{\infty}[\rho(\imath-1)+1][\imath+\sigma|b|-1]|a_\imath|\leq\sigma|b| \qquad (b\in\mathbb{C}^*;\ 0\leq\rho\leq 1;\ 0<\sigma\leq 1).$$

Lemma 2 ([8] [with $n=1$)**.** *] A function $f\in\mathfrak{TK}_b(\rho,\sigma)$ is of the form* (2) *if and only if*

$$\sum_{\imath=2}^{\infty}\imath[\rho(\imath-1)+1]|a_\imath|\leq\sigma|b| \qquad (b\in\mathbb{C}^*;\ 0\leq\rho\leq 1;\ 0<\sigma\leq 1).$$

Lemma 3 ([17])**.** *If $f\in\mathfrak{R}_\eta(D,E)$ is of the form* (1)*, then*

$$|a_\imath|\leq\frac{(D-E)|\eta|}{\imath} \qquad (\imath\geq 2;\ \eta\in\mathbb{C}^*;\ -1\leq E<D\leq 1).$$

The important area of study is the use of special functions in geometric function theory. It is applied in fields including physics, engineering, and mathematics. Several types of special functions, including generalized hypergeometric Gaussian functions [25] and references, are cited therein. In fact, after the appearance of Porwal [26], several researchers familiarized themselves with the the Poisson distribution series [27,28], Mittag-Leffler-type Poisson distribution series [29], Pascal distribution series [30], generalized distribution series [31,32], and binomial distribution series [33], and provide applications for certain classes of univalent functions. In 1939, Jacques Touchard [20] studied the Touchard polynomials. Volterra integral equations, both linear and nonlinear, have been solved using the Touchard polynomial approach. In 2022, Porwal and Murugusundaramoorthy [23] investigated some conditions for Touchard polynomials to be in the subclasses of analytic functions.

The aim of this work is to discuss some conditions for Touchard polynomials to be in the classes $\mathfrak{TB}_b(\rho,\sigma)$ and $\mathfrak{TK}_b(\rho,\sigma)$. Also, we obtain some connection between $\mathfrak{R}_\eta(D,E)$ and $\mathfrak{TK}_b(\rho,\sigma)$ and investigate several mapping properties involving these subclasses. Further, we discuss the geometric properties of an integral operator related to the Touchard polynomial. In addition, briefly mentioned are specific instances of our primary results and several particular examples are presented.

2. Main Results

Unless mentioned, let $0 \le \rho \le 1$, $0 < \sigma \le 1$, $b \in \mathbb{C}^*$, $\eta \in \mathbb{C}^*$, $-1 \le E < D \le 1$, $\gamma > 0$, $\kappa \ge 0$, $z \in \mathbb{D}$ and $\chi'_\jmath = e^{-\gamma}\sum_{\imath=1}^{\infty}\frac{\gamma^\imath \imath^\jmath}{\imath!}$, $\jmath = \kappa, \kappa+1, \kappa+2, \kappa+3$.

Theorem 1. *The function $\Phi^*_\kappa(\mu,\gamma;z) \in \mathfrak{TB}_b(\rho,\sigma)$ if and only if*

$$\begin{cases} \mu\rho\chi'_{\kappa+3} + [\mu+\rho(1+\mu\sigma|b|)]\chi'_{\kappa+2} + [1+\sigma|b|(\rho+\mu)]\chi'_{\kappa+1} + \sigma|b|\chi'_\kappa \le \sigma|b| & (\kappa \ge 1), \\ e^{\gamma}\left[\mu\rho\gamma^3 + [\mu+\rho+\mu\rho(3+\sigma|b|)]\gamma^2 + [1+(\mu+\rho\mu+\rho)(1+\sigma|b|)]\gamma\right] \le \sigma|b| & (\kappa = 0). \end{cases} \tag{9}$$

Proof. To prove that $\Phi^*_\kappa(\mu,\gamma;z) \in \mathfrak{B}_b(\rho,\sigma)$, from Lemma 1 and (6), we have to prove that

$$\begin{aligned}
&\sum_{\imath=2}^{\infty}[\rho(\imath-1)+1][\imath+\sigma|b|-1][1+\mu(\imath-1)]\frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!}e^{-\gamma} \\
&= e^{-\gamma}\sum_{\imath=2}^{\infty}\Big[\mu\rho(\imath-1)^3 + [\mu+\rho(1+\mu\sigma|b|)](\imath-1)^2 + [1+\sigma|b|(\rho+\mu)](\imath-1) + \sigma|b|\Big]\frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!} \\
&= e^{-\gamma}\Bigg[\mu\rho\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^{\kappa+3}}{(\imath-1)!} + [\mu+\rho(1+\mu\sigma|b|)]\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^{\kappa+2}}{(\imath-1)!} \\
&+[1+\sigma|b|(\rho+\mu)]\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^{\kappa+1}}{(\imath-1)!} + \sigma|b|\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^{\kappa}}{(\imath-1)!}\Bigg] \\
&= e^{-\gamma}\Bigg[\mu\rho\sum_{\imath=1}^{\infty}\frac{\gamma^\imath \imath^{\kappa+3}}{\imath!} + [\mu+\rho(1+\mu\sigma|b|)]\sum_{\imath=1}^{\infty}\frac{\gamma^\imath \imath^{\kappa+2}}{\imath!} \\
&+[1+\sigma|b|(\rho+\mu)]\sum_{\imath=1}^{\infty}\frac{\gamma^\imath \imath^{\kappa+1}}{\imath!} + \sigma|b|\sum_{\imath=1}^{\infty}\frac{\gamma^\imath \imath^{\kappa}}{\imath!}\Bigg] \\
&= \begin{cases} \mu\rho\chi'_{\kappa+3} + [\mu+\rho(1+\mu\sigma|b|)]\chi'_{\kappa+2} + [1+\sigma|b|(\rho+\mu)]\chi'_{\kappa+1} + \sigma|b|\chi'_\kappa \le \sigma|b| & (\kappa \ge 1), \\ e^{\gamma}\left[\mu\rho\gamma^3 + [\mu+\rho+\mu\rho(3+\sigma|b|)]\gamma^2 + [1+(\mu+\rho\mu+\rho)(1+\sigma|b|)]\gamma\right] \le \sigma|b| & (\kappa = 0). \end{cases}
\end{aligned}$$

