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Gauging the Maxwell Extended $\mathcal{GL}(n,\mathbb{R})$ and $\mathcal{SL}(n+1,\mathbb{R})$ Algebras

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Abstract: We consider the extension of the general-linear and special-linear algebras by employing the Maxwell symmetry in D space-time dimensions. We show how various Maxwell extensions of the ordinary space-time algebras can be obtained by a suitable contraction of generalized algebras. The extended Lie algebras could be useful in the construction of generalized gravity theories and the objects that couple to them. We also consider the gravitational dynamics of these algebras in the framework of the gauge theories of gravity. By adopting the symmetry-breaking mechanism of the Stelle–West model, we present some modified gravity models that contain the generalized cosmological constant term in four dimensions.

Keywords: cosmology; gauge theory of gravity; Maxwell symmetry

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1. Introduction

The historical developments show that the concept of gauge symmetry is a very powerful principle for constructing theories of fundamental interactions. For example, the electro-weak and strong interactions are described by the gauge theory based on the internal symmetry groups $SU(2) \otimes U(1)$ and $SU(3)$, respectively. In addition to these treatments, inspired by the Yang–Mills gauge theory [1], Einstein's general theory of relativity can be considered as a gauge theory of gravity. In 1956, Utiyama proposed that gravity can be constructed as a gauge theory based on the local homogeneous Lorentz group [2]. Later, in 1961, Kibble and Sciama generalized the gauge group to the Poincaré group and they arrived at what is now known as the Einstein–Cartan gravity [3,4]. After that, many space-time symmetry groups have been used to construct different types of gauge theories of gravity, such as Weyl [5,6], affine [7–9] and conformal groups [10–12].

The invariance of a given system under a certain symmetry transformation helps to find its physical properties. From this idea, if we use new symmetries or extend the well-known symmetries (such as the Poincaré group, the de-Sitter group, etc.) it is expected that one may get more information about any physical systems. Thus, one can say that new or enlarged symmetries may have a great potential to formulate any physical system more accurately. In this context, the Maxwell symmetry introduced by [13,14] is a good example of the symmetry extension, wherein the Poincaré group which describes the symmetries of empty Minkowski space-time is enlarged by six additional abelian anti-symmetric tensor generators satisfying

$$[P_a, P_b] = iZ_{ab}. \tag{1}$$

This enlarged symmetry naturally extends the space-time geometry. Physically, this extension can be considered as the symmetries of a charged particle in a constant electromagnetic field background [15].

Moreover, the Maxwell symmetry has attracted increasing attention after the work of Soroka [16]. Since then a variety of different Maxwell (super) symmetry algebras with interesting geometric and physical properties have been constructed and analyzed. For example, in general, the gauge theory of gravity based on the extended algebras leads to a generalized theory of gravity that includes an additional term to the energy-momentum tensor together with the cosmological constant [17–28] (for vanishing cosmological constant cases, see [20,29–33]). Up to now, the energy-momentum tensor coming from the Maxwell extension has not been extensively analyzed yet, but in this context, a minimal cosmological model has been introduced in [34] and also it is thought that the gauge fields of the Maxwell symmetry may provide a geometric background to describe vector inflatons in cosmological models [35] (for different solutions, see [33,36]). It is well known that such an additional term may be related to dark energy [37,38]. For the non-gravitational case, Maxwell symmetry is also used to describe planar dynamics of the Landau problem [39], higher spin fields [40,41], and applied to the string theory as an internal symmetry of the matter gauge fields [36]. Also, recent papers [42–44] have applied the Maxwell group in a classical form, which is given in [13,14].

Furthermore, it is proposed that the renormalizability and unitarity problems in quantum gravity can be overcome by taking the affine group as the dynamical group in the gauge theory of gravity, with the help of generalized linear connection [45–50]. In this paper, we examine the Maxwell extension for both general-linear and special-linear groups in D dimensions and analyze their gauge theory of gravity, in particular the generalized cosmological constant term. We have already studied these groups in four space-time dimensions in [22,24,28], thus this work will generalize our previous results to D-dimensional space-time.

The organization of the paper is as follows. In Section 2, we study the Maxwell extensions of the general linear group $\mathcal{GL}(n,\mathbb{R})$. Then we show that several Maxwell algebras can be obtained when we choose appropriate subalgebras and reduce it from a 5-dimensional case to 4 dimensions. We also construct the gauge theory of gravity based on the Maxwell extended $\mathcal{GL}(5,\mathbb{R})$. After applying dimensional reduction from 5 to 4 dimensions, we analyze the gravity action for two different cases. In Section 3, we present the special-linear group $\mathcal{SL}(n+1,\mathbb{R})$ and its Maxwell extension. In this framework, we give three examples that this extension leads to derive different generalizations of the Maxwell algebra in 4 dimensions. Similar to the previous section, we present the gauge theory of gravity based on this extended case. Finally, Section 4 concludes the paper with some discussions.

2. Maxwell Extensions of $\mathcal{GL}(n,\mathbb{R})$ Group

The general linear group $\mathcal{GL}(n,\mathbb{R})$ which corresponds to the set of all linear transformations satisfies the following Lie algebra,

$$\left[\mathcal{L}^A_{\ B},\mathcal{L}^C_{\ D}\right] = i\left(\delta^C_B\mathcal{L}^A_{\ D}-\delta^A_D\mathcal{L}^C_{\ B}\right), \tag{2}$$

where $\mathcal{L}^A_{\ B}$ are the generators of the group. This algebra can be extended to Maxwell-general linear algebra by adding an anti-symmetric tensor generator, $\mathcal{Z}_{AB}$, associated with the Maxwell symmetry. The generators $\mathcal{L}^A_{\ B}$ and $\mathcal{Z}_{AB}$ obey the commutators

$$\begin{aligned}\left[\mathcal{L}^A_{\ B},\mathcal{L}^C_{\ D}\right] &= i\left(\delta^C_B\mathcal{L}^A_{\ D}-\delta^A_D\mathcal{L}^C_{\ B}\right),\\ \left[\mathcal{L}^A_{\ B},\mathcal{Z}_{CD}\right] &= i\left(\delta^A_D\mathcal{Z}_{BC}-\delta^A_C\mathcal{Z}_{BD}\right),\\ \left[\mathcal{Z}_{AB},\mathcal{Z}_{CD}\right] &= 0,\end{aligned} \tag{3}$$

where the capital Latin indices run $A,B,C,\ldots=0,1,\ldots,n-1$ and n is the dimension of the group. The algebra with the commutation relations given by Equation (3), is denoted as $\mathcal{MGL}(n,\mathbb{R})$, the Maxwell-general linear algebra.

