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Coefficient Bounds for Symmetric Subclasses of q-Convolution-Related Analytical Functions

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Abstract: By using q-convolution, we determine the coefficient bounds for certain symmetric subclasses of analytic functions of complex order, which are introduced here by means of a certain non-homogeneous Cauchy–Euler-type differential equation of order m.

Keywords: convolution; fractional derivative; coefficients bounds; q-derivative, non-homogeneous Cauchy–Euler-type

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1. Introduction, Definitions and Preliminaries

Assume that $\mathbb{A}$ is the class of analytic functions in the open disc $\Lambda := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ of the form

$$\Upsilon(\zeta) = \zeta + \sum_{t=2}^{+\infty} a_t \zeta^t,\ \zeta \in \Lambda. \tag{1}$$

If the function $h \in \mathbb{A}$ is given by

$$h(\zeta) = \zeta + \sum_{t=2}^{+\infty} c_t \zeta^t,\ \zeta \in \Lambda. \tag{2}$$

The *Hadamard (or convolution) product* of Υ and h is defined by

$$(\Upsilon * h)(\zeta) := \zeta + \sum_{t=2}^{+\infty} a_t c_t \zeta^t,\ \zeta \in \Lambda.$$

A function $\Upsilon \in \mathcal{A}$ belongs to the class $\mathcal{S}^*(\eta)$ if

$$\Re\left\{1 + \frac{1}{\eta}\left(\frac{\zeta \Upsilon'(\zeta)}{\Upsilon(\zeta)} - 1\right)\right\} > 0\ \ (\zeta \in \Lambda;\ \eta \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}). \tag{3}$$

Furthermore, a function $\Upsilon \in \mathcal{A}$ be in the class $\mathcal{C}(\eta)$ if

$$\Re\left\{1 + \frac{1}{\eta}\frac{\zeta \Upsilon''(\zeta)}{\Upsilon'(\zeta)}\right\} > 0\ \ (\zeta \in \Lambda;\ \eta \in \mathbb{C}^*). \tag{4}$$

The classes $\mathcal{S}^*(\eta)$ and $\mathcal{C}(\eta)$ were studied by Nasr and Aouf [1,2] and Wiatrowski [3].

In a wide range of applications in the mathematical, physical, and engineering sciences, the theory of q-calculus is important. Jackson [4,5] was the first to use the q-calculus in various applications and to introduce the q-analogue of the standard derivative and integral operators; see [6–10]. About coefficients' interesting results, see [11–16]. The q-shifted factorial is defined for $\lambda, q \in \mathbb{C}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ as follows

$$(\lambda;q)_t = \begin{cases} 1 & t = 0, \\ (1-\lambda)(1-\lambda q)\ldots\left(1-\lambda q^{t-1}\right) & t \in \mathbb{N}. \end{cases}$$

Using the q-gamma function $\Gamma_q(\zeta)$, we obtain

$$\left(q^{\lambda};q\right)_t = \frac{(1-q)^t\,\Gamma_q(\lambda+t)}{\Gamma_q(\lambda)}, \quad (t \in \mathbb{N}_0),$$

where

$$\Gamma_q(\zeta) = (1-q)^{1-\zeta}\frac{(q;q)_\infty}{\left(q^{\zeta};q\right)_\infty}, \quad (|q|<1).$$

In addition, we note that

$$(\lambda;q)_\infty = \prod_{t=0}^{\infty}\left(1-\lambda q^t\right), \quad (|q|<1),$$

and the q-gamma function $\Gamma_q(\zeta)$ is known

$$\Gamma_q(\zeta+1) = [\zeta]_q\,\Gamma_q(\zeta),$$

where $[t]_q$ denotes the basic q-number defined as follows

$$[t]_q := \begin{cases} \frac{1-q^t}{1-q}, & t \in \mathbb{C}, \\ 1 + \sum\limits_{j=1}^{t-1} q^j, & t \in \mathbb{N} \end{cases}. \tag{5}$$

Using the definition Formula (5), we have the next two products:

(i) For any non negative integer t, the *q-shifted factorial* is given by

$$[t]_q! := \begin{cases} 1, & \text{if} \quad t = 0, \\ \prod\limits_{n=1}^{t}[n]_q, & \text{if} \quad t \in \mathbb{N}. \end{cases}$$

(ii) For any positive number r, the *q-generalized Pochhammer symbol* is defined by

$$[r]_{q,t} := \begin{cases} 1, & \text{if} \quad t = 0, \\ \prod\limits_{n=r}^{r+t-1}[n]_q, & \text{if} \quad t \in \mathbb{N}. \end{cases}$$

