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On Some Bounds for the Gamma Function

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Abstract: Both theoretical and applied mathematics depend heavily on inequalities, which are rich in symmetries. In numerous studies, estimations of various functions based on the characteristics of their symmetry have been provided through inequalities. In this paper, we study the monotonicity of certain functions that involve Gamma functions. We were able to obtain some of the bounds of $\Gamma(v)$ that are more accurate than some recently published inequalities.

Keywords: asymptotic expansion; Gamma function; Windschitl's formula; inequality; best possible constant

MSC: 33B15; 41A60; 41A21

1. Introduction

Mathematicians have made considerable efforts to develop more precise estimates of $n!$ and its natural extension, Gamma function. Scottish mathematician James Stirling (1692–1770) introduced the following formula:

$$n! \sim \sqrt{2\pi n}(n/e)^n, \qquad n \to \infty \tag{1}$$

which is the most widely used and well-known approximation formula for handling large factorials and it bears his name [1–3]. Additionally, Stirling's series [4]

$$\Gamma(v+1) \sim \sqrt{2\pi v}(v/e)^v e^{\sum_{p=1}^{\infty} \frac{B_{2p}}{2p(2p-1)v^{2p-1}}}, \qquad v \to \infty \tag{2}$$

is a generalization of formula (1), where B_p denotes Bernoulli numbers. French scientist Pierre-Simon Laplace (1749–1827) [4] presented

$$\Gamma(1+v) \sim \sqrt{2\pi v}(v/e)^v \left(1 + \frac{1}{12v} + \frac{1}{288v^2} - \frac{139}{51{,}840v^3} - \dots\right), \ v \to \infty. \tag{3}$$

In 1917, Burnside [5] provided a more accurate formula than (1) with

$$\Gamma(v+1) \sim \sqrt{2\pi v}\left(\frac{2v+1}{2e}\right)^{v+1/2}, \qquad v \to \infty. \tag{4}$$

Indian mathematician Srinivasa Ramanujan (1887–1920) [6] presented the asymptotic expansion

$$\Gamma(v+1) \sim \sqrt{\pi}\left(\frac{v}{e}\right)^v \sqrt[6]{8v^3 + 4v^2 + v + 1/30}, \qquad v \to \infty \tag{5}$$

and the following inequality of Gamma function between symmetric bounds

$$\sqrt{\pi}\left(\frac{v}{e}\right)^v \sqrt[6]{8v^3 + 4v^2 + v + 1/100} < \Gamma(v+1) < \sqrt{\pi}\left(\frac{v}{e}\right)^v \sqrt[6]{8v^3 + 4v^2 + v + 1/30}, \quad v \geq 0 \tag{6}$$

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which, according to the book *The Lost Notebook and Other Unpublished Papers*, are conjectures based on some mathematical calculations (see also [7–11]). In 2001, Karatsuba [9] presented

$$\Gamma(v+1) \sim \sqrt{\pi}(v/e)^v \sqrt[6]{8v^3+4v^2+v+1/30+\theta_1(v)}, \quad v \to \infty \tag{7}$$

where

$$\theta_1(v) = -\frac{11}{240v} + \frac{79}{3360v^2} + \frac{3539}{201{,}600v^3} + \ldots,$$

which proves Ramanujan's formula (5). In 2011, Mortici [12] presented

$$\Gamma(v+1) \sim \sqrt{\pi}(v/e)^v \sqrt[6]{8v^3+4v^2+v+1/30}\, e^{\theta_2(v)}, \quad v \to \infty \tag{8}$$

where

$$\theta_2(v) = -\frac{11}{11,520v^4} + \frac{13}{3440v^5} + \frac{1}{691{,}200v^6} + \ldots,$$

which improves Ramanujan formula (5) and is faster than formula (7).

In 2002, in web post, Robert H. Windschitl [13] (see also [14]) presented the important formula

$$\Gamma(v+1) = \sqrt{2\pi v}(v/e)^v \left(v \sinh v^{-1}\right)^{v/2}\left[1+O\left(v^{-5}\right)\right], \qquad v \to \infty \tag{9}$$

which relates the Gamma function and the hyperbolic sine function. He advised using the approximation $\sqrt{2\pi v}(v/e)^v\left(v\sinh v^{-1} + \frac{1}{810}v^{-6}\right)^{v/2}$ to calculate the values of the Gamma function on calculators with limited program or register memory since it is accurate to more than eight decimal places for $v > 8$.

