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On the Padovan Codes and the Padovan Cubes

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Abstract: We present a new interconnection topology called the Padovan cube. Despite their asymmetric and relatively sparse interconnections, the Padovan cubes are shown to possess attractive recurrent structures. Since they can be embedded in a subgraph of the Boolean cube and can have a Fibonacci cube as a subgraph, and since they are also a supergraph of other structures, it is possible that the Padovan cubes can be useful in fault-tolerant computing. For a graph with n vertices, we characterize the Padovan cubes. We also include the number of edges, decompositions, and embeddings, as well as the diameter of the Padovan cubes.

Keywords: Padovan sequence; completeness; Padovan code; Padovan cube

MSC: 05C10; 05C85; 68R10; 94B25; 94C15

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1. Introduction

Many researchers are currently studying the Fibonacci cubes in the field of interconnection topology. The Fibonacci cubes were first introduced by Hsu [1], and many scholars have since studied them, as shown in [2–8]. We will now introduce a new interconnection topology called the Padovan cube.

The Padovan sequence is named after R. Padovan [9,10], and the Padovan sequence has been studied by S. Kritsana, A. Shannon [11–13], and G. Lee [14,15].

The Padovan sequence is the sequence of integers P_n defined by the initial values $P_1 = P_2 = P_3 = 1$ and the recurrence relation

$$P_{n+2} = P_n + P_{n-1}, \quad \text{for all } n \geq 2.$$

The first few numbers of the Padovan sequence are

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37 \cdots$$

Let $\{z_n\}$ be a sequence of positive integers arranged in nondecreasing order. We define $\{z_n\}$ to be *complete* if every positive integer N is the sum of some subsequence of $\{z_n\}$; that is,

$$N = \sum_{i=1}^{\infty} \alpha_i z_i \text{ where } \alpha_i \in \{0, 1\}. \tag{1}$$

In [16], we can obtain the following about the completeness of a sequence: If $z_1 = 1$ and all $n \geq 1$, $z_{n+1} \leq 2z_n$, then the sequence $\{z_n\}$ is complete.

In [17], the author studied the completeness of a generalized Fibonacci sequence, while we can easily prove that the Padovan sequence is complete, some integers cannot be expressed in only one way by using the Padovan sequence $\{P_n\}$. For example, for the integer 8 and Padovan numbers 1, 2, 2, 3, 4, 5, and 7, we can obtain

8 = 1 + 7 = 2 + 2 + 4 = 1 + 3 + 4 = 3 + 5. Thus, to achieve uniqueness, we will deal with this problem with the subsequence of the Padovan sequence. We will also use this subsequence to define new Padovan codes and cubes that have never existed before, and then find some of their properties.

We represent an interconnection topology using a graph $G = (V, E)$, where V denotes the set of vertices and E denotes the set of edges, which are the communication links between vertices. The following terminology and notations will be used:

A *subgraph* of a graph $G = (V, E)$ is any graph $H = (V', E')$, such that $V' \subseteq V$ and $E' \subseteq E$, and we write $H \subseteq G$.

We write $G_1 \cong G_2$ if the two graphs are *isomorphic*.

A graph G_1 is said to be *directly embedded* in G_2, as denoted by $G_1 \sqsubseteq G_2$, if and only if there is a subgraph G_3 of G_2, such that $G_1 \cong G_3$.

We write $G_1 \cup G_2$ to denote the graph $(V_1 \cup V_2, E_1 \cup E_2)$ and $G_1 \cap G_2$ to denote $(V_1 \cap V_2, E_1 \cap E_2)$, and $\cup_{i=1}^{m} G_i = G_1 \cup G_2 \cup \cdots \cup G_m$.

If $G_2 \cap G_3 = (\emptyset, \emptyset)$, i.e., if they are disjoint, then we write $G_1 = G_2 \uplus G_3$ instead of writing $G_2 \cup G_3$ to emphasize that G_1 consists of two disjointed subgraphs. Furthermore, for convenience, we write $\uplus_{i=1}^{m} G_i = m \cdot G$ if the graphs are all isomorphic, i.e., $G_i \cong G$ for all $1 \leq i \leq m$.

The rest of this paper is organized as follows. In Section 2, we define the Padovan codes based on a subsequence of the Padovan sequence. In Section 3, we define the Padovan cubes based on the Padovan codes. In Section 4, we provide some of the topological properties of the Padovan cubes.

2. The Padovan Codes

In this section, we define the Padovan codes by using a subsequence $\{a_n\}$ of the Padovan sequence $\{P_n\}$. The odd-Padovan sequence $\{a_n\}$ is the sequence of integers defined by $a_n = P_{2n+1}$ for all $n \geq 1$. The first few numbers of the odd-Padovan sequence are:

$$1, 2, 4, 7, 12, 21, 37, 65, 114, 200, 351, 616 \cdots$$

The following lemma shows that $\{a_n\}$ has a recurrence relation.

Lemma 1. *Let $\{a_n\}$ be the odd-Padovan sequence. For $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, $a_4 = 7$, and $n \geq 5$, we have $a_n = a_{n-1} + a_{n-2} + a_{n-4}$.*

Proof. Since $a_n = P_{2n+1} = P_{2n-1} + P_{2n-2}$, $a_{n-1} = P_{2n-1}$, and $P_{2n-2} = P_{2n-4} + P_{2n-5}$, we have

$$\begin{aligned} a_n &= P_{2n-1} + P_{2n-2} \\ &= a_{n-1} + (P_{2n-4} + P_{2n-5}) \\ &= a_{n-1} + (P_{2n-6} + P_{2n-7}) + P_{2n-5} \\ &= a_{n-1} + (P_{2n-5} + P_{2n-6}) + P_{2n-7} \\ &= a_{n-1} + P_{2n-3} + P_{2n-7} \\ &= a_{n-1} + a_{n-2} + a_{n-4}. \end{aligned}$$

□

In [18], we can obtain a recurrence relation of $\{a_n\}$: $a_n = 2a_{n-1} - a_{n-2} + a_{n-3}$. To construct the Padovan codes and the Padovan cubes using the odd-Padovan sequence $\{a_n\}$, we must first know if the odd-Padovan sequence $\{a_n\}$ is complete.

Lemma 2. *The odd-Padovan sequence $\{a_n\}$ is complete.*

Proof. Since $a_{n+1} = 2a_n - a_{n-1} + a_{n-2}$,

$$2a_n - a_{n+1} = a_{n-1} - a_{n-2} \geq 0.$$

From the completeness of this sequence, we can see that the odd-Padovan sequence is complete. □

From Lemma 2, we know that every positive integer can be represented as the sum of one or more distinct odd-Padovan numbers. We will now use the odd-Padovan sequence $\{a_n\}$ to represent any positive integer and introduce Padovan coding from these representations. However, there are several ways that an integer can be represented using the odd-Padovan sequence. For example, the odd-Padovan representation of an integer 13 can be obtained as 13 = 12 + 1 or 13 = 7 + 4 + 2. However, when we later construct the Padovan codes and cubes, we try to use the binary digit 1 in a Padovan code as few times as possible. In other words, when expressing a positive integer as the odd-Padovan representation, it is convenient to reduce the number of terms that have been added, if possible. We must have the following Padovan Coding Algorithm to construct the Padovan codes and cubes.

For example, from Algorithm 1, we must choose only 13 = 12 + 1, not 13 = 7 + 4 + 2, as the odd-Padovan representation of 13. The odd-Padovan representation of a number N can be obtained by using the following approach. First, we find the greatest odd-Padovan number a_k that is less than or equal to N, assign a '1' to the bit that corresponds to a_k, then proceed recursively for $N - a_k$. The unassigned bits are 0s. For example, the odd-Padovan representation of an integer 13 can be obtained as 13 = 12 + 1 or 13 = 7 + 4 + 2. However, by the Padovan coding algorithm, the greatest odd-Padovan number available to obtain 13 is $a_5 = 12$; hence, $13 = 12 + 1 = 1 \cdot a_5 + 0 \cdot a_4 + 0 \cdot a_3 + 0 \cdot a_2 + 1 \cdot a_1$. In particular, the three numbers $7 = a_4, 4 = a_3, 2 = a_2$ are three consecutive terms of the odd-Padovan sequence. Thus, we have the following lemma.

