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Certain Inclusion Properties for the Class of q-Analogue of Fuzzy α-Convex Functions

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Abstract: Recently, the properties of analytic functions have been mainly discussed by means of a fuzzy subset and a q-difference operator. We define certain new subclasses of analytic functions by using the fuzzy subordination to univalent functions whose range is symmetric with respect to the real axis. We introduce the family of linear q-operators and define various classes associated with these operators. The inclusion results and various integral properties are the main investigations of this article.

Keywords: analytic functions; fractional derivative, fuzzy q-starlike functions; fuzzy q-convex functions; q-Ruscheweyh derivative operator; q-Srivastava–Attiya operator

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1. Introduction

The phrase "q-calculus" refers to classical calculus without the concept of limits. q-calculus has recently garnered a lot of attention from mathematicians due to its applications in the study of, for example, q-deformed super-algebras, quantum groups, optimal control problems, fractal and multi-fractal measures, and chaotic dynamical systems. Following the introduction of the idea of q-calculus, various authors [1–4] have analyzed classical complex operators in terms of q-calculus. The application of q-calculus involving q-derivatives and q-integrals was initiated by the author of [5,6]. The class of analytic functions $\mathfrak{f}(\varpi)$ in the open unit disk $\Omega = \{\varpi : |\varpi| < 1\}$ is denoted by $\mathbf{X}(\Omega)$. The class $\mathcal{X}_r$ contains the functions $\mathfrak{f} \in \mathbf{X}(\Omega)$ containing a series of the form:

$$\mathfrak{f}(\varpi) = \varpi + \sum_{k=r+1}^{\infty} a_k \varpi^k, \ (\varpi \in \Omega). \tag{1}$$

For $r = 1$, we have $\mathcal{X}_1 = \mathcal{X}$; the class of normalized analytic functions in Ω. We denote the classes of univalent functions, starlike functions, and convex functions by S, S^*, and C, respectively. For $q \in (0,1)$, Jackson [5] introduced and studied the q-difference operator, which is defined by:

$$D_q\mathfrak{f}(\varpi) = \frac{\mathfrak{f}(\varpi) - \mathfrak{f}(q\varpi)}{(1-q)\varpi}; \quad q \neq 1, \ \varpi \neq 0. \tag{2}$$

We note that $\lim\limits_{q\to 1^-} D_q\mathfrak{f}(\varpi) = \mathfrak{f}'(\varpi)$, where $\mathfrak{f}'(\varpi)$ is the usual derivative of the function.

We note that

$$D_q\left\{\sum_{k=1}^{\infty} a_k\varpi^k\right\} = \sum_{k=1}^{\infty}[k]_q a_k\varpi^{k-1}, \tag{3}$$

where

$$[k]_q = \frac{1-q^k}{1-q} = \sum_{k=0}^{\infty} q^k, \, (\varpi \in \Omega). \tag{4}$$

For the following fundamental properties of q-difference operator, we refer to [7,8].

$$D_q(x\mathfrak{f}_1(\varpi) \pm y\mathfrak{f}_2(\varpi)) = xD_q\mathfrak{f}_1(\varpi) \pm yD_q\mathfrak{f}_2(\varpi).$$

$$D_q(\mathfrak{f}_1(\varpi)\mathfrak{f}_2(\varpi)) = \mathfrak{f}_1(q\varpi)D_q(\mathfrak{f}_2(\varpi)) + \mathfrak{f}_2(\varpi)D_q(\mathfrak{f}_1(\varpi)).$$

$$D_q\left(\frac{\mathfrak{f}_1(\varpi)}{\mathfrak{f}_2(\varpi)}\right) = \frac{D_q(\mathfrak{f}_1(\varpi))\mathfrak{f}_2(\varpi) - \mathfrak{f}_1(\varpi)D_q(\mathfrak{f}_2(\varpi))}{\mathfrak{f}_2(q\varpi)\mathfrak{f}_2(\varpi)}, \quad \mathfrak{f}_2(q\varpi)\mathfrak{f}_2(\varpi) \neq 0.$$

$$D_q(\log\mathfrak{f}(\varpi)) = \frac{\ln q D_q(\mathfrak{f}(\varpi))}{(q-1)\mathfrak{f}(\varpi)}.$$

The concepts of geometric function theory and q-theory were connected by introducing a q-analogue of the starlike functions in [9]. Such functions are called q-starlike functions and the class of these functions is denoted by S_q^*. The class C_q stands for the class of q-convex functions. q-Mocanu-type functions were discussed by the authors of [10,11]. The systematic application of the q-difference operator in the framework of geometric function theory was studied by Srivastava [12] in 1989. Furthermore, beneficial for readers who are interested in geometric function theory, is the survey-cum-expository review study by the same author [13]. This review study methodically emphasized several different fractional q-calculus applications in geometric function theory. For more recent contributions associated with the q-difference operator, we refer to [14–19]. The study of linear operators plays a significant role in the theory of functions. Many prominent mathematicians in this field of study are interested in introducing and studying the linear operators in terms of q-analogues.

