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An Application of Poisson Distribution Series on Harmonic Classes of Analytic Functions

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Abstract: Many authors have obtained some inclusion properties of certain subclasses of univalent and functions associated with distribution series, such as Pascal distribution, Binomial distribution, Poisson distribution, Mittag–Leffler-type Poisson distribution, and Geometric distribution. In the present paper, we obtain some inclusion relations of the harmonic class $\mathcal{H}(\alpha,\delta)$ with the classes $\mathcal{S}^*_{\mathcal{H}}$ of starlike harmonic functions and $\mathcal{K}_{\mathcal{H}}$ of convex harmonic functions, also for the harmonic classes $\mathcal{TN}_{\mathcal{H}}(\beta)$ and $\mathcal{TR}_{\mathcal{H}}(\beta)$ associated with the operator Υ defined by applying certain convolution operator regarding Poisson distribution series. Several consequences and corollaries of the main results are also obtained.

Keywords: harmonic; univalent functions; harmonic starlike; harmonic convex; Poisson distribution

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1. Introduction

The focus of this paper is on harmonic analytic functions associated with a convolution operator defined using Poisson distribution series. Connecting certain classes of analytic functions with operators in studies for obtaining properties of the investigated classes follows a line of research popular in geometric function theory. Recently, harmonic classes of analytic functions have been defined and studied for obtaining coefficient estimates and inclusion relations. A generalized linear operator is applied in [1] for defining a new subclass of univalent functions and obtaining some geometrical properties. In [2], two new families of harmonic meromorphically functions are introduced using a certain generalized convolution q-operator and investigations regarding inclusion properties are conducted. A q-derivative operator is used for defining and researching a new class of harmonic functions in [3] and a new class of harmonic functions involving Janowski functions is defined in [4] using symmetric Sălăgean q-differential operator. Studies involving the concept of subordination and Ruscheweyh derivative are performed on a new class of harmonic functions related to starlike harmonic functions and harmonic convex functions in [5]. The concept of subordination is also used for defining the q-analogue of a new subclass of univalent harmonic functions in [6]. The dual concept of superordination is associated with harmonic complex-valued functions in [7]. Special functions continue to be used for the research on harmonic functions, such as hypergeometric functions [8–10].

The presentation of the results obtained in this paper begins by describing the classes of harmonic functions used for the study.

A continuous complex valued function $f = U + iV$ defined in a simply connected complex domain $\mathcal{D}$ is said to be harmonic in $\mathcal{D}$ if both U and V are real harmonic in $\mathcal{D}$. In any simply connected domain we can write $f = w + \overline{v}$, where w and v are analytic in $\mathcal{D}$. We call w the analytic part and v the co-analytic part of f. A necessary and sufficient for f to be locally univalent and sense preserving in $\mathcal{D}$ is that $|w'(z)| > |v'(z)|$ in $\mathcal{D}$.

Let $\mathcal{H}$ be the family of all harmonic functions of the form $f = w + \overline{v}$, where

$$w(\zeta) = \zeta + \sum_{s=2}^{\infty} a_s \zeta^s, \qquad v(\zeta) = \sum_{s=1}^{\infty} b_s \zeta^s, \quad |b_1| < 1. \tag{1}$$

are analytic in the open unit disk $E=\{\zeta : |\zeta| < 1\}$. Furthermore, let $\mathcal{S}_{\mathcal{H}}$ denote the family of functions $f = w + \bar{v}$ that are harmonic univalent and sense preserving in E.

In 1984, Clunie and Sheil-Small [11] studied the class $\mathcal{S}_{\mathcal{H}}$ and its geometric subclasses and obtained some coefficient bounds. This paper opened the way for a prolific research involving harmonic functions. Numerous results related on $\mathcal{S}_{\mathcal{H}}$ and on harmonic functions one may refer to some papers where it was studied harmonic univalent functions with negative coefficients [12], subclasses of harmonic univalent functions [13], starlike harmonic functions [14], Noshiro-type harmonic univalent functions [15], harmonic mappings [16], harmonic univalent functions [17], Planar harmonic mappings [18], harmonic functions with negative coefficients defined by the Dziok–Srivastava operator [19], uniformly harmonic β-starlike functions of complex order [20], and harmonic mappings of bounded boundary rotation [21,22].

Consider the subclass $\mathcal{S}^0_{\mathcal{H}}$ of $\mathcal{S}_{\mathcal{H}}$ as

$$\mathcal{S}^0_{\mathcal{H}} = \{f = w + \overline{v} \in \mathcal{S}_{\mathcal{H}} : v'(0) = b_1 = 0\},$$

first studied in [11].