Algorithm 1 Padovan Coding Algorithm.

An arbitrary positive integer N is represented in the following way using an odd-Padovan sequence.

(1) Select a_{k_1}, the largest odd-Padovan number that does not exceed N (i.e., $1 \leq a_{k_1} \leq N$).
(2) Select a_{k_2}, the largest odd-Padovan number that does not exceed $N - a_{k_1}$ (i.e., $1 \leq a_{k_2} \leq N - a_{k_1}$).
(3) Select a_{k_3}, the largest odd-Padovan number that does not exceed $N - \sum_{j=1}^{2} a_{k_j}$ (i.e., $1 \leq a_{k_3} \leq N - \sum_{j=1}^{2} a_{k_j}$).
(4) Obtain $a_{k_l} (4 \leq l \leq m)$, the odd-Padovan number by repeating the above process (i.e., $1 \leq a_{k_l} \leq N - \sum_{j=1}^{l-1} a_{k_j}$).
(5) The iterative process is stopped, N being $\sum_{j=1}^{m} a_{k_j}$ (i.e., $N = \sum_{j=1}^{m} a_{k_j}$).

Lemma 3. *Every positive integer N can be represented uniquely as the sum of one or more distinct odd-Padovan numbers such that the sum does not include any three consecutive odd-Padovan numbers. That is, if N is any positive integer, then there exists the following unique representation of N:*

$$N = \sum_{i=1}^{\infty} \alpha_i a_i,$$

where each α_i is a binary digit and, for $i \geq 1$, $\alpha_i \alpha_{i+1} \alpha_{i+2} = 0$.

Proof. From Lemma 2, we know that N can be represented as the sum of one or more distinct odd-Padovan numbers. We can then immediately obtain the uniqueness for the odd-Padovan representation of any integer from the Padovan coding algorithm.

Thus, we will prove that the sum does not include any consecutive three odd-Padovan numbers by induction. For n = 1, 2, 4, 7, 12, since the numbers are odd-Padovan numbers, it is clearly true. For n = 3, 5, 6, 8, 9, 10, 11, we have 3 = 2 + 1, 5 = 4 + 1, 6 = 4 + 2, 8 = 7 + 1, 9 = 7 + 2, 10 = 7 + 2 + 1, and 11 = 7 + 4.

Now, we consider $n \geq 13$. If n is an odd-Padovan number then we are finished. Otherwise there exists a j such that $a_j < n < a_{j+1}$. By induction, suppose that each $k < n$ has an odd-Padovan representation and consider $k = n - a_j$. Since $k < n$, k has an odd-Padovan representation. Meanwhile, since $a_{j+1} = a_j + a_{j-1} + a_{j-3}$ and $n < a_{j+1}$, $k = n - a_j < a_{j+1} - a_j = a_{j-1} + a_{j-3} < a_{j-1} + a_{j-2}$. Thus, the odd-Padovan representation of k does not contain $a_{j-1} + a_{j-2}$. As a result, n can be represented as the sum of a_j and the odd-Padovan representation of k.

Therefore, we have $\alpha_i \alpha_{i+1} \alpha_{i+2} = 0$ for all $i \geq 1$. $\square$

Now, we define the order n Padovan code by using the odd-Padovan sequence $\{a_n\}$.

Definition 1 (**Padovan Codes).** *Assume that N is an integer and* $0 \leq N \leq a_n - 1$, *where* $n \geq 1$. *We let* $(b_{n-1}, \cdots, b_2, b_1)_P$ *denote the order n Padovan code of N, where* b_i *is either 0 or 1 for* $1 \leq i \leq (n-1)$ *and*

$$N = \sum_{i=1}^{n-1} b_i \cdot a_i.$$

Let C_n denote the set of all order n Padovan codes. From now on, if n is implicit, we simply represent the Padovan code $(b_{n-1}, \cdots, b_2, b_1)_P$ as $b_{n-1}b_{n-2} \cdots b_2 b_1$.

Example 1. *Here, we present some Padovan coding sets.*

Since $a_1 = 1$ *and* $0 \leq N \leq 1 - 1 = 0$, $C_1 = \{\lambda\}$, *where* λ *denotes the null string.*
Since $a_2 = 2$ *and* $0 \leq N \leq 2 - 1 = 1$, $C_2 = \{0, 1\}$.
Since $a_3 = 4$ *and* $0 \leq N \leq 4 - 1 = 3$, $C_3 = \{00, 01, 10, 11\}$.
Since $a_4 = 7$ *and* $0 \leq N \leq 7 - 1 = 6$, $C_4 = \{000, 001, 010, 011, 100, 101, 110\}$.
Since $a_5 = 12$ *and* $0 \leq N \leq 12 - 1 = 11$,

$$C_5 = \{0000, 0001, 0010, 0011, 0100, 0101, 0110, 1000, 1001, 1010, 1011, 1100\}.$$

Since $a_6 = 21$ *and* $0 \leq N \leq 21 - 1 = 20$,

$$\begin{aligned} C_6 = \ & \{00000, 00001, 00010, 00011, 00100, 00101, 00110, 01000, 01001, 01010, 01011, 01100, 10000, 10001, 10010, \\ & 10011, 10100, 10101, 10110, 11000, 11001\}. \end{aligned}$$

Since $a_7 = 37$ *and* $0 \leq N \leq 37 - 1 = 36$,

$$\begin{aligned} C_7 = \ & \{000000, 000001, 000010, 000011, 000100, 000101, 000110, 001000, 001001, 001010, 001011, 001100, 010000, \\ & 010001, 010010, 010011, 010100, 010101, 010110, 011000, 011001, 100000, 100001, 100010, 100011, 100100, \\ & 100101, 100110, 101000, 101001, 101010, 101011, 101100, 110000, 110001, 110010, 110011\}. \end{aligned}$$

From Lemma 1, the Padovan coding algorithm, and Lemma 3, we can find that a $(n-1)$ binary string α is an order-n Padovan code of some positive integer N if and only if α contains neither 111 nor 1101 as its substring.

Let $\alpha \cdot \beta$ denote the string obtained by concatenating strings α and β. Define $\lambda \cdot \alpha = \alpha \cdot \lambda = \alpha$, where λ denotes the null string. If S is a set of strings, then $\alpha \cdot S = \{\alpha \cdot \beta \,|\, \beta \in S\}$. For example, $0 \cdot C_2 = \{00, 01\}$ and $10 \cdot C_3 = \{1000, 1001, 1010, 1011\}$. Furthermore, for the null string λ, since $1100 \cdot \lambda = 1100$, we have $1100 \cdot C_1 = \{1100\}$.

From Example 1, we know that $C_6 = 0 \cdot C_5 \cup 10 \cdot C_4 \cup 1100 \cdot C_2$ and $C_7 = 0 \cdot C_6 \cup 10 \cdot C_5 \cup 1100 \cdot C_3$. This can be generalized in the following theorem.

Theorem 1. *Let* C_n *denote the set of order-n Padovan codes. For all* $n \geq 5$, *we have*

$$C_n = 0 \cdot C_{n-1} \cup 10 \cdot C_{n-2} \cup 1100 \cdot C_{n-4},$$

and $|C_n| = a_n$ *for all* $n \geq 1$.

Proof. For a non-negative integer N and the nth odd-Padovan number a_n, $0 \leq N = \sum_{i=1}^{n-1} \alpha_i \cdot a_i \leq a_n - 1$. If $\alpha \in 0 \cdot C_{n-1} \cup 10 \cdot C_{n-2} \cup 1100 \cdot C_{n-4}$, then, clearly, $\alpha \in C_n$.

