symmetry

MDPI

Article

Ricci Soliton of $\mathcal{CR}$-Warped Product Manifolds and Their Classifications

Yanlin Li [1], Sachin Kumar Srivastava [2], Fatemah Mofarreh [3], Anuj Kumar [2] and Akram Ali [4,*]

[1] School of Mathematics, Hangzhou Normal University, Hangzhou 311121, China; liyl@hznu.edu.cn
[2] Department of Mathematics, Central University of Himachal Pradesh, Dharamshala 176215, Himachal Pradesh, India; sachin@cuhimachal.ac.in (S.K.S.); kumaranuj9319@gmail.com (A.K.)
[3] Mathematical Science Department, Faculty of Science, Princess Nourah bint Abdulrahman University, Riyadh 11546, Saudi Arabia; fyalmofarrah@pnu.edu.sa
[4] Department of Mathematics, College of Science, King Khalid University, Abha 61413, Saudi Arabia
* Correspondence: akali@kku.edu.sa

Abstract: In this article, we derived an equality for $\mathcal{CR}$-warped product in a complex space form which forms the relationship between the gradient and Laplacian of the warping function and second fundamental form. We derived the necessary conditions of a $\mathcal{CR}$-warped product submanifolds in K*ä*hler manifold to be an Einstein manifold in the impact of gradient Ricci soliton. Some classification of $\mathcal{CR}$-warped product submanifolds in the K*ä*hler manifold by using the Euler–Lagrange equation, Dirichlet energy and Hamiltonian is given. We also derive some characterizations of Einstein warped product manifolds under the impact of Ricci Curvature and Divergence of Hessian tensor.

Keywords: Ricci solitons; warped products; geometric inequalities; complex space from; Dirichlet energy

check for updates

Citation: Li, Y.; Srivastava, S.K.; Mofarreh, F.; Kumar, A.; Ali, A. Ricci Soliton of $\mathcal{CR}$-Warped Product Manifolds and Their Classifications. *Symmetry* **2023**, *15*, 976. https://doi.org/10.3390/sym15050976

Academic Editor: Calogero Vetro

Received: 28 March 2023
Revised: 13 April 2023
Accepted: 14 April 2023
Published: 25 April 2023

1. Introduction

The evolution equation for a one-parameter family of a Riemannian metric g is characterized

$$\frac{\partial(g(t))}{\partial t} = -Ric(g(t)), \tag{1}$$

where Ric indicates the Ricci curvature. Equation (1) is known as the Ricci flow Equation [1]. The Ricci flow equation is a nonlinear partial differential equation that is highly nonlinear and weakly parabolic. This is strictly parabolic to the group of the diffeomorphism of smooth manifold M, termed as a Gauge group [1–3] which has several applications in quantum physics, particle physics, and general relativity. Gauge groups such as neutrinos and leptons characterize numerous standard models in particle physics. The Yang–Mills, general relativity, and electromagnetism are our greatest theories of nature. Each is based on a gauge symmetry at its foundation. The concept of a gauge symmetry is undoubtedly complicated. However, at its core, it is just an ambiguity in the words we use to explain physics. Why should nature enjoy such uncertainty? Understanding nature as a redundant set of variables is helpful for two reasons. First, although gauge symmetry makes our explanation of physics redundant, it seems concise. Ricci flow has several applications in relativity and physics [4]. The fixed point of Ricci flow is known as a Ricci soliton (for more details, see [5]), which is characterized by the following relation

$$Ric + \frac{1}{2}\mathcal{L}_X g = \lambda g, \tag{2}$$

where $\mathcal{L}_X$ denotes the Lie derivative and $\lambda \in \mathbb{R}$ can be any constant. The nature of Ricci soliton depends on the scalar λ. If $\lambda > 0$, $\lambda < 0$ and $\lambda = 0$, thus the Ricci soliton is known as expanding, shrinking and steady, respectively. In the relation (2), if we replace vector field X by gradient of some smooth function f then Equation (2) is transformed as follows:

$$Ric + \nabla^2 f = \lambda g, \tag{3}$$

where $\nabla^2 f$ is a Hessian of f. Equation (3) is known as gradient Ricci soliton equation, and is equivalent to

$$Ric = \lambda g + Hess^f. \tag{4}$$

The Equation (4) is the fundamental equation relating the Ricci tensor and Hessian tensor. As a special case, if f is a constant function on a smooth manifold M admitting gradient Ricci soliton, then (3) reduces to $Ric = \lambda g$. This expression leads to M as an Einstein manifold.

Furthermore, the geometry of warped products has significant use in mathematics and physics. In physics, several solutions of the Einstein field equation have warped product structures, for example, Schwarzschild's solution and Friedmann–Lemaitre–Robertson–Walker's solution. In string theory, the RS model has a significant role which is a five-dimensional anti-de Sitter warped product manifold. Such manifolds were first realized in 1969 by R. L. Bishop and B. O'Neill when they studied manifolds of negative curvature. They proved that there does not exist any Riemannian product manifold whose curvature is negative. After that, several authors studied warped product manifolds under different circumstances. This was taking more attention at the beginning of the twenty-first century, when authors such as [6–16] studied relevant topics and discussed some geometric properties related to singularity theory and submanifolds theory, etc. In 2002, B. Y. Chen derived the existence of the $\mathcal{CR}$-warped product of the form $M_T \times_f M_\perp$ in a K*ä*hler manifold, where M_T is real submanifold and $M_\perp$ is a complex submanifold. After that, numerous geometers studied $\mathcal{CR}$-warped product manifolds and their generalized classes in different ambient spaces [6,7,17–20].

The warped product manifold denoted by $M \times_f N$ is a product of Riemannian manifolds M_1 and M_2, which furnished a Riemannian metric g gratifying

$$g = g_M + f^2 g_N, \tag{5}$$

where g_M and g_N are the Riemannian metric of the smooth manifolds M and N, respectively, and $f : M \longrightarrow (0, \infty)$ bis a smooth function known as a warping function [21]. If $M \times_f N$ is a warped product manifold, then the following relations hold in

$$\nabla_{X_1} Z_1 = \nabla_{Z_1} X_1 = X_1(\ln f) Z_1, \tag{6}$$

$$\nabla_{Z_1} Z_2 = \nabla'_{Z_1} Z_2 - \nabla(\ln f) g(Z_1, Z_2). \tag{7}$$

From the above relation, we deduce that M and N are totally geodesic and totally umbilical manifolds in $M \times_f N$.