In 2009, Alzer [15] presented the following double inequality with a symmetrical bounds structure:

$$\left(v \sinh v^{-1}\right)^{v/2}\left[1+A_1 v^{-5}\right] < \frac{\Gamma(v+1)}{\sqrt{2\pi v}(v/e)^v} < \left(v \sinh v^{-1}\right)^{v/2}\left[1+A_2 v^{-5}\right], \quad v > 0 \tag{10}$$

with the best possible constants $A_1 = 0$ and $A_2 = \frac{1}{1620}$. Numerical calculations show that the lower bound in inequality (10) is superior to that of its counterpart in inequality (6) for $v \geq 2.07$. Additionally, the upper bound in inequality (10) is superior to that of its counterpart in inequality (6) for $v \geq 0.992$. In 2010, Nemes [16] presented

$$\Gamma(v+1) = \sqrt{2\pi v}(v/e)^v \left(1 + \frac{10}{120v^2-1}\right)^v \left[1+O\left(v^{-5}\right)\right], \qquad v \to \infty \tag{11}$$

which is considerably easier than (9) and has exactly the same number of exact digits. Formulas (9) and (11) are more accurate than Ramanujan's formula. In 2014, Lu, Song and Ma [17] deduced that there exists an n, such that for every $v > n$, the double inequality

$$\left[v \sinh\left(\frac{1}{v} + \frac{1}{810v^7} - \frac{67}{42{,}525v^9}\right)\right]^{v/2} < \frac{\Gamma(v+1)}{\sqrt{2\pi v}(v/e)^v} < \left[v \sinh\left(\frac{1}{v} + \frac{1}{810v^7}\right)\right]^{v/2}$$

holds. Additionally, they provided some numerical comparisons to show how much better their approximations were than others such as Nemes' formula (11). In 2022, Mahmoud and Almuashi [18] presented the new asymptotic formulas

$$\Gamma(v+1) \sim \sqrt{2\pi v}(v/e)^v \left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2} \exp\left(\sum_{r=1}^{\infty}\frac{\mu_r}{v^r}\right), \quad v \to \infty \tag{12}$$

and

$$\Gamma(v+1) \sim \sqrt{2\pi v}(v/e)^v \left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2} \left(\sum_{r=0}^{\infty}\frac{\lambda_r}{v^r}\right), \qquad v \to \infty \tag{13}$$

where

$$\begin{cases} \mu_r = \frac{B_{r+1}}{r(r+1)} - \frac{1}{2}\rho_{r+1}, \\ \rho_r = \chi_r - \frac{1}{r}\sum_{j=1}^{r-1} j\rho_j\chi_{r-j}, \\ \chi_0 = 1, \ \ \chi_{2r} = \frac{10}{3(20)^r}, \ \ \chi_{2r-1} = 0, \\ \lambda_0 = 1, \ \ \lambda_r = \frac{1}{r}\sum_{j=1}^{r} j\lambda_{r-j}\mu_j, \end{cases} \qquad r = 1, 2, 3, \ldots .$$

Both the two formulas (9) and (13) have the same rate of convergence, but the second is simpler.

For more details about asymptotic formulas and bounds of $\Gamma(v)$, please see [17,19–23] and the references therein.

In the rest of this paper, and motivated by formula (13), we will prove the following double symmetric inequality:

$$\sqrt{2\pi v}(v/e)^v\left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2}\left(1+\frac{\lambda}{v^5}\right) < \Gamma(v+1) < \sqrt{2\pi v}(v/e)^v\left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2}\left(1+\frac{\mu}{v^5}\right), \qquad v \geq 1 \tag{14}$$

with $\lambda = 0$ and the best possible constant $\mu = \frac{461}{907200}$. Additionally, we will present comparisons between this inequality and the inequalities (6) and (10) presented by Ramanujan and Alzer, respectively, to clarify the superiority of our new results.

2. Main Results

Now, we will present new bounds of the Gamma function depending on the asymptotic formulas (12) and (13).