Suppose that $\alpha \in C_n$. Then, a Padovan code α has $(n-1)$ binary digits. Thus, without a loss of generality, we may assume that $\alpha = 0 \cdot \beta$ or $\alpha = 1 \cdot \beta$, where β is a Padovan code with $(n-2)$ binary digits. If $\alpha = 0 \cdot \beta$, then $\beta \in C_{n-1}$ because α has $(n-1)$ binary digits and C_{n-1} is the set of order $(n-1)$ Padovan codes with $(n-2)$ binary digits. That is, $\alpha \in 0 \cdot C_{n-1}$. Now, we assume that $\alpha = 1 \cdot \beta$, where β is a Padovan code with $(n-2)$ binary digits. If $\beta = 0 \cdot \beta_1$ where β_1 is a Padovan code with $(n-3)$ binary digits, then $\alpha = 10 \cdot \beta_1 \in 10 \cdot C_{n-2}$. Thus, we assume that $\beta = 1 \cdot \beta_1$, i.e., $\alpha = 11 \cdot \beta_1$. If $\beta_1 = 1 \cdot \beta_2$, where β_2 is a Padovan code with $(n-4)$ binary digits, then $\alpha = 111 \cdot \beta_2$. From Lemma 3, this is impossible. Thus, $\beta_1 = 0 \cdot \beta_2$. Again, if $\beta_2 = 1 \cdot \beta_3$, where β_3 is a Padovan code with $(n-5)$ binary digits, then $\alpha = 1101 \cdot \beta_3$. From Lemma 1, since $a_n = a_{n-1} + a_{n-2} + a_{n-4}$, the integer N, which is made of the string of binary digits α, satisfies $N > a_n - 1$. However, this is impossible. Thus, we have $\beta_2 = 0 \cdot \beta_3$, where $\beta_3 \in C_{n-4}$., i.e., $\alpha = 1100 \cdot \beta_3 \in 1100 \cdot C_{n-4}$. Thus, we have $\alpha \in 0 \cdot C_{n-1} \cup 10 \cdot C_{n-2} \cup 1100 \cdot C_{n-4}$.

Therefore,

$$C_n = 0 \cdot C_{n-1} \cup 10 \cdot C_{n-2} \cup 1100 \cdot C_{n-4}$$

Moreover, by induction, we have

$$\begin{aligned} |C_n| &= |0 \cdot C_{n-1} \cup 10 \cdot C_{n-2} \cup 1100 \cdot C_{n-4}| \\ &= a_{n-1} + a_{n-2} + a_{n-4} = a_n. \end{aligned}$$

□

Let $I = (b_{n-1}, \cdots, b_2, b_1)$ and $J = (c_{n-1}, \cdots, c_2, c_1)$ be two binary numbers. The Hamming distance between I and J, denoted by $H(I, J)$, is the number of bits where the two binary numbers differ. For example, if $I = (1,1,0,1)$ and $J = (1,0,1,1)$, then $H(I, J) = 2$.

Corollary 1. *Let* $A(n-1, n-2)$ *be the number of codes for which the Hamming distance is 1 when comparing the codes in* $0 \cdot C_{n-1}$ *with those in* $10 \cdot C_{n-2}$*. Let* $A(n-1, n-4)$ *be the number of codes for which the Hamming distance is 1 when comparing the codes in* $0 \cdot C_{n-1}$ *with those in* $1100 \cdot C_{n-4}$*. Let* $A(n-2, n-4)$ *be the number of codes for which the Hamming distance is 1 when comparing the codes in* $10 \cdot C_{n-2}$ *with those in* $1100 \cdot C_{n-4}$*. Then, we have*

$$A(n-1, n-2) = a_{n-2}, \quad A(n-1, n-4) = a_{n-4}, \quad A(n-2, n-4) = a_{n-4}.$$

Proof. From Theorem 1, we know that $|C_n| = a_n$. For some $\alpha \in 0 \cdot C_{n-1}$ and $\beta \in 10 \cdot C_{n-2}$, if $H(\alpha, \beta) = 1$, then it can only be $\alpha = 00 \cdot \alpha_1$ and $\beta = 10 \cdot \alpha_1$ for $\alpha_1 \in C_{n-2}$. Since $|C_{n-2}| = a_{n-2}$, we have $A(n-1, n-2) = a_{n-2}$.

For some $\alpha \in 0 \cdot C_{n-1}$ and $\gamma \in 1100 \cdot C_{n-4}$, if $H(\alpha, \gamma) = 1$, then it can only be $\alpha = 0100 \cdot \alpha_2$ and $\gamma = 1100 \cdot \alpha_2$ for $\alpha_2 \in C_{n-4}$. Since $|C_{n-4}| = a_{n-4}$, we have $A(n-1, n-4) = a_{n-4}$.

For some $\beta \in 10 \cdot C_{n-2}$ and $\gamma \in 1100 \cdot C_{n-4}$, if $H(\beta, \gamma) = 1$, then it can only be $\beta = 1000 \cdot \beta_1$ and $\gamma = 1100 \cdot \beta_1$ for $\beta_1 \in C_{n-4}$. Since $|C_{n-4}| = a_{n-4}$, we have $A(n-2, n-4) = a_{n-4}$. □

3. Padovan Cubes

It is straightforward to define the Padovan cubes based on the Padovan codes.

Definition 2 (Padovan Cubes). *Let N denote an integer, where $1 \leq N \leq a_n$ for some n. Let I_P and J_P denote the Padovan codes of i and j, $0 \leq i, j \leq N-1$. The Padovan cube of order N is a graph $(V(N), E(N))$, where $V(N) = \{0, 1, 2, \cdots, N-1\}$ and $\{i, j\} \in E(N)$ if and only if $H(I_P, J_P) = 1$.*

Definition 3 (Padovan Cube of Order *n*). *The Padovan cube of order n, denoted by $\mathcal{P}_n$, is a Padovan cube with a_n nodes. Define $\mathcal{P}_0 = (\emptyset, \emptyset)$.*

Example 2. *Figure 1 shows examples of Padovan cubes of order N, where N = 1, 2, 4, 7, 12, 21, and 37, respectively. Note that since each bit in the Padovan code corresponds to an odd-Padovan number, two vertices, i and j, are adjacent only if $|i-j| = a_k$ for some k. For example, in the Padovan cube of size 12, the vertex $0 = (0000)_P$ has a link with each of the following vertices: $1 = (0001)_P$, $2 = (0010)_P$, $4 = (0100)_P$, and $7 = (1000)_P$, which all have a difference of 0 from an odd-Padovan number.*

Note that, for example, $\mathcal{P}_6$, contains three subgraphs that are isomorphic to $\mathcal{P}_5$, $\mathcal{P}_4$, and $\mathcal{P}_2$, respectively. There are $a_4 + a_2 + a_2 = 7 + 2 + 2 = 11$ edges connecting the three subgraphs.

Lemma 4. *Let $\mathcal{P}_n = (V_n, E_n)$ be the Padovan cube of order n for all $n \geq 5$. Let $S_1(n)$, $S_2(n)$, and $S_3(n)$ denote the subgraphs of $\mathcal{P}_n$ induced by the set of vertices in $V_{n-1} = \{0, 1, 2, \cdots, a_{n-1} - 1\}$, $V_{n-2} = \{a_{n-1}, a_{n-1}+1, \cdots, a_{n-1}+a_{n-2}-1\}$, and $V_{n-4} = \{a_{n-1}+a_{n-2}, a_{n-1}+a_{n-2}+1, \cdots, a_{n-1}+a_{n-2}+a_{n-4}-1\}$, respectively. Then, we have $S_1(n) \cong \mathcal{P}_{n-1}$, $S_2(n) \cong \mathcal{P}_{n-2}$, and $S_3(n) \cong \mathcal{P}_{n-4}$.*

Proof. If n = 5, 6, then, from Figure 1, it is easy to verify the lemma.