Recently, the Ricci soliton of warped products is taking more attention from the geometers. A Ricci soliton with warped product structure and gradient Ricci soliton with warped product structure were classified by different authors [22–28]. The author of [23] derived that if warped product manifold admits a gradient Ricci soliton, then the fiber is necessarily Einstein, and the potential function depends on the base manifold. In [24], the authors considered the Ricci soliton of a warped product. They derived useful results for such a manifold and applied them to the different spacetime models. Recently, the authors of [29] derived some useful results from Sasakian manifolds that admit an almost $\star$-Ricci soliton structure, and the authors of [30] extended a $\star$-Ricci soliton to a $\star$-η-Ricci soliton in Kenmotsu manifold. Furthermore, the Ricci curvature of warped products is utilized in string theory and the general theory of relativity. In the well-known Einstein's field equations (for more details, see [31]), the Ricci tensor establishes the connection to the matter distribution in the universe. Furthermore, the Ricci tensor is a part of the curvature of spacetime, which represents gravity's general relativity and also examines the degree to which matter will tend to converge or diverge concerning time. More generally, the Riemannian curvature is not more important than the Ricci curvature in physics. Suppose the solution of the Einstein field equation is a Ricci flat Riemannian (pseudo-Riemannian) manifold. In that case, it indicates that the cosmological constant is zero (for more details, see [32–34]). Due to the huge application of the Ricci soliton or, more generally, the Ricci tensor in physics, we are motivated to study the Ricci soliton of $\mathcal{CR}$-warped product submanifold in a complex space form. We consider the question and provide a partial answer to it. The question is that what are necessary and sufficient conditions for warped product immersions in complex space forms to be an Einstein warped product manifold with the impact of a gradient Ricci soliton?

This article is arranged as follows: Second 2 includes some necessary information about the K*ä*hler manifold and its submanifolds. In Section 3, we derived an equality for a $\mathcal{CR}$-warped product in a complex space form, forming the relationship between the gradient and Laplacian of the warping function and second fundamental form. The vector field X is considered as the gradient of the warping function in the relation (2) and derived the condition for such warped product to be Einsteinian and also derived some useful results in this article as applications of Theorem 1 and Lemma 2 into the Euler–Lagrange equation, in the Dirichlet energy and in the Ricci curvature in Section 4.

2. Preliminaries

From the well-known literature of complex geometry, an almost Hermitian manifold is a smooth manifold that admits an almost complex structure $\mathcal{J}$ and Hermitian metric g satisfying

$$\mathcal{J}^2 = -\mathcal{I},\ g(U_1, U_2) = g(\mathcal{J}U_1, \mathcal{J}U_2), \tag{8}$$

for every $U_1, U_2 \in \Gamma(T\widetilde{M}^{2n})$, $\Gamma(T\widetilde{M}^{2n})$ and $\mathcal{I}$ indicate for the section of tangent bundle $\widetilde{M}^{2n}$ and identity transformation, respectively, (see, [6,35–37]). The metric g is skew-symmetric

$$g(\mathcal{J}U_1, U_2) = -g(U_1, \mathcal{J}U_2). \tag{9}$$

Definition 1. *A Kähler manifold [6,35–37] is almost a Hermitian manifold, $\widetilde{M}^{2n}$ satisfies*

$$(\overline{\nabla}_{U_1}\mathcal{J})U_2 = 0, \tag{10}$$

$\forall U_1, U_2 \in \Gamma(T\widetilde{M}^{2n})$*. Here, $\widetilde{\nabla}$ indicates the Levi-Civita connection on $\widetilde{M}^{2n}$.*

Moreover, if the holomorphic sectional curvature of a non-flat K*ä*hler manifold is constant, then $\widetilde{M}^{2n}$ is termed as a complex space form. In this article, we denote $\widetilde{M}(c)$ for a complex space form. The the curvature tensor $\widetilde{R}$ of $\widetilde{M}(c)$ is characterized by

$$\begin{aligned} \widetilde{R}(U_1, U_2, U_3, U_4) \quad &= \tfrac{c}{4}(g(U_1, U_3)g(U_2, U_4) + g(U_1, \mathcal{J}U_3)g(\mathcal{J}U_2, U_4)) \\ &- \tfrac{c}{4}(g(U_2, U_3)g(U_1, U_4) + g(U_1, \mathcal{J}U_4)g(U_2, \mathcal{J}U_3)) \\ &+ \tfrac{c}{2}g(U_1, \mathcal{J}U_2)g(\mathcal{J}U_3, U_4), \end{aligned} \tag{11}$$

where c is holomorphic sectional curvature.

Let us assume that M is an m-dimensional Riemannian submanifold of a K*ä*hler manifold $\widetilde{M}^{2n}$. Let us denote $\Gamma(TM)$ for the section of the tangent bundle of M and $\Gamma(TM^{\perp})$ for the set of all normal vector fields of M, respectively, and also ∇ for the Levi-Civita connection on tangent bundle TM, $\nabla^{\perp}$ for the Levi-Civita connection on normal bundle $TM^{\perp}$, respectively. Thus, the Gauss and Weingarten formulas are described as follows:

$$\overline{\nabla}_{U_1}U_2 = \nabla_{U_1}U_2 + h(U_1, U_2), \tag{12}$$
$$\overline{\nabla}_{U_1}\xi = -A_\xi U_1 + \nabla^{\perp}_{U_1}\xi, \tag{13}$$

for all $U_1, U_2 \in \Gamma(TM)$ and $\xi \in \Gamma(TM^{\perp})$, where A_ξ and h are shape operator and second fundamental form, aided by

$$g(A_\xi U_1, U_2) = g(h(U_1, U_2), \xi). \tag{14}$$

The submanifold M is totally umbilical [6,7,19,20] if H satisfies $h(U_1, U_2) = g(U_1, U_2)H$, is totally geodesic if $h \equiv 0$ and minimal if $H = 0$, where H is the mean curvature vector described by $H = \frac{1}{m} trace(h)$. The covariant derivative of σ is computed by the following relation

$$(\nabla_{U_3}\sigma)(U_1, U_2) = \nabla^{\perp}_{U_3}\sigma(U_1, U_2) - \sigma(\nabla_{U_3}U_1, U_2) - \sigma(U_1, \nabla_{U_3}U_2). \tag{15}$$

The following relation characterizes the Gauss and Codazzi equations

$$\begin{aligned} \overline{R}(U_1, U_2, U_3, U_4) = \quad & R(U_1, U_2, U_3, U_4) + g(\sigma(U_1, U_3), \sigma(U_2, U_4)) \\ & - g(\sigma(U_1, U_4), \sigma(U_2, U_3)), \end{aligned} \tag{16}$$

$$(\overline{R}(U_1, U_2)U_3)^{\perp} = (\nabla_{U_1}\sigma)(U_2, U_3) - (\nabla_{U_2}\sigma)(U_1, U_3), \tag{17}$$

for every $U_1, U_2, U_3, U_4 \in \Gamma(TM)$. Moreover, for each $U_1 \in \Gamma(TM)$ and $\xi \in \Gamma(TM^{\perp})$, we have

$$\mathcal{J}U_1 = tU_1 + nU_1, \tag{18}$$
$$\mathcal{J}\xi = t^{'}\xi + n^{'}\xi, \tag{19}$$

where $tU_1 = tan(\mathcal{J}U_1)$ (resp. $t^{'}\xi = tan(\mathcal{J}\xi)$) and $nU_1 = nor(\mathcal{J}U_1)$ (resp. $n^{'}\xi = nor(\mathcal{J}\xi)$) are the tangential and normal parts of $\mathcal{J}U_1$ (resp.$\mathcal{J}\xi$), and tan and nor are orthogonal projections on TM and $TM^{\perp}$. With the help of (9), (18) and (19), we have

$$g(tU_1, U_1) = -g(U_1, tU_1),\ g(\xi, n^{'}\zeta) = -g(n^{'}\xi, \zeta), \tag{20}$$
$$g(nU_1, \xi) = g(U_1, t^{'}\xi). \tag{21}$$