In [20], the authors introduced an operator $R_q^\lambda : \mathcal{X} \to \mathcal{X}$ defined by:

$$R_q^\lambda\mathfrak{f}(\varpi) = \varpi + \sum_{k=1}^{\infty}\frac{[k+\lambda-1]_q}{[\lambda]_q![k-1]_q!}a_k\varpi^k, \ (\lambda > -1), \tag{5}$$

where $\mathfrak{f} \in \mathcal{X}$ and

$$[k]_q! = \begin{cases} [k]_q[k-1]_q \ldots\ldots [1]_q; & k = 1,2,\ldots \\ 1; & k = 0. \end{cases}$$

For $\lambda = m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we have

$$R_q^m\mathfrak{f}(\varpi) = \frac{\varpi D_q^m\left(\varpi^{m-1}\mathfrak{f}(\varpi)\right)}{[m]_q!}.$$

From this, we can easily deduce that:

$$R_q^0\mathfrak{f}(\varpi) = \mathfrak{f}(\varpi) \text{ and } R_q^1\mathfrak{f}(\varpi) = \varpi D_q\mathfrak{f}(\varpi).$$

Particularly, for $q \to 1^-$, the operator R^λ, known as the Ruscheweyh derivative operator, is implied, for detail see [21].

The authors in [22] introduced the q-Srivastava–Attiya operator. First, for $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $s \in \mathbb{C}$ when $|\varpi| < 1$ and $\Re(s) > 1$ when $|\varpi| = 1$, they defined the q-Hurwitz–Lerch zeta function as the following:

$$\phi_q(s,b;\varpi) = \sum_{k=0}^{\infty}\frac{\varpi^k}{[n+b]_q^s}.$$

Equivalently, we have

$$\begin{aligned}\psi_q(s,b;\varpi) &= [k+b]_q^s\left\{\phi_q(s,b;\varpi)-[b]_q^s\right\}\\ &= \varpi+\sum_{k=2}^{\infty}\left(\frac{[1+b]_q}{[k+b]_q}\right)^s\varpi^k. \end{aligned}\tag{6}$$

Then, by making use of (6) and (1), they defined the q-Srivastava–Attiya operator, $J_{q,b}^s:\mathcal{X}\to\mathcal{X}$, as

$$\begin{aligned}J_{q,b}^s\mathfrak{f}(\varpi) &= \psi_q(s,b;\varpi)*\mathfrak{f}(\varpi)\\ &= \varpi+\sum_{k=2}^{\infty}\left(\frac{[1+b]_q}{[k+b]_q}\right)^s a_k\varpi^k. \end{aligned}\tag{7}$$

In particular, if we take $q\to 1^-$, then this operator, $J_{q,b}^s$, reduces to the Srivastava–Attiya operator [23]. We use (5) and (7) to define $RJ_{q,b}^{s,\lambda}:\mathcal{X}\to\mathcal{X}$ by

$$RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)=\varpi+\sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q}{[\lambda]_q![k-1]_q!}\left(\frac{[1+b]_q}{[k+b]_q}\right)^s a_k\varpi^k. \tag{8}$$

The following identities can easily be deduced from (8):

$$\varpi D_q\left(RJ_{q,b}^{\lambda,s+1}\mathfrak{f}(\varpi)\right)=\left(1+\frac{[b]_q}{q^b}\right)RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)-\frac{[b]_q}{q^b}RJ_{q,b}^{\lambda,s+1}\mathfrak{f}(\varpi). \tag{9}$$

$$\varpi D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)=\left(1+\frac{[\lambda]_q}{q^\lambda}\right)RJ_{q,b}^{\lambda+1,s}\mathfrak{f}(\varpi)-\frac{[\lambda]_q}{q^\lambda}RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi). \tag{10}$$

The subordination of analytic functions $\mathcal{P}$ and $\mathcal{Q}$ denoted by $\mathcal{P}\prec\mathcal{Q}$ are defined as $\mathcal{P}(\varpi)=\mathcal{Q}(w(\varpi))$, where $w(\varpi)$ is Schwartz function in Ω (see [24]). Moreover, the idea of differential subordination was introduced and investigated by the authors in [25,26]. G.I. Oros and Gh. Oros were the first to study fuzzy subordination and differential subordination. For more information, see [27,28]. The study of fuzzy differential subordination involved the work of several scholars, for example, see [29–38]. Here, we provide a brief review of a few key fundamental ideas pertaining to the fuzzy differential subordination and q-calculus.