□

Corollary 1. *Let $\mu = 0$ in Theorem 1, then $\mathcal{TF}_\kappa(\gamma; z) \in \mathfrak{TB}_b(\rho, \sigma)$ if and only if*

$$\begin{cases} \rho\chi'_{\kappa+2} + [1+\rho\sigma|b|]\chi'_{\kappa+1} + \sigma|b|\chi'_\kappa \leq \sigma|b| & (\kappa \geq 1), \\ e^\gamma\left[\rho\gamma^2 + [1+\rho(1+\sigma|b|)]\gamma\right] \leq \sigma|b| & (\kappa = 0). \end{cases} \tag{10}$$

Remark 1. *Putting $b = 1-\alpha$ $(0 \leq \alpha < 1)$ and $\sigma = 1$ in Corollary 1, we improve the result due to Porwal and Murugusundaramoorthy [23] [Theorem 2.2]*

Corollary 2. *Let $\rho = 0$ in Theorem 1, then $\Phi^*_\kappa(\mu, \gamma; z) \in \mathcal{TS}_b(\sigma)$ if and only if*

$$\begin{cases} \mu\chi'_{\kappa+2} + [1+\mu\sigma|b|]\chi'_{\kappa+1} + \sigma|b|\chi'_\kappa \leq \sigma|b| & (\kappa \geq 1), \\ e^\gamma\left[\mu\gamma^2 + [1+\mu(1+\sigma|b|)]\gamma\right] \leq \sigma|b| & (\kappa = 0). \end{cases}$$

Corollary 3. *Let $\rho = 0$ and $\mu = 0$ in Theorem 1, then $\mathcal{TF}_\kappa(\gamma; z) \in \mathcal{TS}_b(\sigma)$ if and only if*

$$\begin{cases} \chi'_{\kappa+1} + \sigma|b|\chi'_\kappa \leq \sigma|b| & (\kappa \geq 1), \\ e^\gamma\gamma \leq \sigma|b| & (\kappa = 0). \end{cases}$$

Example 1. *(1) Let $b = 1-\alpha$ $(0 \leq \alpha < 1)$ and $\sigma = 1$ in Corollary 3, then $\mathcal{TF}_\kappa(\gamma; z) \in \mathcal{S}^*(\alpha)$ if and only if*

$$\begin{cases} \chi'_{\kappa+1} + (1-\alpha)\chi'_\kappa \leq 1-\alpha & (\kappa \geq 1), \\ e^\gamma\gamma \leq 1-\alpha & (\kappa = 0). \end{cases}$$

(2) Let $b = 1$ and $\sigma = 1$ in Corollary 3, then $\mathcal{TF}_\kappa(\gamma; z) \in \mathcal{S}^*$ if

$$\begin{cases} \chi'_{\kappa+1} + \chi'_\kappa \leq 1 & (\kappa \geq 1), \\ e^\gamma\gamma \leq 1 & (\kappa = 0). \end{cases}$$

Corollary 4. *Let $\rho = 1$ in Theorem 1, then $\Phi^*_\kappa(\mu, \gamma; z) \in \mathcal{TC}_b(\sigma)$ if and only if*

$$\begin{cases} \mu\chi'_{\kappa+3} + [1+\mu(1+\sigma|b|)]\chi'_{\kappa+2} + [1+\sigma|b|(1+\mu)]\chi'_{\kappa+1} + \sigma|b|\chi'_\kappa \leq \sigma|b| & (\kappa \geq 1), \\ e^\gamma\left[\mu\gamma^3 + [\mu+1+\mu(3+\sigma|b|)]\gamma^2 + [1+(2\mu+1)(1+\sigma|b|)]\gamma\right] \leq \sigma|b| & (\kappa = 0). \end{cases}$$

Corollary 5. *Let $\rho = 1$ and $\mu = 0$ in Theorem 1, then $\mathcal{TF}_\kappa(\gamma; z) \in \mathcal{TC}_b(\sigma)$ if and only if*

$$\begin{cases} \chi'_{\kappa+2} + [1+\sigma|b|]\chi'_{\kappa+1} + \sigma|b|\chi'_\kappa \leq \sigma|b| & (\kappa \geq 1), \\ e^\gamma\left[\gamma^2 + [2+\sigma|b|]\gamma\right] \leq \sigma|b| & (\kappa = 0). \end{cases}$$

Example 2. *(1) Let $b = 1-\alpha$ $(0 \leq \alpha < 1)$ and $\sigma = 1$ in Corollary 5, then $\mathcal{TF}_\kappa(\gamma; z) \in \mathcal{C}(\alpha)$ if and only if*

$$\begin{cases} \chi'_{\kappa+2} + [2-\alpha]\chi'_{\kappa+1} + (1-\alpha)\chi'_\kappa \leq 1-\alpha & (\kappa \geq 1), \\ e^\gamma\left[\gamma^2 + [3-\alpha]\gamma\right] \leq 1-\alpha & (\kappa = 0). \end{cases}$$

(2) Let $b = 1$ and $\sigma = 1$ in Corollary 5, then $\mathcal{TF}_\kappa(\gamma; z) \in \mathcal{C}$ if

$$\begin{cases} \chi'_{\kappa+2} + 2\chi'_{\kappa+1} + \chi'_\kappa \leq 1 & (\kappa \geq 1), \\ e^\gamma\left[\gamma^2 + 3\gamma\right] \leq 1 & (\kappa = 0). \end{cases}$$

Theorem 2. *The function* $\Phi^*_\kappa(\mu,\gamma;z) \in \mathfrak{T}\mathfrak{K}_b(\rho,\sigma)$ *if and only if*

$$\begin{cases} \mu\rho\chi'_{\kappa+3} + [\mu+\rho(1+\mu)]\chi'_{\kappa+2} + [1+\rho+\mu]\chi'_{\kappa+1} + \chi'_\kappa \le \sigma|b| & (\kappa \ge 1), \\ \mu\rho\gamma^3 + [4\mu\rho+\mu+\rho]\gamma^2 + [1+2(\mu+\rho(\mu+1))]\gamma + (1-e^{-\gamma}) \le \sigma|b| & (\kappa = 0). \end{cases} \tag{11}$$