Theorem 1. *The function*

$$P_1(v) = \frac{\Gamma(v+1)}{\sqrt{2\pi v}\left(\frac{v}{e}\right)^v\left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2}}$$

is strictly decreasing for $v \geq 1$. Furthermore,

$$\Gamma(v+1) > \sqrt{2\pi v}(v/e)^v\left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2}, \qquad v \geq 1. \tag{15}$$

Proof. The function $T_1(v) = \log P_1(v) - \log P_1(v+1)$ satisfies $T_1''(v) = \frac{F_1(v)}{F_2(v)}$, where

$$\begin{aligned} F_1(v+1) = \quad & 4.4256\times 10^{11}v^{12} + 7.96608\times 10^{12}v^{11} + 6.48621\times 10^{13}v^{10} + 3.15729\times 10^{14}v^9 \\ & +1.02268\times 10^{15}v^8 + 2.32054\times 10^{15}v^7 + 3.77909\times 10^{15}v^6 + 4.44603\times 10^{15}v^5 \\ & +3.74551\times 10^{15}v^4 + 2.19984\times 10^{15}v^3 + 8.53104\times 10^{14}v^2 + 1.95514\times 10^{14}v \\ & +1.99325\times 10^{13} \end{aligned}$$

and

$$F_2(v) = 2v^2(v+1)^2\left(20v^2-1\right)^2\left(20v^2+40v+19\right)^2\left(60v^2+7\right)^2\left(60v^2+120v+67\right)^2.$$

Then, $T_1(v)$ is a convex function for $v \geq 1$, and hence $T_1'(v)$ is an increasing function for $v \geq 1$. Using the asymptotic expansion (12), we have

$$\ln P_1(v) \sim \frac{461}{907{,}200v^5} - \frac{5197}{9{,}072{,}000v^7} + \frac{1{,}436{,}249}{1{,}710{,}720{,}000v^9} - \frac{26{,}863{,}154{,}077}{14{,}010{,}796{,}800{,}000v^{11}} + \ldots, \quad v \to \infty$$

and

$$\frac{P_1'(v)}{P_1(v)} \sim -\frac{461}{181{,}440v^6} + \frac{5197}{1{,}296{,}000v^8} - \frac{1{,}436{,}249}{190{,}080{,}000v^{10}} + \frac{26{,}863{,}154{,}077}{1{,}273{,}708{,}800{,}000v^{12}} + \ldots, \quad v \to \infty.$$

Then,
$$\lim_{v\to\infty} T_1'(v) = 0$$
and $T_1'(v)$ is negative for $v \geq 1$. So,
$$\frac{P_1'(v)}{P_1(v)} - \frac{P_1'(v+1)}{P_1(v+1)} < 0, \quad v \geq 1$$
and hence
$$\frac{P_1'(v)}{P_1(v)} < \frac{P_1'(v+1)}{P_1(v+1)} < \frac{P_1'(v+2)}{P_1(v+2)} < \ldots\ldots < \lim_{n\to\infty} \frac{P_1'(v+n)}{P_1(v+n)}, \quad v \geq 1.$$
However,
$$\lim_{n\to\infty} \frac{P_1'(v+n)}{P_1(v+n)} = 0.$$
Therefore, $P_1'(v) < 0$ or $P_1(v)$ is decreasing function for $v \geq 1$ with $\lim_{v\to\infty} P_1(v) = 1$, where we use the asymptotic expansion (13) to obtain
$$P_1(v) \sim 1 + \frac{461}{907{,}200v^5} - \frac{5197}{9{,}072{,}000v^7} + \frac{1{,}436{,}249}{1{,}710{,}720{,}000v^9} + \ldots, \quad v \to \infty.$$
Then, $P_1(v) > 1$ for $v \geq 1$ or
$$\Gamma(v+1) > \sqrt{2\pi v}(v/e)^v \left(\frac{v^2 + \frac{7}{60}}{v^2 - \frac{1}{20}}\right)^{v/2}, \quad v \geq 1.$$
□

Theorem 2. *The function*
$$P_2(v) = \frac{\Gamma(v+1)}{\sqrt{2\pi v}\left(\frac{v}{e}\right)^v \left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2} \left(\frac{461}{907{,}200v^5} + 1\right)}$$
is strictly increasing for $v \geq 1$. Furthermore,
$$\Gamma(v+1) < \sqrt{2\pi v}(v/e)^v \left(\frac{v^2 + \frac{7}{60}}{v^2 - \frac{1}{20}}\right)^{v/2} \left(1 + \frac{\mu}{v^5}\right), \quad v \geq 1 \tag{16}$$
with the best possible constant $\mu = \frac{461}{907200}$.