Definition 1 ([39])**.** *Let $\mathcal{S}\neq\phi$. When F maps from $\mathcal{S}$ to $[0,1]$, F is referred to as a fuzzy subset of $\mathcal{S}$.*

The fuzzy subset can also be defined as the following.

Definition 2 ([39])**.** *A Fuzzy subset of $\mathcal{S}$ is a pair $((\mathfrak{I},F_{\mathfrak{I}})$, where $F_{\mathfrak{I}}:\mathcal{S}\to[0,1]$ is known as the membership function of the fuzzy set $(\mathfrak{I},F_{\mathfrak{I}})$ and $\mathfrak{I}=\{x\in\mathcal{S}:0<F_{\mathfrak{I}}(x)\leq 1\}=\sup(\mathfrak{I},F_{\mathfrak{I}})$ is called the support of fuzzy set $(\mathfrak{I},F_{\mathfrak{I}})$.*

Definition 3 ([39])**.** *Fuzzy subsets $(\mathfrak{I}_1,F_{\mathfrak{I}_1})$ and $(\mathfrak{I}_2,F_{\mathfrak{I}_2})$ of $\mathcal{S}$ are equal if and only if $\mathfrak{I}_1=\mathfrak{I}_2$, whereas $(\mathfrak{I}_1,F_{\mathfrak{I}_1})\subseteq(\mathfrak{I}_2,F_{\mathfrak{I}_2})$ if and only if $F_{\mathfrak{I}_1}(\eta)\leq F_{\mathfrak{I}_2}(\eta)$, $\eta\in\mathcal{S}$.*

Definition 4 ([28])**.** *The fuzzy subordination of analytic functions $\mathfrak{f}$ and $\mathfrak{g}$ is denoted by $\mathfrak{f}\prec_F\mathfrak{g}$ (or $\mathfrak{f}(\varpi)\prec_F\mathfrak{g}(\varpi)$) if:*

$$\mathfrak{f}(\varpi_0)=\mathfrak{g}(\varpi_0)\ \textit{and}\ F(\mathfrak{f}(\varpi))\leq F(\mathfrak{g}(\varpi)),\varpi\in\mathfrak{D},$$

where $\mathfrak{D} \subset \mathbb{C}$ and ϖ_0 are a fixed point in $\mathfrak{D}$.

Remark 1. *One of the following function $\mathfrak{F}_i : \mathbb{C} \to [0,1]$, $(i = 1,2,3,4)$, may be used as an example.*

$$\mathfrak{F}_1(\varpi) = \frac{|\varpi|}{1+|\varpi|}, \mathfrak{F}_2(\varpi) = \frac{1}{1+|\varpi|}, \mathfrak{F}_3(\varpi) = |\sin|\varpi||, \mathfrak{F}_4(\varpi) = |\cos|\varpi||.$$

Remark 2. *The notions of classical subordination and the fuzzy subordination coincides when $\mathfrak{D} = \Omega$ in Definition 4.*

After the authors of [40] established the idea, numerous prominent researchers in [41–43] have contributed to this topic by employing the fuzzy subordination connected to specific operators. We mention here a few recent contributions that are published in the same direction [32,44–49]. In many diverse areas of study, including engineering, biological systems with memory, electric networks, computer graphics, physics, turbulence, etc., the operators connected to fuzzy differential subordination have a wide range of applications. Using the Caputo–Fabrizio fractional derivative in the context of biological systems, Baleanu et al. [50] proposed a novel study on the mathematical modeling of the human liver. Additionally, Srivastava et al. [51] examined the analysis of the transmission dynamics of the dengue infection in terms of the fractional calculus. The authors in [52] used a new integral transform to study the Korteweg–de Vries equation, where the fractional derivative is proposed in the Caputo sense. This equation was developed to represent a broad spectrum of physical behaviors of the evolution and association of nonlinear waves. One can refer to [30,35,53] for more applications. Now, by using the concepts of the q-difference operator and the fuzzy subordination, we define the following classes:

Let T be the class of analytic functions $\varphi(\varpi)$ which are univalent convex functions in Ω with $\varphi(0) = 1$ and $\Re(\varphi(\varpi)) > 0$ in Ω and where $\varphi(\Omega)$ is symmetric with respect to the real axis. Now, for $\varphi(\varpi) \in T$ and $q \in (0,1)$ with $F : \mathbb{C} \to [0,1]$, $0 \neq \eta \in \mathbb{C}$, $s \geq 0$ and $b \in \mathbb{N}$, we define the following.