Proof. To prove that $\Phi^*_\kappa(\mu,\gamma;z) \in \mathfrak{T}\mathfrak{K}_b(\rho,\sigma)$, from Lemma 2 and (6), we have to prove that

$$\begin{aligned}
&\sum_{\imath=2}^{\infty}\imath[\rho(\imath-1)+1][1+\mu(\imath-1)]\frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!}e^{-\gamma} \\
&= e^{-\gamma}\sum_{\imath=2}^{\infty}\Big[\mu\rho(\imath-1)^3 + [\mu+\rho(1+\mu)](\imath-1)^2 + [1+\rho+\mu](\imath-1)+1\Big]\frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!} \\
&= e^{-\gamma}\left[\mu\rho\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^{\kappa+3}}{(\imath-1)!} + [\mu+\rho(1+\mu)]\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^{\kappa+2}}{(\imath-1)!}\right. \\
&\left. +[1+\rho+\mu]\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^{\kappa+1}}{(\imath-1)!} + \sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^{\kappa}}{(\imath-1)!}\right] \\
&= e^{-\gamma}\left[\mu\rho\sum_{\imath=1}^{\infty}\frac{\gamma^\imath\imath^{\kappa+3}}{\imath!} + [\mu+\rho(1+\mu)]\sum_{\imath=1}^{\infty}\frac{\gamma^\imath\imath^{\kappa+2}}{\imath!} + [1+\rho+\mu]\sum_{\imath=1}^{\infty}\frac{\gamma^\imath\imath^{\kappa+1}}{\imath!} + \sum_{\imath=1}^{\infty}\frac{\gamma^\imath\imath^{\kappa}}{\imath!}\right] \\
&= \begin{cases} \mu\rho\chi'_{\kappa+3} + [\mu+\rho(1+\mu)]\chi'_{\kappa+2} + [1+\rho+\mu]\chi'_{\kappa+1} + \chi'_\kappa \le \sigma|b| & (\kappa \ge 1), \\ \mu\rho\gamma^3 + [4\mu\rho+\mu+\rho]\gamma^2 + [1+2(\mu+\rho(\mu+1))]\gamma + (1-e^{-\gamma}) \le \sigma|b| & (\kappa = 0). \end{cases}
\end{aligned}$$

□

Corollary 6. *Let* $\mu = 0$ *in Theorem 2, then* $\mathcal{TF}_\kappa(\gamma;z) \in \mathfrak{T}\mathfrak{K}_b(\rho,\sigma)$ *if and only if*

$$\begin{cases} \rho\chi'_{\kappa+2} + [1+\rho]\chi'_{\kappa+1} + \chi'_\kappa \le \sigma|b| & (\kappa \ge 1), \\ \rho\gamma^2 + [1+2\rho]\gamma + (1-e^{-\gamma}) \le \sigma|b| & (\kappa = 0). \end{cases} \tag{12}$$

Corollary 7. *Let* $\rho = 0$ *in Theorem 2, then* $\Phi^*_\kappa(\mu,\gamma;z) \in \mathcal{TK}_b(\sigma)$ *if and only if*

$$\begin{cases} \mu\chi'_{\kappa+2} + [1+\mu]\chi'_{\kappa+1} + \chi'_\kappa \le \sigma|b| & (\kappa \ge 1), \\ \mu\gamma^2 + [1+2\mu]\gamma + (1-e^{-\gamma}) \le \sigma|b| & (\kappa = 0). \end{cases} \tag{13}$$

Corollary 8. *Let* $\rho = 0$ *and* $\mu = 0$ *in Theorem 2, then* $\mathcal{TF}_\kappa(\gamma;z) \in \mathcal{TK}_b(\sigma)$ *if and only if*

$$\begin{cases} \chi'_{\kappa+1} + \chi'_\kappa \le \sigma|b| & (\kappa \ge 1), \\ \gamma + (1-e^{-\gamma}) \le \sigma|b| & (\kappa = 0). \end{cases} \tag{14}$$

Example 3. *(1) Let* $b = 1-\alpha$ *and* $\sigma = 1$, $(0 \le \alpha < 1)$ *in Corollary 8, then* $\mathcal{TF}_\kappa(\gamma;z) \in \mathcal{TK}(\alpha)$ *if and only if*

$$\begin{cases} \chi'_{\kappa+1} + \chi'_\kappa \le 1-\alpha & (\kappa \ge 1), \\ \gamma + (1-e^{-\gamma}) \le 1-\alpha & (\kappa = 0). \end{cases}$$

(2) Let $b = 1$ *and* $\sigma = 1$ *in Corollary 8, then* $\mathcal{TF}_\kappa(\gamma;z) \in \mathcal{TK}$ *if and only if*

$$\begin{cases} \chi'_{\kappa+1} + \chi'_\kappa \le 1 & (\kappa \ge 1), \\ \gamma + (1-e^{-\gamma}) \le 1 & (\kappa = 0). \end{cases}$$

Theorem 3. *Let $f \in \mathfrak{R}_\eta(D,E)$, $\mathcal{Q}_\kappa(\mu,\gamma;z)$ defined by (7) be in $\mathfrak{T}\mathfrak{K}_b(\rho,\sigma)$ if and only if*

$$\begin{cases} (D-E)|\eta|\left[\mu\rho\chi'_{\kappa+2}+[\mu+\rho]\chi'_{\kappa+1}+\chi'_\kappa\right]\le\sigma|b| & (\kappa\ge1), \\ (D-E)|\eta|\left[\mu\rho\gamma^2+[\mu\rho+\mu+\rho]\gamma+(1-e^{-\gamma})\right]\le\sigma|b| & (\kappa=0). \end{cases}$$