2.1. Decomposition of $\mathcal{MGL}(5,\mathbb{R})$

In this section, we start with the 35-dimensional $\mathcal{MGL}(5,\mathbb{R})$ algebra in the 5-dimensional space-time and carry out the dimensional reduction to the 4 dimensions. For this purpose, if we define the following generators

$$L^a{}_b = \mathcal{L}^a{}_b, \quad P_a = \left(\mathcal{L}^4{}_a - \frac{\lambda}{2}\mathcal{Z}_{4a}\right), \quad Z_{ab} = \mathcal{Z}_{ab}, \tag{4}$$

and this definition yields the 26-dimensional subalgebra of $\mathcal{MGL}(5,\mathbb{R})$ as,

$$\begin{aligned}
[L^a{}_b, L^c{}_d] &= i(\delta^c{}_b L^a{}_d - \delta^a{}_d L^c{}_b), \\
[L^a{}_b, P_c] &= -i\delta^a{}_c P_b, \\
[P_a, P_b] &= i\lambda Z_{ab}, \\
[L^a{}_b, Z_{cd}] &= i(\delta^a{}_d Z_{bc} - \delta^a{}_c Z_{bd}),
\end{aligned} \tag{5}$$

where the generators $L^a{}_b$, P_a, Z_{ab} correspond to the general linear transformation, translation, and Maxwell symmetry transformation, respectively. Here, the constant λ has the unit of L^{-2} which will be related to the cosmological constant where L is considered as the unit of length. We note that the small Latin indices are $a, b, c, \ldots = 0, 1, 2, 3$ and the remaining commutators are zero. This algebra is the Maxwell extension of general affine $\mathcal{GA}(4,\mathbb{R})$ algebra which is the semi-direct product of the general-linear group $\mathcal{GL}(4,\mathbb{R})$ with the group of translation T_4 (for more details, see [9,45,51]). This 26-dimensional extended group is denoted by $\mathcal{MGA}(4,\mathbb{R})$. The method of nonlinear realization [52–55], allows us to obtain a differential realization of the generators as [22]

$$\begin{aligned}
P_a &= i\left(\partial_a - \frac{\lambda}{2}x^b\partial_{ab}\right), \\
Z_{ab} &= i\partial_{ab}, \\
L^a{}_b &= i(x^a\partial_b + 2\theta^{ac}\partial_{bc}),
\end{aligned} \tag{6}$$

where $\partial_a = \frac{\partial}{\partial x^a}$, $\partial_{ab} = \frac{\partial}{\partial \theta^{ab}}$, and $\partial_{ab}\theta^{cd} = \frac{1}{2}\left(\delta^c_a\delta^d_b - \delta^c_b\delta^d_a\right)$. One can check that these differential operators fulfill the Maxwell–affine algebra and verify the self-consistency of Jacobi identities.

Taking the tangent space (Minkowski metric), $\eta_{ab} = diag(+,-,-,-)$ into consideration, we can define the following generators,

$$M_{ab} = \eta_{[ac}L^c{}_{b]}, \quad T_{ab} = \eta_{(ac}L^c{}_{b)}, \quad P_a = \left(\mathcal{L}^4{}_a - \frac{\lambda}{2}\mathcal{Z}_{4a}\right), \quad Z_{ab} = \mathcal{Z}_{ab}, \tag{7}$$

where the antisymmetrization and symmetrization of the objects are defined by $A_{[a}B_{b]} = A_aB_b - A_bB_a$ and $A_{(a}B_{b)} = A_aB_b + A_bB_a$, respectively. Thus, the Lie algebra of these generators can be given as,

$$\begin{aligned}
[M_{ab}, M_{cd}] &= i(\eta_{ad}M_{bc} + \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac}), \\
[M_{ab}, T_{cd}] &= i(-\eta_{ad}T_{bc} + \eta_{bc}T_{ad} - \eta_{ac}T_{bd} + \eta_{bd}T_{ac}), \\
[T_{ab}, T_{cd}] &= i(\eta_{ad}M_{bc} + \eta_{bc}M_{ad} + \eta_{ac}M_{bd} - \eta_{bd}M_{ac}), \\
[M_{ab}, P_c] &= i(\eta_{bc}P_a - \eta_{ac}P_b), \\
[T_{ab}, P_c] &= -i(\eta_{bc}P_a + \eta_{ac}P_b), \\
[P_a, P_b] &= i\lambda Z_{ab}, \\
[M_{ab}, Z_{cd}] &= i(\eta_{ad}Z_{bc} + \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac}), \\
[T_{ab}, Z_{cd}] &= i(\eta_{ad}Z_{bc} - \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} + \eta_{bd}Z_{ac}),
\end{aligned} \tag{8}$$

where M_{ab} is the anti-symmetric Lorentz generator, T_{ab} is the symmetric deformation generator, P_a is the translation generator, and Z_{ab} is the Maxwell symmetry generator. The algebra spanned by $\{M_{ab}, T_{ab}, P_a, Z_{ab}\}$ is the Maxwell–affine algebra introduced in Equation (5). This is also a minimal Maxwell extension of $\mathcal{GA}(4,\mathbb{R})$ group. The differential realization of the generators are

$$\begin{aligned}
M_{ab} &= i\Big(x_{[a}\partial_{b]} + 2\theta_{[a}^{\;c}\partial_{b]c}\Big),\\
T_{ab} &= i\Big(x_{(a}\partial_{b)} + 2\theta_{(a}^{\;c}\partial_{b)c}\Big),\\
P_a &= i\Big(\partial_a - \frac{\lambda}{2}x^b\partial_{ab}\Big),\\
Z_{ab} &= i\partial_{ab}.
\end{aligned} \tag{9}$$

Moreover, if we consider the following definitions with the Minkowski metric η_{ab},

$$M_{ab} = \eta_{[ac}\mathcal{L}^c{}_{b]}, \quad P_a = \Big(\mathcal{L}^4{}_a - \frac{\lambda}{2}\mathcal{Z}_{4a}\Big), \quad D = \mathcal{L}^4{}_4 \quad Z_{ab} = \mathcal{Z}_{ab}, \tag{10}$$

then we get the 17-dimensional subalgebra with following non-zero commutation relations,

$$\begin{aligned}
[M_{ab}, M_{cd}] &= i(\eta_{ad}M_{bc} + \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac}),\\
[M_{ab}, P_c] &= i(\eta_{bc}P_a - \eta_{ac}P_b),\\
[P_a, P_b] &= i\lambda Z_{ab},\\
[M_{ab}, Z_{cd}] &= i(\eta_{ad}Z_{bc} + \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac}),\\
[P_a, D] &= iP_a,\\
[Z_{ab}, D] &= 2iZ_{ab}.
\end{aligned} \tag{11}$$

This is the Maxwell–Weyl algebra introduced in [56] and studied in the context of the gauge theory of gravity in [21]. The differential realization of the generators $\{M_{ab}, P_a, D, Z_{ab}\}$ is given by

$$\begin{aligned}
M_{ab} &= i\Big(x_{[a}\partial_{b]} + 2\theta_{[a}^{\;c}\partial_{b]c}\Big),\\
P_a &= i\Big(\partial_a - \frac{\lambda}{2}x^b\partial_{ab}\Big),\\
D &= i\Big(x^a\partial_a + 2\theta^{ab}\partial_{ab}\Big),\\
Z_{ab} &= i\partial_{ab},
\end{aligned} \tag{12}$$

and they correspond to the generalized Lorentz transformations, space-time translations, dilatation, and the Maxwell symmetry transformations, respectively. In the absence of the dilatation symmetry, $\mathcal{L}^4{}_4 = 0$, the algebra in Equation (11) reduces to the well-known Maxwell algebra [16,17]. Herein we can conclude that $\mathcal{MGL}(n,\mathbb{R})$ algebra is a comprehensive symmetry algebra which has the great potential for obtaining different kinds of Maxwell extended algebras.