A sense-preserving harmonic mapping $f \in \mathcal{S}^0_{\mathcal{H}}$ is in the class $\mathcal{S}^*$ if the range $f(E)$ is starlike with respect to the origin. A function $f \in \mathcal{S}^*_{\mathcal{H}}$ is called a harmonic starlike mapping in E. Additionally, a function f defined in E is included in the class $\mathcal{K}_{\mathcal{H}}$ if $f \in \mathcal{S}^0_{\mathcal{H}}$ and if $f(E)$ is a convex domain. A function $f \in \mathcal{K}_{\mathcal{H}}$ is called convex harmonic in E. Analytically, we have

$$f \in \mathcal{S}^*_{\mathcal{H}} \text{ iff } arg\left(\frac{\partial}{\partial\theta} f\left(re^{i\theta}\right)\right) \geq 0,$$

and

$$f \in \mathcal{K}_{\mathcal{H}} \text{ iff } \frac{\partial}{\partial\theta}\left\{arg\left(arg\left(\frac{\partial}{\partial\theta} f\left(re^{i\theta}\right)\right)\right)\right\} \geq 0,$$

$$\zeta = re^{i\theta} \in E,\ 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi.$$

These classes and their properties are described in [15].

Let $\mathcal{T}_{\mathcal{H}}$ be the class of functions in $\mathcal{S}_{\mathcal{H}}$ that may be expressed as $f = w + \bar{v}$, where

$$w(\zeta) = \zeta - \sum_{s=2}^{\infty} |a_s| \zeta^s,\ v(\zeta) = \sum_{s=1}^{\infty} |b_s| \zeta^s,\ |b_1| < 1. \tag{2}$$

For $0 \leq \beta < 1$, let

$$\mathcal{N}_{\mathcal{H}}(\beta) = \left\{f \in \mathcal{H} : \mathrm{Re}\left(\frac{f'(\zeta)}{\zeta'}\right) \geq \beta,\ \zeta = re^{i\theta} \in E\right\},$$

and

$$\mathcal{R}_{\mathcal{H}}(\beta) = \left\{f \in \mathcal{H} : \mathrm{Re}\left(\frac{f''(\zeta)}{\zeta''}\right) \geq \beta,\ \zeta = re^{i\theta} \in E\right\}$$

where

$$\zeta' = \frac{\partial}{\partial\theta}\left(\zeta = re^{i\theta}\right),\ \zeta'' = \frac{\partial}{\partial\theta}(\zeta'),\ f'(\zeta) = \frac{\partial}{\partial\theta} f\left(re^{i\theta}\right),\ f'' = \frac{\partial}{\partial\theta}(f'(\zeta)).$$

Define

$$\mathcal{TN}_{\mathcal{H}}(\beta) = \mathcal{N}_{\mathcal{H}}(\beta) \cap \mathcal{T}_{\mathcal{H}} \quad \text{and} \quad \mathcal{TR}_{\mathcal{H}}(\beta) = \mathcal{R}_{\mathcal{H}}(\beta) \cap \mathcal{T}_{\mathcal{H}}.$$

The classes $\mathcal{T}_{\mathcal{H}}$, $\mathcal{N}_{\mathcal{H}}(\beta)$, $\mathcal{TN}_{\mathcal{H}}(\beta)$, $\mathcal{R}_{\mathcal{H}}(\beta)$ and $\mathcal{TR}_{\mathcal{H}}(\beta)$ were defined and studied in [12,23].

Sokòl et al. [24] defined and studied the class $\mathcal{H}(\alpha,\delta)$ of functions of the form (1) that satisfy the condition

$$\mathrm{Re}\Big\{w'(\zeta)+v'(\zeta)+3\alpha\zeta(w''(\zeta)+v''(\zeta))+\alpha\zeta^3(w'''(\zeta)+v'''(\zeta))\Big\}>\delta,$$

for some $\alpha \geq 0$ and $0 \leq \delta < 1$. In particular, for $\alpha = 0$, we obtain the class $\mathcal{H}(\delta)$ which satisfy the condition

$$\mathrm{Re}\{w'(\zeta)+v'(\zeta)\}>\delta.$$

A discrete random variable X is said to have a Poisson distribution, with parameter m if it has a probability mass function given by

$$\Pr(X=\kappa)=\frac{e^{-m}}{\kappa!}m^{\kappa},\ \kappa=0,1,2,\ldots$$

and m is the parameter of the distribution.

Very recently, Porwal [25] (see also, [26,27]) defined a Poisson distribution series as

$$\mathfrak{G}(m,\zeta)=\zeta+\sum_{s=2}^{\infty}\frac{m^{s-1}}{(s-1)!}e^{-m}\zeta^s,$$

where m is called the parameter.