Proof. The function $T_2(v) = \ln P_2(v) - \ln P_2(v+1)$ satisfies $T_2''(v) = \frac{F_3(v)}{F_4(v)}$, where
$$\begin{aligned} F_3(v+1) = \quad & -8.11049 \times 10^{35}v^{30} - 3.64972 \times 10^{37}v^{29} - 7.91358 \times 10^{38}v^{28} - 1.10102 \times 10^{40}v^{27} \\ & -1.10437 \times 10^{41}v^{26} - 8.50657 \times 10^{41}v^{25} - 5.23342 \times 10^{42}v^{24} - 2.64083 \times 10^{43}v^{23} \\ & -1.11381 \times 10^{44}v^{22} - 3.98112 \times 10^{44}v^{21} - 1.21847 \times 10^{45}v^{20} - 3.21822 \times 10^{45}v^{19} \\ & -7.37762 \times 10^{45}v^{18} - 1.47414 \times 10^{46}v^{17} - 2.5746 \times 10^{46}v^{16} - 3.93687 \times 10^{46}v^{15} \\ & -5.27342 \times 10^{46}v^{14} - 6.18444 \times 10^{46}v^{13} - 6.3396 \times 10^{46}v^{12} - 5.66425 \times 10^{46}v^{11} \\ & -4.39262 \times 10^{46}v^{10} - 2.93963 \times 10^{46}v^{9} - 1.68451 \times 10^{46}v^{8} - 8.18045 \times 10^{45}v^{7} \\ & -3.32038 \times 10^{45}v^{6} - 1.10538 \times 10^{45}v^{5} - 2.93909 \times 10^{44}v^{4} - 6.00088 \times 10^{43}v^{3} \\ & -8.83256 \times 10^{42}v^{2} - 8.34133 \times 10^{41}v - 3.79528 \times 10^{40} \end{aligned}$$

and

$$\begin{aligned} F_4(v) = \quad & 2v^2\Big(907{,}200v^5 + 4{,}536{,}000v^4 + 9{,}072{,}000v^3 + 9{,}072{,}000v^2 + 4{,}536{,}000v + 907{,}661\Big)^2 \\ & (v+1)^2\Big(20v^2-1\Big)^2\Big(20v^2+40v+19\Big)^2\Big(60v^2+7\Big)^2 \\ & \Big(60v^2+120v+67\Big)^2\Big(907{,}200v^5+461\Big)^2. \end{aligned}$$

Then $T_2(v)$ is a concave function for $v \geq 1$ and hence $T_2'(v)$ is a decreasing function for $v \geq 1$. Using the asymptotic expansion (12), we have

$$\ln P_2(v) \sim -\frac{5197}{9{,}072{,}000v^7} + \frac{1{,}436{,}249}{1{,}710{,}720{,}000v^9} + \frac{212{,}521}{1{,}646{,}023{,}680{,}000v^{10}} + \ldots, \quad v \to \infty$$

and

$$\frac{P_2'(v)}{P_2(v)} \sim \frac{5197}{1{,}296{,}000v^8} - \frac{1{,}436{,}249}{190{,}080{,}000v^{10}} - \frac{212{,}521}{164{,}602{,}368{,}000v^{11}} + \ldots, \quad v \to \infty.$$

Then

$$\lim_{v\to\infty} T_2'(v) = 0$$

and $T_2'(v)$ is positive for $v \geq 1$. So,

$$\frac{P_2'(v)}{P_2(v)} - \frac{P_2'(v+1)}{P_2(v+1)} > 0, \quad v \geq 1$$

and hence

$$\frac{P_2'(v)}{P_2(v)} > \frac{P_2'(v+1)}{P_2(v+1)} > \frac{P_2'(v+2)}{P_2(v+2)} > \ldots\ldots > \lim_{n\to\infty} \frac{P_2'(v+n)}{P_2(v+n)}, \quad v \geq 1.$$