Definition 5. *Let $\mathfrak{f} \in \mathcal{X}$, $\varphi \in T$, $0 \leq \alpha \leq 1$ and $q \in (0,1)$. Then, $\mathfrak{f} \in FM_q(\alpha;\varphi)$ if and only if*

$$(1-\alpha)\frac{\varpi D_q \mathfrak{f}(\varpi)}{\mathfrak{f}(\varpi)} + \alpha\frac{D_q(\varpi D_q \mathfrak{f}(\varpi))}{D_q \mathfrak{f}(\varpi)} \prec_F \varphi(\varpi).$$

Moreover, let us denote

$$FM_q(0;\varphi) = FST_q(\varphi), \qquad FM_q(1;\varphi) = FC_q(\varphi).$$

A function $\mathfrak{f} \in \mathcal{X}$ is in $FST_q(\varphi)$ and $FC_q(\varphi)$ if and only if

$$\frac{\varpi D_q \mathfrak{f}(\varpi)}{\mathfrak{f}(\varpi)} \prec_F \varphi(\varpi) \text{ and } \frac{D_q(\varpi D_q \mathfrak{f}(\varpi))}{D_q \mathfrak{f}(\varpi)} \prec_F \varphi(\varpi),$$

respectively.

Special cases:

(i) For $q \to 1^-$, we have the class $FM_q(\alpha;\varphi) = FM_\alpha(\varphi)$ introduced in [36].

(ii) For $q \to 1^-$ and $\alpha = 0$, we have the class $FM_q(\alpha;\varphi) = FS^*(\varphi)$ studied by Shah et al. [36].

(iii) If $q \to 1^-$ and $\alpha = 1$, then we have the class $FM_q(\alpha;\varphi) = FC(\varphi)$ introduced by Shah et al. [36].

Here, some new classes are defined by applying the q-linear operator given by (8):

Definition 6. *Let $\mathfrak{f} \in \mathcal{X}$, $\varphi \in T$, $0 \leq \alpha \leq 1$, $q \in (0,1)$, $\lambda > -1$, $b > -1$ and s be real. Then,*

$$\mathfrak{f} \in FM_{q,b}^{\lambda,s}(\alpha;\varphi) \text{ if and only if } RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi) \in FM_q(\alpha;\varphi).$$

Furthermore,

$$\mathfrak{f} \in FST_{q,b}^{\lambda,s}(\varphi) \text{ if and only if } RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi) \in FST_q(\varphi)$$

and

$$\mathfrak{f} \in FC_{q,b}^{\lambda,s}(\varphi) \text{ if and only if } RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi) \in FC_q(\varphi).$$

We note that

$$\mathfrak{f} \in FC_{q,b}^{\lambda,s}(\varphi) \text{ if and only if } \varpi\left(D_q\mathfrak{f}\right) \in FST_{q,b}^{\lambda,s}(\varphi). \tag{11}$$

Special cases:

(i) If $s = 0 = \lambda$, then $FM_{q,b}^{\lambda,s}(\alpha;\varphi) = FM_q(\alpha;\varphi)$, $FST_{q,b}^{\lambda,s}(\varphi) = FST_q(\varphi)$ and $FC_{q,b}^{\lambda,s}(\varphi) = FC_q(\varphi)$.

(ii) If $q \to 1^-$ and $\lambda = 0$, then the classes $FM_{q,b}^{\lambda,s}(\alpha;\varphi)$, $FST_{q,b}^{\lambda,s}(\varphi)$ and $FC_{q,b}^{\lambda,s}(\varphi)$ are reduced to the classes $FM_{\alpha}^{s,b}(\varphi)$, $FST_b^s(\varphi)$ and $FC_b^s(\varphi)$ introduced by Shah et al. [36].

(iii) If $q \to 1^-$ and $s = 0 = \lambda$, then $FM_{q,b}^{\lambda,s}(\alpha;\varphi) = FM_{\alpha}(\varphi)$, $FST_{q,b}^{\lambda,s}(\varphi) = FST(\varphi)$ and $FC_{q,b}^{\lambda,s}(\varphi) = FC(\varphi)$, we refer to [36].

2. Main Results

The following lemma is needed to prove our investigations.

Lemma 1 ([54]). *Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$, and let $h(\varpi) \in T$ with*

$$\Re\{\beta h(\varpi) + \gamma\} > 0. \tag{12}$$

If $p(\varpi) = 1 + p_1\varpi + p_2\varpi^2 + \ldots$ is analytic in Ω, then

$$p(\varpi) + \frac{\varpi D_q p(\varpi)}{\beta p(\varpi) + \gamma} \prec_F h(\varpi) \text{ implies } p(\varpi) \prec_F h(\varpi),$$

where $F : \mathbb{C} \to [0,1]$.