Now, for $\varepsilon_1,\varepsilon_2>0$, Porwal and Srivastava [28] introduced the operator $\Upsilon(\varepsilon_1,\varepsilon_2)$ for $f(\zeta)\in\mathcal{S}_{\mathcal{H}}$ as

$$\Upsilon(f)=\Upsilon(\varepsilon_1,\varepsilon_2)f(\zeta)=\mathfrak{G}(\varepsilon_1,\zeta)*w(\zeta)+\overline{\mathfrak{G}(\varepsilon_2,\zeta)*v(\zeta)}=\Phi(\zeta)+\overline{\Psi(\zeta)},\tag{3}$$

where

$$\Phi(\zeta)=\zeta+\sum_{s=2}^{\infty}\frac{\varepsilon_1^{s-1}}{(s-1)!}e^{-\varepsilon_1}a_s\zeta^s,\Psi(\zeta)=b_1\zeta+\sum_{s=2}^{\infty}\frac{\varepsilon_2^{s-1}}{(s-1)!}e^{-\varepsilon_2}b_s\zeta^s.\tag{4}$$

for $f=w+\overline{v}$ in $\mathcal{H}$.

Following the work of Porwal and Srivastava [28] (see also, [29–37]), and by applying the convolution operator Υ,we obtain some inclusion relations of the harmonic classes $\mathcal{H}(\alpha,\delta)$, $\mathcal{S}^*_{\mathcal{H}}$, $\mathcal{K}_{\mathcal{H}}$, $\mathcal{TN}_{\mathcal{H}}(\beta)$ and $\mathcal{TR}_{\mathcal{H}}(\beta)$.

2. Preliminary Lemmas

Before starting and proving our main results, we need several lemmas to be used in the sequel.

Lemma 1 ([24])**.** *Consider $f=w+\overline{v}$, where w and v are given by (1) and suppose that $\alpha\geq 0$, $0\leq\delta<1$ and*

$$\sum_{s=2}^{\infty}s[1+\alpha\Big(s^2-1\Big)]|a_s|+\sum_{s=1}^{\infty}s[1+\alpha\Big(s^2-1\Big)]|b_s|\leq 1-\delta.\tag{5}$$

then $f\in\mathcal{H}(\alpha,\delta)$.

When $f\in\mathcal{H}(\alpha,\delta)$, then

$$|a_s|\leq\frac{1-\delta}{s[1+\alpha(s^2-1)]},\ s\geq 2,\tag{6}$$

and

$$|b_s|\leq\frac{1-\delta}{s[1+\alpha(s^2-1)]},\ s\geq 1.\tag{7}$$

Lemma 2 ([15]). *Consider $f = w + \overline{v}$, where w and v are given by (2) and assume that $0 \leq \beta < 1$. Then $f \in \mathcal{TN}_{\mathcal{H}}(\beta)$ if, and only if,*

$$\sum_{s=2}^{\infty} s|a_s| + \sum_{s=1}^{\infty} s|b_s| \leq 1 - \beta. \tag{8}$$

When $f \in \mathcal{TN}_{\mathcal{H}}(\beta)$, then

$$|a_s| \leq \frac{1-\beta}{s}, \; s \geq 2, \tag{9}$$

and

$$|b_s| \leq \frac{1-\beta}{s}, \; s \geq 1. \tag{10}$$

Lemma 3 ([23]). *Consider $f = w + \overline{v}$, where w and v are given by (2), and assume that $0 \leq \beta < 1$. Then $f \in \mathcal{TR}_{\mathcal{H}}(\beta)$ if, and only if,*

$$\sum_{s=2}^{\infty} s^2|a_s| + \sum_{s=1}^{\infty} s^2|b_s| \leq 1 - \beta. \tag{11}$$

When $f \in \mathcal{TR}_{\mathcal{H}}(\beta)$, then

$$|a_s| \leq \frac{1-\beta}{s^2}, \; s \geq 2 \tag{12}$$

and

$$|b_s| \leq \frac{1-\beta}{s^2}, \; s \geq 1. \tag{13}$$

Lemma 4 ([11]). *If $f = w + \overline{v} \in \mathcal{S}^*_{\mathcal{H}}$, where w and v are given by (1) with $b_1 = 0$, then*

$$|a_s| \leq \frac{(2s+1)(s+1)}{6} \text{ and } |b_s| \leq \frac{(2s-1)(s-1)}{6}. \tag{14}$$

Lemma 5 ([11]). *If $f = w + \overline{v} \in \mathcal{K}_{\mathcal{H}}$, where w and v are given by (1) with $b_1 = 0$, then*

$$|a_s| \leq \frac{s+1}{2} \text{ and } |b_s| \leq \frac{s-1}{2}. \tag{15}$$

For convenience throughout in the sequel, we use the following notations:

$$\sum_{s=2}^{\infty} \frac{t^{s-1}}{(s-1)!} = e^t - 1$$

and

$$\sum_{s=j}^{\infty} \frac{t^{s-1}}{(s-j)!} = t^{j-1}e^t, \quad j \geq 2.$$

3. Inclusion Relations of the Class $\mathcal{H}(\alpha, \delta)$

In this section we will prove the inclusion relations of the harmonic class $\mathcal{H}(\alpha, \delta)$ with the classes $\mathcal{S}^*_{\mathcal{H}}$ and $\mathcal{K}_{\mathcal{H}}$ associated of the operator Υ defined by (3).