However,

$$\lim_{n\to\infty} \frac{P_2'(v+n)}{P_2(v+n)} = 0.$$

Therefore, $P_2'(v) > 0$ or $P_2(v)$ is an increasing function for $v \geq 1$ with $\lim_{v\to\infty} P_2(v) = 1$, where we use the asymptotic expansion (13) to obtain

$$P_2(v) \sim \frac{1}{1+\frac{461}{907{,}200v^5}}\left(1 + \frac{461}{907{,}200v^5} - \frac{5197}{9{,}072{,}000v^7} + \frac{1{,}436{,}249}{1{,}710{,}720{,}000v^9} + \ldots\right), \quad v \to \infty.$$

Then, $P_2(v) < 1$ for $v \geq 1$ or

$$\Gamma(v+1) < \sqrt{2\pi v}(v/e)^v \left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2}\left(1+\frac{461}{907{,}200v^5}\right), \quad v \geq 1.$$

This inequality is equivalent to $T_3(v) < \mu$ for all $v \geq 1$, where

$$T_3(v) = v^5\left[\frac{\Gamma(v+1)}{\sqrt{2\pi v}(v/e)^v\left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2}} - 1\right].$$

However, using the asymptotic expansion (13), we obtain

$$\lim_{v\to\infty} T_3(v) = \frac{461}{907{,}200}.$$

This implies that $\mu \geq \frac{461}{907{,}200}$, or the best possible value of μ is $\frac{461}{907{,}200}$. $\square$

3. Comparison between Previous and New Results

Remark 1. *Using the expansion*

$$\left(v^2+\frac{7}{60}\right)-v\left(v^2-\frac{1}{20}\right)\sinh v^{-1}=\sum_{p=2}^{\infty}\frac{(p-1)(2p+7)}{10(2p+3)!v^{2p}},$$

then we obtain

$$\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}>v\sinh v^{-1},\quad v>\frac{1}{\sqrt{20}}$$

and our new lower bound of inequality (15) is better than the lower bound of Alzer's inequality (10) for $v \geq 1$.

Remark 2. *Consider the function*

$$T_4(v)=\ln\left(\left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2}\left(\frac{461}{907{,}200v^5}+1\right)\right)-\ln\left(\left(\frac{v^4+\frac{31}{294}v^2}{v^4-\frac{3}{49}v^2+\frac{11}{5880}}\right)^{v/2}\left(\frac{1}{1620v^5}+1\right)\right),$$

then we have $T_4''(v)=-\frac{F_5(v)}{F_6(v)}$, where

$$\begin{aligned}F_5(v+1) \quad &= 3.14926\times10^{33}v^{32}+1.00776\times10^{35}v^{31}+1.56197\times10^{36}v^{30}+1.56184\times10^{37}v^{29}\\&+1.13218\times10^{38}v^{28}+6.339\times10^{38}v^{27}+2.85184\times10^{39}v^{26}+1.05892\times10^{40}v^{25}\\&+3.30784\times10^{40}v^{24}+8.81677\times10^{40}v^{23}+2.02672\times10^{41}v^{22}+4.05079\times10^{41}v^{21}\\&+7.08347\times10^{41}v^{20}+1.0888\times10^{42}v^{19}+1.47617\times10^{42}v^{18}+1.76937\times10^{42}v^{17}\\&+1.87751\times10^{42}v^{16}+1.76448\times10^{42}v^{15}+1.46798\times10^{42}v^{14}+1.07968\times10^{42}v^{13}\\&+7.00358\times10^{41}v^{12}+3.99296\times10^{41}v^{11}+1.99146\times10^{41}v^{10}+8.63451\times10^{40}v^{9}\\&+3.22806\times10^{40}v^{8}+1.02952\times10^{40}v^{7}+2.76159\times10^{39}v^{6}+6.11223\times10^{38}v^{5}\\&+1.08669\times10^{38}v^{4}+1.49173\times10^{37}v^{3}+1.48399\times10^{36}v^{2}+9.52017\times10^{34}v\\&+2.95688\times10^{33}\end{aligned}$$

and

$$\begin{aligned}F_6(v) \quad = \quad & v\left(1-20v^2\right)^2\left(60v^2+7\right)^2\left(294v^2+31\right)^2\left(5880v^4-360v^2+11\right)^2\\&\left(1620v^5+1\right)^2\left(907{,}200v^5+461\right)^2.\end{aligned}$$