Recall that the set of order-n Padovan codes C_n denotes the set of labels of vertices in $\mathcal{P}_n$. Thus, the interconnections of $\mathcal{P}_n$ are based on the Hamming distance of these codes. From Theorem 1, we know that $C_n = 0 \cdot C_{n-1} \cup 10 \cdot C_{n-2} \cup 1100 \cdot C_{n-4}$. Note that $0 \cdot C_{n-1}$, $10 \cdot C_{n-2}$, and $1100 \cdot C_{n-4}$ are exactly the set of labels of the vertices in $S_1(n)$, $S_2(n)$, and $S_3(n)$, respectively. Moreover, $\{i, j\} \in \mathcal{P}_{n-1}$ if and only if $\{i, j\} \in S_1(n)$ because both i and j have the same prefix of '0' in the order n codes, which does not affect their Hamming distance. In other words, $S_1(n) \cong \mathcal{P}_{n-1}$. Similarly, $S_2(n) \cong \mathcal{P}_{n-2}$ and $S_3(n) \cong \mathcal{P}_{n-4}$. $\square$

Let $\mathcal{P}_n = (V_n, E_n)$ denote the Padovan cube of order n, $n \geq 5$. The edges set $Link_1(n)$ is defined as the edge $\{i, j\} \in Link_1(n)$ if and only if $i \in V_{n-1}$ and $j \in V_{n-2}$. The edges set $Link_2(n)$ is defined as the edge $\{i, j\} \in Link_2(n)$ if and only if $i \in V_{n-1}$ and $j \in V_{n-4}$. The edges set $Link_3(n)$ is defined as the edge $\{i, j\} \in Link_3(n)$ if and only if $i \in V_{n-2}$ and $j \in V_{n-4}$, where the vertices sets V_{n-1}, V_{n-2}, and V_{n-4} are the same as defined in Lemma 4.

For example, from Figure 1, it can be seen that for $\mathcal{P}_6 = (V_6, E_6)$, we have $V_5 = \{0, 1, 2, \cdots, 10, 11\}$, $V_4 = \{12, 13, \cdots, 17, 18\}$, and $V_2 = \{19, 20\}$. Furthermore, then $Link_1(6) = \{\{0, 12\}, \{1, 13\}, \{2, 14\}, \{3, 15\}, \{4, 16\}, \{5, 17\}, \{6, 18\}\}$, $Link_2(6) = \{\{7, 19\}, \{8, 20\}\}$, and $Link_3(6) = \{\{12, 19\}, \{13, 20\}\}$.

Lemma 5. *For the Padovan cube of order n, $n \geq 5$, $\mathcal{P}_n = (V_n, E_n)$, we have $|Link_1(n)| = a_{n-2}$, $|Link_2(n)| = a_{n-4}$, and $|Link_3(n)| = a_{n-4}$.*

Proof. Let $A(n-1, n-2)$, $A(n-1, n-4)$, and $A(n-2, n-4)$ be the same as defined in Corollary 1. Since $\{i, j\} \in E_n$ if and only if $H(I_P, J_P) = 1$, we have $|Link_1(n)| = A(n-1, n-2)$, $|Link_2(n)| = A(n-1, n-4)$, and $|Link_3(n)| = A(n-2, n-4)$. Therefore, from Corollary 1, the proof is completed. $\square$

We know that for the Padovan sequence $\{P_n\}$, $a_n = P_{2n+1} = \sum_{m=0}^{n-1} P_{2m} + 1$. Thus, we have $|Link_1(n)| = \sum_{m=0}^{n-3} P_{2m} + 1$ and $|Link_2(n)| = |Link_3(n)| = \sum_{m=0}^{n-5} P_{2m} + 1$.

We now characterize the Padovan cubes.

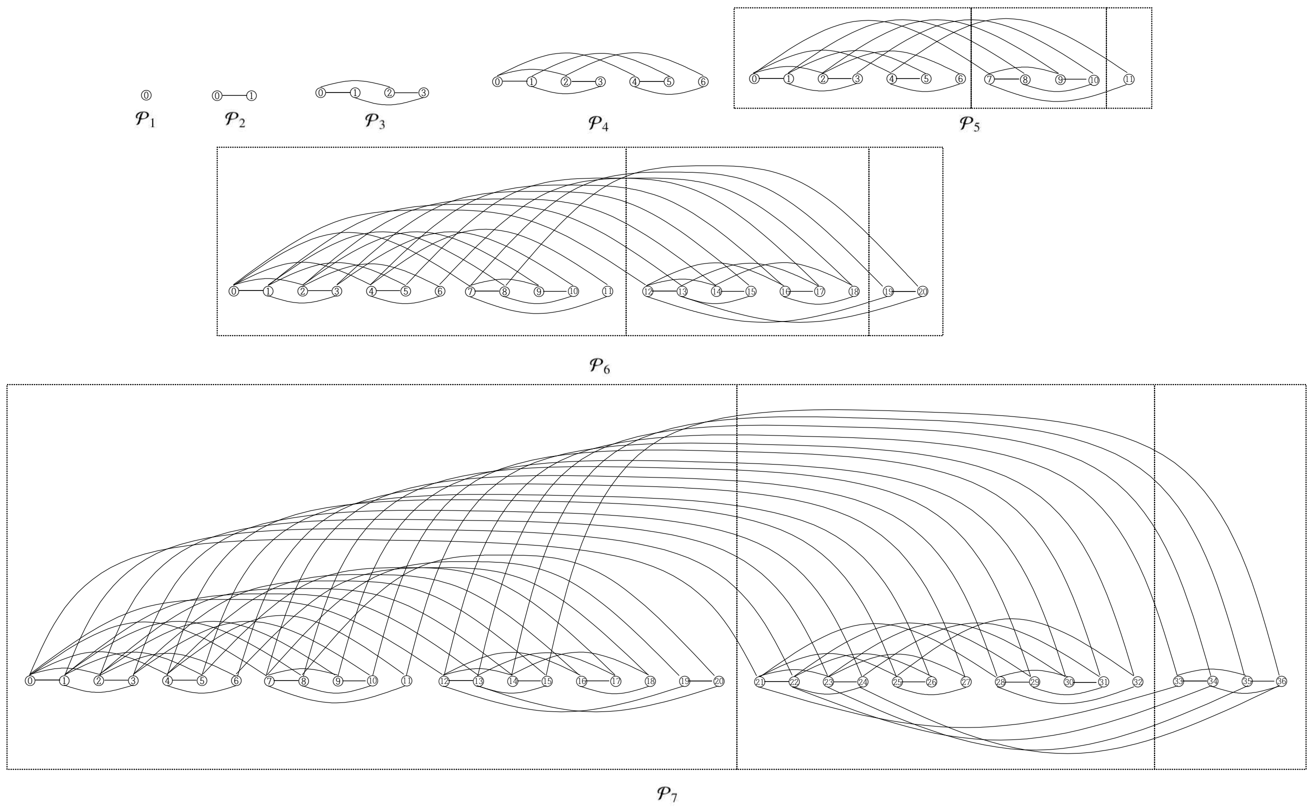

Figure 1. Padovan cubes from $\mathcal{P}_1$ to $\mathcal{P}_7$.

Theorem 2. *For $n \geq 5$, the Padovan cube $\mathcal{P}_n$ can be decomposed into $\mathcal{P}_{n-1}$, $\mathcal{P}_{n-2}$, and $\mathcal{P}_{n-4}$; the three subgraphs are pairwise disjointed and are connected exactly by the set of edges in $Link_1(n)$, $Link_2(n)$, and $Link_3(n)$.*

Proof. Let $S_1(n)$, $S_2(n)$, and $S_3(n)$ be the same as defined in Lemma 4. By Lemma 4, $S_1(n) \cong \mathcal{P}_{n-1}$, $S_2(n) \cong \mathcal{P}_{n-2}$, and $S_3(n) \cong \mathcal{P}_{n-4}$. Moreover, based on the definition of three induced subgraphs, we have $S_1(n) \cap S_2(n) = (\emptyset, \emptyset)$, $S_1(n) \cap S_3(n) = (\emptyset, \emptyset)$, and $S_2(n) \cap S_3(n) = (\emptyset, \emptyset)$. It should be clear that each edge $\{i, j\}$ in $Link_1(n)$ connects a vertex j in $S_2(n)$ to a vertex $i = j - a_{n-1}$ in $S_1(n)$. Furthermore, each edge $\{i, j\}$ in $Link_2(n)$ connects a vertex j in $S_3(n)$ to a vertex $i = j - a_{n-1}$ in $S_1(n)$. Moreover, each edge $\{i, j\}$ in $Link_3(n)$ connects a vertex j in $S_3(n)$ to a vertex $i = j - a_{n-2}$ in $S_2(n)$.

We will now show that no other edges exist between $S_1(n)$ and $S_2(n)$.