Theorem 1. *Let $0 \leq \alpha \leq 1$, $\varphi \in T$, $q \in (0,1)$, $\lambda > -1$, s be real and $b > -1$. Then,*

(i) $FM_{q,b}^{\lambda,s}(\alpha;\varphi) \subset FST_{q,b}^{\lambda,s}(\varphi)$ *for* $0 \leq \alpha \leq 1$.

(ii) $FM_{q,b}^{\lambda,s}(\alpha;\varphi) \subset FST_{q,b}^{\lambda,s}(\varphi)$ *for* $\alpha \geq 1$.

(iii) $FM_{q,b}^{\lambda,s}(\alpha_2;\varphi) \subset FM_{q,b}^{\lambda,s}(\alpha_1;\varphi)$ *for* $0 \leq \alpha_1 < \alpha_2 < 1$.

Proof. (i) Let $\mathfrak{f} \in FM_{q,b}^{\lambda,s}(\alpha;\varphi)$. We set

$$\frac{\varpi D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)}{RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)} = p(\varpi), \tag{13}$$

for analytic $p(\varpi)$ in Ω with $p(0) = 1$.

The q-logarithmic differentiation of (13) yields:

$$\frac{D_q\left(\varpi D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)\right)}{\varpi D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)} - \frac{D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)}{RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)} = \frac{D_q p(\varpi)}{p(\varpi)}.$$

Equivalently,

$$\frac{D_q\left(\varpi D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)\right)}{D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)} = p(\varpi) + \frac{\varpi D_q p(\varpi)}{p(\varpi)}.$$

Since $\mathfrak{f} \in FM_{q,b}^{\lambda,s}(\alpha;\varphi)$, we obtain:

$$\begin{aligned}(1-\alpha)\frac{\varpi D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)}{RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)} + \alpha\frac{D_q\left(\varpi D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)\right)}{D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)} &= p(\varpi) + \alpha\frac{\varpi D_q p(\varpi)}{p(\varpi)} \\ &\prec_F \varphi(\varpi). \end{aligned} \tag{14}$$

We use Lemma 1 to obtain $p(\varpi) \prec_F \varphi(\varpi)$. Consequently, $\mathfrak{f} \in FST_{q,b}^{\lambda,s}(\varphi)$.
(ii) Suppose that $\mathfrak{f} \in FM_{q,b}^{\lambda,s}(\alpha;\varphi)$. Then,

$$(1-\alpha)\frac{\varpi D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)}{RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)} + \alpha\frac{D_q\left(\varpi D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)\right)}{D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)} = p_1(\varpi) \prec_F \varphi(\varpi).$$

Now,

$$\begin{aligned}\alpha\frac{D_q\left(\varpi D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)\right)}{D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)} &= (1-\alpha)\frac{\varpi D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)}{RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)} + \alpha\frac{D_q\left(\varpi D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)\right)}{D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)} \\ &\quad +(\alpha-1)\frac{\varpi D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)}{RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)} \\ &= (\alpha-1)\frac{\varpi D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)}{RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)} + p_1(\varpi).\end{aligned}$$

This implies

$$\begin{aligned}\frac{D_q\left(\varpi D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)\right)}{D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)} &= \left(\frac{1}{\alpha}-1\right)\frac{\varpi D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)}{RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)} + \frac{1}{\alpha}p_1(\varpi) \\ &= \left(\frac{1}{\alpha}-1\right)p_2(\varpi) + \frac{1}{\alpha}p_1(\varpi).\end{aligned}$$

Since $p_1, p_2 \prec_F \varphi(\varpi)$, we can write $\frac{\varpi D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)}{RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)} \prec_F \varphi(\varpi)$. This completes the proof of (ii).
(iii) For $\alpha_1 = 0$, the result from part (i) is true.

Now, we suppose that $\mathfrak{f} \in FM^{\lambda,s}_{q,b}(\alpha_2;\varphi)$. Then,

$$(1-\alpha_2)\frac{\varpi D_q\left(RJ^{\lambda,s}_{q,b}\mathfrak{f}(\varpi)\right)}{RJ^{\lambda,s}_{q,b}\mathfrak{f}(\varpi)}+\alpha_2\frac{D_q\left(\varpi D_q\left(RJ^{\lambda,s}_{q,b}\mathfrak{f}(\varpi)\right)\right)}{D_q\left(RJ^{\lambda,s}_{q,b}\mathfrak{f}(\varpi)\right)}=q_1(\varpi)\prec_F\varphi(\varpi). \tag{15}$$