Furthermore, the covariant derivative of $\mathcal{J}$, t and n are described by

$$(\nabla_{U_1}\mathcal{J})U_2 = \nabla_{U_1}\mathcal{J}U_2 - \mathcal{J}\nabla_{U_1}U_2, \tag{22}$$
$$(\nabla_{U_1}t)U_2 = \nabla_{U_1}tU_2 - t\nabla_{U_1}U_2, \tag{23}$$
$$(\nabla_{U_1}n)U_2 = \nabla^{\perp}_{U_1}nU_2 - n\nabla_{U_1}U_2, \tag{24}$$

$\forall\, U_1, U_2 \in \Gamma(TM)$. By the utilization of (8), (10), (12), (13), (18), (19) and (22)–(24), we obtain

$$(\nabla_{U_1}t)U_2 = A_{nU_2}U_1 + t^{'}h(U_1, U_2), \tag{25}$$
$$(\nabla_{U_1}n)U_2 = -h(U_1, tU_2) + n^{'}h(U_1, U_2). \tag{26}$$

for every $U_1, U_2 \in \Gamma(TM)$. Consider a smooth function $f : M \longrightarrow \mathbb{R}$, thus, the gradient and Laplacian are described by

$$\|\nabla f\|^2 = \sum_{q=1}^{n}(E_q(f))^2 \text{ and } g(\nabla f, U_1) = U_1 f, \tag{27}$$

$$\Delta f = \sum_{q=1}^{n}(\nabla_{E_q}E_q)f - E_q(E_q(f)), \tag{28}$$

for any $U_1 \in \Gamma(TM)$. The relation (28) can be expressed as

$$\Delta f = -\sum_{q=1}^{n} g(\nabla_{E_q} grad(f), E_q). \tag{29}$$

Let H^f be the Hessian of f, thus the Laplacian and Hessian are ailed by

$$\Delta f = -TraceH^f = -\sum_{q=1}^{n} H^f(E_q, E_q). \tag{30}$$

We will recall the above results for later use.

3. Curvature Inequality

In [36], Chen derived the general inequality for $\mathcal{CR}$-warped product submanifold $M_T \times_f M_\perp$ of K*ä*hler manifold which forms a connection between the gradient of the warping function and second fundamental form via the accompanying relation

$$\|h\|^2 \geq 2\beta\|\nabla \ln f\|^2, \tag{31}$$

where $\beta = dim(M_\perp)$. The above relation establishes a relationship between intrinsic invariant and extrinsic invariant. He also derived the classification of such types of warped products by the solution of a special system of partial differentials in a complex space form. After some time, he derived the curvature type inequality for a $\mathcal{CR}$-warped product submanifold $M_T \times_f M_\perp$ in a complex space theorem which is expressed as the following:

Theorem 1 ([35])**.** *Let $M = M_T \times_f M_\perp$ is $\mathcal{CR}$-warped product in $\widetilde{M}(c)$. Then the following inequality holds in $M = M_T \times_f M_\perp$*

$$\|h\|^2 \geq c\alpha\beta + 2\beta\|\nabla(\ln f)\|^2 - \beta\Delta(\ln f). \tag{32}$$

where $\Delta(\ln f)$ denotes the Laplacian of $\ln f$.

By the inequality (1), he classified $\mathcal{CR}$-warped product manifolds in complex Euclidean space satisfying the equality case of (1) up to rigid motion by the partial Serge embedding defined as $\phi_a^{pk} : \mathbb{C}_*^k \times \mathbb{S}^p \longrightarrow \mathbb{C}^{ap+k}$, where $\mathbb{C}_*^F = \mathbb{C} \sim \{0\}$ and $a, p, k \in \mathbb{N}$. With the help of Hopf fibration, he derived some conditions for $\mathcal{CR}$-warped products in complex projective space $\mathbb{CP}^n(4)$ and in complex hyperbolic space $\mathbb{CH}^n(-4)$ to satisfy the equality sign in (1) (for more details, see Theorem 5.1 [35]). Thereafter, several authors studied $\mathcal{CR}$-warped products in different ambient manifolds.

Definition 2. *Let M be a Riemannian submanifold of a Kähler manifold $\widetilde{M}^{2n}$. Then M is real submanifold if $\mathcal{J}(TM) \subset TM$ and M is complex submanifold if $\mathcal{J}(TM) \subset TM^\perp$.*

Definition 3. *A $\mathcal{CR}$-submanifold of Kähler manifold $\widetilde{M}^{2n}$ whose tangent bundle decomposed as $TM = \mathfrak{D} \oplus \mathfrak{D}_\perp$, where $\mathfrak{D}$ is a real distribution and $\mathfrak{D}_\perp$ is a complex distribution. Moreover, if there is a Riemannian metric on M of the form $g = g_{M_T} + f^2 g_{M_\perp}$ then M is a $\mathcal{CR}$-warped product of the form $M = M_T \times_f M_\perp$.*

Example 1. *Let us consider* 10*-dimensional Euclidean space* $\mathbb{R}^{10}$ *with coordinate* $(x_1, \cdots, x_5, y_1, \cdots, y_5)$ *and Euclidean metric* g*. An almost complex structure on* $\mathbb{R}^{10}$ *is defined by*

$$\mathcal{J}\left(\sum_{q=1}^{5} X^q \frac{\partial}{\partial x_q} + \sum_{s=1}^{5} Y^s \frac{\partial}{\partial y_s}\right) = \sum_{q=1}^{5} X^q \frac{\partial}{\partial y_q} - \sum_{s=1}^{5} Y^s \frac{\partial}{\partial x_s}. \tag{33}$$

Consider a subset $M \subset \mathbb{R}^{10}$ *immersed as a submanifold by the following immersion*

$$x_1 = u,\ x_2 = w,\ x_3 = u\sin\theta,\ x_4 = \delta,\ x_5 = v\cos\theta, \tag{34}$$

$$y_1 = v,\ y_2 = \delta,\ y_3 = v\cos\theta,\ y_4 = 2w,\ y_5 = u\sin\theta. \tag{35}$$

The basis spans the tangent subspace of M *at each point*

$$Z_u = \frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial x_3} + \cos\theta\frac{\partial}{\partial y_5},\ Z_\delta = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial y_2}, \tag{36}$$

$$Z_v = \frac{\partial}{\partial y_1} + \cos\theta\frac{\partial}{\partial x_5} + \sin\theta\frac{\partial}{\partial y_3},\ Z_w = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_4}, \tag{37}$$

$$Z_\theta = u\cos\theta\frac{\partial}{\partial x_3} - v\sin\theta\frac{\partial}{\partial x_5} + v\cos\theta\frac{\partial}{\partial y_3} - u\sin\theta\frac{\partial}{\partial y_5}. \tag{38}$$

With straightforward computation, we observed that the distribution spanned by $\{Z_u, Z_v, Z_w, Z_\delta\}$ *and distribution spanned by* Z_θ *are invariant and anti-invariant distribution, respectively. This shows that* M *is a warped product manifold with warping function* $f = \sqrt{u^2 + v^2}$*.*