Proof. From Lemmas 2, 3, and (7), we have to prove that

$$\begin{aligned}
&\sum_{\imath=2}^{\infty}\imath[\rho(\imath-1)+1][1+\mu(\imath-1)]\left[\frac{(D-E)|\eta|}{\imath}\right]\frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!}e^{-\gamma} \\
&=(D-E)|\eta|e^{-\gamma}\sum_{\imath=2}^{\infty}\left[\mu\rho(\imath-1)^2+[\mu+\rho](\imath-1)+1\right]\frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!} \\
&=(D-E)|\eta|e^{-\gamma}\left[\mu\rho\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^{\kappa+2}}{(\imath-1)!}+[\mu+\rho]\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^{\kappa+1}}{(\imath-1)!}+\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^{\kappa}}{(\imath-1)!}\right] \\
&=(D-E)|\eta|e^{-\gamma}\left[\mu\rho\sum_{\imath=1}^{\infty}\frac{\gamma^{\imath}\imath^{\kappa+2}}{\imath!}+[\mu+\rho]\sum_{\imath=1}^{\infty}\frac{\gamma^{\imath}\imath^{\kappa+1}}{\imath!}+\sum_{\imath=1}^{\infty}\frac{\gamma^{\imath}\imath^{\kappa}}{\imath!}\right] \\
&=\begin{cases} (D-E)|\eta|\left[\mu\rho\chi'_{\kappa+2}+[\mu+\rho]\chi'_{\kappa+1}+\chi'_\kappa\right]\le\sigma|b| & (\kappa\ge1), \\ (D-E)|\eta|\left[\mu\rho\gamma^2+[\mu\rho+\mu+\rho]\gamma+(1-e^{-\gamma})\right]\le\sigma|b| & (\kappa=0). \end{cases}
\end{aligned}$$

□

Corollary 9. *Let $\mu=0$ in Theorem 3 and $f\in\mathfrak{R}_\eta(D,E)$, then $\mathcal{P}_\kappa(\gamma;z)$ is in $\mathfrak{T}\mathfrak{K}_b(\rho,\sigma)$ if and only if*

$$\begin{cases} (D-E)|\eta|\left[\rho\chi'_{\kappa+1}+\chi'_\kappa\right]\le\sigma|b| & (\kappa\ge1), \\ (D-E)|\eta|[\rho\gamma+(1-e^{-\gamma})]\le\sigma|b| & (\kappa=0). \end{cases}$$

Corollary 10. *Let $\rho=0$ in Theorem 3 and $f\in\mathfrak{R}_\eta(D,E)$, then $\mathcal{Q}_\kappa(\mu,\gamma;z)$ be in $\mathcal{TK}_b(\sigma)$ if and only if*

$$\begin{cases} (D-E)|\eta|\left[\mu\chi'_{\kappa+1}+\chi'_\kappa\right]\le\sigma|b| & (\kappa\ge1), \\ (D-E)|\eta|[\mu\gamma+(1-e^{-\gamma})]\le\sigma|b| & (\kappa=0). \end{cases}$$

Corollary 11. *Let $\rho=0$ and $\mu=0$ in Theorem 3 and $f\in\mathfrak{R}_\eta(D,E)$, then $\mathcal{P}_\kappa(\gamma;z)$ is in $\mathcal{TK}_b(\sigma)$ if and only if*

$$\begin{cases} (D-E)|\eta|\chi'_\kappa\le\sigma|b| & (\kappa\ge1), \\ (D-E)|\eta|(1-e^{-\gamma})\le\sigma|b| & (\kappa=0). \end{cases}$$

Theorem 4. *The function $\mathcal{Q}_\kappa(\mu,\gamma;z)$ maps $f(z)\in\mathcal{S}^*$ to $\mathfrak{T}\mathfrak{B}_b(\rho,\sigma)$ if and only if*

$$\begin{cases} \mu\rho\chi'_{\kappa+4}+[\rho\mu(1+\sigma|b|)+\mu+\rho]\chi'_{\kappa+3}+[1+\sigma|b|(\rho\mu+\rho+\mu)+\rho+\mu]\chi'_{\kappa+2} \\ \qquad +[1+\sigma|b|(\rho+\mu+1)]\chi'_{\kappa+1}+\sigma|b|\chi'_\kappa\le\sigma|b| \qquad (\kappa\ge1), \\ e^{\gamma}[\mu\rho\gamma^4+[\rho\mu(7+\sigma|b|)+\mu+\rho]\gamma^3+[\rho\mu(10+4\sigma|b|)+(\rho+\mu)(4+\sigma|b|)+1]\gamma^2 \\ \qquad +[2\rho\mu(1+\sigma|b|)+2(\mu+\rho)(1+\sigma|b|)+2+\sigma|b|]\gamma]\le\sigma|b| \qquad (\kappa=0). \end{cases} \tag{15}$$

Proof. From Lemma 1 and (7), we have to prove that

$$\sum_{\imath=2}^{\infty}[\rho(\imath-1)+1][\imath+\sigma|b|-1][1+\mu(\imath-1)]\frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!}|a_\imath|e^{-\gamma}\le\sigma|b|$$

Since $f(z) \in \mathcal{S}^*$, then $|a_\imath| \leq \imath$. So

$$\sum_{\imath=2}^{\infty}[\rho(\imath-1)+1][\imath+\sigma|b|-1][1+\mu(\imath-1)]\frac{\gamma^{\imath-1}(\imath-1)^{\kappa}}{(\imath-1)!}|a_\imath|e^{-\gamma}$$