Theorem 1. *Let $\varepsilon_1, \varepsilon_2 > 0$, $\alpha \geq 0$ and $\delta \in [0, 1)$. If*

$$\begin{aligned}&\big[2\alpha\big(\varepsilon_1^5 + \varepsilon_2^5\big) + 3\alpha\big(11\varepsilon_1^4 + 9\varepsilon_2^4\big) + (159\alpha + 2)\varepsilon_1^3 + (246\alpha + 15)\varepsilon_1^2 + (90\alpha + 24)\varepsilon_1 \\ &+6(1 - e^{-\varepsilon_1}) + (99\alpha + 2)\varepsilon_2^3 + (102\alpha + 9)\varepsilon_2^2 + (18\alpha + 6)\varepsilon_2\big] \leq 6(1 - \delta),\end{aligned} \tag{16}$$

then

$$\Upsilon(\mathcal{S}^*_{\mathcal{H}}) \subset \mathcal{H}(\alpha, \delta).$$

Proof. Let $f = w + \bar{v} \in \mathcal{S}^*_{\mathcal{H}}$ so that w and v are given by (1) with $b_1 = 0$. We have to show that $\Upsilon(f) = \Phi + \Psi \in \mathcal{H}(\alpha, \delta)$, where Φ and Ψ are analytic functions in E defined by (4) with $b_1 = 0$. In view of Lemma 1**,** we need to prove that

$$Q(\varepsilon_1, \varepsilon_2, \alpha) \leq 1 - \delta,$$

where

$$Q(\varepsilon_1, \varepsilon_2, \alpha) = \sum_{s=2}^{\infty} s\left(1 + \alpha\left(s^2 - 1\right)\right)\left|\frac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-1)!}a_s\right| + \sum_{s=2}^{\infty} s\left(1 + \alpha\left(s^2 - 1\right)\right)\left|\frac{e^{-\varepsilon_2}\varepsilon_2^{s-1}}{(s-1)!}b_s\right|. \tag{17}$$

Using the inequalities (14) of Lemma 4**,** we obtain

$$\begin{aligned} Q(\varepsilon_1, \varepsilon_2, \alpha) &\leq \tfrac{1}{6}\Bigg[\sum_{s=2}^{\infty}(2s+1)(s+1)\left(s + \alpha s\left(s^2 - 1\right)\right)\tfrac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-1)!} \\ &\quad + \sum_{s=2}^{\infty}(2s-1)(s-1)\left(s + \alpha s\left(s^2 - 1\right)\right)\tfrac{e^{-\varepsilon_2}\varepsilon_2^{s-1}}{(s-1)!}\Bigg] \\ &= \tfrac{1}{6}\Bigg[\sum_{s=2}^{\infty}\left[2\alpha s^5 + 3\alpha s^4 + (2-\alpha)s^3 + (3-3\alpha)s^2 + (1-\alpha)s\right]\tfrac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-1)!} \\ &\quad + \sum_{s=2}^{\infty}\left[2\alpha s^5 - 3\alpha s^4 + (2-\alpha)s^3 + (3\alpha-3)s^2 + (1-\alpha)s\right]\tfrac{e^{-\varepsilon_2}\varepsilon_2^{s-1}}{(s-1)!}\Bigg]. \end{aligned} \tag{18}$$

Writing

$$s = (s-1) + 1, \tag{19}$$

$$s^2 = (s-1)(s-2) + 3(s-1) + 1, \tag{20}$$

$$s^3 = (s-1)(s-2)(s-3) + 6(s-1)(s-2) + 7(s-1) + 1, \tag{21}$$

$$s^4 = (s-1)(s-2)(s-3)(s-4) + 10(s-1)(s-2)(s-3) + 25(s-1)(s-2) + 15(s-1) + 1, \tag{22}$$

and

$$\begin{aligned} s^5 = &(s-1)(s-2)(s-3)(s-4)(s-5) + 15(s-1)(s-2)(s-3)(s-4) \\ &+65(s-1)(s-2)(s-3) + 90(s-1)(s-2) + 31(s-1) + 1, \end{aligned} \tag{23}$$