Now, we recall one lemma related to $\mathcal{CR}$-warped product of K*ä*hler manifold for further use:

Lemma 1. *For a* $\mathcal{CR}$*-warped product* $M = M_T \times_f M_\perp$ *of Kähler manifold* $\widetilde{M}^{2n}$*, we obtain*

$$g(h(X_1, X_2), \mathcal{J}Z_1) = 0, \tag{39}$$

$$g(h(X_1, Z_1), \mathcal{J}Z_2) = -\mathcal{J}X_1(\ln f)g(Z_1, Z_2), \tag{40}$$

$$g(h(\mathcal{J}X_1, Z_1), \mathcal{J}Z_2) = X_1(\ln f)g(Z_1, Z_2), \tag{41}$$

$$g(h(\mathcal{J}X_1, Z_1), \mathcal{J}h(X_1, Z_1)) = \|h_\nu(X_1, Z_1)\|^2, \tag{42}$$

for all $X_1, X_1 \in \Gamma(TM_T)$ *and* $Z_1, Z_2 \in \Gamma(TM_\perp)$ *.*

Lemma 2. *Let* $M = M_T \times_f M_\perp$ *be a* $\mathcal{CR}$*-warped product in* $\widetilde{M}(c)$*. Then, we obtain*

$$\sum_{q=1}^{2\alpha}\sum_{s=1}^{\beta} \|h(E_q, E_s^*)\|^2 + \beta\Delta(\ln f) = c\alpha\beta + \beta\|\nabla(\ln f)\|^2, \tag{43}$$

$$\begin{aligned} \textstyle\sum_{q=1}^{2\alpha}\sum_{s=1}^{\beta} \|h(E_q, E_s^*)\|^2 - \beta\|\nabla(\ln f)\|^2 &= c\alpha\beta + \beta\Big(\textstyle\sum_{q=1}^{\alpha} Hess^{\ln f}(E_q, E_q)\Big) \\ &\quad + \beta\Big(\textstyle\sum_{q=1}^{\alpha} Hess^{\ln f}(\mathcal{J}E_q, \mathcal{J}E_q)\Big). \end{aligned} \tag{44}$$

Proof. By the application of (17), we have

$$\begin{aligned} \overline{R}(X_1, \mathcal{J}X_1, Z_1, \mathcal{J}Z_1) =& g((\nabla_{X_1}h)(\mathcal{J}X_1, Z_1), \mathcal{J}Z_1) \\ &- g((\nabla_{\mathcal{J}X_1}h)(X_1, Z_1), \mathcal{J}Z_1). \end{aligned} \tag{45}$$

By the definition of (15), we obtain $(\nabla_{X_1}h)(\mathcal{J}X_1, Z_1) = \nabla^{\perp}_{X_1}h(\mathcal{J}X_1, Z_1) - h(\nabla_{X_1}\mathcal{J}X_1, Z_1) - h(\mathcal{J}X_1, \nabla_{X_1}Z_1)$. By the application of covariant differentiation property into the last expression, we have

$$\begin{aligned} g((\nabla_{X_1}h)(\mathcal{J}X_1, Z_1), \mathcal{J}Z_1) &= X_1 g(h(\mathcal{J}X_1, Z_1), \mathcal{J}Z_1) - g(h(\mathcal{J}X_1, Z_1), \nabla^{\perp}_{X_1}\mathcal{J}Z_1) \\ &\quad - g(h(\nabla_{X_1}\mathcal{J}X_1, Z_1), \mathcal{J}Z_1) - g(h(\mathcal{J}X_1, \nabla_{X_1}Z_1), \mathcal{J}Z_1). \end{aligned}$$

With the help of Definition 10, (12), (13) and (14) and (39), the above expression takes the following form

$$\begin{aligned} g((\nabla_{X_1}h)(\mathcal{J}X_1, Z_1), \mathcal{J}Z_1) = \;& X_1^2(\ln f)g(Z_1, Z_1) - 2(X_1(\ln f))^2 g(Z_1, Z_1) \\ & -\nabla_{X_1}X_1(\ln f)g(Z_1, Z_1) - \|h_\nu(X_1, Z_1)\|^2. \end{aligned} \tag{46}$$

Similarly, we have

$$\begin{aligned} g((\nabla_{\mathcal{J}X_1}h)(X, Z_1), \mathcal{J}Z_1) = \;& -\mathcal{J}X_1^2(\ln f)g(Z_1, Z_1) + \|h_\nu(\mathcal{J}X_1, Z_1)\|^2 \\ & +\nabla_{\mathcal{J}X_1}\mathcal{J}X_1(\ln f)g(Z_1, Z_1) + 2(\mathcal{J}X_1(\ln f))^2 g(Z_1, Z_1). \end{aligned} \tag{47}$$

From (45)–(47), we have

$$\begin{aligned} -\tfrac{c}{2}\|X_1\|^2\|Z_1\|^2 = \;& X_1^2(\ln f)g(Z_1, Z_1) - 2(X_1(\ln f))^2 g(Z_1, Z_1) \\ & -\|h_\nu(\mathcal{J}X_1, Z_1)\|^2 - 2(\mathcal{J}X_1(\ln f))^2 g(Z_1, Z_1) \\ & -\nabla_{\mathcal{J}X_1}\mathcal{J}X_1(\ln f)g(Z_1, Z_1) - \nabla_{X_1}X_1(\ln f)g(Z_1, Z_1) \\ & -\|h_\nu(X_1, Z_1)\|^2 + \mathcal{J}X_1^2(\ln f)g(Z_1, Z_1). \end{aligned} \tag{48}$$

Now, consider $\{E_1, E_2, \cdots, E_m, E_{m+1}, \cdots, E_{2n}\}$ to be an orthonormal frame for a tangent bundle of $\widetilde{M}(c)$, in which $\{E_1, E_2, \cdots, E_m\}$ are tangent to the $\mathcal{CR}$-warped product M and $\{E_{m+1}, \cdots, E_{2n}\}$ are normal to M. By using the fact that M is a $\mathcal{CR}$-warped product submanifold in $\widetilde{M}(c)$ then, we observe that $m = 2\alpha + \beta$ and $2n - m = \beta + 2k$. Let $\{E_1, E_2, \cdots, E_\alpha, E_{\alpha+1} = \mathcal{J}E_1, E_{\alpha+2} = \mathcal{J}E_2, \cdots, E_{2\alpha} = \mathcal{J}E_\alpha\}$ be a basis of $\mathfrak{D}$, $\{E_{2\alpha+1} = E_1^*, E_{2\alpha+2} = E_2^*, \cdots, E_m = E_\beta^*,\}$ be a basis of $\mathfrak{D}_\perp$, $\{E_{m+1} = \mathcal{J}E_1^*, E_{m+2} = \mathcal{J}E_2^*, \cdots, E_{m+\beta} = \mathcal{J}E_\beta^*,\}$ be a basis of $n\mathfrak{D}_\perp$ and $\{E_{m+\beta+1} = \hat{E}_1, E_{m+\beta+2} = \hat{E}_2, \cdots, E_{m+\beta+k} = \hat{E}_k, E_{m+\beta+k+1} = \hat{E}_{k+1}, \cdots, E_{2n} = \hat{E}_{2k}\}$ be a basis of ν. Using above frame in the relation of (48), we find

$$\sum_{q=1}^{2\alpha}\sum_{s=1}^{\beta}\|h_\nu(E_q, E_s^*)\|^2 = c\alpha\beta - \beta\Delta(\ln f). \tag{49}$$