Now, we can easily write

$$J_q(\alpha_1,\mathfrak{f}(\varpi))=\frac{\alpha_1}{\alpha_2}q_1(\varpi)+\left(1-\frac{\alpha_1}{\alpha_2}\right)q_2(\varpi), \tag{16}$$

with

$$J_q(\alpha_1,\mathfrak{f}(\varpi))=(1-\alpha_1)\frac{\varpi D_q\left(RJ^{\lambda,s}_{q,b}\mathfrak{f}(\varpi)\right)}{RJ^{\lambda,s}_{q,b}\mathfrak{f}(\varpi)}+\alpha_1\frac{D_q\left(\varpi D_q\left(RJ^{\lambda,s}_{q,b}\mathfrak{f}(\varpi)\right)\right)}{D_q\left(RJ^{\lambda,s}_{q,b}\mathfrak{f}(\varpi)\right)},$$

where we have used (15) and $\frac{\varpi D_q\mathfrak{f}(\varpi)}{\mathfrak{f}(\varpi)}=q_2(\varpi)\prec_F\varphi(\varpi)$. (16) implies our required result. □

Corollary 1. *For $\lambda=0=s$, we have $FM_q(\alpha;\varphi)\subset FST_q(\varphi)$. Furthermore, for $q\to1^-$, $FM_\alpha(\varphi)\subset FST(\varphi)$, see [36].*

Corollary 2. *For $q\to1^-$ and $\lambda=0$, we have $FM^s_b(\alpha;\varphi)\subset FST^s_b(\varphi)$. Moreover, for $s=0$ and $\alpha=1$, we have $FC(\varphi)\subset FST(\varphi)$ and $FC\subset FST$ when $\varphi(\varpi)=\frac{1+\varpi}{1-\varpi}$. We refer to [36].*

Corollary 3. *For $s=0=\lambda$, we have $FM_q(\alpha;\varphi)\subset FC_q(\varphi)$. Moreover, for $q\to1^-$, we have $FM_\alpha(\varphi)\subset FC(\varphi)$. We refer to [36].*

Corollary 4. *For $s=0=\lambda$, we have $FM_q(\alpha_2;\varphi)\subset FM_q(\alpha_1;\varphi)$. Moreover, for $q\to1^-$, we have $FM_{\alpha_2}(\varphi)\subset FM_{\alpha_1}(\varphi)$, see [36].*

Remark 3. *If $\alpha_2=1$ and letting $\mathfrak{f}\in FM^{\lambda,s}_{q,b}(1;\varphi)=FC^{\lambda,s}_{q,b}(\varphi)$. Then, by Theorem 1(iii), we have:*

$$\mathfrak{f}\in FM^{\lambda,s}_{q,b}(\alpha_1;\varphi),\text{ for }\ 0\le\alpha_1<1.$$

We use Theorem 1(i), to obtain $\mathfrak{f}\in FST^{\lambda,s}_{q,b}(\varphi)$. Consequently, $FC^{\lambda,s}_{q,b}(\varphi)\subset FST^{\lambda,s}_{q,b}(\varphi)$.

Theorem 2. *Let $\varphi\in T$, $0\le\alpha\le1$, $q\in(0,1)$, $\lambda\in\mathbb{N}_0$, s be real and $b>-1$. Then,*
(i) $FST^{\lambda+1,s}_{q,b}(\varphi)\subset FST^{\lambda,s}_{q,b}(\varphi)$.
(ii) $FST^{\lambda,s}_{q,b}(\varphi)\subset FST^{\lambda,s+1}_{q,b}(\varphi)$.

Proof. (i) Let $\mathfrak{f}\in FST^{\lambda+1,s}_{q,b}(\varphi)$ and let $\mathfrak{f}_{\lambda+1,q}(\varpi)=RJ^{\lambda+1,s}_{q,b}\mathfrak{f}(\varpi)$. Then,

$$\frac{\varpi D_q\mathfrak{f}_{\lambda+1,q}(\varpi)}{\mathfrak{f}_{\lambda+1,q}(\varpi)}\prec_F\varphi(\varpi).$$

Now, let

$$\frac{\varpi D_q\mathfrak{f}_{\lambda,q}(\varpi)}{\mathfrak{f}_{\lambda,q}(\varpi)}=h(\varpi) \tag{17}$$

for analytic $h(\varpi)$ in Ω with $h(0)=1$. Using (10) and (17), we obtain

$$\frac{\varpi D_q(\mathfrak{f}_{\lambda,q}(\varpi))}{\mathfrak{f}_{\lambda,q}(\varpi)}=(1+L_q)\frac{\mathfrak{f}_{\lambda+1,q}(\varpi)}{\mathfrak{f}_{\lambda,q}(\varpi)}-L_q,$$

equivalently,

$$(1+L_q)\frac{\mathfrak{f}_{\lambda+1,q}(\varpi)}{\mathfrak{f}_{\lambda,q}(\varpi)} = h(\varpi)+L_q, \quad \left(\text{for } L_q = \frac{[\lambda]_q}{q^n}\right).$$