in (18), we have

$$\begin{aligned} Q(\varepsilon_1, \varepsilon_2, \alpha) \leq \frac{1}{6}\Bigg[&\sum_{s=2}^{\infty}[2\alpha(s-1)(s-2)(s-3)(s-4)(s-5) + \\ &33\alpha(s-1)(s-2)(s-3)(s-4) + (159\alpha+2)(s-1)(s-2)(s-3) + \\ &(246\alpha+15)(s-1)(s-2) + (90\alpha+24)(s-1) + 6]\frac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-1)!} \\ &+ \sum_{s=2}^{\infty}[2\alpha(s-1)(s-2)(s-3)(s-4)(s-5) + 27\alpha(s-1)(s-2)(s-3)(s-4) \\ &+(99\alpha+2)(s-1)(s-2)(s-3) + (102\alpha+9)(s-1)(s-2) \\ &+(18\alpha+6)(s-1)]\frac{e^{-\varepsilon_2}\varepsilon_2^{s-1}}{(s-1)!}\Bigg] \end{aligned}$$

$$= \frac{1}{6}\Bigg[2\alpha\sum_{s=6}^{\infty}\frac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-6)!} + 33\alpha\sum_{s=5}^{\infty}\frac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-5)!} + (159\alpha+2)\sum_{s=4}^{\infty}\frac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-4)!}$$

$$+(246\alpha+15)\sum_{s=3}^{\infty}\frac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-3)!} + (90\alpha+24)\sum_{s=2}^{\infty}\frac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-2)!} + 6\sum_{s=2}^{\infty}\frac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-1)!}$$

$$+2\alpha\sum_{s=6}^{\infty}\frac{e^{-\varepsilon_2}\varepsilon_2^{s-1}}{(s-6)!} + 27\alpha\sum_{s=5}^{\infty}\frac{e^{-\varepsilon_2}\varepsilon_2^{s-1}}{(s-5)!} + (99\alpha+2)\sum_{s=4}^{\infty}\frac{e^{-\varepsilon_2}\varepsilon_2^{s-1}}{(s-4)!}$$

$$+(102\alpha+9)\sum_{s=3}^{\infty}\frac{e^{-\varepsilon_2}\varepsilon_2^{s-1}}{(s-3)!} + (18\alpha+6)\sum_{s=2}^{\infty}\frac{e^{-\varepsilon_2}\varepsilon_2^{s-1}}{(s-2)!}\Bigg]$$

$$= \frac{1}{6}\Big[2\alpha\varepsilon_1^5 + 33\alpha\varepsilon_1^4 + (159\alpha+2)\varepsilon_1^3 + (246\alpha+15)\varepsilon_1^2 + (90\alpha+24)\varepsilon_1 + 6\big(1-e^{-\varepsilon_1}\big)$$

$$2\alpha\varepsilon_2^5 + 27\alpha\varepsilon_2^4 + (99\alpha+2)\varepsilon_2^3 + (102\alpha+9)\varepsilon_2^2 + (18\alpha+6)\varepsilon_2\Big].$$

The last relation is bounded above by $1-\delta$ if condition (16) holds.
□

Theorem 2. *Let* $\varepsilon_1, \varepsilon_2 > 0$, $\alpha \geq 0$, *and* $\delta \in [0,1)$. *If*

$$\begin{array}{c}\big[\alpha\big(\varepsilon_1^4+\varepsilon_2^4\big) + 11\alpha\varepsilon_1^3 + (30\alpha+1)\varepsilon_1^2 + (18\alpha+4)\varepsilon_1 + 2(1-e^{-\varepsilon_1}) \\ +\alpha\varepsilon_2^4 + 9\alpha\varepsilon_2^3 + (18\alpha+1)\varepsilon_2^2 + (6\alpha+2)\varepsilon_2\big] \leq 2(1-\delta),\end{array} \tag{24}$$

then

$$\Upsilon(\mathcal{K}_{\mathcal{H}}) \subset \mathcal{H}(\alpha,\delta).$$

Proof. Let $f = w + \bar{v} \in \mathcal{K}_{\mathcal{H}}$ so that w and v are given by (1) with $b_1 = 0$. We have to prove that $\Upsilon(f) = \Phi + \Psi \in \mathcal{H}(\alpha,\delta)$, where Φ and Ψ are analytic functions in E defined by (4) with $b_1 = 0$. We have to show, in view of Lemma 1, that

$$Q(\varepsilon_1,\varepsilon_2,\alpha) \leq 1-\delta,$$

where $Q(\varepsilon_1,\varepsilon_2,\alpha)$ as given in (17). Using the inequalities (15) of Lemma 5, we obtain

$$\begin{aligned} Q(\varepsilon_1,\varepsilon_2,\alpha) &\leq \tfrac{1}{2}\Bigg[\sum_{s=2}^{\infty}(s+1)\big(s+\alpha s\big(s^2-1\big)\big)\tfrac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-1)!} + \\ &\quad \sum_{s=2}^{\infty}(s-1)\big(s+\alpha s\big(s^2-1\big)\big)\tfrac{e^{-\varepsilon_2}\varepsilon_2^{s-1}}{(s-1)!}\Bigg] \\ &= \tfrac{1}{2}\Bigg[\sum_{s=2}^{\infty}\big[\alpha s^4+\alpha s^3+(1-\alpha)s^2+(1-\alpha)s\big]\tfrac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-1)!} \\ &\quad + \sum_{s=2}^{\infty}\big[\alpha s^4-\alpha s^3+(1-\alpha)s^2+(\alpha-1)s\big]\tfrac{e^{-\varepsilon_2}\varepsilon_2^{s-1}}{(s-1)!}\Bigg]. \end{aligned} \tag{25}$$