$$\begin{aligned}
&\leq \sum_{\imath=2}^{\infty}\imath[\rho(\imath-1)+1][\imath+\sigma|b|-1][1+\mu(\imath-1)]\frac{\gamma^{\imath-1}(\imath-1)^{\kappa}}{(\imath-1)!}e^{-\gamma}\\
&\leq e^{-\gamma}\sum_{\imath=2}^{\infty}\Big[\mu\rho(\imath-1)^4+[\rho\mu(1+\sigma|b|)+\mu+\rho](\imath-1)^3+[1+\sigma|b|(\rho\mu+\rho+\mu)+\rho+\mu](\imath-1)^2\\
&+[1+\sigma|b|(\rho+\mu+1)](\imath-1)+\sigma|b|]\frac{\gamma^{\imath-1}(\imath-1)^{\kappa}}{(\imath-1)!}\\
&\leq e^{-\gamma}\left[\mu\rho\sum_{\imath=2}^{\infty}(\imath-1)^4\frac{\gamma^{\imath-1}(\imath-1)^{\kappa}}{(\imath-1)!}+[\rho\mu(1+\sigma|b|)+\mu+\rho]\sum_{\imath=2}^{\infty}(\imath-1)^3\frac{\gamma^{\imath-1}(\imath-1)^{\kappa}}{(\imath-1)!}\right.\\
&+[1+\sigma|b|(\rho\mu+\rho+\mu)+\rho+\mu]\sum_{\imath=2}^{\infty}(\imath-1)^2\frac{\gamma^{\imath-1}(\imath-1)^{\kappa}}{(\imath-1)!}\\
&\left.+[1+\sigma|b|(\rho+\mu+1)]\sum_{\imath=2}^{\infty}(\imath-1)\frac{\gamma^{\imath-1}(\imath-1)^{\kappa}}{(\imath-1)!}+\sigma|b|\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^{\kappa}}{(\imath-1)!}\right]
\end{aligned}$$

$$\begin{aligned}
&\leq e^{-\gamma}\left[\mu\rho\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^{\kappa+4}}{(\imath-1)!}+[\rho\mu(1+\sigma|b|)+\mu+\rho]\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^{\kappa+3}}{(\imath-1)!}\right.\\
&+[1+\sigma|b|(\rho\mu+\rho+\mu)+\rho+\mu]\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^{\kappa+2}}{(\imath-1)!}\\
&\left.+[1+\sigma|b|(\rho+\mu+1)]\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^{\kappa+1}}{(\imath-1)!}+\sigma|b|\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^{\kappa}}{(\imath-1)!}\right]\\
&\leq e^{-\gamma}\left[\mu\rho\sum_{\imath=1}^{\infty}\frac{\gamma^{\imath}\imath^{\kappa+4}}{\imath!}+[\rho\mu(1+\sigma|b|)+\mu+\rho]\sum_{\imath=1}^{\infty}\frac{\gamma^{\imath}\imath^{\kappa+3}}{\imath!}\right.\\
&+[1+\sigma|b|(\rho\mu+\rho+\mu)+\rho+\mu]\sum_{\imath=1}^{\infty}\frac{\gamma^{\imath}\imath^{\kappa+2}}{\imath!}\\
&\left.+[1+\sigma|b|(\rho+\mu+1)]\sum_{\imath=1}^{\infty}\frac{\gamma^{\imath}\imath^{\kappa+1}}{\imath!}+\sigma|b|\sum_{\imath=1}^{\infty}\frac{\gamma^{\imath}\imath^{\kappa}}{\imath!}\right]
\end{aligned}$$

$$\leq\begin{cases}\mu\rho\chi'_{\kappa+4}+[\rho\mu(1+\sigma|b|)+\mu+\rho]\chi'_{\kappa+3}+[1+\sigma|b|(\rho\mu+\rho+\mu)+\rho+\mu]\chi'_{\kappa+2}\\ \quad+[1+\sigma|b|(\rho+\mu+1)]\chi'_{\kappa+1}+\sigma|b|\chi'_{\kappa}\leq\sigma|b| \qquad (\kappa\geq 1),\\[2ex] e^{\gamma}[\mu\rho\gamma^4+[\rho\mu(7+\sigma|b|)+\mu+\rho]\gamma^3+[\rho\mu(10+4\sigma|b|)+(\rho+\mu)(4+\sigma|b|)+1]\gamma^2\\ \quad+[2\rho\mu(1+\sigma|b|)+2(\mu+\rho)(1+\sigma|b|)+2+\sigma|b|]\gamma]\leq\sigma|b| \qquad (\kappa=0).\end{cases}$$

□

Corollary 12. *Let $\mu=0$ in Theorem 4, then $\mathcal{P}_{\kappa}(\gamma;z)$ maps $f(z)\in\mathcal{S}^*$ to $\mathfrak{TB}_b(\rho,\sigma)$ if and only if*

$$\begin{cases}\rho\chi'_{\kappa+3}+[1+\rho(1+\sigma|b|)]\chi'_{\kappa+2}+[1+\sigma|b|(\rho+1)]\chi'_{\kappa+1}+\sigma|b|\chi'_{\kappa}\leq\sigma|b| & (\kappa\geq 1),\\ e^{\gamma}\left[\rho\gamma^3+[\rho(4+\sigma|b|)+1]\gamma^2+[2\rho(1+\sigma|b|)+2+\sigma|b|]\gamma\right]\leq\sigma|b| & (\kappa=0).\end{cases}$$

Corollary 13. *Let $\rho = 0$ in Theorem 4, then $\mathcal{Q}_\kappa(\mu,\gamma;z)$ maps $f(z) \in \mathcal{S}^*$ to $\mathcal{TS}_b(\sigma)$ if and only if*

$$\begin{cases} \mu\chi'_{\kappa+3} + [1+\mu\sigma|b|+\mu]\chi'_{\kappa+2} + [1+\sigma|b|(\mu+1)]\chi'_{\kappa+1} + \sigma|b|\chi'_\kappa \leq \sigma|b| & (\kappa \geq 1), \\ e^\gamma\left[\mu\gamma^3 + [\mu(4+\sigma|b|)+1]\gamma^2 + [2\mu(1+\sigma|b|)+2+\sigma|b|]\gamma\right] \leq \sigma|b| & (\kappa = 0). \end{cases}$$

Corollary 14. *Let $\rho = 0$ and $\mu = 0$ in Theorem 4, then $\mathcal{P}_\kappa(\gamma;z)$ maps $f(z) \in \mathcal{S}^*$ to $\mathcal{TS}_b(\sigma)$ if and only if*

$$\begin{cases} \chi'_{\kappa+2} + [1+\sigma|b|]\chi'_{\kappa+1} + \sigma|b|\chi'_\kappa \leq \sigma|b| & (\kappa \geq 1), \\ e^\gamma[\gamma^2 + [2+\sigma|b|]\gamma] \leq \sigma|b| & (\kappa = 0). \end{cases}$$