The q-logarithmic differentiation yields:

$$\frac{\varpi D_q\left(\mathfrak{f}_{\lambda+1,q}(\varpi)\right)}{\mathfrak{f}_{\lambda+1,q}(\varpi)} = p(\varpi) + \frac{\varpi D_q h(\varpi)}{h(\varpi)+L_q}. \tag{18}$$

Since $\mathfrak{f} \in FST^{\lambda+1,s}_{q,b}(\varphi)$, (18) implies

$$p(\varpi) + \frac{\varpi D_q h(\varpi)}{h(\varpi)+L_q} \prec_F \varphi(\varpi).$$

We assume that $\Re\{\varphi(\varpi)+L_q\} > 0$ and we use Lemma 1 to obtain $h(\varpi) \prec_F \varphi(\varpi)$. Consequently, $\mathfrak{f} \in FST^{\lambda,s}_{q,b}(\varphi)$.

To prove part (ii), we follow a similar technique to that used in part (i) by taking $\mathfrak{f}^{s,b}_q(\varpi) = RJ^{\lambda,s}_{q,b}\mathfrak{f}(\varpi)$ along with identity (9). □

Corollary 5. *For $\lambda = 0$ and $q \to 1^-$ in part (ii) of the above theorem, we obtain the inclusion relation as Theorem 2.5, proven in [36].*

Theorem 3. *Let $\varphi \in T$, $0 \le \alpha \le 1$, $q \in (0,1)$, $\lambda \in \mathbb{N}_0$, s be real and $b > -1$. Then,*
(i) $FC^{\lambda+1,s}_{q,b}(\varphi) \subset FC^{\lambda,s}_{q,b}(\varphi)$.
(ii) $FC^{\lambda,s}_{q,b}(\varphi) \subset FC^{\lambda,s+1}_{q,b}(\varphi)$.

Proof. (i) Let $\mathfrak{f} \in FC^{\lambda+1,s}_{q,b}(\varphi)$. Then, by (11),

$$\varpi\left(D_q\mathfrak{f}\right) \in FST^{\lambda+1,s}_{q,b}(\varphi).$$

We use (i) of Theorem 2 to obtain:

$$\varpi\left(D_q\mathfrak{f}\right) \in FST^{\lambda,s}_{q,b}(\varphi)$$

Again, by using relation (11), we obtain

$$\mathfrak{f} \in FC^{\lambda,s}_{q,b}(\varphi).$$

In similar way, one can prove part (ii) by applying part (ii) of Theorem 2 along with the relation (11). □

Corollary 6. *For $\lambda = 0$ and $q \to 1^-$ in part (ii) of the above theorem, we obtain the inclusion relation as Theorem 2.6, proven in [36].*

Remark 4. *From Theorem 1, Theorem 2 and Theorem 3, we can extend the inclusions as the following.*

$$FM^{\lambda+1,s}_{q,b}(\alpha;\varphi) \subset FST^{\lambda+1,s}_{q,b}(\varphi) \subset FST^{\lambda,s}_{q,b}(\varphi) \subset \ldots \subset FST^{s}_{q,b}(\varphi).$$

$$FC^{\lambda+1,s}_{q,b}(\varphi) \subset FC^{\lambda,s}_{q,b}(\varphi) \subset \ldots \subset FC^{s}_{q,b}(\varphi).$$

Theorem 4. *Let a function $\mathfrak{f} \in \mathcal{X}$. Then, $\mathfrak{f} \in FM_{q,b}^{\lambda,s}(\alpha;\varphi)$ if and only if there exists $\mathfrak{g} \in FST_{q,b}^{\lambda,s}(\varphi)$ such that*

$$\mathfrak{f}(\varpi) = \left[\frac{1}{\alpha}\right]_q \left[\int_0^{\varpi} \tau^{\frac{1}{\alpha}-1}\left(\frac{\mathfrak{g}(\tau)}{\tau}\right)^{\frac{1}{\alpha}} d_q\tau\right]^{\alpha}, (\alpha \neq 0). \tag{19}$$

Proof. Let $\mathfrak{f} \in FM_{q,b}^{\lambda+1,s}(\alpha;\varphi)$. Then, by Definition 6,

$$(1-\alpha)\frac{\varpi D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)}{RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)} + \alpha\frac{D_q\left(\varpi D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)\right)}{D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)} \prec_F \varphi(\varpi). \tag{20}$$

By some simple calculations in (19), we obtain:

$$\varpi D_q\mathfrak{f}(\varpi).(\alpha\mathfrak{f}(\varpi))^{\frac{1}{\alpha}-1} = (\mathfrak{g}(\varpi))^{\frac{1}{\alpha}} \tag{21}$$