By Theorem 1, all vertices in $S_2(n)$ have a prefix of '10' in their labels, while all vertices in $S_1(n)$ have a prefix of '00' or '01'. If i is the vertex with the prefix of '01' and j is the vertex with the prefix of '10', then $H(I_P, J_P) \geq 2$. Thus, there is no such edge $\{i, j\}$ in $\mathcal{P}_n$ that exists between a vertex $i \in S_1(n)$ and a vertex $j \in S_2(n)$.The only possibility is to have the edges between $S_1(n)$ and $S_2(n)$, as given by $Link_1(n)$.

Similarly, for two vertices, i and j, in $\mathcal{P}_n$, these properties are obtained for $Link_2(n)$ and $Link_3(n)$, even if the vertices are the same as in the two cases $a_{n-1} \leq i, j \leq a_{n-2} + a_{n-1} - 1$ and $a_{n-2} + a_{n-1} \leq i, j \leq a_n - 1$, $n \geq 5$.

Therefore, the proof is completed. □

Let ϵ_n be the number of edges of the Padovan cube $\mathcal{P}_n$ for $n \geq 1$. From Figure 1, we know that $\epsilon_1 = 0$, $\epsilon_2 = 1$, $\epsilon_3 = 4$, $\epsilon_4 = 9$, $\epsilon_5 = 19$, $\epsilon_6 = 40$, and $\epsilon_7 = 83$.

Corollary 2. *If ϵ_n is the number of edges of the Padovan cube $\mathcal{P}_n$ for $n \geq 5$, then we have*

$$\epsilon_n = \epsilon_{n-1} + \epsilon_{n-2} + \epsilon_{n-4} + a_{n-2} + 2a_{n-4}.$$

Proof. From Theorem 2, we have

$$\epsilon_n = \epsilon_{n-1} + \epsilon_{n-2} + \epsilon_{n-4} + |Link_1(n)| + |Link_2(n)| + |Link_3(n)|.$$

From Lemma 5, since $|Link_1(n)| = a_{n-2}$, $|Link_2(n)| = a_{n-4}$, and $|Link_3(n)| = a_{n-4}$, the proof is completed. □

In particular, the generating function of $\{\epsilon_n\}$ is

$$\frac{x(1+x-x^2)}{(1+x)(1-2x+x^2-x^3)^2}.$$

4. Decompositions and Topological Properties of Padovan Cubes

Now, we define a sequence $\{r_n\}$ of positive integers by using the odd-Padovan sequence $\{a_n\}$. Let $\{r_n\}$ be defined as follows: $r_1 = a_1 = 1$, $r_2 = a_2 = 2$, $r_3 = a_1 + a_2 = 3$, $r_4 = a_2 + a_3 = 6$, and for $n \geq 5$, $r_n = r_{n-1} + r_{n-2} + r_{n-4}$. Then, we have

$$\{r_n\} := 1, 2, 3, 6, 10, 18, 31, 55, 96, 169, 296, \cdots.$$

Since $a_n = a_{n-1} + a_{n-2} + a_{n-4}$, we know that the sequence $\{r_n\}$ is defined in the same way as the odd-Padovan sequence $\{a_n\}$.

In Figure 1, we see that $\mathcal{P}_6$ contains three subgraphs: $\mathcal{P}_5$, $\mathcal{P}_4$, and $\mathcal{P}_2$. Moreover, $\mathcal{P}_5$ can be decomposed into three subgraphs, $\mathcal{P}_4$, $\mathcal{P}_3$, and $\mathcal{P}_1$; therefore, $\mathcal{P}_6$ contains two disjointed $\mathcal{P}_4$, one $\mathcal{P}_3$, one $\mathcal{P}_2$, and one $\mathcal{P}_1$. Using the introduced notations, we write $\mathcal{P}_6 \sqsupseteq 2\mathcal{P}_4 \uplus \mathcal{P}_3 \uplus \mathcal{P}_2 \uplus \mathcal{P}_1$ to denote that $\mathcal{P}_6$ contains two disjointed subgraphs that are isomorphic to $\mathcal{P}_4$, one subgraph that is isomorphic to $\mathcal{P}_3$, one subgraph that is isomorphic to $\mathcal{P}_2$, and one subgraph that is isomorphic to $\mathcal{P}_1$.

Theorem 3. *For $4 \leq k \leq n$ and $n \geq 8$, the Padovan cube $\mathcal{P}_n$ admits the following decomposition: for $n - k \geq 4$,*

$$\mathcal{P}_n \sqsupseteq r_k\mathcal{P}_{n-k} \uplus (r_{k-1} + r_{k-3})\mathcal{P}_{n-k-1} \uplus r_{k-2}\mathcal{P}_{n-k-2} \uplus r_{k-1}\mathcal{P}_{n-k-3}.$$

Proof. From Theorem 2, we know that $\mathcal{P}_n \sqsupseteq \mathcal{P}_{n-1} \uplus \mathcal{P}_{n-2} \uplus \mathcal{P}_{n-4}$, $\mathcal{P}_{n-1} \sqsupseteq \mathcal{P}_{n-2} \uplus \mathcal{P}_{n-3} \uplus \mathcal{P}_{n-5}$, $\mathcal{P}_{n-2} \sqsupseteq \mathcal{P}_{n-3} \uplus \mathcal{P}_{n-4} \uplus \mathcal{P}_{n-6}$, and $\mathcal{P}_{n-3} \sqsupseteq \mathcal{P}_{n-4} \uplus \mathcal{P}_{n-5} \uplus \mathcal{P}_{n-7}$. Since $r_1 = 1$, $r_2 = 2$, $r_3 = 3$, and $r_4 = 6$, we have, for $n \geq 8$,

$$\begin{aligned} \mathcal{P}_n &\sqsupseteq 6\mathcal{P}_{n-4} \uplus 4\mathcal{P}_{n-5} \uplus 2\mathcal{P}_{n-6} \uplus 3\mathcal{P}_{n-7} \\ &= r_4\mathcal{P}_{n-4} \uplus (r_3 + r_1)\mathcal{P}_{n-5} \uplus r_2\mathcal{P}_{n-6} \uplus r_3\mathcal{P}_{n-7}. \end{aligned}$$

We now assume that the statement holds true for $n \leq N$ and $k \leq K$, where $4 \leq K \leq N$. By induction, we consider the case $n = N+1$ and $k = K+1$. According to Theorem 2, $\mathcal{P}_{N+1}$ consists of one $\mathcal{P}_N$, one $\mathcal{P}_{N-1}$, and one $\mathcal{P}_{N-3}$. Based on the hypothesis, $\mathcal{P}_N$ is divided into r_K of $\mathcal{P}_{N-K}$, $(r_{K-1} + r_{K-3})$ of $\mathcal{P}_{N-K-1}$, r_{K-2} of $\mathcal{P}_{N-K-2}$, and r_{K-1} of $\mathcal{P}_{N-K-3}$, $\mathcal{P}_{N-1}$ is divided into r_{K-1} of $\mathcal{P}_{(N-1)-(K-1)}$, $(r_{(K-1)-1} + r_{(K-1)-3})$ of $\mathcal{P}_{(N-1)-(K-1)-1}$, $r_{(K-1)-2}$ of $\mathcal{P}_{(N-1)-(K-1)-2}$, and $r_{(K-1)-1}$ of $\mathcal{P}_{(N-1)-(K-1)-3}$, and $\mathcal{P}_{N-3}$ is divided into r_{K-3} of $\mathcal{P}_{(N-3)-(K-3)}$, $(r_{(K-3)-1} + r_{(K-3)-3})$ of $\mathcal{P}_{(N-3)-(K-3)-1}$, $r_{(K-3)-2}$ of $\mathcal{P}_{(N-3)-(K-3)-2}$, and $r_{(K-3)-1}$ of $\mathcal{P}_{(N-3)-(K-3)-3}$. Together, the total number of $\mathcal{P}_{(N+1)-(K+1)}$ is $r_K + r_{K-1} + r_{K-3} = r_{K+1}$, the total number of $\mathcal{P}_{(N+1)-(K+1)-1}$ is $(r_{K-1} + r_{K-2} + r_{K-4}) + (r_{K-3} + r_{K-4} + r_{K-6}) = r_K + r_{K-2}$, the total number of $\mathcal{P}_{(N+1)-(K+1)-2}$ is $r_{K-2} + r_{K-3} + r_{K-5} = r_{K-1}$, and the total number of $\mathcal{P}_{(N+1)-(K+1)-3}$ is $r_{K-1} + r_{K-2} + r_{K-4} = r_K$. That is,

$$\begin{aligned} \mathcal{P}_{N+1} \sqsupseteq\ & r_{K+1}\mathcal{P}_{(N+1)-(K+1)} \uplus (r_{(K+1)-1} + r_{(K+1)-3})\mathcal{P}_{(N+1)-(K+1)-1} \\ & \uplus r_{(K+1)-2}\mathcal{P}_{(N+1)-(K+1)-2} \uplus r_{(K+1)-1}\mathcal{P}_{(N+1)-(K+1)-3}. \end{aligned}$$