It is known in terms of the classical (Euler's) gamma function $\Gamma(\zeta)$, that

$$\Gamma_q(\zeta) \to \Gamma(\zeta) \qquad \text{as } q \to 1^-.$$

In addition, we observe that

$$\lim_{q\to 1^-}\left\{\frac{\left(q^{\lambda};q\right)_t}{(1-q)^t}\right\} = (\lambda)_t,$$

where $(\lambda)_t$ is given by

$$(\lambda)_t = \begin{cases} 1, & \text{if} \quad t = 0, \\ \lambda(\lambda+1)\ldots(\lambda+t-1), & \text{if} \quad t \in \mathbb{N}. \end{cases}.$$

For $0 < q < 1$. El-Deeb et al. [17] defined that the *q-derivative operator* for $\Upsilon * h$ is defined by

$$\begin{aligned}\mathcal{D}_q(\Upsilon * h)(\zeta) &:= \mathcal{D}_q\left(\zeta + \sum_{t=2}^{+\infty} a_t c_t \zeta^t\right)\\ &= \frac{(\Upsilon * h)(\zeta) - (\Upsilon * h)(q\zeta)}{\zeta(1-q)} = 1 + \sum_{t=2}^{+\infty} [t]_q a_t c_t \zeta^{t-1},\ \zeta \in \Lambda,\end{aligned}$$

Let $\vartheta > -1$ and $0 < q < 1$; El-Deeb et al. [17] defined the linear operator $\mathcal{R}_h^{\vartheta,q} : \mathbb{A} \to \mathbb{A}$ as follows:

$$\mathcal{R}_h^{\vartheta,q}\Upsilon(\zeta) * \mathcal{N}_{q,\vartheta+1}(\zeta) = \zeta\, \mathcal{D}_q(\Upsilon * h)(\zeta),\ \zeta \in \Lambda,$$

where the function $\mathcal{M}_{q,\vartheta+1}$ is given by

$$\mathcal{N}_{q,\vartheta+1}(\zeta) := \zeta + \sum_{t=2}^{+\infty} \frac{[\vartheta+1]_{q,t-1}}{[t-1]_q!}\zeta^t,\ \zeta \in \Lambda.$$

A simple computation shows that

$$\mathcal{R}_h^{\vartheta,q}\Upsilon(\zeta) := \zeta + \sum_{t=2}^{+\infty} \frac{[t]_q!}{[\vartheta+1]_{q,t-1}} a_t c_t \zeta^t,\ \zeta \in \Lambda \quad (\vartheta > -1,\ 0 < q < 1). \tag{6}$$

Remark 1 ([17])**.** *From the definition relation* (6)*, we can obtain that the next relations hold for all* $\Upsilon \in \mathcal{A}$*:*

(i) $[\vartheta+1]_q \mathcal{R}_h^{\vartheta,q}\Upsilon(\zeta) = [\vartheta]_q \mathcal{R}_h^{\vartheta+1,q}\Upsilon(\zeta) + q^\vartheta\, \zeta \mathcal{D}_q\left(\mathcal{R}_h^{\vartheta+1,q}\Upsilon(\zeta)\right),\ \zeta \in \Lambda;$

(ii) $$\mathcal{I}_h^\vartheta\Upsilon(\zeta) := \lim_{q\to 1^-} \mathcal{R}_h^{\vartheta,q}\Upsilon(\zeta) = \zeta + \sum_{t=2}^{+\infty} \frac{t!}{(\vartheta+1)_{t-1}} a_t c_t \zeta^t, \quad \zeta \in \Lambda. \tag{7}$$

Remark 2 ([17])**.** *By taking different particular cases for the coefficients* c_t*, El-Deeb et al.* [17] *observed the following special cases for the operator* $\mathcal{R}_h^{\vartheta,q}$*:*