Corollary 15. *Let $\rho = 1$ in Theorem 4, then $\mathcal{Q}_\kappa(\mu,\gamma;z)$ maps $f(z) \in \mathcal{S}^*$ to $\mathcal{TC}_b(\sigma)$ if and only if*

$$\begin{cases} \mu\chi'_{\kappa+4} + [\mu(2+\sigma|b|)+1]\chi'_{\kappa+3} + [2+\sigma|b|(2\mu+1)+\mu]\chi'_{\kappa+2} \\ \qquad + [1+\sigma|b|(2+\mu)]\chi'_{\kappa+1} + \sigma|b|\chi'_\kappa \leq \sigma|b| \qquad (\kappa \geq 1), \\ e^\gamma[\mu\gamma^4 + [\mu(8+\sigma|b|)+1]\gamma^3 + [\mu(10+4\sigma|b|)+(1+\mu)(4+\sigma|b|)+1]\gamma^2 \\ \qquad + [2\mu(1+\sigma|b|)+2(1+\mu)(1+\sigma|b|)+2+\sigma|b|]\gamma] \leq \sigma|b| \qquad (\kappa = 0). \end{cases}$$

Corollary 16. *Let $\rho = 1$ and $\mu = 0$ in Theorem 4, then $\mathcal{P}_\kappa(\gamma;z)$ maps $f(z) \in \mathcal{S}^*$ to $\mathcal{TC}_b(\sigma)$ if and only if*

$$\begin{cases} \chi'_{\kappa+3} + [2+\sigma|b|]\chi'_{\kappa+2} + [1+2\sigma|b|]\chi'_{\kappa+1} + \sigma|b|\chi'_\kappa \leq \sigma|b| & (\kappa \geq 1), \\ e^\gamma[\gamma^3 + [5+\sigma|b|]\gamma^2 + [4+3\sigma|b|]\gamma] \leq \sigma|b| & (\kappa = 0). \end{cases}$$

Theorem 5. *The function $\mathcal{Q}_\kappa(\mu,\gamma;z)$ maps $f(z) \in \mathcal{S}^*$ to $\mathfrak{TR}_b(\rho,\sigma)$ if and only if*

$$\begin{cases} \mu\rho\chi'_{\kappa+4} + [\mu+\rho(1+2\mu)]\chi'_{\kappa+3} + [1+2(\rho+\mu)+\rho\mu]\chi'_{\kappa+2} \\ \qquad + [\rho+\mu+2]\chi'_{\kappa+1} + \chi'_\kappa \leq \sigma|b| \qquad (\kappa \geq 1), \\ \mu\rho\gamma^4 + [\mu+\rho(1+8\mu)]\gamma^3 + [1+5(\rho+\mu)+14\rho\mu]\gamma^2 \\ \qquad + [4(\mu\rho+\rho+\mu)+3]\gamma + (1-e^{-\gamma}) \leq \sigma|b| \qquad (\kappa = 0). \end{cases} \tag{16}$$

Proof. From Lemma 2 and (7), we have to prove that

$$\sum_{\imath=2}^{\infty} \imath[\rho(\imath-1)+1][1+\mu(\imath-1)]\frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!}|a_\imath|e^{-\gamma} \leq \sigma|b|$$

Since $f(z) \in \mathcal{S}^*$, then $|a_\imath| \leq \imath$. So

$$\begin{aligned} &\sum_{\imath=2}^{\infty} \imath[\rho(\imath-1)+1][1+\mu(\imath-1)]\frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!}|a_\imath|e^{-\gamma} \\ &\leq \sum_{\imath=2}^{\infty} \imath^2[\rho(\imath-1)+1][1+\mu(\imath-1)]\frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!}e^{-\gamma} \\ &\leq e^{-\gamma}\sum_{\imath=2}^{\infty}\Big[\mu\rho(\imath-1)^4 + [\mu+\rho(1+2\mu)](\imath-1)^3 + [1+2(\rho+\mu)+\rho\mu](\imath-1)^2 \\ &+[\rho+\mu+2](\imath-1)+1\Big]\frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!} \end{aligned}$$

$$\leq e^{-\gamma}\Bigg[\mu\rho\sum_{\imath=2}^{\infty}(\imath-1)^4\frac{\gamma^{\imath-1}(\imath-1)^{\kappa}}{(\imath-1)!}+[\mu+\rho(1+2\mu)]\sum_{\imath=2}^{\infty}(\imath-1)^3\frac{\gamma^{\imath-1}(\imath-1)^{\kappa}}{(\imath-1)!}$$
$$+[1+2(\rho+\mu)+\rho\mu]\sum_{\imath=2}^{\infty}(\imath-1)^2\frac{\gamma^{\imath-1}(\imath-1)^{\kappa}}{(\imath-1)!}+[\rho+\mu+2]\sum_{\imath=2}^{\infty}(\imath-1)\frac{\gamma^{\imath-1}(\imath-1)^{\kappa}}{(\imath-1)!}$$
$$+\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^{\kappa}}{(\imath-1)!}\Bigg]$$
$$\leq e^{-\gamma}\Bigg[\mu\rho\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^{\kappa+4}}{(\imath-1)!}+[\mu+\rho(1+2\mu)]\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^{\kappa+3}}{(\imath-1)!}$$
$$+[1+2(\rho+\mu)+\rho\mu]\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^{\kappa+2}}{(\imath-1)!}+[\rho+\mu+2]\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^{\kappa+1}}{(\imath-1)!}+\sum_{\imath=2}^{\infty}\frac{\gamma^{\imath-1}(\imath-1)^{\kappa}}{(\imath-1)!}\Bigg]$$
$$\leq e^{-\gamma}\Bigg[\mu\rho\sum_{\imath=1}^{\infty}\frac{\gamma^{\imath}\imath^{\kappa+4}}{\imath!}+[\mu+\rho(1+2\mu)]\sum_{\imath=1}^{\infty}\frac{\gamma^{\imath}\imath^{\kappa+3}}{\imath!}$$
$$+[1+2(\rho+\mu)+\rho\mu]\sum_{\imath=1}^{\infty}\frac{\gamma^{\imath}\imath^{\kappa+2}}{\imath!}+[\rho+\mu+2]\sum_{\imath=1}^{\infty}\frac{\gamma^{\imath}\imath^{\kappa+1}}{\imath!}+\sum_{\imath=1}^{\infty}\frac{\gamma^{\imath}\imath^{\kappa}}{\imath!}\Bigg]$$
$$\leq\begin{cases}\mu\rho\chi'_{\kappa+4}+[\mu+\rho(1+2\mu)]\chi'_{\kappa+3}+[1+2(\rho+\mu)+\rho\mu]\chi'_{\kappa+2}\\ \quad+[\rho+\mu+2]\chi'_{\kappa+1}+\chi'_{\kappa}\leq\sigma|b| \qquad (\kappa\geq 1),\\ \\ \mu\rho\gamma^4+[\mu+\rho(1+8\mu)]\gamma^3+[1+5(\rho+\mu)+14\rho\mu]\gamma^2\\ \quad+[4(\mu\rho+\rho+\mu)+3]\gamma+(1-e^{-\gamma})\leq\sigma|b| \qquad (\kappa=0).\end{cases}$$
□