2.2. Gauging the $\mathcal{MGL}(5,\mathbb{R})$ Algebra

In this part, we consider the gauge theory of the $\mathcal{MGL}(5,\mathbb{R})$ algebra. To gauge this algebra, we follow the methods presented in [17,22]. At first, we need to introduce gauge potentials, i.e., a vector-valued 1-form $\mathcal{A}(x) = \mathcal{A}^{\mathcal{A}}(x)X_{\mathcal{A}}$ as

$$\mathcal{A}(x) = \tilde{\omega}^B{}_A\mathcal{L}^A{}_B + B^{AB}\mathcal{Z}_{AB}, \tag{13}$$

where $\mathcal{A}^{\mathcal{A}}(x) = \{\tilde{\omega}^B{}_A, B^{AB}\}$ are the gauge fields corresponding to the generators $X_{\mathcal{A}} = \{\mathcal{L}^A_B, \mathcal{Z}_{AB}\}$. The variation of the gauge field $\mathcal{A}(x)$ under infinitesimal gauge transformation in tangent space can be calculated by using the following formula

$$\delta\mathcal{A} = -d\zeta - i[\mathcal{A}, \zeta], \tag{14}$$

with the gauge generator

$$\zeta(x) = \tilde{\lambda}^B{}_A \mathcal{L}^A_B + \varphi^{AB}\mathcal{Z}_{AB}, \tag{15}$$

where, $\tilde{\lambda}^B{}_A(x)$ and $\varphi^{AB}(x)$, are the parameters of the corresponding generators. The transformation properties of the gauge fields under infinitesimal action of the $\mathcal{MGL}(5,\mathbb{R})$ are

$$\begin{aligned} \delta\tilde{\omega}^A_B &= -d\tilde{\lambda}^A_B + \tilde{\lambda}^C_B\omega^A_C - \tilde{\lambda}^A_C\omega^C_B, \\ \delta B^{AB} &= -d\varphi^{AB} + \tilde{\lambda}^{[A}{}_C B^{CB]} - \tilde{\omega}^{[A}{}_C\varphi^{CB]}. \end{aligned} \tag{16}$$

The curvature 2-form $\mathcal{F}(x)$ is given by the structure equation

$$\mathcal{F} = d\mathcal{A} + \frac{i}{2}[\mathcal{A}, \mathcal{A}], \tag{17}$$

written in terms of components

$$\mathcal{F}(x) = \tilde{\mathcal{R}}^B{}_A\mathcal{L}^A_B + \mathcal{F}^{AB}\mathcal{Z}_{AB}, \tag{18}$$

one can calculate the curvature 2-forms of the associated gauge fields as

$$\begin{aligned} \tilde{\mathcal{R}}^A_B &= d\tilde{\omega}^A_B + \tilde{\omega}^A_C \wedge \tilde{\omega}^C_B = \mathcal{D}\tilde{\omega}^A_B, \\ \mathcal{F}^{AB} &= dB^{AB} + \tilde{\omega}^{[A}{}_C \wedge B^{CB]} = \mathcal{D}B^{AB}, \end{aligned} \tag{19}$$

where $\mathcal{D} = d + \tilde{\omega}$ is the exterior covariant derivative with respect to $\mathcal{MGL}(5,\mathbb{R})$ connection $\tilde{\omega}^A_B$. These is the $\mathcal{GL}(5,\mathbb{R})$ curvature 2-form and a new curvature 2-form for the tensor generator $\mathcal{Z}_{AB}$. Under an infinitesimal gauge transformation with parameters $\zeta(x)$, the change in curvature is given by

$$\delta\mathcal{F} = i[\zeta, \mathcal{F}], \tag{20}$$

and hence one gets

$$\begin{aligned} \delta\tilde{\mathcal{R}}^A_B &= \tilde{\lambda}^A_C\tilde{\mathcal{R}}^C_B - \tilde{\lambda}^C_B\tilde{\mathcal{R}}^A_C, \\ \delta\mathcal{F}^{AB} &= \tilde{\lambda}^{[A}{}_C\mathcal{F}^{CB]} - \tilde{\mathcal{R}}^{[A}{}_C\varphi^{CB]}, \end{aligned} \tag{21}$$

the gauge variations of the curvatures. By taking the exterior covariant derivatives of the curvatures, the Bianchi identities become

$$\begin{aligned} \mathcal{D}\tilde{\mathcal{R}}^A_B &= 0, \\ \mathcal{D}\mathcal{F}^{AB} &= \tilde{\mathcal{R}}^{[A}{}_C B^{CB]}. \end{aligned} \tag{22}$$

Having found the transformations of the gauge fields and the curvatures, we are ready to look for an invariant gravitational Lagrangian under these transformations.

2.3. Gravitational Action

In this section, we follow the approach of Stelle and West (SW) [57,58]. Our starting point is the local $\mathcal{SO}(2,3)$ symmetry with the metric signature as $(+,-,-,-,+)$ on a 4-dimensional Minkowski space-time. It is well known that the symmetry-breaking mechanism of the Stelle–West model provides a physically realistic mechanism for obtaining gravity as a gauge theory with a spontaneously broken local symmetry. Stelle and West considered an action where a symmetry-breaking mechanism is induced by introducing a non-dynamical vector field V^A in order to promote local $\mathcal{SO}(2,3)$ transformations to gauge symmetries, which is constrained by the condition $V_A V^A = c^2$.

The SW action can be given by

$$S_{SW} = \sigma \int V^E \epsilon_{ABCDE} \mathcal{R}^{AB} \wedge \mathcal{R}^{CD} + \alpha\left(c^2 - V_A V^A\right), \tag{23}$$

where σ is a constant and α is an arbitrary 4-form serving as a Lagrange multiplier. The totally anti-symmetric symbol ϵ_{ABCDE} is an invariant tensor of the algebra $so(2,3)$. We can also note that $\mathcal{A}^{AB}(x)$ is a de-Sitter connection 1-form, and $\mathcal{R}^{AB}(x)$ is its curvature 2-form. Choosing

$$V^A = (c, 0, 0, 0, 0), \tag{24}$$

$$e^a(x) = -lDV^a = -lc\mathcal{A}^a_4, \quad DV^4 = 0, \tag{25}$$

where $e^a(x)$ corresponds to the vierbein field, D is the Lorentz covariant derivative and l is related to the cosmological constant according to $l = \sqrt{3/\Lambda}$, $\mathcal{SO}(2,3)$ symmetry is broken spontaneously to $\mathcal{SO}(1,3)$, and we obtain the action

$$S_{SW} = \frac{1}{2\kappa} \int \epsilon_{abcd} \left(R^{ab} \wedge e^c \wedge e^d - \frac{\Lambda}{6} e^a \wedge e^b \wedge e^c \wedge e^d\right), \tag{26}$$

where $R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}$ is the Ricci curvature 2-form, and we identify $\mathcal{A}^{ab}(x) = \omega^{ab}(x)$ and also set $\sigma c = -\frac{3}{4\kappa\Lambda}$ together with κ is being Einstein's gravitational constant.