(i) *For* $c_t = \dfrac{(-1)^{t-1}\Gamma(\rho+1)}{4^{t-1}(t-1)!\Gamma(t+\rho)}$, $\rho > 0$*, El-Deeb and Bulboacă* [18] *and El-Deeb* [19] *obtained the operator* $\mathcal{N}_{\rho,q}^{\vartheta}$ *studied by:*

$$\begin{aligned}\mathcal{N}_{\rho,q}^{\vartheta}\Upsilon(\zeta) &:= \zeta + \sum_{t=2}^{+\infty} \frac{(-1)^{t-1}\Gamma(\rho+1)}{4^{t-1}(t-1)!\Gamma(t+\rho)} \cdot \frac{[t]_q!}{[\vartheta+1]_{q,t-1}} a_t \zeta^t\\ &= \zeta + \sum_{t=2}^{+\infty} \frac{[t]_q!}{[\vartheta+1]_{q,t-1}} \psi_t a_t \zeta^t,\ \zeta \in \Lambda,\ (\rho > 0,\ \vartheta > -1,\ 0 < q < 1),\end{aligned} \tag{8}$$

where

$$\psi_t := \frac{(-1)^{t-1}\Gamma(\rho+1)}{4^{t-1}(t-1)!\Gamma(t+\rho)}; \tag{9}$$

(ii) *For* $c_t = \left(\dfrac{m+1}{m+t}\right)^\alpha$, $\alpha > 0$, $m \geq 0$*, El-Deeb and Bulboacă* [20] *and Srivastava and El-Deeb* [21] *obtained the operator* $\mathcal{N}_{m,1,q}^{\vartheta,\alpha} =: \mathcal{M}_{m,q}^{\vartheta,\alpha}$ *studied by:*

$$\mathcal{M}_{m,q}^{\vartheta,\alpha}\Upsilon(\zeta) := \zeta + \sum_{t=2}^{+\infty} \left(\frac{m+1}{m+t}\right)^\alpha \cdot \frac{[t]_q!}{[\vartheta+1]_{q,t-1}} a_t \zeta^t,\ \zeta \in \Lambda; \tag{10}$$

(iii) *For* $c_t = \dfrac{n^{t-1}}{(t-1)!}e^{-n}$, $n > 0$*, El-Deeb et al.* [17] *obtained the q-analogue of Poisson operator defined by:*

$$\mathcal{I}_q^{\vartheta,n}\Upsilon(\zeta) := \zeta + \sum_{t=2}^{+\infty} \frac{n^{t-1}}{(t-1)!}e^{-n} \cdot \frac{[t]_q!}{[\vartheta+1]_{q,t-1}} a_t \zeta^t,\ \zeta \in \Lambda; \tag{11}$$

(iv) *For* $c_t = \left[\frac{1+\ell+\lambda(t-1)}{1+\ell}\right]^n$, $n \in \mathbb{Z}$, $\ell \geq 0$, $\lambda \geq 0$, *El-Deeb et al.* [17] *obtained the q-analogue of Prajapat operator defined by*

$$\mathcal{J}_{q,\ell,\lambda}^{\vartheta,n}\Upsilon(\zeta) := \zeta + \sum_{t=2}^{+\infty}\left[\frac{1+\ell+\lambda(t-1)}{1+\ell}\right]^n \cdot \frac{[t]_q!}{[\vartheta+1]_{q,t-1}} a_t \zeta^t,\ \zeta \in \Lambda. \tag{12}$$

In this paper, we define the following subclasses $\mathcal{SC}_h^{\vartheta,q}(\eta,\gamma,\beta)$ and $\mathcal{N}_h^{\vartheta,q}(\eta,\gamma,\beta,m,\mu)$ $(\eta \in \mathbb{C}^*,\ 0 \leq \gamma \leq 1,\ 0 \leq \beta < 1,\ \vartheta > -1,\ 0 < q < 1,\ m \in \mathbb{N}^* = \mathbb{N}\backslash\{1\} = \{2,3,4,\ldots\},\ \mu \in \mathbb{R}\backslash(-\infty,-1])$ as follows:

Definition 1. *For a function* Υ *has the form* (1) *and* h *is defined by* (2)*, the function* Υ *belongs to the class* $\mathcal{SC}_h^{\vartheta,q}(\eta,\gamma,\beta)$ *if*

$$\Re\left\{1 + \frac{1}{\eta}\left[\frac{\zeta\left[(1-\gamma)\mathcal{R}_h^{\vartheta,q}\Upsilon(\zeta) + \gamma\zeta\left(\mathcal{R}_h^{\vartheta,q}\Upsilon(\zeta)\right)'\right]'}{(1-\gamma)\mathcal{R}_h^{\vartheta,q}\Upsilon(\zeta) + \gamma\zeta\left(\mathcal{R}_h^{\vartheta,q}\Upsilon(\zeta)\right)'} - 1\right]\right\} > \beta$$

$$(\eta \in \mathbb{C}^*;\ 0 \leq \gamma \leq 1;\ 0 \leq \beta < 1;\ \vartheta > -1,\ 0 < q < 1;\ \zeta \in \Lambda). \tag{13}$$

Remark 3.