Then $T_4(v)$ is concave for $v \geq 1$ or $T_4'(v)$ is decreasing for $v \geq 1$. However,

$$T_4'(v)\sim\frac{1793}{42{,}336{,}000v^8}-\frac{793{,}529}{96{,}808{,}320{,}000v^{10}}-\frac{11{,}231}{18{,}289{,}152{,}000v^{11}}+\frac{1{,}328{,}563{,}181}{1{,}280{,}774{,}073{,}600{,}000v^{12}}+\ldots,\quad v\to\infty.$$

Then $\lim_{v\to\infty}T_4'(v)=0$ and $T_4'(v)>0$ for $v\geq1$ or $T_4(v)$ is increasing for $v\geq1$ with $\lim_{v\to\infty}T_4(v)=0$, where

$$T_4(v)\sim-\frac{1793}{296{,}352{,}000v^7}+\frac{793{,}529}{871{,}274{,}880{,}000v^9}+\frac{11{,}231}{182{,}891{,}520{,}000v^{10}}+\ldots,\quad v\to\infty.$$

Therefore, $T_4(v)<0$ for $v\geq1$ or

$$\left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2}\left(\frac{461}{907{,}200v^5}+1\right)<\left(\frac{v^4+\frac{31}{294}v^2}{v^4-\frac{3}{49}v^2+\frac{11}{5880}}\right)^{v/2}\left(\frac{1}{1620v^5}+1\right),\quad v\geq1. \quad (17)$$

Using the following expansion for $v>0$

$$\left(\frac{11v^4}{5880}-\frac{3v^2}{49}+1\right)\sinh(v)-\left(\frac{31v^3}{294}+v\right)=\sum_{p=4}^{\infty}\frac{((p-3)(p-2)(44p(p+4)+245))v^{2p+1}}{1470(2p+1)!}>0$$

we have

$$\left(v\sinh v^{-1}\right)^{v/2}\left(\frac{1}{1620v^5}+1\right)>\left(\frac{v^4+\frac{31}{294}v^2}{v^4-\frac{3}{49}v^2+\frac{11}{5880}}\right)^{v/2}\left(\frac{1}{1620v^5}+1\right),\quad v\geq 1. \tag{18}$$

Using inequalities (17) and (18), we conclude that our new upper bound of inequality (15) is better than the upper bound of Alzer's inequality (10) for $v\geq 1$.

Remark 3. *Consider the function*

$$T_5(v)=\frac{1}{6}\ln\left(8v^3+4v^2+v+\frac{1}{100}\right)-\ln\left(\sqrt{2v}\left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2}\right),$$

then we have $T_5''(v)=\frac{F_7(v)}{6v(20v^2-1)(60v^2+7)(800v^3+400v^2+100v+1)}$, where

$$\begin{aligned}F_7(v+3.4)\quad &=-3.2256\times 10^{10}v^{11}-1.10615\times 10^{12}v^{10}-1.70749\times 10^{13}v^9-1.5627\times 10^{14}v^8\\&-9.39317\times 10^{14}v^7-3.87682\times 10^{15}v^6-1.11372\times 10^{16}v^5-2.20314\times 10^{16}v^4\\&-2.88463\times 10^{16}v^3-2.28577\times 10^{16}v^2-8.809\times 10^{15}v-6.29583\times 10^{14}.\end{aligned}$$

Then $T_5(v)$ is concave for $v\geq 3.4$ or $T_5'(v)$ is decreasing for $v\geq 3.4$. However,

$$T_5'(v)\sim-\frac{30{,}847}{103{,}680{,}000v^8}+\frac{1}{1{,}280{,}000v^7}+\frac{539}{207{,}360v^6}-\frac{23}{4800v^5}+\frac{7}{4800v^4}+\ldots,\quad v\to\infty.$$