In analogy with the Stelle–West action in Equation (23), we will try to construct a gravitational action generalized to the case of the $\mathcal{MGL}(5,\mathbb{R})$ symmetry group. For this purpose, we combine the curvature 2-forms $\tilde{\mathcal{R}}^A_B$ and $\mathcal{F}^{AB}$ into an anti-symmetric shifted curvature 2-form as follows

$$\mathcal{J}^{AB} = \tilde{\mathcal{R}}^{AB} - \mu\mathcal{F}^{AB}, \tag{27}$$

where μ is a dimensionless constant which will be employed in the definition of the cosmological constant $\Lambda = \mu\lambda$ and we define a new object as $\tilde{\mathcal{R}}^{AB} = \tilde{\mathcal{R}}^{[A}{}_C g^{CB]}$. Here we introduced an additional symmetric tensor field g^{AB}, which is called the premetric tensor field. It does not represent the metric tensor of space-time but it helps to construct an invariant Lagrangian (for more details, see [22,59]). The components of the premetric field are $\mathcal{GL}(5,\mathbb{R})$ tensor valued 0-forms and the infinitesimal transformation of the premetric tensor field under the local $\mathcal{GL}(5,\mathbb{R})$ symmetry group is

$$\delta g^{AB} = \tilde{\lambda}^{(A}{}_C g^{CB)}, \quad \delta g_{AB} = -\tilde{\lambda}^C{}_{(A} g_{CB)}, \tag{28}$$

and its covariant derivative is given as

$$\mathcal{Q}^{AB} = \mathcal{D}g^{AB} = dg^{AB} + \tilde{\omega}^{(A}{}_C \wedge B^{CB)}. \tag{29}$$

We note that if one identifies g^{AB} as the space-time metric tensor, then $\mathcal{Q}^{AB}$ becomes the nonmetricity. On the other hand, this kind of additional metric-like fields may be discussed in the context of the metric-affine gravity [28,49,60].

By using Equation (28), we can find the gauge transformation of the new object as $\delta\tilde{\mathcal{R}}^{AB} = \tilde{\lambda}^{[A}{}_C \tilde{\mathcal{R}}^{CB]}$. Moreover, since the tensorial translation being traded for diffeomorphism invariance is not symmetric of the action [18], omitting the tensorial-space translations, the transformation rules for the curvature $\mathcal{F}^{AB}$ in Equation (21) can be rewritten as $\delta\mathcal{F}^{AB} = \tilde{\lambda}^{[A}{}_C \mathcal{F}^{CB]}$. Then using this background, one can show that the shifted curvature in Equation (27) has the following gauge transformation,

$$\delta\mathcal{J}^{AB} = \tilde{\lambda}^{[A}{}_C \mathcal{J}^{CB]}. \tag{30}$$

Now, we can start with the following generalized Stelle–West action

$$S_{MGL} = \sigma \int V^E \eta_{ABCDE} \mathcal{J}^{AB} \wedge \mathcal{J}^{CD} + \alpha\Big(c^2 - V_A V^A\Big), \tag{31}$$

where V^A is a 0-form non-dynamical $\mathcal{GL}(5,\mathbb{R})$ five-vector field with dimensions of the length and it satisfies the transformation $\delta V^A = \tilde{\lambda}^A{}_C V^C$. We also use another definition $V_A = V^B g_{AB}$ which obeys the transformation rule as $\delta V_A = -\tilde{\lambda}^C{}_A V_C$. Note that the totally anti-symmetric symbol η_{ABCDE} is an invariant tensor under $gl(5,\mathbb{R})$ algebra with the following transformation rule,

$$\delta\eta_{ABCDE} = -\tilde{\lambda}^F{}_A \eta_{FBCDE} - \tilde{\lambda}^F{}_B \eta_{AFCDE} - \tilde{\lambda}^F{}_C \eta_{ABFDE} - \tilde{\lambda}^F{}_D \eta_{ABCFE} - \tilde{\lambda}^F{}_E \eta_{ABCDF} \tag{32}$$

With the help of Equation (30), one can easily check that the action Equation (31) is gauge invariant. We note that $\mathcal{J}^{AB}$ is an asymmetric under the interchange of indices due to the definition of $\tilde{\mathcal{R}}^{AB}$. Thus, only the anti-symmetric part of $\tilde{\mathcal{R}}^{AB}$ contributes to the gravitational dynamics in the action (31).

Then by the variation of the action with respect to $\tilde{\omega}^A_F(x)$, $B^{AB}(x)$ and $g^{FG}(x)$, we obtain the field equations

$$\begin{aligned} \mathcal{D}\Big(\eta_{ABCDE} V^E g^{FB} \mathcal{J}^{CD}\Big) - \mu\eta_{ABCDE} V^E B^{FB} \wedge \mathcal{J}^{CD} &= 0, \\ \mathcal{D}\Big(V^E \eta_{ABCDE} \mathcal{J}^{CD}\Big) &= 0, \\ V^E \eta_{ACDE(F} \tilde{\mathcal{R}}^A{}_{G)} \wedge \mathcal{J}^{CD} + \frac{1}{2} V^E g_{FG} \eta_{ABCDE} \mathcal{J}^{AB} \wedge \mathcal{J}^{CD} &= 0, \end{aligned} \tag{33}$$

and they are invariant under local $\mathcal{MGL}(5,\mathbb{R})$ transformations.

Four-Dimensional Case

If we take $V^A = (c, 0, 0, 0, 0)$ and fix the constant as $\sigma c = -\frac{3}{4\kappa\Lambda}$, the gravitational action Equation (31) spontaneously breaks down to,

$$\begin{aligned} S_{MGL} &= -\frac{3}{4\kappa\Lambda} \int \eta_{abcd} \mathcal{J}^{ab} \wedge \mathcal{J}^{cd} \\ &= -\frac{3}{4\kappa\Lambda} \int \eta_{abcd} \Big(\tilde{\mathcal{R}}^{[a}_{e} g^{eb]} \wedge \tilde{\mathcal{R}}^{[c}_{f} g^{fd]} - 2\mu \tilde{\mathcal{R}}^{[a}_{e} g^{eb]} \wedge \mathcal{F}^{cd} + \mu^2 \mathcal{F}^{ab} \wedge \mathcal{F}^{cd}\Big), \end{aligned} \tag{34}$$

where $\eta_{abcd} = e\epsilon_{abcd}$ obey the same transformation rule given in Equation (32) and e is the determinant of the vierbein field and ϵ_{abcd} is the Levi–Civita symbol. This action has a structural similarity to the Maxwell–Affine gravity action which was given in [22] but includes more general curvature 2-forms. So we can say that if we construct the Stelle–West-like action by using curvature 2-forms which come from $\mathcal{MGL}(5,\mathbb{R})$ we can obtain a generalized framework for Maxwell gravity. Furthermore, if we assume the premetric field $g^{ab}(x)$ as the tangent space metric tensor, then the action (31) can be written as follows

$$\begin{aligned} S_{MGL} &= -\frac{3}{4\kappa\Lambda} \int \eta_{abcd} \mathcal{J}^{ab} \wedge \mathcal{J}^{cd} \\ &= -\frac{3}{4\kappa\Lambda} \int \eta_{abcd} \Big(\tilde{\mathcal{R}}^{ab} \wedge \tilde{\mathcal{R}}^{cd} - 2\mu \tilde{\mathcal{R}}^{ab} \wedge \mathcal{F}^{cd} + \mu^2 \mathcal{F}^{ab} \wedge \mathcal{F}^{cd}\Big). \end{aligned} \tag{35}$$