The proof is completed by an induction on $n = N + 1$ and $k = K + 1$. $\square$

For example, for $k = 10$, since $r_{10} = 169$, $r_9 = 96$, $r_8 = 55$, and $r_7 = 31$, we have

$$\mathcal{P}_n \sqsupseteq r_{10}\mathcal{P}_{n-10} \uplus (r_9 + r_7)\mathcal{P}_{n-11} \uplus r_8\mathcal{P}_{n-12} \uplus r_9\mathcal{P}_{n-13} = 169\mathcal{P}_{n-10} \uplus 127\mathcal{P}_{n-11} \uplus 55\mathcal{P}_{n-12} \uplus 96\mathcal{P}_{n-13}.$$

Thus, if $n = 14$, then we have

$$\mathcal{P}_{14} \sqsupseteq 169\mathcal{P}_4 \uplus 127\mathcal{P}_3 \uplus 55\mathcal{P}_2 \uplus 96\mathcal{P}_1.$$

That is, the Padovan cube $\mathcal{P}_{14}$ consists of 169 $\mathcal{P}_4$, 127 $\mathcal{P}_3$, 55 $\mathcal{P}_2$, and 96 $\mathcal{P}_1$.

Let $\mathcal{B}_n$ and Γ_n, respectively, denote the Boolean cube and the Fibonacci cube of order n. It is well known that the Boolean cube $\mathcal{B}_n$ can be decomposed into two isomorphic $\mathcal{B}_{n-1}$ plus a set $link_{\mathcal{B}}(n)$ of 2^{n-1} edges connecting all corresponding vertices in the two subgraphs, while the Fibonacci cube Γ_n can be decomposed into two subgraphs Γ_{n-1} and Γ_{n-2} plus a set $link_{\Gamma}(n)$ of F_{n-2} edges connecting all corresponding vertices in the two subgraphs, where F_{n-2} is the $(n-2)$nd Fibonacci number.

In [1], we have the following lemma.

Lemma 6 ([1])**.** *Let Γ_n denote the Fibonacci cube of order n. Assume that $n \geq 1$. Then,*

(1) $\Gamma_{2n} \sqsupseteq (\Gamma_1 \uplus \Gamma_3 \uplus \cdots \uplus \Gamma_{2n-1})$;
(2) $\Gamma_{2n+1} \sqsupseteq (\Gamma_2 \uplus \Gamma_4 \uplus \cdots \uplus \Gamma_{2n})$;
(3) $\Gamma_{n+2} \sqsupseteq (\Gamma_1 \uplus \Gamma_2 \uplus \Gamma_3 \uplus \cdots \uplus \Gamma_n)$.

Theorem 4. *Let Γ_n, $\mathcal{P}_n$, and $\mathcal{B}_n$ denote the Fibonacci cube, the Padovan cube, and the Boolean cube, respectively, of order n. Then,*

(1) $\Gamma_{n+1} \sqsubseteq \mathcal{P}_n \sqsubseteq \mathcal{B}_{n-1}$ *for* $n \geq 2$;
(2) $\mathcal{B}_n \sqsubseteq \mathcal{P}_{2n-1}$ *for* $n \geq 2$;
(3) $\mathcal{P}_n \sqsubseteq \Gamma_{2n-1}$ *for* $n \geq 3$.

Proof. (1) We will show that the set of vertices in $\mathcal{P}_n$ is a subset of $\mathcal{B}_{n-1}$. If this is true, then we can safely claim that $\mathcal{P}_n \sqsubseteq \mathcal{B}_{n-1}$, because the interconnections are based on the same Hamming distance. The key observation is as follows. With the Boolean cube $\mathcal{B}_n$, all 2^n combinations of 0s and 1s are permitted in an n-bit binary code. By contrast, from Lemma 3, there is no case wherein three 1s appear consecutively in the $(n-1)$-bit Padovan representation for $\mathcal{P}_n$. Thus, $\mathcal{P}_n \sqsubseteq \mathcal{B}_{n-1}$. Similarly, since no consecutive 1s appeared in the Fibonacci codes, we have $\Gamma_{n+1} \sqsubseteq \mathcal{P}_n$. Thus, $\Gamma_{n+1} \sqsubseteq \mathcal{P}_n \sqsubseteq \mathcal{B}_{n-1}$.

(2) The embedding of $\mathcal{B}_n$ in $\mathcal{P}_{2n-1}$ can be proved by induction. As the basis, $\mathcal{B}_2 \cong \mathcal{P}_3$, so $\mathcal{B}_2 \sqsubseteq \mathcal{P}_3$ (refer to Figure 1). Assume that the statement is true for $n \leq N$, where $N \geq 2$ denotes an integer. Now, consider $n = N + 1$.

By Theorem 2, $\mathcal{P}_{2(N+1)-1}$ can be decomposed into $\mathcal{P}_{2N}$, $\mathcal{P}_{2N-1}$, and $\mathcal{P}_{2N-3}$. Furthermore, there are $|Link_1(2N+1)| = a_{2N-1}$ edges connecting each vertex i in $\mathcal{P}_{2N}$ to its corresponding vertex $i + a_{2N-1}$ in $\mathcal{P}_{2N-1}$, $|Link_2(2N+1)| = a_{2N-3}$ edges connecting each vertex i in $\mathcal{P}_{2N}$ to its corresponding vertex $i + a_{2N-1} + a_{2N-2}$ in $\mathcal{P}_{2N-3}$, and $|Link_3(n)| = a_{2N-3}$ edges connecting each vertex $i + a_{2N-1}$ in $\mathcal{P}_{2N-1}$ to its corresponding vertex $i + a_{2N-1} + a_{2N-2}$ in $\mathcal{P}_{2N-3}$, where $0 \leq i \leq a_{2N-1} - 1$.

At this point, $\mathcal{B}_{N+1}$ can also be decomposed into two isomorphic $\mathcal{B}_N$ plus a set $link_{\mathcal{B}}(N+1)$ of 2^N edges connecting all corresponding vertices in the two subgraphs. By the induction hypothesis, each $\mathcal{B}_N$ can be embedded in $\mathcal{P}_{2N-1}$, and therefore, in $\mathcal{P}_{2N+1}$. On the other hand, it can be seen that $link_{\mathcal{B}}(N+1)$ is a subset of $Link_1(2N+1) \cup Link_2(2N+1) \cup Link_3(2N+1)$. Thus, $\mathcal{B}_{N+1} \sqsubseteq \mathcal{P}_{2(N+1)-1}$ and the proof is therefore completed.

(3) The embedding of $\mathcal{P}_n$ in Γ_{2n-1} can be proved by induction on n. Since Γ_5 has five vertices, which are 000, 001, 010, 100, and 101, and five edges, and $\mathcal{P}_3$ has four vertices, which are 00, 01, 10, and 11, and four edges, it can clearly be seen that we have $\mathcal{P}_3 \sqsubseteq \Gamma_5$ (refer to Figures 1 and 2). Assume that the statement is true for $n \leq N$, where $N \geq 3$ denotes an integer. Now, consider $n = N + 1$.

It is well known that the Fibonacci cube Γ_{2N+1} can be decomposed into Γ_{2N} and Γ_{2N-1} and there are $|link_\Gamma(2N+1)| = F_{2N-1}$ edges connecting each vertex i in Γ_{2N} to its corresponding vertex $i + F_{2N-1}$ in Γ_{2N-1} where $0 \le i \le F_{2N-1} - 1$.