Corollary 17. *Let $\mu=0$ in Theorem 5, then $\mathcal{P}_\kappa(\gamma;z)$ maps $f(z)\in\mathcal{S}^*$ to $\mathfrak{TK}_b(\rho,\sigma)$ if and only if*

$$\begin{cases}\rho\chi'_{\kappa+3}+[1+2\rho]\chi'_{\kappa+2}+[\rho+2]\chi'_{\kappa+1}+\chi'_{\kappa}\leq\sigma|b| & (\kappa\geq 1),\\ \\ \rho\gamma^3+[1+5\rho]\gamma^2+[4\rho+3]\gamma+(1-e^{-\gamma})\leq\sigma|b| & (\kappa=0).\end{cases}$$

Corollary 18. *Let $\rho=0$ in Theorem 5, then $\mathcal{Q}_\kappa(\mu,\gamma;z)$ maps $f(z)\in\mathcal{S}^*$ to $\mathcal{TK}_b(\sigma)$ if and only if*

$$\begin{cases}\mu\chi'_{\kappa+3}+(1+2\mu)\chi'_{\kappa+2}+[\mu+2]\chi'_{\kappa+1}+\chi'_{\kappa}\leq\sigma|b| & (\kappa\geq 1),\\ \\ \mu\gamma^3+[1+5\mu]\gamma^2+[4\mu+3]\gamma+(1-e^{-\gamma})\leq\sigma|b| & (\kappa=0).\end{cases}$$

Corollary 19. *Let $\rho=0$ and $\mu=0$ in Theorem 5, then $\mathcal{P}_\kappa(\gamma;z)$ maps $f(z)\in\mathcal{S}^*$ to $\mathcal{TK}_b(\sigma)$ if and only if*

$$\begin{cases}\chi'_{\kappa+2}+2\chi'_{\kappa+1}+\chi'_{\kappa}\leq\sigma|b| & (\kappa\geq 1),\\ \\ \gamma^2+3\gamma+(1-e^{-\gamma})\leq\sigma|b| & (\kappa=0).\end{cases}$$

Theorem 6. *The inequality (11) satisfies if and only if*
(i) the function $Q_\kappa(\mu,\gamma;z)$ maps $f(z)\in C$ to $TK_b(\rho,\sigma)$;
(ii) the function $G_\kappa(\mu,\gamma;z)$ maps $f(z)\in S^*$ to $TK_b(\rho,\sigma)$.

Proof. (i) From Lemma 2 and (7), we have to prove that

$$\sum_{\imath=2}^{\infty}\imath[\rho(\imath-1)+1][1+\mu(\imath-1)]\frac{\gamma^{\imath-1}(\imath-1)^{\kappa}}{(\imath-1)!}|a_\imath|e^{-\gamma}\leq\sigma|b|.$$

Since $f(z) \in \mathcal{C}$, then $|a_\imath| \leq 1$. The proof of Theorem 6 is similar to Theorem 2, so we omit it.
(ii) The proof is similar to Theorem 2, using $|a_\imath| \leq \imath$ and (8), so we omit it. □

Corollary 20. *Let* $\mu = 0$ *in Theorem 6, then*
(i) the function $P_\kappa(\gamma; z)$ *maps* $f(z) \in C$ *to* $TK_b(\rho, \sigma)$*;*
(ii) the function $R_\kappa(\gamma; z)$ *maps* $f(z) \in S^*$ *to* $TK_b(\rho, \sigma)$*,*
if the inequality (12) holds.

Corollary 21. *Let* $\rho = 0$ *in Theorem 6, then*
(i) the function $Q_\kappa(\mu, \gamma; z)$ *maps* $f(z) \in C$ *to* $TK_b(\sigma)$*;*
(ii) the function $G_\kappa(\mu, \gamma; z)$ *maps* $f(z) \in S^*$ *to* $TK_b(\sigma)$*.*
if the inequality (13) holds.

Corollary 22. *Let* $\rho = 0$ *and* $\mu = 0$ *in Theorem 6, then*
(i) the function $P_\kappa(\gamma; z)$ *maps* $f(z) \in C$ *to* $TK_b(\sigma)$*;*
(ii) the function $R_\kappa(\gamma; z)$ *maps* $f(z) \in S^*$ *to* $TK_b(\sigma)$*.*
if the inequality (14) holds.