By the virtue of (40) and (41), we have

$$\sum_{q=1}^{2\alpha}\sum_{s=1}^{\beta}\|h_{n\mathfrak{D}_\perp}(E_q, E_s^*)\|^2 = \beta\|\nabla \ln f\|^2. \tag{50}$$

By the utilization of (49) and (50), we have (43). Since the Laplacian of some functions is a trace of Hessian, by using this fact, we obtain (44). □

Theorem 2. *If $M = M_T \times_f M_\perp$ is $\mathcal{CR}$-warped product in $\widetilde{M}(c)$. Then the second fundamental form satisfies*

$$\begin{aligned} \|h\|^2 \geq \;& c\alpha\beta + \beta\textstyle\sum_{q=1}^{\alpha} Hess^{\ln f}(E_q, E_q) \\ & +\beta\textstyle\sum_{q=1}^{\alpha} Hess^{\ln f}(\mathcal{J}E_q, \mathcal{J}E_q) + 2\beta\|\nabla(\ln f)\|^2. \end{aligned} \tag{51}$$

Proof. By the direct use of (30) into (32), we obtain (51). □

4. Applications

The study of curvatures in differential geometry and physics has great importance. The curvatures of the Riemannian (or pseudo-Riemannian) manifold are determined intrinsically and extrinsically. In curvatures, the Ricci curvature and scalar curvature are more applicable in physics. The Ricci curvature Ric and scalar curvature ρ of M is defined as

$$Ric = \sum_{q} R(X, E_q)E_q, \tag{52}$$

$$\tau(TM) = \sum_{1 \leq q \neq s \leq m} \mathcal{K}(E_q \wedge E_s), \tag{53}$$

where $\mathcal{K}(E_q \wedge E_s)$ is the sectional curvature of the plane spanned by E_q and E_s. Let G_k be the k-plane section of TM spanned by the orthonormal basis $\{E_1, E_2, \cdots, E_k\}$. Thus, the scalar curvature $\rho(G_k)$ of G_k is described by

$$\tau(G_k) = \sum_{1 \leq q \neq s \leq r} \mathcal{K}(E_q \wedge E_s), \tag{54}$$

In the potential theory, Dirichlet energies have significant use. If $f : M \longrightarrow \mathbb{R}$ is a smooth function, then the Dirichlet energy is defined by

$$E(f) = \frac{1}{2} \int_M \|\nabla f\|^2 dV. \tag{55}$$

where $E(f)$ and dV indicate Dirichlet energy and volume element, respectively. If $\mathbb{R}$ is replaced by a smooth manifold, then the Sigma model evaluates the Dirichlet energy. The Lagrange equation of the Sigma model has solutions that provide extreme Dirichlet energies. In Lagrangian mechanics, the Lagrangian L of a mechanical system is $T - V$, where T is kinetic energy and V is the system's potential energy, respectively. As a generalization to smooth manifolds, the Lagrangian of the smooth function f is determined by

$$L = \frac{1}{2}\|f\|^2. \tag{56}$$

The Euler–Lagrange equation for a Lagrangian L is $\Delta f = 0$. Now, we recall some useful results for further use:

Lemma 3 ([7])**.** *Let M be a compact, connected Riemannian manifold without boundary and f be a smooth function on M such that $\Delta f \geq 0$ ($\Delta f \leq 0$). Then f is a constant function.*

Moreover, if we apply the Green Theorem on a compact orientable Riemannian manifold without boundary, then we obtain

$$\int_M \Delta f dV = 0, \tag{57}$$

by using $\Delta f = div(X)$, for $X = \nabla(f)$, it immediately follows that

$$\int_M div(X) dV = 0, \tag{58}$$

where $div(X)$ indicates the divergence [7] for a connected, compact Riemannian manifold with a boundary. The well-known Hopf lemma takes the following form:

Theorem 3 ([7])**.** *Let M be a compact, connected Riemannian manifold with boundary and f be a smooth function on M such that $\Delta f = 0$ on M and $f_{\partial M} = 0$, then $f = 0$.*

The above theorem is also known as the uniqueness theorem of the Dirichlet problem. The Hamiltonian, denoted by H, represents the mechanical system's total energy. On the smooth even-dimensional manifold, the Hamiltonian induces a symplectic structure. The Hamiltonian [7] on the manifold is characterized by

$$\mathcal{H}(df, x) = \frac{1}{2}\sum_{q=1}^{n}(df(E_q))^2 = \frac{1}{2}\sum_{q=1}^{n}(E_q(f))^2 = \frac{1}{2}\|\nabla f\|^2. \tag{59}$$

4.1. Application to Euler–Lagrange Equation

Theorem 4. *Let $M = M_T \times_f M_\perp$ be a $\mathcal{CR}$-warped product in $\widetilde{M}(c)$ and warping function is a solution of the Euler–Lagrange equation, then M is a Riemannian $\mathcal{CR}$-product if*

$$\|h\|^2 \geq c\alpha\beta. \tag{60}$$

Proof. Suppose that $\ln f$ satisfies the Euler–Lagrange equation, i.e., $\Delta f = 0$. This implies that

$$\Delta(\ln f) = 0. \tag{61}$$

Since $\|h\|^2 \geq c\alpha\beta + 2\beta\|\nabla(\ln f)\|^2 - \beta\Delta(\ln f)$, therefore using the above equation, we have

$$\|h\|^2 \geq c\alpha\beta + 2\beta\|\nabla(\ln f)\|^2. \tag{62}$$

Using the given relation (60), the above expression is reduced to $\|\nabla(\ln f)\|^2 \leq 0$. However, $\|\nabla(\ln f)\|^2$ is always positive. Therefore, we must have $\nabla(\ln f) = 0$. This implies that f is constant; thus, the warped product is a Riemannian $\mathcal{CR}$-product. This accomplishes the proof. □

Corollary 1. *Let $M = M_T \times_f M_\perp$ be a $\mathcal{CR}$-warped product in $\widetilde{M}(c)$ and warping function is a solution of Euler–Lagrange equation, then M is a Riemannian $\mathcal{CR}$-product if*

$$\sum_{q=1}^{2\alpha}\sum_{s=1}^{\beta}\|h(E_q, E_s^*)\|^2 = c\alpha\beta. \tag{63}$$

Proof. By the direct application of (43) and proceeding same steps as Theorem 4, we achieve (63). □

Theorem 5. *Let $M = M_T \times_f M_\perp$ be a compact, orientable $\mathcal{CR}$-warped product submanifold in $\widetilde{M}(c)$ such that $\partial M = \phi$. Then, M is a Riemannian $\mathcal{CR}$-product if and only if*

$$\|h\|^2 \leq c\alpha\beta. \tag{64}$$

Proof. By taking integration of (32), we have

$$\int_M \beta\Delta(\ln f)dV \geq \int_M c\alpha\beta dV + 2\int_M \beta\|\nabla(\ln f)\|^2 dV - \int_M \|h\|^2 dV. \tag{65}$$