Using the Equations (19)–(22) in (25), we have

$$Q(\varepsilon_1,\varepsilon_2,\alpha) \leq \frac{1}{2}\Bigg[\sum_{s=2}^{\infty}[\alpha(s-1)(s-2)(s-3)(s-4) + 11\alpha(s-1)(s-2)(s-3)+$$

$$(30\alpha+1)(s-1)(s-2) + (18\alpha+4)(s-1) + 2]\frac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-1)!}\Bigg]$$

$$+\frac{1}{2}\Bigg[\sum_{s=2}^{\infty}[\alpha(s-1)(s-2)(s-3)(s-4) + 9\alpha(s-1)(s-2)(s-3)+$$

$$(18\alpha+1)(s-1)(s-2)+(6\alpha+2)(s-1)]\frac{e^{-\varepsilon_2}\varepsilon_2^{s-1}}{(s-1)!}\Bigg]$$

$$=\frac{1}{2}\Bigg[\alpha\sum_{s=5}^{\infty}\frac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-5)!}+11\alpha\sum_{s=4}^{\infty}\frac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-4)!}+(30\alpha+1)\sum_{s=3}^{\infty}\frac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-3)!}$$

$$+(18\alpha+4)\sum_{s=2}^{\infty}\frac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-2)!}+2\sum_{s=2}^{\infty}\frac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-1)!}+\alpha\sum_{s=5}^{\infty}\frac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-5)!}+$$

$$9\alpha\sum_{s=4}^{\infty}\frac{e^{-\varepsilon_2}\varepsilon_2^{s-1}}{(s-4)!}+(18\alpha+1)\sum_{s=3}^{\infty}\frac{e^{-\varepsilon_2}\varepsilon_2^{s-1}}{(s-3)!}+(6\alpha+2)\sum_{s=2}^{\infty}\frac{e^{-\varepsilon_2}\varepsilon_2^{s-1}}{(s-2)!}\Bigg]$$

$$=\frac{1}{2}\Big[\alpha\varepsilon_1^4+11\alpha\varepsilon_1^3+(30\alpha+1)\varepsilon_1^2+(18\alpha+4)\varepsilon_1+2\big(1-e^{-\varepsilon_1}\big)+$$

$$\alpha\varepsilon_2^4+9\alpha\varepsilon_2^3+(18\alpha+1)\varepsilon_2^2+(6\alpha+2)\varepsilon_2].$$

The last relation is bounded above by $1-\delta$ if condition (24) holds.
□

Next, we determine the connection between the classes $\mathcal{TN}_{\mathcal{H}}(\beta)$ and $\mathcal{H}(\alpha,\delta)$.

Theorem 3. *Let $\varepsilon_1,\varepsilon_2>0$, $\alpha\geq 0$ and $\delta,\beta\in[0,1)$. If*

$$(1-\beta)[\alpha\left(\varepsilon_1^2+\varepsilon_2^2\right)+3\alpha(\varepsilon_1+\varepsilon_2)-\left(e^{-\varepsilon_1}+e^{-\varepsilon_2}\right)+2]\leq 1-\delta-|b_1|,$$

then

$$\Upsilon(\mathcal{TN}_{\mathcal{H}}(\beta))\subset\mathcal{H}(\alpha,\delta).$$

Proof. Let $f=w+\bar{v}\in\mathcal{TN}_{\mathcal{H}}(\beta)$ so that w and v are given by (2). Using Lemma 1, we have to prove that $L(\varepsilon_1,\varepsilon_2,\alpha)\leq 1-\delta$, where

$$\begin{aligned}L(\varepsilon_1,\varepsilon_2,\alpha)=&\sum_{s=2}^{\infty}\left(s+\alpha s\left(s^2-1\right)\right)\left|\tfrac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-1)!}a_s\right|\\&+|b_1|+\sum_{s=2}^{\infty}\left(s+\alpha s\left(s^2-1\right)\right)\left|\tfrac{e^{-\varepsilon_2}\varepsilon_2^{s-1}}{(s-1)!}b_s\right|.\end{aligned}\tag{26}$$