where $\tilde{\mathcal{R}}^{ab}$ and $\mathcal{F}^{ab}$ are anti-symmetric curvature 2-forms. This action generalizes the minimal Maxwell gravity which was discussed in [17]. Let us expand this action to a more explicit form. Similar to the definition in Equation (25), we define the following vector fields,

$$e^a(x) = -\sqrt{\frac{3}{\Lambda}}\omega^{a5}, \quad b^a(x) = -\sqrt{\frac{3}{\Lambda}} B^{a5} \tag{36}$$

where $e^a(x)$ can be considered as the vierbein vector field and $b^a(x)$ is an additional vector field as an effect of the Maxwell symmetry. Then the shifted curvature becomes

$$\mathcal{J}^{ab} = R^{ab}(\omega) - \mu DB^{ab} - \frac{\Lambda}{3}\left(e^a \wedge e^b - \mu e^{[a} \wedge b^{b]}\right) \tag{37}$$

where D is the Lorentz covariant derivative and $R^{ab}(\omega)$ is the Riemann curvature 2-form. From this background, neglecting the total derivatives, the action (35) reduces

$$\begin{aligned} S_{MGL} \;=\; & \frac{1}{2\kappa}\int \eta_{abcd}\left(R^{ab} \wedge e^c \wedge e^d - \frac{\Lambda}{6} e^a \wedge e^b \wedge e^c \wedge e^d\right) \\ & -2\mu\eta_{abcd}\left(R^{ab} \wedge e^c \wedge b^d + \frac{1}{2} DB^{ab} \wedge e^c \wedge e^d - \mu DB^{ab} \wedge e^c \wedge b^d\right) \\ & +\frac{2\Lambda\mu}{3}\eta_{abcd}\left(e^a \wedge e^b \wedge e^c \wedge b^d - \mu e^a \wedge b^b \wedge e^c \wedge b^d\right). \end{aligned} \tag{38}$$

The first line includes the Einstein–Hilbert-like term together with the cosmological constant term, the second line contains mixed terms, and the last line corresponds to the generalized cosmological terms with the Maxwell symmetry contributions.

Thus we can say that the gauge theory of gravity based on $\mathcal{MGL}(5,\mathbb{R})$ extends the geometrical framework of Einstein's gravitational theory to be included the generalized cosmological constant term. So this theory provides an alternative way to introduce the cosmological term.

3. Maxwell Extensions of the $\mathcal{SL}(n+1,\mathbb{R})$ Group

The group of special-linear transformations (also known as the metalinear group) $\mathcal{SL}(n+1,\mathbb{R})$, being the subgroup of $\mathcal{GL}(n,\mathbb{R})$, consists of matrices of determinant unity and it is generated by $n(n+2)$ trace-free generators,

$$\mathring{\mathcal{L}}^A{}_B = \mathcal{L}^A{}_B - \frac{1}{n}\delta^A_B \mathcal{L}^C{}_C, \tag{39}$$

and satisfies the following Lie algebra [61–63] (for more details, see [64]),

$$\left[\mathring{\mathcal{L}}^A{}_B, \mathring{\mathcal{L}}^C{}_D\right] = i\left(\delta^C_B \mathring{\mathcal{L}}^A{}_D - \delta^A_D \mathring{\mathcal{L}}^C{}_B\right), \tag{40}$$

where $\mathring{\mathcal{L}}^A_B$ are the $\mathcal{SL}(n+1,\mathbb{R})$ generators and the index structure is essentially the same as that of the previous section.

We extend this algebra by an anti-symmetric tensor generator $\mathcal{Z}_{AB}$ associated with the Maxwell symmetry as,

$$\begin{aligned} \left[\mathring{\mathcal{L}}^A{}_B, \mathring{\mathcal{L}}^C{}_D\right] &= i\left(\delta^C_B \mathring{\mathcal{L}}^A{}_D - \delta^A_D \mathring{\mathcal{L}}^C{}_B\right), \\ \left[\mathring{\mathcal{L}}^A{}_B, \mathcal{Z}_{CD}\right] &= i\left(\delta^A_D \mathcal{Z}_{BC} - \delta^A_C \mathcal{Z}_{BD} + \frac{2}{n}\delta^A_B \mathcal{Z}_{CD}\right), \\ \left[\mathcal{Z}_{AB}, \mathcal{Z}_{CD}\right] &= 0. \end{aligned} \tag{41}$$

This algebra can be named as the Maxwell-special-linear algebra and denoted with $\mathcal{MSL}(n+1,\mathbb{R})$.

3.1. Decomposition of $\mathcal{MSL}(n+1,\mathbb{R})$

Let us now analyze the algebra (41) in four space-time dimensions. As a first example, if we define the generators as

$$\mathring{L}^a{}_b = \mathring{\mathcal{L}}^a{}_b, \quad P_a = \left(\mathring{\mathcal{L}}^4{}_a - \frac{\lambda}{2}\mathcal{Z}_{4a}\right), \quad Z_{ab} = \mathcal{Z}_{ab}, \tag{42}$$

then this definition leads to the following 25-dimensional subalgebra

$$
\begin{aligned}
[\mathring{L}^a{}_b, \mathring{L}^c{}_d] &= i(\delta^c{}_b \mathring{L}^a{}_d - \delta^a{}_d \mathring{L}^c{}_b), \\
[\mathring{L}^a{}_b, P_c] &= -i\left(\delta^a{}_c P_b - \frac{1}{4}\delta^a_b P_c\right), \\
[P_a, P_b] &= i\lambda Z_{ab}, \\
[\mathring{L}^a{}_b, Z_{cd}] &= i\left(\delta^a{}_d Z_{bc} - \delta^a{}_c Z_{bd} + \frac{1}{2}\delta^a_b Z_{cd}\right),
\end{aligned}
\tag{43}
$$

where the generators $\mathring{L}^a{}_b$, P_a, Z_{ab} correspond to the generalized special-linear, translation, and the Maxwell symmetries, respectively. This algebra is the Maxwell extended special-affine algebra $\mathcal{SA}(4,\mathbb{R})$ which is the semi-direct product of the 15-dimensional $\mathcal{SL}(4,\mathbb{R})$ transformation and 4-dimensional translation T_4. It is also known as the Maxwell-special-linear algebra and denoted by $\mathcal{MSA}(4,\mathbb{R})$ (for more details, see [24]). The differential realization of the generators can also be found as follows

$$
\begin{aligned}
P_a &= i\left(\partial_a - \frac{\lambda}{2}x^b\partial_{ab}\right), \\
Z_{ab} &= i\partial_{ab}, \\
L^a{}_b &= i(x^a\partial_b + 2\theta^{ac}\partial_{bc}) - \frac{i}{4}\delta^a{}_b\left(x^c\partial_c + 2\theta^{cd}\partial_{cd}\right),
\end{aligned}
\tag{44}
$$