Now, utilizing the relations of (57) into Equation (65), we receive

$$0 \geq \int_M c\alpha\beta dV + 2\int_M \beta\|\nabla(\ln f)\|^2 dV - \int_M \|h\|^2 dV. \tag{66}$$

By the application of (64) and (66), we observe $\int_M \beta\|\nabla(\ln f)\|^2 dV \leq 0$. This implies that $\|\nabla(\ln f)\|^2$ is negative, i.e., $\|\nabla(\ln f)\|^2 \leq 0$. The last expression leads to f being a constant. This completes the proof. □

Theorem 6. *Let $M = M_T \times_f M_\perp$ be a compact, orientable $\mathcal{CR}$-warped product submanifold in $\widetilde{M}(c)$ such that $\partial M \neq \phi$. Then M is a Riemannian $\mathcal{CR}$-product if and only if*

$$E(\ln f) = \frac{1}{2\beta}\left(\int_M \left(\sum_{q=1}^{2\alpha}\sum_{s=1}^{\beta} \|h(E_q, E_s^*)\|^2 - c\alpha\beta\right) dV\right). \tag{67}$$

Proof. By taking integration of (43), we have

$$\begin{aligned} \int_M \sum_{q=1}^{2\alpha}\sum_{s=1}^{\beta}\left(\|h(E_q, E_s^*)\|^2\right)dV + \beta\int_M \Delta(\ln f)dV = \quad & \beta\int_M \|\nabla(\ln f)\|^2 dV \\ & + \int_M c\alpha\beta dV. \end{aligned} \tag{68}$$

Now utilizing the relations (55) into Equation (68), we receive

$$\begin{aligned} \beta\int_M \Delta(\ln f)dV = \quad & \int_M c\alpha\beta dV + 2\beta E(\ln f) \\ & - \int_M \sum_{q=1}^{2\alpha}\sum_{s=1}^{\beta}\left(\|h(E_q, E_s^*)\|^2\right)dV. \end{aligned} \tag{69}$$

By the virtue of (69) and (67), we have $\int_M \Delta(\ln f)dV = 0$. By applying the Hopf lemma, we achieve that f is constant. □

Theorem 7. *Let $M = M_T \times_f M_\perp$ be a compact, orientable $\mathcal{CR}$-warped product submanifold in $\widetilde{M}(c)$ such that $\partial M \neq \phi$. Then the M is a Riemannian $\mathcal{CR}$-product if and only if Hamiltonian H satisfies*

$$\mathcal{H}(df, x) = -\frac{1}{2\beta}\left(c\alpha\beta - \sum_{q=1}^{2\alpha}\sum_{s=1}^{\beta}\|h(E_q, E_s^*)\|^2\right). \tag{70}$$

Proof. By the utilization of (59) into (43), we have

$$\sum_{q=1}^{2\alpha}\sum_{s=1}^{\beta}\left(\|h(E_q, E_s^*)\|^2\right) = c\alpha\beta + 2\beta\mathcal{H}(df, x) - \beta\Delta(\ln f). \tag{71}$$

By virtue of (71), we observe that the relation (70) holds if and only if $\Delta(\ln f) = 0$. Since M is a compact, orientable Riemannian manifold, then by the application of Hopf lemma, we have f as constant. This completes the proof. □

4.2. Application to Gradient Ricci Soliton

The Ricci soliton is a natural generalization of Einstein manifolds. Such manifolds are important to study warped product manifolds because any regular surface is Einsteinian, and the surface of revolution is a warped product manifold. Moreover, they apply to each other in a more general setting, which can be realized in current times. Another generalization of the Einstein manifold is an almost Ricci soliton and quasi-Einstein manifold. In this paper, we derive some characterization of Einstein manifolds with the impact of Ricci soliton.

Theorem 8. *Let $M = M_T \times_f M_\perp$ be a $\mathcal{CR}$-warped product in $\widetilde{M}(c)$ admitting a shrinking gradient Ricci soliton. Then the following inequality holds*

$$\|h\|^2 \geq c\alpha\beta + \beta\|\nabla(\ln f)\|^2 + \beta\sum_{q=1}^{2\alpha} Ric(E_q, E_q). \tag{72}$$

Additionally, the equality holds if M_T is a totally geodesic submanifold and $M_\perp$ is totally umbilical submanifolds of $M(c)$.

Proof. If $M = M_T \times_f M_\perp$ is a $\mathcal{CR}$-warped product submanifold of a complex space form $\overline{M}(c)$ admitting a shrinking gradient Ricci soliton. Then, M fulfils the following relation

$$Ric(X_1, X_2) = \lambda g(X_1, X_2) + Hess^{\ln f}(X_1, X_2), \tag{73}$$

for every $X_1, X_2 \in \Gamma(TM_T)$. By using the above defined frame for M_T in (73), we have

$$\sum_{q=1}^{\alpha} Ric(E_q, E_q) = \lambda\alpha + \sum_{q=1}^{\alpha} Hess^{\ln f}(E_q, E_q). \tag{74}$$

By replacing E_q by $\mathcal{J}E_q$ into (74), we have

$$\sum_{q=1}^{\alpha} Ric(\mathcal{J}E_q, \mathcal{J}E_q) = \lambda\alpha + \sum_{q=1}^{\alpha} Hess^{\ln f}(\mathcal{J}E_q, \mathcal{J}E_q). \tag{75}$$

By the consequence of (74) and (75), we have

$$\begin{aligned} \textstyle\sum_{q=1}^{\alpha} Hess^{\ln f}(\mathcal{J}E_q, \quad & \mathcal{J}E_q) + \textstyle\sum_{q=1}^{\alpha} Hess^{\ln f}(E_q, E_q) \\ & = \textstyle\sum_{q=1}^{\alpha} Ric(E_q, E_q) - 2\lambda\alpha + \textstyle\sum_{q=1}^{\alpha} Ric(\mathcal{J}E_q, \mathcal{J}E_q). \end{aligned} \tag{76}$$

Applying (76) into the relation (51), we have

$$\|h\|^2 \geq c\alpha\beta + \beta\|\nabla(\ln f)\|^2 - 2\alpha\lambda + \sum_{q=1}^{2\alpha} Ric(E_q, E_q). \tag{77}$$

This accomplishes the proof. □

Theorem 9. *Let $M = M_T \times_f M_\perp$ be a $\mathcal{CR}$-warped product in $\widetilde{M}(c)$ admitting a shrinking gradient Ricci soliton. Then M is Einsteinian if the following equality holds*

$$c - 2\lambda + \frac{1}{\alpha}\sum_{q=1}^{2\alpha} Ric(E_q, E_q) = \frac{1}{\alpha\beta}\sum_{q=1}^{2\alpha}\sum_{s=1}^{\beta} \|h(E_q, E_s^*)\|^2. \tag{78}$$

Proof. By the virtue of (44), (75) and (76), we receive that

$$\begin{aligned} \textstyle\sum_{q=1}^{2\alpha}\sum_{s=1}^{\beta} \|h(E_q, E_s^*)\|^2 = \quad & \alpha\beta(c - 2\lambda) + \beta\|\nabla(\ln f)\|^2 \\ & + \beta\Big(\textstyle\sum_{q=1}^{\alpha}\big[Ric(E_q, E_q) + Ric(\mathcal{J}E_q, \mathcal{J}E_q)\big]\Big). \end{aligned} \tag{79}$$