Then $\lim_{v\to\infty}T_5'(v)=0$ and $T_5'(v)>0$ for $v\geq 3.4$, or $T_5(v)$ is increasing for $v\geq 3.4$ with $\lim_{v\to\infty}T_5(v)=0$, where

$$T_5(v)\sim-\frac{529}{61{,}440{,}000v^8}+\frac{30{,}847}{725{,}760{,}000v^7}-\frac{1}{7{,}680{,}000v^6}-\frac{539}{1{,}036{,}800v^5}+\frac{23}{19{,}200v^4}-\frac{7}{14{,}400v^3}+\ldots,\quad v\to\infty.$$

Therefore, $T_5(v)<0$ for $v\geq 3.4$ or

$$\sqrt[6]{8v^3+4v^2+v+\frac{1}{100}}<\sqrt{2v}\left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2},\quad v\geq 3.4.$$

Then our new lower bound of inequality (15) is superior to that of its counterpart in inequality (6) for $v\geq 3.4$.

Remark 4. *Consider the function*

$$T_6(v)=\frac{1}{6}\ln\left(8v^3+4v^2+v+\frac{1}{30}\right)-\ln\left(\sqrt{2v}\left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2}\left(\frac{461}{907{,}200v^5}+1\right)\right)$$

then we have $T_6''(v)=\frac{F_8(v)}{2v^2(1-20v^2)^2(60v^2+7)^2(240v^3+120v^2+30v+1)^2(907{,}200v^5+461)^2)}$, where

$$\begin{aligned}F_8(v+1.5)\quad &=2.6073\times 10^{21}v^{20}+7.68646\times 10^{22}v^{19}+1.0737\times 10^{24}v^{18}+9.44632\times 10^{24}v^{17}\\&+5.86836\times 10^{25}v^{16}+2.73516\times 10^{26}v^{15}+9.91887\times 10^{26}v^{14}+2.86407\times 10^{27}v^{13}\\&+6.68251\times 10^{27}v^{12}+1.27107\times 10^{28}v^{11}+1.97924\times 10^{28}v^{10}+2.52336\times 10^{28}v^9\\&+2.62358\times 10^{28}v^8+2.20574\times 10^{28}v^7+1.47838\times 10^{28}v^6+7.72565\times 10^{27}v^5\\&+3.04037\times 10^{27}v^4+8.51281\times 10^{26}v^3+1.5292\times 10^{26}v^2+1.39397\times 10^{25}v\\&+2.20424\times 10^{23}.\end{aligned}$$

Then $T_6(v)$ is convex for $v \geq 1.5$ or $T_6'(v)$ is increasing for $v \geq 1.5$. However,

$$T_6'(v) \sim -\frac{2629}{10{,}368{,}000v^8} + \frac{1}{115{,}200v^7} + \frac{13}{2688v^6} - \frac{11}{2880v^5} + \ldots, \quad v \to \infty.$$

Then $\lim_{v\to\infty} T_6'(v) = 0$ and $T_6'(v) < 0$ for $v \geq 1.5$ or $T_6(v)$ is decreasing for $v \geq 1.5$ with $\lim_{v\to\infty} T_6(v) = 0$, where

$$T_6(v) \sim -\frac{121}{22{,}118{,}400v^8} + \frac{2629}{72{,}576{,}000v^7} - \frac{1}{691{,}200v^6} - \frac{13}{13{,}440v^5} + \frac{11}{11{,}520v^4} + \ldots, \quad v \to \infty.$$

Therefore, $T_6(v) > 0$ for $v \geq 1.5$ or

$$\sqrt[6]{8v^3 + 4v^2 + v + \frac{1}{30}} > \sqrt{2v}\left(\frac{v^2 + \frac{7}{60}}{v^2 - \frac{1}{20}}\right)^{v/2}\left(\frac{461}{907{,}200v^5} + 1\right), \quad v \geq 1.5.$$

Then our new upper bound of inequality (16) is superior to that of its counterpart in inequality (6) for $v \geq 1.5$.

4. Conclusions

The main conclusions of this paper are stated in Theorems (1) and (2). Concretely speaking, we studied the monotonicity of two functions involving the Gamma function to introduce the double inequality (14). We proved that our new inequality is better than Alzer's double inequality (10) for $v \geq 1$. Additionally, our new lower (upper) bound is better than the lower (upper) bound of Ramanujan's inequality (6) for $v \geq 3.4$ ($v \geq 1.5$), respectively. Our results demonstrate that the approximation formula (12) had some advantages over Windschitl's formula (9) in producing more precise inequalities for the Gamma function.