As a second example, this time we combine the algebras given in Equations (3) and (41) in four dimensions. Defining the following generators,

$$
L^a{}_b = \mathcal{L}^a{}_b, \quad \mathring{L}^a{}_b = \mathring{\mathcal{L}}^a{}_b, \quad P_a = \mathcal{L}^4{}_a - \frac{1}{2}\mathcal{Z}_{4a}, \quad P^a_* = \mathcal{L}^a{}_4, \quad Z_{ab} = \mathcal{Z}_{ab}, \quad Z_a = \mathcal{Z}_{4a}, \tag{45}
$$

We get the Lie algebra as,

$$
\begin{aligned}
[L^a{}_b, L^c{}_d] &= i(\delta^c{}_b L^a{}_d - \delta^a{}_d L^c{}_b), \\
[L^a{}_b, P_c] &= -i\delta^a{}_c P_b, \\
[L^a{}_b, P^c_*] &= i\delta^c{}_b P^a_*, \\
[P_a, P_b] &= iZ_{ab}, \\
\left[P^a_*, P^b_*\right] &= 0, \\
[P^a_*, P_b] &= i\left(L^a{}_b - \delta^a{}_b L^5{}_5\right) = i\mathring{L}^a{}_b, \\
[L^a{}_b, Z_{cd}] &= i(\delta^a{}_d Z_{bc} - \delta^a{}_c Z_{bd}), \\
[L^a{}_b, Z_c] &= -i\delta^a{}_c Z_b, \\
[P_a, Z_{cd}] &= 0, \\
[P_a, Z_c] &= -iZ_{ac}, \\
[P^a_*, Z_{cd}] &= i(\delta^a_d Z_c - \delta^a_c Z_d), \\
[P^a_*, Z_c] &= 0,
\end{aligned}
\tag{46}
$$

where we have used the expression $L^4{}_4 = \frac{1}{4}L^c{}_c$. Here, P^a_* is a vector and P_b is a co-vector with respect to $\mathcal{GL}(n,\mathbb{R})$ transformation and they also generate pseudo-translations. Moreover, there is an additional vector generator Z_a that comes from the Maxwell symmetry extension. This algebra is a novel Maxwell extension of the meta-linear group studied in [62,63] in the four-dimensional space-time.

Furthermore, if the tangent (flat) space carries a metric with the component η_{ab}, one can lower the indices and a finer splitting of $\mathcal{MSA}(4,\mathbb{R})$ algebra can be achieved:

$$M_{ab} = \eta_{[ac}\mathring{\mathcal{L}}^{c}{}_{b]}, \quad T_{ab} = \eta_{(ac}\mathring{\mathcal{L}}^{c}{}_{b)}, \quad P_a = \left(\mathring{\mathcal{L}}^{4}{}_{a} - \frac{\lambda}{2}\mathcal{Z}_{5a}\right), \quad Z_{ab} = \mathcal{Z}_{ab}, \tag{47}$$

and the commutation relations given by Equation (43) become

$$\begin{aligned}
[M_{ab}, M_{cd}] &= i(\eta_{ad}M_{bc} + \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac}), \\
[M_{ab}, T_{cd}] &= i(-\eta_{ad}T_{bc} + \eta_{bc}T_{ad} - \eta_{ac}T_{bd} + \eta_{bd}T_{ac}), \\
[T_{ab}, T_{cd}] &= i(\eta_{ad}M_{bc} + \eta_{bc}M_{ad} + \eta_{ac}M_{bd} - \eta_{bd}M_{ac}), \\
[M_{ab}, P_c] &= i(\eta_{bc}P_a - \eta_{ac}P_b), \\
[T_{ab}, P_c] &= -i\left(\eta_{bc}P_a + \eta_{ac}P_b - \frac{1}{2}\eta_{ab}P_c\right), \\
[P_a, P_b] &= i\lambda Z_{ab}, \\
[M_{ab}, Z_{cd}] &= i(\eta_{ad}Z_{bc} + \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac}), \\
[T_{ab}, Z_{cd}] &= i(\eta_{ad}Z_{bc} - \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} + \eta_{bd}Z_{ac} + \eta_{ab}Z_{cd}),
\end{aligned} \tag{48}$$

where M_{ab} generates the metric-preserving Lorentz subgroup, T_{ab} generates the (nontrivial) relativistic four-volume-preserving transformations (shear generator) with $\mathrm{tr}T_{ab} = 0$, P_a is the translation generator, and Z_{ab} is the Maxwell generator. The differential realizations of these generators are given by

$$\begin{aligned}
M_{ab} &= i\left(x_{[a}\partial_{b]} + 2\theta^{c}_{[a}\partial_{b]c}\right), \\
T_{ab} &= i\left(x_{(a}\partial_{b)} + 2\theta^{c}_{(a}\partial_{b)c}\right) + \frac{i}{2}\eta_{ab}\left(x^{c}\partial_{c} + 2\theta^{cd}\partial_{cd}\right), \\
P_a &= i\left(\partial_a - \frac{\lambda}{2}x^{b}\partial_{ab}\right), \\
Z_{ab} &= i\partial_{ab}.
\end{aligned} \tag{49}$$

Thus, we derived the Maxwell extension of $\mathcal{SA}(4,\mathbb{R})$ algebra in the presence of a metric [45].

3.2. The Gauge Theory of the $\mathcal{MSL}(n+1,\mathbb{R})$ Group

In the gauging of the $\mathcal{MSL}(n+1,\mathbb{R})$ symmetry group, we adopt the same construction procedures given in the previous section. For this purpose, we start by writing down an algebra-valued gauge field

$$\mathcal{A}(x) = \mathcal{A}^{A}X_{A} = \mathring{\omega}^{B}{}_{A} \wedge \mathring{\mathcal{L}}^{A}{}_{B} + B^{AB}\mathcal{Z}_{AB}, \tag{50}$$

where $\mathcal{A}^{\mathcal{A}}(x) = \{\mathring{\omega}^{B}{}_{A}, B^{AB}\}$ are the gauge fields corresponding to the generators $X_A = \left\{\mathring{\mathcal{L}}^{A}{}_{B}, \mathcal{Z}_{AB}\right\}$, respectively. Using the Lie algebra valued gauge parameters

$$\zeta(x) = \zeta^{A}X_{A} = \varphi^{AB}\mathcal{Z}_{AB} + \mathring{\lambda}^{B}{}_{A} \wedge \mathring{\mathcal{L}}^{A}{}_{B}, \tag{51}$$

where $\varphi^{AB}(x)$ and $\mathring{\lambda}^{B}{}_{A}(x)$ are the Maxwell and the special-linear transformation parameters, respectively. By using Equations (14), (50), and (51), the transformation properties of the gauge fields under the infinitesimal $\mathcal{MSL}(n+1,\mathbb{R})$ are

$$\begin{aligned}
\delta\mathring{\omega}^{A}{}_{B} &= -d\mathring{\lambda}^{A}{}_{B} + \mathring{\lambda}^{C}{}_{B}\mathring{\omega}^{A}{}_{C} - \mathring{\lambda}^{A}{}_{C}\mathring{\omega}^{C}{}_{B}, \\
\delta B^{AB} &= -d\varphi^{AB} + \mathring{\lambda}^{[A}{}_{C}B^{CB]} - \mathring{\omega}^{[A}{}_{C}\varphi^{CB]} \\
&= -d\varphi^{AB} + \tilde{\lambda}^{[A}{}_{C}B^{CB]} - \tilde{\omega}^{[A}{}_{C}\varphi^{CB]} - \frac{2}{n}\tilde{\lambda}B^{AB} + \frac{2}{n}\tilde{\omega}\varphi^{AB},
\end{aligned} \tag{52}$$

where $\tilde{\lambda}$ and $\tilde{\omega}$ are the trace parts of the $\mathcal{GL}(n,\mathbb{R})$ valued parameter and gauge field, respectively. To find the curvature 2-forms, we make use of Equation (17) and recall the following definition

$$\mathcal{F}(x) = \mathcal{F}^A X_A = \mathring{\mathcal{R}}^B{}_A \wedge \mathring{\mathcal{L}}^A_B + \mathcal{F}^{AB}\mathcal{Z}_{AB}. \tag{53}$$