Suppose (78) satisfies in M then from (79), we have $\|\nabla(\ln f)\|^2 = 0$. From the above expression, we receive that $\nabla(\ln f) = 0$. Therefore f is constant. Therefore, by the gradient Ricci soliton equation, we easily observe that M is Einstein's warped product. This finishes the proof. □

Corollary 2. *Let $M = M_T \times_f M_\perp$ be a $\mathcal{CR}$-warped product in $\mathbb{C}^n$ admitting a shrinking gradient Ricci soliton. Then M is Einsteinian if the following equality holds*

$$\lambda = \frac{1}{2\alpha}\sum_{q=1}^{2\alpha} Ric(E_q, E_q) - \frac{1}{2\alpha\beta}\sum_{q=1}^{2\alpha}\sum_{s=1}^{\beta} \|h(E_q, E_s^*)\|^2. \tag{80}$$

Proof. Since $\mathbb{C}^n$ is flat space, therefore $c = 0$, and by the direct application of (78), we obtain the result. □

Theorem 10. *Let $M = M_T \times_f M_\perp$ be a $\mathcal{CR}$-warped product in $\widetilde{M}(c)$ admitting a steady gradient Ricci soliton. Then M is Einsteinian if the following equality holds*

$$\frac{1}{\beta}\sum_{q=1}^{2\alpha}\sum_{s=1}^{\beta}\|h(E_q, E_s^*)\|^2 = c\alpha + \sum_{q=1}^{\alpha} Ric(E_q, E_q). \tag{81}$$

Proof. For the steady gradient Ricci soliton $\lambda = 0$, we achieve the result by proceeding with similar steps as the proof of Theorem 9. □

Theorem 11. *The necessary condition for a compact $\mathcal{CR}$-warped product submanifold $M = M_T \times_f M_\perp$ in $\widetilde{M}(c)$ to be a $\mathcal{CR}$-product is that*

$$\sum_{q=1}^{2\alpha}\sum_{s=1}^{\beta}\|h(E_q, E_s^*)\|^2 = c\alpha\beta - \beta\int_M Ric(\nabla(\tau), -)dV. \tag{82}$$

Proof. Since warping function f is smooth, $\ln f = \tau$ is also a smooth function, applying τ to the well known Ricci identity, we have

$$d(\tau)\mathcal{R}(X_1, X_2)X_3 = \nabla^2 d(\tau)(X_2, X_1, X_3) - \nabla^2 d(\tau)(X_1, X_2, X_3), \tag{83}$$

for any vector fields X_1, X_2, X_3 that are tangent to M_T. Because f is a smooth function and $\nabla^2_{X_1X_2} = \nabla_{X_1X_2} - \nabla_{\nabla_{X_1X_2}}$, then the curvature tensor behaves like the derivative defined by $\mathcal{R}_{X_1}X_2 = \nabla_{X_2}\nabla_{X_1} - \nabla_{X_1}\nabla_{X_2}$. With the help of the property that τ is closed we can easily prove $\nabla^2 d(\tau)(X_2, X_1, X_3) = \nabla^2 d(\tau)(X_1, X_2, X_3)$. Now, for an orthonormal frame $\{E_1, E_2, \cdots, E_{2\alpha}\}$ on M_T, we have $\nabla_{E_i}E_j(p) = 0$, for fixed point $p \in M_T$. If we describe $\nabla_{E_i}X_1 = 0$ for any $X_1 \in \Gamma(TM_T)$, and considering the trace with respect to X_2 and X_3 in the relation $\nabla^2 d(\tau)(X_2, X_1, X_3) = \nabla^2 d(\tau)(X_1, X_2, X_3)$, then using (83), we concede that

$$div(Hess^\tau) = Ric(\nabla(\tau), -) - d(\Delta(\tau)). \tag{84}$$

Since M is a compact $\mathcal{CR}$-warped product manifold with boundary then by taking the integration of (84), we have

$$\int_M div(Hess^\tau)dV = \int_M Ric(\nabla(\tau), -)dV - \Delta(\tau), \tag{85}$$

where dV is the volume element. From the relation (43), we have

$$\sum_{q=1}^{2\alpha}\sum_{s=1}^{\beta}\|h(E_q, E_s^*)\|^2 = c\alpha\beta - \beta\Delta(\ln f) + \beta\|\nabla(\ln f)\|^2. \tag{86}$$

Now, employing Green Theorem in (84), we have $\int_M div(Hess^\tau)dV = 0$, therefore (84) reduces into the following form

$$\Delta(\tau) = \int_M Ric(\nabla(\tau), -)dV. \tag{87}$$

In view of (86) and (87), we obtain

$$\sum_{q=1}^{2\alpha}\sum_{s=1}^{\beta}\|h(E_q, E_s^*)\|^2 = c\alpha\beta - \beta\int_M Ric(\nabla(\tau), -)dV + \beta\|\nabla(\ln f)\|^2. \tag{88}$$

If (82) holds in M then by relation (88), we have $\|\nabla(\ln f)\|^2 = 0$. With the help of the last expression, we conclude that f is constant. This accomplishes the proof. □

5. Conclusions

In short, this article includes some characterization results of Riemannian $\mathcal{CR}$-product and Einstein warped product in Kahler manifold under the impact of Euler–Lagrange equation, Dirichlet energy, Hamiltonian, gradient Ricci soliton and divergence of the Hessian operator. In the future, these characterizations will be studied under the effect of $\star$-Ricci soliton, $\star$-η-Ricci soliton and some other types of solitons.

Author Contributions: Conceptualization, S.K.S. and A.K.; methodology, A.A. and Y.L.; software, F.M.; validation, A.A., F.M. and A.K.; formal analysis, A.A.; investigation, S.K.S.; resources, A.K.; data curation, A.A.; writing—original draft preparation, A.K.; writing—review and editing, A.A.; visualization, Y.L.; supervision, S.K.S.; project administration, F.M. and Y.L.; funding acquisition, A.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Natural Science Foundation of China (Grant No. 12101168) and Zhejiang Provincial Natural Science Foundation of China (Grant No. LQ22A010014). The author (Akram Ali) would like to express their gratitude to the Deanship of Scientific Research at King Khalid University, Saudi Arabia, for providing a funding research group under the research grant R. G. P.1/87/43. Furthermore, The authors express their gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R27), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Data Availability Statement: There are no data used for the above study.