We use the linear operator given by (8) in (21), and then take q-logarithmic differentiation to obtain:

$$(1-\alpha)\frac{\varpi D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)}{RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)} + \alpha\frac{D_q\left(\varpi D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)\right)}{D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{f}(\varpi)\right)} = \frac{\varpi D_q\left(RJ_{q,b}^{\lambda,s}\mathfrak{g}(\varpi)\right)}{RJ_{q,b}^{\lambda,s}\mathfrak{g}(\varpi)}. \tag{22}$$

From (20) and (22), we conclude our required result. □

Corollary 7. *For $\lambda = 0$ and $q \to 1^-$, we obtain Theorem 2.7, proven in [36].*

Theorem 5. *Let $\mathfrak{f} \in FM_{q,b}^{\lambda,s}(\alpha;\varphi)$. Then,*

$$F_{m,q}(\varpi) = \frac{[m+1]_q}{\varpi^m}\int_0^{\varpi} t^{m-1}\mathfrak{f}(t)d_qt \tag{23}$$

is in $FST_{q,b}^{\lambda,s}(\varphi)$.

Proof. Let $\mathfrak{f} \in FM_{q,b}^{\lambda,s}(\alpha;\varphi)$. If we set, for $F_{m,q}^{\lambda}(\varpi) = RJ_{q,b}^{\lambda,s}(F_{m,q}(\varpi))$,

$$\frac{\varpi D_q\left(F_{m,q}^{\lambda}(\varpi)\right)}{F_{m,q}^{\lambda}(\varpi)} = q(\varpi), \tag{24}$$

for analytic $q(\varpi)$ in Ω with $q(0) = 1$.

Simple calculations (23) imply that

$$\frac{D_q(\varpi^m F_{m,q}(\varpi))}{[m+1]_q} = \varpi^{m-1}\mathfrak{f}(\varpi).$$

This implies

$$\varpi D_q F_{m,q}(\varpi) = \left(1+\frac{[m]_q}{q^m}\right)\mathfrak{f}(\varpi) - \frac{[m]_q}{q^m}F_{m,q}(\varpi). \tag{25}$$

From (24), (25) and (8), we obtain

$$q(\varpi) = \left(1+\frac{[m]_q}{q^m}\right)\frac{\varpi(\mathfrak{f}_{\lambda,q}(\varpi))}{F_{m,q}^{\lambda}(\varpi)} - \frac{[m]_q}{q^m},$$

where $F_{m,q}^{\lambda}(\varpi) = RJ_{q,b}^{\lambda,s}(F_{m,q}(\varpi))$ and $\mathfrak{f}_{\lambda,q}(\varpi) = RJ_{q,b}^{\lambda,s}(\mathfrak{f}(\varpi))$. We take q-logarithmic differentiation:

$$\frac{\varpi D_q(\mathfrak{f}_{\lambda,q}(\varpi))}{(\mathfrak{f}_{\lambda,q}(\varpi))} = q(\varpi) + \frac{\varpi D_q q(\varpi)}{q(\varpi) + L_q}, \quad \left(\text{for } L_q = \frac{[m]_q}{q^m}\right). \tag{26}$$

Since $\mathfrak{f} \in FM_{q,b}^{\lambda,s}(\alpha;\varphi) \subset FST_{q,b}^{\lambda,s}(\varphi)$, (26) implies

$$q(\varpi) + \frac{\varpi D_q q(\varpi)}{q(\varpi) + L_q} \prec_F \varphi(\varpi).$$

Now, we apply Lemma 1 to conclude $q(\varpi) \prec_F \varphi(\varpi)$. Consequently, $\frac{\varpi D_q\left(F_{m,q}^{\lambda}(\varpi)\right)}{F_{m,q}^{\lambda}(\varpi)} \prec_F \varphi(\varpi)$. Hence, $F_{m,q} \in FST_{q,b}^{\lambda,s}(\varphi)$. □

Corollary 8. *For $\lambda = 0$ and $q \to 1^-$, we obtain Theorem 2.8, proven in [36].*

3. Conclusions

We successfully defined and studied the class of fuzzy q-Mocanu-type functions associated with the family of linear operators. The main results of our work are the generalization of various classical results in terms of the fuzzy subordination and q-theory. In this article, we studied the concepts of a fuzzy differential subordination associated with q-theory. First, we introduced the q-linear operator by combining two well-known q-operators and then, by using this operator, we defined various subclasses of analytic functions. For the newly defined classes, we investigated certain inclusion results and integral properties. As corollaries, some well-known conclusions were also mentioned.

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