Now, $\mathcal{P}_{N+1}$ can also be decomposed into $\mathcal{P}_N$, $\mathcal{P}_{N-1}$, and $\mathcal{P}_{N-3}$ plus three sets $Link_1(N+1)$, $Link_2(N+1)$, and $Link_3(N+1)$ of a_{N-1}, a_{N-3}, and a_{N-3}, respectively, with edges connecting all corresponding vertices in the three subgraphs. By the induction hypothesis, $\mathcal{P}_N$, $\mathcal{P}_{N-1}$, and $\mathcal{P}_{N-3}$ can be embedded in Γ_{2N-1}, Γ_{2N-3}, and Γ_{2N-7}, respectively. Furthermore, from Lemma 6, we know that $\Gamma_{2n} \sqsupseteq (\Gamma_1 \uplus \Gamma_3 \uplus \cdots \uplus \Gamma_{2n-1})$. On the other hand, Γ_{2N-3} and Γ_{2N-7} already have been embedded into Γ_{2N} and it can therefore be seen that $Link_1(N+1) \cup Link_2(N+1) \cup Link_3(N+1)$ is a subset of $link_\Gamma(2N+1)$.

Therefore, $\mathcal{P}_{N+1} \sqsubseteq \Gamma_{2(N+1)-1}$ and hence the proof is completed. □

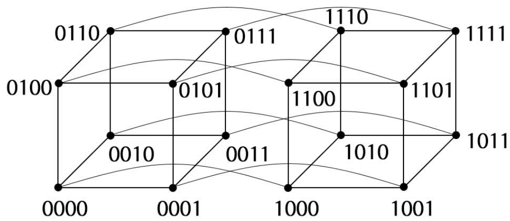

Figure 2. Binary cube $\mathcal{B}_4$.

For example, the graph $\mathcal{B}_4$ is shown in Figure 2. In Figure 3, an embedding of $\mathcal{P}_5$ in $\mathcal{B}_4$ is shown, where vertices marked with a tiny circle correspond to the 'forbidden' codes and vertices marked with a black point correspond to the 'embedding' codes in the Padovan representation. Similarly, in Figure 4, an embedding of Γ_6 in $\mathcal{P}_5$ is shown, where vertices marked with a tiny cross correspond to the 'forbidden' codes and vertices marked with a black point correspond to the 'embedding' codes in the Fibonacci representation.

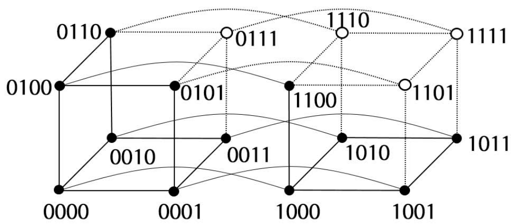

Figure 3. Padovan cube $\mathcal{P}_5$.

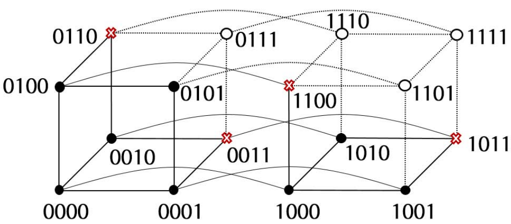

Figure 4. Fibonacci cube Γ_6.

In [19], we can find the backgrounds and application prospects of the hypercube interconnection network. That is, in the field of supercomputing computer architecture, the hypercube interconnection network is defined as a binary n-cube multiprocessor. The hypercube is treated as a loosely coupled system that is composed of $N = 2n$ processors that are linked in an n-dimensional binary cube. Each processor denotes a node of the cube. In particular, in [19], the author briefly introduced the fields where the Fibonacci cube is used. For example, in mathematical chemistry, this concept is used in the study of hexagonal graphs. Furthermore, in computer science, Fibonacci cubes are interesting from an algorithmic point of view. Therefore, research into the Padovan cubes that considers the Fibonacci cubes is interesting.

We now consider the degree of the vertices in the Padovan cube $\mathcal{P}_n$. The floor function $\lfloor x \rfloor$, also called the greatest integer function or integer value, gives the largest integer less than or equal to x. Similarly, the ceiling function $\lceil x \rceil$ gives the smallest integer greater than or equal to x. For example, $\lfloor 2.4 \rfloor = 2$, $\lceil 2.4 \rceil = 3$.

Theorem 5. *Let $d_n(i)$ denote the degree of the vertex i in the Padovan cube $\mathcal{P}_n$, for $n \geq 3$. Then, we can obtain $\lceil \frac{n}{3} \rceil \leq d_n(i) \leq n-1$.*

Proof. From Figure 1, we have $d_2(i) = 1$, $1 \leq d_3(i) \leq 2$, $2 \leq d_4(i) \leq 3$, and $2 \leq d_5(i) \leq 4$. That is, for $n = 2, 3, 4$, and 5, the inequality $\lceil \frac{n}{3} \rceil \leq d_n(i) \leq n-1$ is established.

We now consider $n \geq 6$. Recall that the Padovan code that represents a vertex in $\mathcal{P}_n$ has $(n-1)$ bits. The neighbors of a vertex i all have a Hamming distance of 1 with I_P, the Padovan code of i. Clearly, the vertex 0 has exactly $(n-1)$ neighbors in $\mathcal{P}_n$, and hence $d_n(i) \leq n-1$ for all other vertices.

Let δ_n denote the minimum degree in $\mathcal{P}_n$. From Figure 1, we know that $\delta_1 = 0$, $\delta_2 = 1$, $\delta_3 = \delta_4 = \delta_5 = \delta_6 = 2$, and $\delta_7 = 3$. The Padovan cube $\mathcal{P}_n$ can be decomposed into three subgraphs $\mathcal{P}_{n-1}$, $\mathcal{P}_{n-2}$, and $\mathcal{P}_{n-4}$, which are induced by the three sets of vertices $\{0, 1, 2, \cdots, a_{n-1} - 1\}$, $\{a_{n-1}, a_{n-1} + 1, \cdots, a_{n-1} + a_{n-2} - 1\}$, and $\{a_{n-1} + a_{n-2}, a_{n-1} + a_{n-2} + 1, \cdots, a_{n-1} + a_{n-2} + a_{n-4} - 1\}$, respectively. Theorem 2 is applied again and $\mathcal{P}_{n-1}$ decomposed into three subgraphs, $G_1 \cong \mathcal{P}_{n-2}$, $G_2 \cong \mathcal{P}_{n-3}$, and $G_3 \cong \mathcal{P}_{n-5}$, induced by the vertices $\{0, 1, 2, \cdots, a_{n-2} - 1\}$, $\{a_{n-2}, a_{n-2} + 1, \cdots, a_{n-2} + a_{n-3} - 1\}$, and $\{a_{n-2} + a_{n-3}, a_{n-2} + a_{n-3} + 1, \cdots, a_{n-2} + a_{n-3} + a_{n-5} - 1\}$, respectively. Let $G_4 = \mathcal{P}_{n-2}$ and $G_5 = \mathcal{P}_{n-4}$, which are induced by the two sets of vertices $\{a_{n-1}, a_{n-1} + 1, \cdots, a_{n-1} + a_{n-2} - 1\}$ and $\{a_{n-1} + a_{n-2}, a_{n-1} + a_{n-2} + 1, \cdots, a_{n-1} + a_{n-2} + a_{n-4} - 1\}$, respectively. It can be seen that $\min\{\delta_{n-2} + 1, \delta_{n-3} + 1, \delta_{n-5} + 2, \delta_{n-2} + 1, \delta_{n-4} + 2\} \leq \delta_n \leq \min\{\delta_{n-2} + 2, \delta_{n-3} + 2, \delta_{n-5} + 2, \delta_{n-2} + 2, \delta_{n-4} + 2\}$, where the first δ_{n-2} corresponds to the minimum degree of vertices in G_1; the second term δ_{n-3} to G_2; the third term δ_{n-4} to G_3; the fourth term δ_{n-2} to G_4; and the fifth term δ_{n-5} to G_5. Since $\delta_k \geq \delta_{k-1}$ for all $k \geq 2$, we have $\min\{\delta_{n-3} + 1, \delta_{n-5} + 2\} \leq \delta_n \leq \delta_{n-5} + 2$.