Acknowledgments: The author (Akram Ali) would like to express their gratitude to the Deanship of Scientific Research at King Khalid University, Saudi Arabia, for providing funding research group under the research grant R. G. P.1/87/43. Furthermore, The author (Fatemah Mofarreh) expressed their gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R27), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Hamilton, R.S. The Ricci flow in dimension three. *J. Differ. Geom.* **1982**, *17*, 255–306.
2. Hamilton, R.S. Eternal solutions to the Ricci flow. *J. Differ. Geom.* **1993**, *38*, 1–11. [CrossRef]
3. Hamilton, R.S. The Harnack estimate for the Ricci flow. *J. Differ. Geom.* **1993**, *37*, 225–243. [CrossRef]
4. Hamilton, R.S. The Ricci flow on surfaces, Mathematics, and general relativity. *Contemp. Math.* **1988**, *71*, 237–262.
5. Chow, B.; Knopf, D. *The Ricci Flow: An Introduction.*; American Mathematical Society: Providence, RI, USA, 2004.
6. Srivastava, S.K.; Sharma, A. Pointwise pseudo-slant warped product submanifolds in a Kähler manifold. *Mediterr. J. Math.* **2017**, *14*, 20. [CrossRef]
7. Ali, A.; Ozel, C. Geometry of warped product pointwise semi-slant submanifolds of cosymplectic manifolds and its applications. *Int. J. Geom. Methods Mod. Phys.* **2017**, *14*, 1750042. [CrossRef]
8. Li, Y.; Tuncer, O.O. On (contra)pedals and (anti)orthotomics of frontals in de Sitter 2-space. *Math. Meth. Appl. Sci.* **2023**, *1*, 1–15. [CrossRef]
9. Li, Y.; Erdoğdu, M.; Yavuz, A. Differential Geometric Approach of Betchow-Da Rios Soliton Equation. *Hacet. J. Math. Stat.* **2022**, *1*, 1–12. [CrossRef]
10. Li, Y.; Abolarinwa, A.; Alkhaldi, A.; Ali, A. Some Inequalities of Hardy Type Related to Witten-Laplace Operator on Smooth Metric Measure Spaces. *Mathematics* **2022**, *10*, 4580. [CrossRef]
11. Li, Y.; Aldossary, M.T.; Abdel-Baky, R.A. Spacelike Circular Surfaces in Minkowski 3-Space. *Symmetry* **2023**, *15*, 173. [CrossRef]
12. Li, Y.; Chen, Z.; Nazra, S.H.; Abdel-Baky, R.A. Singularities for Timelike Developable Surfaces in Minkowski 3-Space. *Symmetry* **2023**, *15*, 277. [CrossRef]
13. Li, Y.; Alkhaldi, A.; Ali, A.; Abdel-Baky, R.A.; Saad, M.K. Investigation of ruled surfaces and their singularities according to Blaschke frame in Euclidean 3-space. *AIMS Math.* **2023**, *8*, 13875–13888. [CrossRef]
14. Qian, Y.; Yu, D. Rates of approximation by neural network interpolation operators. *Appl. Math. Comput.* **2022**, *41*, 126781. [CrossRef]
15. Li, Y.; Eren, K.; Ayvacı, K.H.; Ersoy, S. The developable surfaces with pointwise 1-type Gauss map of Frenet type framed base curves in Euclidean 3-space. *AIMS Math.* **2023**, *8*, 2226–2239. [CrossRef]
16. Li, Y.; Abdel-Salam, A.A.; Saad, M.K. Primitivoids of curves in Minkowski plane. *AIMS Math.* **2023**, *8*, 2386–2406. [CrossRef]
17. Mihai, I. Contact $\mathcal{CR}$-warped product submanifolds in Sasakian space forms. *Geom. Dedicata* **2004**, *109*, 165–173. [CrossRef]
18. Hasegawa, I.; Mihai, I. Contact $\mathcal{CR}$-warped product submanifolds in Sasakian manifolds. *Geom. Dedicata* **2003**, *102*, 143–150. [CrossRef]

19. Srivastava, S.K.; Mofarreh, F.; Kumar, A.; Ali, A. Characterizations of $\mathcal{PR}$-Pseudo-Slant Warped Product Submanifold of Para-Kenmotsu Manifold with Slant Base. *Symmetry* **2022**, *14*, 1001. [CrossRef]
20. Dhiman, M.; Kumar, A.; Srivastava, S.K. $\mathcal{PR}$-Semi Slant Warped Product Submanifold of ParaKenmotsu Manifolds. *Results Math.* **2022**, *77*, 1–30. [CrossRef]
21. Chen, B.Y. *Differential Geometry of Warped Product Manifolds and Submanifolds*; World Scientific: Singapore, 2017.
22. Feitosa, F.E.S.; Freitas Filho, A.A.; Gomes, J.N.V. On the construction of gradient Ricci soliton warped product. *Nonlinear Anal.* **2017**, *161*, 30–43. [CrossRef]
23. Lemes de Sousa, M.; Pina, R. Gradient Ricci solitons with the structure of the warped product. *Results Math.* **2017**, *71*, 825–840. [CrossRef]
24. De, U.C.; Mantica, C.A.; Shenawy, S.; Ünal, B. Ricci solitons on singly warped product manifolds and applications. *J. Geom. Phys.* **2021**, *166*, 104257. [CrossRef]
25. Tokura, W.; Adriano, L.; Pina, R.; Barboza, M. The nonexistence of gradient almost Ricci solitons warped product. *Differ. Geom. Its Appl.* **2022**, *82*, 101884. [CrossRef]
26. Alkhaldi, A.H.; Ali, A. Classification of warped product submanifolds in Kenmotsu space forms admitting gradient Ricci solitons. *Mathematics* **2019**, *7*, 112. [CrossRef]
27. Karaca, F.; Özgür, C. Gradient Ricci solitons on multiply warped product manifolds. *Filomat* **2018**, *32*, 4221–4228. [CrossRef]
28. Kaya, D.A.; Leyla, O.N.A.T. Ricci Solitons on Multiply Warped Product Manifolds. *Int. Electron. J. Geom.* **2020**, *13*, 152–159. [CrossRef]
29. Rovenski, V.; Patra, D.S. Characteristics of Sasakian Manifolds Admitting Almost $\star$-Ricci Solitons. *Fractal Fract.* **2023**, *7*, 156. [CrossRef]
30. Yoldaş, H.I.; Haseeb, A.; Mofarreh, F. Certain Curvature Conditions on Kenmotsu Manifolds and $\star$-η-Ricci Solitons. *Axioms* **2023**, *12*, 140. [CrossRef]
31. Wald, R.M. *General Relativity*; University of Chicago Press: Chicago, IL, USA, 2010.
32. Siddiqi, M.D.; Siddiqui, S.A. Conformal Ricci soliton and Geometrical structure in a perfect fluid spacetime. *Int. J. Geom. Methods Mod. Phys.* **2020**, *17*, 2050083. [CrossRef]
33. Duggal, K.L. Almost Ricci solitons and physical applications. *Int. Electron. J. Geom.* **2017**, *10*, 1–10.
34. Akbar, M.M.; Eric, W. Ricci solitons and Einstein-scalar field theory. *Class. Quantum Gravity* **2009**, *26*, 055015. [CrossRef]
35. Chen, B.Y. Another general inequality for CR-warped products in complex space forms. *Hokkaido Math. J.* **2003**, *32*, 415–444. [CrossRef]
36. Chen, B.Y. Geometry of warped product CR-submanifolds in Kaehler manifolds. *Monatshefte Math.* **2001**, *133*, 177–195. [CrossRef]
37. Chen, B.Y. Geometry of warped product CR-submanifolds in Kaehler manifolds, II. *Monatshefte Math.* **2001**, *134*, 103–119. [CrossRef]