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References

1. Batir, N. Very accurate approximations for the factorial function. *J. Math. Inequal.* **2010**, *4*, 335–344. [CrossRef]
2. Gosper, R.W. Decision procedure for indefinite hypergeometric summation. *Proc. Natl. Acad. Sci. USA* **1978**, *75*, 40–42. [CrossRef] [PubMed]
3. Mortici, C. On Gospers formula for the Gamma function. *J. Math. Inequal.* **2011**, *5*, 611–614. [CrossRef]
4. Abramowitz, M.; Stegun, I.A. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th ed.; Nation Bureau of Standards, Applied Mathematical Series; Dover Publications: New York, NY, USA; Washington, DC, USA, 1972; Volume 55.
5. Burnside, W. A rapidly convergent series for log $N!$. *Messenger Math.* **1917**, *46*, 157–159.
6. Andrews, G.E.; Berndt, B.C. *Ramanujan's Lost Notebook: Part IV*; Springer Science+ Business Media: New York, NY, USA, 2013.
7. Chen, C.-P.; Elezović, N.; Vukšić, L. Asymptotic formulae associated with the Wallis power function and digamma function. *J. Class. Anal.* **2013**, *2*, 151–166. [CrossRef]
8. Berndt, B.C.; Choi, Y.-S.; Kang, S.-Y. The problems submitted by Ramanujan. *J. Indian Math. Soc., Contemp. Math.* **1999**, *236*, 15–56.
9. Karatsuba, E.A. On the asymptotic representation of the Euler Gamma function by Ramanujan. *J. Comput. Appl. Math.* **2001**, *135*, 225–240. [CrossRef]
10. Mortici, C. On Ramanujan's large argument formula for the Gamma function. *Ramanujan J.* **2011**, *26*, 185–192. [CrossRef]
11. Ramanujan, S. *The Lost Notebook and Other Unpublished Papers*; Andrews, G.E., Intr.; Narosa Publ. H.-Springer: New Delhi, India; Berlin/Heidelberg, Germany, 1988.
12. Mortici, C. Improved asymptotic formulas for the Gamma function. *Comput. Math. Appl.* **2011**, *61*, 3364–3369. [CrossRef]
13. Programmable Calcualtors. Available online: http://www.rskey.org/CMS/the-library/11 (accessed on 20 April 2020).

14. Smith, W.D. The Gamma Function Revisited. 2006. Available online: http://schule.bayernport.com/gamma/gamma05.pdf (accessed on 20 April 2020).
15. Alzer, H. Sharp upper and lower bounds for the Gamma function. *Proc. R. Soc. Edinb.* **2009**, *139A*, 709–718. [CrossRef]
16. Nemes, G. New asymptotic expansion for the Gamma function. *Arch. Math.* **2010**, *95*, 161–169. [CrossRef]
17. Lu, D.; Song, L.; Ma, C. A generated approximation of the Gamma function related to Windschitl's formula. *J. Number Theory* **2014**, *140*, 215–225. [CrossRef]
18. Mahmoud, M.; Almuashi, H. On Some Asymptotic Expansions for the Gamma Function. *Symmetry* **2022**, *14*, 2459. [CrossRef]
19. Chen, C.-P. Asymptotic expansions of the Gamma function related to Windschitl's formula. *Appl. Math. Comput.* **2014**, *245*, 174–180. [CrossRef]
20. Yang, Z.-H.; Tian, J.-F. An accurate approximation formula for Gamma function. *J. Inequal. Appl.* **2018**, *2018*, 56. [CrossRef] [PubMed]
21. Yang, Z.-H.; Tian, J.-F. Two asymptotic expansions for Gamma function developed by Windschitl's formula. *Open Math.* **2018**, *16*, 1048–1060. [CrossRef]
22. Yang, Z.-H.; Tian, J.-F. A family of Windschitl type approximations for Gamma function. *J. Math. Inequal.* **2018**, *12*, 889–899. [CrossRef]
23. Yang, Z.-H.; Tian, J.-F. Windschitl type approximation formulas for the Gamma function. *J. Inequal. Appl.* **2018**, *2018*, 272. [CrossRef] [PubMed]