Then it yields the curvature 2-forms of the $\mathcal{MSL}(n+1,\mathbb{R})$ algebra as

$$\begin{aligned}
\mathring{\mathcal{R}}^A_B &= d\mathring{\omega}^A_B + \mathring{\omega}^A_C \wedge \mathring{\omega}^C_B \\
&= \mathcal{D}\mathring{\omega}^A_B, \\
\mathcal{F}^{AB} &= dB^{AB} + \mathring{\omega}^{[A}_C \wedge B^{CB]} \\
&= dB^{AB} + \tilde{\omega}^{[A}_C \wedge B^{CB]} - \frac{2}{n}\tilde{\omega}B^{AB} \\
&= \mathcal{D}B^{AB},
\end{aligned} \tag{54}$$

where the exterior covariant derivative $\mathcal{D}$ is defined with respect to $\mathcal{SL}(n+1,\mathbb{R})$ connection. The infinitesimal variation of the curvature 2-forms under the gauge transformations can be obtained by using Equation (20),

$$\begin{aligned}
\delta\mathring{\mathcal{R}}^A_B &= \mathring{\lambda}^A_C\mathring{\mathcal{R}}^C_B - \mathring{\lambda}^C_B\mathring{\mathcal{R}}^A_C \\
\delta\mathcal{F}^{AB} &= \mathring{\lambda}^{[A}_C\mathcal{F}^{CB]} - \mathring{\mathcal{R}}^{[A}_C\varphi^{CB]}, \\
&= \tilde{\lambda}^{[A}_C\mathcal{F}^{CB]} - \mathring{\mathcal{R}}^{[A}_C\varphi^{CB]} + \frac{2}{n}\mathring{\mathcal{R}}\varphi^{AB} - \frac{2}{n}\tilde{\lambda}\mathcal{F}^{AB}.
\end{aligned} \tag{55}$$

As before, we again introduce an additional symmetric tensor field, $g^{AB}(x)$ as a premetric tensor field. The components of the premetric tensor field are $\mathcal{SL}(n+1,\mathbb{R})$ tensor valued 0-forms. The infinitesimal transformation of the premetric field under local $\mathcal{SL}(n+1,\mathbb{R})$ is given by

$$\delta g^{AB} = \mathring{\lambda}^{(A}_C g^{CB)}, \quad \delta g_{AB} = \mathring{\lambda}^C{}_{(A}g_{CB)}. \tag{56}$$

With the help of the premetric tensor field, we can define the following combined structure,

$$\mathring{\mathcal{R}}^{AB} = \mathring{\mathcal{R}}^{[A}_C g^{CB]}, \tag{57}$$

and one can easily show that the gauge variation of $\mathring{\mathcal{R}}^{AB}$ is given by

$$\delta\mathring{\mathcal{R}}^{AB} = \mathring{\lambda}^{[A}_E\mathring{\mathcal{R}}^{EB]}. \tag{58}$$

Now, we can define a shifted curvature as follows

$$\mathcal{Y}^{AB} = \mathring{\mathcal{R}}^{AB} - \mu\mathcal{F}^{AB}, \tag{59}$$

where μ is a dimensionless arbitrary constant. This object also has the following gauge transformation,

$$\delta\mathcal{Y}^{AB} = \mathring{\lambda}^{[A}_E\mathcal{Y}^{EB]}. \tag{60}$$

At this point, to find a gravitational action, similar to the previous section, we again consider the Stelle–West model. Thus, we write the following action in 5 dimensions,

$$S_{MSL} = \sigma\int V^E\eta_{ABCDE}\mathcal{Y}^{AB}\wedge\mathcal{Y}^{CD} + \alpha\Big(c^2 - V_A V^A\Big), \tag{61}$$

where, similar to the previous section, V^A and $V_A = V^C g_{CA}$ are 0-form non-dynamical five-vector fields with respect to the special-linear group in 5 dimensions and satisfy

$\delta V^A = \mathring{\lambda}^A{}_C V^C$ and $\delta V_A = \mathring{\lambda}^C{}_A V_C$, respectively. Moreover, the transformation of the fully anti-symmetric tensor η_{ABCDE} can be given as

$$\delta\eta_{ABCDE} = -\mathring{\lambda}^F{}_A\eta_{FBCDE} - \mathring{\lambda}^F{}_B\eta_{AFCDE} - \mathring{\lambda}^F{}_C\eta_{ABFDE} - \mathring{\lambda}^F{}_D\eta_{ABCFE} - \mathring{\lambda}^F{}_E\eta_{ABCDF}. \tag{62}$$

Here, making use of Equation (39), one can decompose the special linear transformation parameter as $\mathring{\lambda}^A_B = \tilde{\lambda}^A_B - \frac{1}{n}\delta^A_B\tilde{\lambda}$. Thus, it can be easily shown that $\delta\eta_{ABCDE} = 0$.

The action in Equation (61) is a slightly modified version of Equation (31) due to the special-linear group symmetry and its equations of motion being the same as that of Equation (33).

A Gravitational Action in Four Dimensions

To find a gravitational model in four dimensions, we will use the action in Equation (61). First of all, we assume that the premetric tensor field g^{AB} is diagonal unless otherwise indicated. The special-linear connection tensor then decomposes into anti-symmetric and symmetric parts via the premetric tensor as

$$\mathring{\omega}^A_B = \omega^{AC}g_{CB} + v^{AC}g_{CB}, \tag{63}$$

where $\omega^{AC}(x)$ is anti-symmetric and $v^{AC}(x)$ is symmetric, with respect to the indices. Then we can decompose the curvature 2-forms as follows

$$\mathring{\mathcal{R}}^{AB} = \mathring{R}^{AB} + E^{AB}, \tag{64}$$

where R^{AB} is the anti-symmetric part and E^{AB} is the symmetric part and their explicit expressions are given by

$$\begin{aligned} \mathring{R}^{AB} &= d\omega^{AB} + \omega^A_C \wedge \omega^{CB} + v^A_C \wedge v^{CB}, \\ E^{AB} &= dv^{AB} + \omega^{(A}_C \wedge v^{CB)}. \end{aligned} \tag{65}$$

In this regard, the Maxwell curvature 2-form can also be written as

$$\mathcal{F}^{AB} = dB^{AB} + \omega^{[A}_C \wedge B^{CB]} + v^{[A}_C \wedge B^{CB]}, \tag{66}$$

and the shifted curvature 2-form takes the following form

$$\mathcal{Y}^{AB} = \mathring{R}^{AB} + E^{AB} - \mu\mathcal{F}^{AB}. \tag{67}$$