We now have two cases:

(i) Since $\delta_{n-3} + 1 \leq \delta_n$ and (ii);
(ii) $\delta_n = \delta_{n-5} + 2$.

(i) Since $\delta_1 = 0$, $\delta_2 = 1$, $\delta_3 = \delta_4 = \delta_5 = \delta_6 = 2$, $\delta_7 = 3$, and $\delta_{n-3} + 1 \leq \delta_n$, we know that $3 \leq \delta_8$, $3 \leq \delta_9$, $4 \leq \delta_{10}$, $4 \leq \delta_{11}$, and $4 \leq \delta_{12}$, which implies that $\lceil \frac{n}{3} \rceil \leq \delta_n$;
(ii) Since $\delta_n = \delta_{n-5} + 2$, we have that $\delta_8 = 4, \cdots, \delta_{11} = 4$, $\delta_{12} = 5$, $\delta_{13} = 6, \cdots, \delta_{16} = 6$, $\delta_{17} = 7$, and so on. Thus, this fact can be changed as follows: $4 \leq \delta_8, \cdots, 4 \leq \delta_{11}$, $4 \leq \delta_{12}$, $6 \leq \delta_{13}, \cdots, 6 \leq \delta_{16}$, $6 \leq \delta_{17}$, and so on. This implies that $\lceil \frac{n-1}{5} \rceil + \lceil \frac{n-2}{5} \rceil$.

From the above two cases, we have $\min\{\lceil \frac{n}{3} \rceil, \lceil \frac{n-1}{5} \rceil + \lceil \frac{n-2}{5} \rceil\} \leq \delta_n$. Since $\lceil \frac{n-1}{5} \rceil + \lceil \frac{n-2}{5} \rceil - \lceil \frac{n}{3} \rceil \geq 0$ for $n \geq 6$, we have $\lceil \frac{n}{3} \rceil \leq \delta_n$. □

Let $I_P = (b_{n-1}, \cdots, b_2, b_1)_P$ and $J_P = (c_{n-1}, \cdots, c_2, c_1)_P$, respectively, denote the Padovan codes of two vertices i and j in $\mathcal{P}_n$. Note that a path connecting the vertex i and the vertex j corresponds to a sequence of codes that transforms I_P to J_P, where two consecutive codes differ by exactly one bit. The following lemma shows that the Padovan cube is connected, i.e., that a path always exists between any pair of vertices i and j. Moreover, the length of the shortest path is determined by the Hamming distance between I_P and J_P.

Lemma 7. *Let $D(i, j)$ denote the distance between the vertex i and the vertex j in the Padovan cube $\mathcal{P}_n$, $n \geq 2$. Let I_P and J_P denote the Padovan codes of two vertices, i and j, respectively, in $\mathcal{P}_n$. We then have $D(i, j) = H(I_P, J_P)$.*

Proof. To prove that $H(I_P, J_P)$ is the distance between the vertex i and vertex j, we first demonstrate that it is possible to construct a path of that length. We then show that there are no shorter paths between the vertex i and the vertex j. Therefore, $D(i, j) = H(I_P, J_P)$.

Assume that $I_P \neq J_P$. Now, we will construct a path from the vertex i to the vertex j. The construction is as follows. Find the most significant bit where I_P and J_P differ. Let the bit position found be k. Without a loss of generality, let $b_k = 1$ and $c_k = 0$. Now, by definition, I_P has a link with $I'_P = (b_{n-1}, b_{n-2}, \cdots, \bar{b_k}, b_{k-1}, \cdots, b_1)_P$, where $\bar{b}_k$ denotes the complement of b_k. Note that $H(I'_P, J_P) = H(I_P, J_P) - 1$. The construction ends if $I'_P = J_P$; otherwise, we simply proceed recursively for I'_P and J_P. By using a simple inductive argument, it is easy to show that the construction will connect the vertex i and the vertex j and the length of the path is equal to $H(I_P, J_P)$.

Now, we show that there are no shorter paths between the vertex i and the vertex j in $\mathcal{P}_n$. Note that $\mathcal{P}_n$ is a subgraph of $\mathcal{B}_{n-1}$. It is known that in $\mathcal{B}_{n-1}$, the shortest path between the vertex k and the vertex l is of length $H(K, L)$, where K and L denote the binary codes of k and l, respectively. Since the two Padovan codes I_P and J_P, respectively, represent the two vertices i' and j' in $\mathcal{B}_{n-1}$, the shortest path between the vertex i' and the vertex j' in $\mathcal{B}_{n-1}$ must be greater than or equal to $H(I_P, J_P)$. Therefore, we have $D(i, j) = H(I_P, J_P)$. □

Theorem 6. *Let Δ_n denote the diameter of the Padovan cube $\mathcal{P}_n$ for all $n \geq 2$. Then, the Padovan cube $\mathcal{P}_n$ is a connected graph and $\Delta_n = n - 1$ for all $n \geq 2$.*

Proof. The graph $\mathcal{P}_n$ is connected by Lemma 7. Clearly, from Figure 1, we know that $\Delta_1 = 0$, $\Delta_2 = 1$, $\Delta_3 = 2$, and $\Delta_4 = 3$.

For $n \geq 5$ and positive integer $k \geq 1$, note that

$$a_n - 1 = \begin{cases} (1100 \cdots 1100)_P, & n = 4k + 1, \\ (1100 \cdots 11001)_P, & n = 4k + 2, \\ (1100 \cdots 110011)_P, & n = 4k + 3, \\ (1100 \cdots 1100110)_P, & n = 4k + 4, \end{cases}$$

and

$$a_{n-2} - 1 = \begin{cases} (0011 \cdots 0011)_P, & n = 4k + 1, \\ (0011 \cdots 00110)_P, & n = 4k + 2, \\ (0011 \cdots 001100)_P, & n = 4k + 3, \\ (0011 \cdots 0011001)_P, & n = 4k + 4, \end{cases}$$

respectively, differ in exactly $(n - 1)$ places in their Padovan codes. Hence, the distance between the vertex $a_n - 1$ and the vertex $a_{n-2} - 1$ is $(n - 1)$ by Lemma 7. It should be clear that no other pairs of vertices in $\mathcal{P}_n$ can be of greater distance.

Therefore $\Delta_n = n - 1$. □

5. Conclusions

Here, we presented the Padovan cube and analyzed its structural properties. The structural issues that were analyzed include the recursive decompositions, boundary conditions of the degree, diameter, and relations with other structures of the cubes. We also showed that various types of structures can be directly embedded in the Padovan cube. These preliminary results show that the Padovan cubes have very attractive recurrent structures.

However, compared with the Boolean cube and Fibonacci cube, the Padovan cube is hard to handle. The first few terms of the Padovan sequence overlap in the same number, so achieving the completeness of the Padovan sequence is easy but uniqueness cannot be guaranteed. To ensure uniqueness, the Padovan cube was only constructed with the odd-term subsequences, so the properties of the existing Padovan sequence could not be used as they were.

We can consider the communication primitives. On a Padovan cube of order n, we determine that the send/receive operation between any two vertices can be completed in at most $n-1$ steps, which is optimal. The Padovan cube may also be useful for a reconfigurable distributed network, because recursive message-routing algorithms can be designed for such networks, and the cost of interconnections is less than that of the Boolean cube.

We can construct a Padovan cube $\mathcal{P}_n$ of order n with Algorithm 2 (Padovan cubes algorithm).

Algorithm 2 Padovan Cubes Algorithm.

(1) Choose n according to the length of the binary codes you want to construct. Then, the binary codes are all $n-1$ in length.
(2) For the nth odd-Padovan number a_n, the set C_n of Padovan codes is obtained using the Padovan Coding Algorithm.
(3) The a_n vertices are numbered sequentially from 0 to $a_n - 1$.
(4) If the Hamming distance between the codes of any two vertices is 1, then the two vertices are connected by an edge. Otherwise, the two vertices are not connected.

Various properties related to Padovan cubes that were not covered in this paper should continue to be studied in the future. For example, we may consider a Padovan cube polynomial. Furthermore, as in the Lucas sequence in the Fibonacci sequence, the Lucas–Padovan sequence for the Padovan sequence can be considered, and it is expected that another cube can be constructed using this approach.

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