Theorem 7. *The function* $\mathcal{G}_\kappa(\mu, \gamma; z)$ *maps* $f(z) \in \mathcal{C}$ *to* $\mathfrak{TK}_b(\rho, \sigma)$ *if and only if*

$$\begin{cases} \mu\rho\chi'_{\kappa+2} + [\mu+\rho]\chi'_{\kappa+1} + \chi'_\kappa \leq \sigma|b| & (\kappa \geq 1), \\ \mu\rho\gamma^2 + [\rho\mu + \mu + \rho]\gamma + (1 - e^{-\gamma}) \leq \sigma|b| & (\kappa = 0). \end{cases} \tag{17}$$

Proof. From Lemma 2 and (8), we have to prove that

$$\sum_{\imath=2}^{\infty} \imath[\rho(\imath-1)+1][1+\mu(\imath-1)] \frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!} \frac{|a_\imath|}{\imath} \leq \sigma|b|.e^{-\gamma}$$

Since $f(z) \in \mathcal{C}$, then $|a_\imath| \leq 1$. So

$$\begin{aligned}
&\sum_{\imath=2}^{\infty}[\rho(\imath-1)+1][1+\mu(\imath-1)] \frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!} e^{-\gamma} \\
&\leq e^{-\gamma} \sum_{\imath=2}^{\infty} \left[\mu\rho(\imath-1)^2 + [\mu+\rho](\imath-1) + 1\right] \frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!} \\
&\leq e^{-\gamma} \left[\mu\rho \sum_{\imath=2}^{\infty} (\imath-1)^2 \frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!} + [\mu+\rho] \sum_{\imath=2}^{\infty} (\imath-1) \frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!} + \sum_{\imath=2}^{\infty} \frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!}\right] \\
&\leq e^{-\gamma} \left[\mu\rho \sum_{\imath=2}^{\infty} \frac{\gamma^{\imath-1}(\imath-1)^{\kappa+2}}{(\imath-1)!} + [\mu+\rho] \sum_{\imath=2}^{\infty} \frac{\gamma^{\imath-1}(\imath-1)^{\kappa+1}}{(\imath-1)!} + \sum_{\imath=2}^{\infty} \frac{\gamma^{\imath-1}(\imath-1)^\kappa}{(\imath-1)!}\right] \\
&\leq e^{-\gamma} \left[\mu\rho \sum_{\imath=1}^{\infty} \frac{\gamma^\imath \imath^{\kappa+2}}{\imath!} + [\mu+\rho] \sum_{\imath=1}^{\infty} \frac{\gamma^\imath \imath^{\kappa+1}}{\imath!} + \sum_{\imath=1}^{\infty} \frac{\gamma^\imath \imath^\kappa}{\imath!}\right] \\
&\qquad \leq \begin{cases} \mu\rho\chi'_{\kappa+2} + [\mu+\rho]\chi'_{\kappa+1} + \chi'_\kappa \leq \sigma|b| & (\kappa \geq 1), \\ \mu\rho\gamma^2 + [\mu\rho + \mu + \rho]\gamma + (1 - e^{-\gamma}) \leq \sigma|b| & (\kappa = 0). \end{cases}
\end{aligned}$$

□

Corollary 23. *Let* $\mu = 0$ *in Theorem 7, then* $\mathcal{R}_\kappa(\gamma; z)$ *maps* $f(z) \in \mathcal{C}$ *to* $\mathfrak{TK}_b(\rho, \sigma)$ *if and only if*

$$\begin{cases} \rho\chi'_{\kappa+1} + \chi'_\kappa \leq \sigma|b| & (\kappa \geq 1), \\ \rho\gamma + (1 - e^{-\gamma}) \leq \sigma|b| & (\kappa = 0). \end{cases} \tag{18}$$

Corollary 24. *Let $\rho = 0$ in Theorem 7, then $\mathcal{G}_\kappa(\mu, \gamma; z)$ maps $f(z) \in \mathcal{C}$ to $\mathcal{TK}_b(\sigma)$ if and only if*

$$\begin{cases} \mu\chi'_{\kappa+1} + \chi'_\kappa \leq \sigma|b| & (\kappa \geq 1), \\ \mu\gamma + (1 - e^{-\gamma}) \leq \sigma|b| & (\kappa = 0). \end{cases}$$

Corollary 25. *Let $\rho = 0$ and $\mu = 0$ in Theorem 7, then $\mathcal{R}_\kappa(\gamma; z)$ maps $f(z) \in \mathcal{C}$ to $\mathcal{TK}_b(\sigma)$ if and only if*

$$\begin{cases} \chi'_\kappa \leq \sigma|b| & (\kappa \geq 1), \\ (1 - e^{-\gamma}) \leq \sigma|b| & (\kappa = 0). \end{cases}$$

Theorem 8. *The function $\mathfrak{L}_\kappa(\gamma; z) = \int_0^z \left[2 - \frac{\mathcal{T}_\kappa(\gamma, s)}{s}\right] ds$ is in $\mathfrak{TK}_b(\rho, \sigma)$ if and only if (18) holds.*

Proof. It is easy to see that

$$\mathfrak{L}_\kappa(\gamma; z) = z - \sum_{\imath=2}^{\infty} \frac{\gamma^{\imath-1}(\imath - 1)^\kappa}{\imath!} e^{-\gamma} z^\imath.$$

Using Lemma 2, we only need to show that

$$\sum_{\imath=2}^{\infty} \imath[\rho(\imath - 1) + 1] \frac{\gamma^{\imath-1}(\imath - 1)^\kappa}{\imath!} e^{-\gamma} \leq \sigma|b|.$$

The proof is similar to Theorem 7, so we omit it. □

3. Conclusions

Several applications of analytic functions have been studied by several authors. In our study, we apply some applications to investigate some conditions of a power series associated with Touchard polynomials belonging to classes of analytic functions of complex order b, such as $\mathfrak{TB}_b(\rho, \sigma)$ and $\mathfrak{TK}_b(\rho, \sigma)$. Next, we obtain some inclusion relations between the classes $\mathfrak{R}_\eta(D, E)$ and $\mathfrak{TK}_b(\rho, \sigma)$. Also, we investigate several mapping properties involving these subclasses. Further, we discuss the geometric properties of an integral operator related to the Touchard polynomial. Additionally, briefly mentioned are specific instances of our primary results. Also, several particular examples are presented. In the future, we can study Touchard polynomials with several subclasses of analytic functions.

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