Now, we will reduce the space-time dimension from 5 to 4. Since $g^{A4} = 0$, one can identify the field $g^{ab}(x)$ as the metric tensor in the four dimensions with the variation $\delta g^{ab} = \mathring{\lambda}^{(a}{}_c g^{cb)}$, under infinitesimal gauge transformation. Under these circumstances, the action in Equation (61) spontaneously breaks down to the MacDowell–Mansouri-like action [65]

$$\begin{aligned} S_{MSL} &= -\frac{1}{4\kappa\Lambda}\int \eta_{abcd}\mathcal{Y}^{ab}\wedge\mathcal{Y}^{cd} \\ &= -\frac{1}{4\kappa\Lambda}\int \eta_{abcd}\Big(\mathring{\mathcal{R}}^{ab}\wedge\mathring{\mathcal{R}}^{cd} - 2\mu\mathring{\mathcal{R}}^{ab}\wedge\mathcal{F}^{cd} + 4\mu\mathcal{F}^{ab}\wedge\mathcal{F}^{cd}\Big). \end{aligned} \tag{68}$$

where we again used $V^A = (c, 0, 0, 0, 0)$ and $\sigma c = -\frac{1}{4\kappa\Lambda}$. We also note that $\mathring{\mathcal{R}}^{ab} = \mathring{R}^{ab} + E^{ab}$ is asymmetric under the interchange of indices, so only the anti-symmetric part of $\mathring{\mathcal{R}}^{ab}$ contributes to the equations of motion in the presence of the fully anti-symmetric tensor η_{abcd}. At this point, one can redefine the shifted curvature in terms of an anti-symmetric part of $\mathring{\mathcal{R}}^{ab}$ by excluding the symmetric E^{ab} part as

$$Y^{ab} = \boldsymbol{R}^{ab} - \mu F^{ab}, \tag{69}$$

where the new objects $\boldsymbol{R}^{ab}$ and F^{ab} are given by

$$\begin{aligned}\boldsymbol{R}^{ab} &= R^{ab}(\omega) + v^{a}{}_{c} \wedge v^{cb}, \\ F^{ab} &= dB^{ab} + \omega^{[a}_{c} \wedge B^{cb]} + v^{[a}_{c} \wedge B^{cb]} - \lambda e^{a} \wedge e^{b} - \lambda r^{a} \wedge r^{b} + 2\lambda b^{a} \wedge b^{b},\end{aligned} \tag{70}$$

with the following linear combinations

$$e^{a}(x) = \frac{1}{\sqrt{|\Lambda|}}(\omega^{a5} - \mu B^{a5}), \tag{71}$$

$$r^{a}(x) = \frac{1}{\sqrt{|\Lambda|}}(v^{a5} - \mu B^{a5}), \tag{72}$$

$$b^{a}(x) = \frac{\mu}{\sqrt{|\Lambda|}}B^{a5}. \tag{73}$$

The field $e^{a}(x)$ may be identified as the generalized vierbein field and the remaining objects are additional vector fields. If one assumes $\omega^{ab}(x)$ to be the Riemannian connection 1-form, then one can identify $R^{ab}(\omega) = d\omega^{ab} + \omega^{a}{}_{c} \wedge \omega^{cb}$ to be a Riemann curvature 2-form. Making use of the new definition of the shifted curvature Equation (69), the action Equation (68) can be written as follows,

$$\begin{aligned}S_{MSL} = &-\frac{1}{2\kappa}\int \eta_{abcd}\left[\frac{1}{2\Lambda}R^{ab}(\omega) \wedge R^{cd}(\omega) + \frac{1}{\Lambda}R^{ab}(\omega) \wedge v^{c}{}_{e} \wedge v^{ed}\right] \\ &-\eta_{abcd}\left[\frac{\mu}{\Lambda}R^{ab}(\omega) \wedge F^{cd} + \frac{\mu}{\Lambda}F^{ab} \wedge v^{c}{}_{e} \wedge v^{ed} - \frac{\mu^{2}}{2\Lambda}F^{ab} \wedge F^{cd}\right].\end{aligned} \tag{74}$$

The first term in the first line is the topological Gauss–Bonnet term with respect to $R^{ab}(\omega)$, which does not contribute to the equations of motion. In the second line, the first term contains generalized Einstein–Hilbert action together with some additional interaction terms. The third term contains the generalized cosmological constant term and additional interaction terms between $B^{ab}(x)$, $e^{a}(x)$, $v^{a}(x)$, and $c^{a}(x)$ fields.

4. Conclusions

In this work, we investigated the Maxwell extensions of the general-linear group $\mathcal{GL}(n,\mathbb{R})$ and the special-linear group $\mathcal{SL}(n+1,\mathbb{R})$. Firstly, we presented the Maxwell extension of the $\mathcal{GL}(n,\mathbb{R})$ group and we showed that this extension leads to the Maxwell algebra [16,17], Maxwell–Weyl algebra [21], and Maxwell–Affine algebra [22] when we chose appropriate subalgebras of $\mathcal{GL}(5,\mathbb{R})$. In this context, we also derived a new type of Maxwell–Affine algebra endowed with a metric tensor in Equation (8). Moreover, we constructed the gauge theory of gravity based on the extended case (3) and we wrote down a Stelle–West-like gauge invariant gravitational action in 5 dimensions.

Then we analyzed the action in Equation (23) for two conditions under the dimensional reduction from 5 to 4 dimensions (an alternative approach to the Stelle–West method, one can use the coset space dimensional reduction method [66,67]). In the first one, we kept the affine characteristics of the construction and we found an action in Equation (34) which has a similar structure as that of the action given in [22], but with a more general gravitational action. In the second condition, we employed the premetric tensor as a diagonal metric tensor for the tangent space and found a generalized gravitational theory which includes the Einstein–Hilbert-like action together with the generalized cosmological term in Equation (38).

Secondly, we demonstrated the tensor extension of the $\mathcal{SL}(n+1,\mathbb{R})$ group in the context of Maxwell symmetry. By using suitable subalgebras of $\mathcal{SL}(n+1,\mathbb{R})$, we obtained the Maxwell-special-affine algebra [24] and additional new kinds of extended algebras.

The algebra given in Equation (46) corresponds to the Maxwell extension of the meta-linear algebra [62,63] and in Equation (48) we presented a new kind of Maxwell-special-affine algebra endowed with a metric tensor. Moreover, we constructed the gauge theory of gravity and we derived a modified gravitation theory in five dimensions from the Stelle–West-like action. We then reduced the dimension of the action in Equation (61) from 5 to 4 dimensions, and we derived a gravitational action which contains the Einstein–Hilbert term, a generalized cosmological term together with additional terms. This result generalizes the results given in [24].

Finally, we can infer that the Maxwell extension of $\mathcal{GL}(n,\mathbb{R})$ and $\mathcal{SL}(n+1,\mathbb{R})$ algebras lead to the derivation of richer gravity theories which may include generalized cosmological constant terms and some additional terms in 4-dimensional space-time. It is well-known that dark energy may be described by adding the cosmological constant term to the standard Einstein–Hilbert action, so the gauge theory of the Maxwell extended algebras may play an important role to explain dark energy phenomenon.

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