Using the inequalities (9) and (10) of Lemma 2, it follows that

$$L(\varepsilon_1,\varepsilon_2,\alpha)\leq(1-\beta)\left[\sum_{s=2}^{\infty}\left(1-\alpha+\alpha s^2\right)\frac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-1)!}\right.$$

$$\left.+\sum_{s=2}^{\infty}\left(1-\alpha+\alpha s^2\right)\frac{e^{-\varepsilon_2}\varepsilon_2^{s-1}}{(s-1)!}\right]+|b_1|$$

$$=(1-\beta)\left[\sum_{s=2}^{\infty}[\alpha(s-1)(s-2)+3\alpha(s-1)+1]\frac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-1)!}\right.$$

$$\left.+\sum_{s=2}^{\infty}[\alpha(s-1)(s-2)+3\alpha(s-1)+1]\frac{e^{-\varepsilon_2}\varepsilon_2^{s-1}}{(s-1)!}\right]+|b_1|$$

$$=(1-\beta)\left[\alpha\left(\varepsilon_1^2+\varepsilon_2^2\right)+3\alpha(\varepsilon_1+\varepsilon_2)-\left(e^{-\varepsilon_1}+e^{-\varepsilon_2}\right)+2\right]+|b_1|\leq 1-\delta,$$

by the given hypothesis, which completes the proof of Theorem 3. □

Next, we find the relationship between the classes $\mathcal{TR}_{\mathcal{H}}(\beta)$ and $\mathcal{H}(\alpha,\delta)$.

Theorem 4. *Let $\varepsilon_1, \varepsilon_2 > 0$, $\alpha \geq 0$ and $\delta, \beta \in [0,1)$. If*

$$(1-\beta)\left[\alpha(\varepsilon_1+\varepsilon_2)+\alpha\left(2-e^{-\varepsilon_1}-e^{-\varepsilon_2}\right)+\frac{1}{\varepsilon_1}(1-e^{-\varepsilon_1}-\varepsilon_1 e^{-\varepsilon_1})+\right.$$

$$\left.\frac{1}{\varepsilon_2}(1-e^{-\varepsilon_2}-\varepsilon_2 e^{-\varepsilon_2})\right] \leq 1-\delta-|b_1|,$$

then

$$\Upsilon(\mathcal{TR}_{\mathcal{H}}(\beta)) \subset \mathcal{H}(\alpha,\delta).$$

Proof. Applying Lemma 1, we need only to show that $L(\varepsilon_1,\varepsilon_2,\alpha) \leq 1-\delta$, where $L(\varepsilon_1,\varepsilon_2,\alpha)$ as given in (26). Using the inequalities (12) and (13) of Lemma 3**,** it follows that

$$L(\varepsilon_1,\varepsilon_2,\alpha) = \sum_{s=2}^{\infty}\left(s+\alpha s\left(s^2-1\right)\right)\left|\frac{e^{-\varepsilon 1}\varepsilon_1^{s-1}}{(s-1)!}a_s\right|$$

$$+|b_1|+\sum_{s=2}^{\infty}\left(s+\alpha s\left(s^2-1\right)\right)\left|\frac{e^{-\varepsilon_2}\varepsilon_2^{s-1}}{(s-1)!}b_s\right|$$

$$\leq (1-\beta)\left[\sum_{s=2}^{\infty}\left(\alpha s+\frac{1-\alpha}{s}\right)\frac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-1)!}\right.$$

$$\left.+\sum_{s=2}^{\infty}\left(\alpha s+\frac{1-\alpha}{s}\right)\frac{e^{-\varepsilon_2}\varepsilon_2^{s-1}}{(s-1)!}\right]+|b_1|$$

$$=(1-\beta)\left[\sum_{s=2}^{\infty}\alpha\frac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-2)!}+\sum_{s=2}^{\infty}\alpha\frac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-1)!}+\sum_{s=0}^{\infty}\left(\frac{1-\alpha}{s+2}\right)\frac{e^{-\varepsilon_1}\varepsilon_1^{s+1}}{(s+1)!}\right.$$

$$\left.+\sum_{s=2}^{\infty}\alpha\frac{e^{-\varepsilon_2}\varepsilon_2^{s-1}}{(s-2)!}+\sum_{s=2}^{\infty}\alpha\frac{e^{-\varepsilon_2}\varepsilon_2^{s-1}}{(s-1)!}+\sum_{s=0}^{\infty}\left(\frac{1-\alpha}{s+2}\right)\frac{e^{-\varepsilon_1}\varepsilon_1^{s+1}}{(s+1)!}\right]+|b_1|$$

$$=(1-\beta)\left[\alpha\varepsilon_1+\alpha\left(1-e^{-\varepsilon_1}\right)+\frac{1}{\varepsilon_1}(1-e^{-\varepsilon_1}-\varepsilon_1 e^{-\varepsilon_1})\right.$$

$$\left.+\alpha\varepsilon_2+\alpha\left(1-e^{-\varepsilon_2}\right)+\frac{1}{\varepsilon_2}(1-e^{-\varepsilon_2}-\varepsilon_2 e^{-\varepsilon_2})\right]+|b_1| \leq 1-\delta,$$

by given hypothesis. □

Theorem 5. *Let $\varepsilon_1, \varepsilon_2 > 0$, $\alpha \geq 0$ and $\delta \in [0,1)$. If*

$$e^{-\varepsilon_1}+e^{-\varepsilon_2} \geq 1+\frac{|b_1|}{1-\delta}, \tag{27}$$

then

$$\Upsilon(\mathcal{H}(\alpha,\delta)) \subset \mathcal{H}(\alpha,\delta).$$

Proof. Using the inequalities (6) and (7) of Lemma 1, we obtain

$$L(\varepsilon_1,\varepsilon_2,\alpha) \leq (1-\delta)\left[\sum_{s=2}^{\infty}\frac{e^{-\varepsilon_1}\varepsilon_1^{s-1}}{(s-1)!}+\sum_{s=2}^{\infty}\frac{e^{-\varepsilon_2}\varepsilon_2^{s-1}}{(s-1)!}\right]+|b_1|$$

$$=(1-\delta)\left[1-e^{-\varepsilon_1}+1-e^{-\varepsilon_2}\right]+|b_1|$$

$$=(1-\delta)[2-e^{-\varepsilon_1}-e^{-\varepsilon_2}]+|b_1| \leq 1-\delta,$$

by the given condition (7). □

4. Corollaries and Consequences

By specializing the parameter $\alpha = 0$ in main results, we obtain the following special cases for the subclass $\mathcal{H}(\delta)$.

Corollary 1. *Let* $\varepsilon_1, \varepsilon_2 > 0$ *and* $\delta \in [0,1)$. *If*

$$2\varepsilon_1^3 + 15\varepsilon_1^2 + 24\varepsilon_1 + 2\varepsilon_2^3 + 9\varepsilon_2^2 + 6\varepsilon_2 + 6\left(1 - e^{-\varepsilon_1}\right) \leq 6(1-\delta),$$

then

$$\Upsilon(\mathcal{S}^*_{\mathcal{H}}) \subset \mathcal{H}(\delta).$$

Corollary 2. *Let* $\varepsilon_1, \varepsilon_2 > 0$ *and* $\delta \in [0,1)$. *If*

$$\varepsilon_1^2 + 4\varepsilon_1 + 2\left(1 - e^{-\varepsilon_1}\right) + \varepsilon_2^2 + 2\varepsilon_2 \leq 2(1-\delta),$$

then

$$\Upsilon(\mathcal{K}_{\mathcal{H}}) \subset \mathcal{H}(\delta).$$

Corollary 3. *Let* $\varepsilon_1, \varepsilon_2 > 0$ *and* $\delta, \beta \in [0,1)$. *If*

$$(1-\beta)\left(2 - e^{-\varepsilon_1} - e^{-\varepsilon_2}\right) \leq 1 - \delta - |b_1|,$$

then

$$\Upsilon(\mathcal{TN}_{\mathcal{H}}(\beta)) \subset \mathcal{H}(\delta).$$

Corollary 4. *Let* $\varepsilon_1, \varepsilon_2 > 0$ *and* $\delta, \beta \in [0,1)$. *If*

$$(1-\beta)\left[\left(\frac{1}{\varepsilon_1}(1 - e^{-\varepsilon_1} - \varepsilon_1 e^{-\varepsilon_1}) + \frac{1}{\varepsilon_2}(1 - e^{-\varepsilon_2} - \varepsilon_2 e^{-\varepsilon_2})\right)\right] \leq 1 - \delta - |b_1|,$$

then

$$\Upsilon(\mathcal{TR}_{\mathcal{H}}(\beta)) \subset \mathcal{H}(\delta).$$

5. Conclusions

This paper deals with the applications of the Poisson distribution on some subclasses of harmonic functions. The main scope of this paper is to find some inclusion relations of the harmonic class $\mathcal{H}(\alpha, \delta)$ with the classes $\mathcal{S}^*_{\mathcal{H}}$ of starlike harmonic functions and $\mathcal{K}_{\mathcal{H}}$ of convex harmonic functions, also for the harmonic classes $\mathcal{TN}_{\mathcal{H}}(\beta)$ and $\mathcal{TR}_{\mathcal{H}}(\beta)$ associated with the operator Υ defined by Poisson distribution series. Further by specializing the parameter $\alpha = 0$, several consequences of the main results are mentioned.

Making use of the operator Υ researchers could be inspired to find new inclusion relations for new harmonic classes of analytic functions with the classes $\mathcal{S}^*_{\mathcal{H}}$, $\mathcal{K}_{\mathcal{H}}$, $\mathcal{TN}_{\mathcal{H}}(\beta)$ and $\mathcal{TR}_{\mathcal{H}}(\beta)$.

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