(i) *For* $q \to 1^-$, *we obtain that* $\lim_{q\to 1^-} \mathcal{SC}_h^{\vartheta,q}(\eta,\gamma,\beta) =: \mathcal{G}_h^{\vartheta}(\eta,\gamma,\beta)$, *where* $\mathcal{G}_h^{\vartheta}(\eta,\gamma,\beta)$ *represents the functions* $\Upsilon \in \mathbb{A}$ *that satisfies* (13) *for* $\mathcal{R}_h^{\vartheta,q}$ *replaced with* $\mathcal{I}_h^{\vartheta}$ (7).

(ii) *For* $c_t = \frac{(-1)^{t-1}\Gamma(\rho+1)}{4^{t-1}(t-1)!\Gamma(t+\rho)}$, $\rho > 0$, *we obtain the subclass* $\mathcal{B}_\rho^{\vartheta,q}(\eta,\gamma,\beta)$, *that represents the functions* $\Upsilon \in \mathbb{A}$ *that satisfies* (13) *for* $\mathcal{R}_h^{\vartheta,q}$ *replaced with* $\mathcal{N}_{\rho,q}^{\vartheta}$ (8).

(iii) *For* $c_t = \left(\frac{m+1}{m+t}\right)^\alpha$, $\alpha > 0$, $m \geq 0$, *we obtain the class* $\mathcal{M}_{m,\alpha}^{\vartheta,q}(\eta,\gamma,\beta)$, *that represents the functions* $\Upsilon \in \mathbb{A}$ *that satisfies* (13) *for* $\mathcal{R}_h^{\vartheta,q}$ *replaced with* $\mathcal{M}_{m,q}^{\vartheta,\alpha}$ (10).

(iv) *For* $c_t = \frac{n^{t-1}}{(t-1)!}e^{-n}$, $n > 0$, *we obtain the class* $\mathcal{I}_t^{\vartheta,q}(\eta,\gamma,\beta)$, *that represents the functions* $\Upsilon \in \mathbb{A}$ *that satisfies* (13) *for* $\mathcal{R}_h^{\vartheta,q}$ *replaced with* $\mathcal{I}_q^{\vartheta,t}$ (11).

(v) *For* $c_t = \left[\frac{1+\ell+\lambda(t-1)}{1+\ell}\right]^n$, $n \in \mathbb{Z}$, $\ell \geq 0$, $\lambda \geq 0$, *we obtain the class* $\mathcal{J}_{n,\ell,\lambda}^{\vartheta,q}(\eta,\gamma,\beta)$, *that represents the functions* $\Upsilon \in \mathbb{A}$ *that satisfies* (13) *for* $\mathcal{R}_h^{\vartheta,q}$ *replaced with* $\mathcal{J}_{q,\ell,\lambda}^{\vartheta,n}$ (12).

The following lemma must be used in to show our study results:

Definition 2. *A function* $\Upsilon \in \mathbb{A}$ *belongs to the class* $\mathcal{N}_h^{\vartheta,q}(\eta,\gamma,\beta,m,\mu)$ *if it satisfies the following non-homogeneous Cauchy–Euler type differential equation of order m:*

$$\zeta^m \frac{d^m w}{d\zeta^m} + \binom{m}{1}(\mu+m-1)\zeta^{m-1}\frac{d^{m-1}w}{d\zeta^{m-1}} + \cdots + \binom{m}{m} w \prod_{j=0}^{m-1}(\mu+j) = g(\zeta)\prod_{j=0}^{m-1}(\mu+j+1)$$

$$\Big(w = \Upsilon(\zeta);\ g(\zeta) \in \mathcal{SC}_h^{\vartheta,q}(\eta,\gamma,\beta);\ \eta \in \mathbb{C}^*,\ 0 \leq \gamma \leq 1,\ 0 \leq \beta < 1;\ \vartheta > -1;\ 0 < q < 1;$$
$$m \in \mathbb{N}^*;\ \mu \in \mathbb{R}\backslash(-\infty,-1]\Big).$$

Remark 4.

(i) *Putting* $q \to 1^-$, *we obtain that* $\lim_{q\to 1^-} \mathcal{N}_h^{\vartheta,q}(\eta,\gamma,\beta,m,\mu) =: \mathcal{T}_h^{\vartheta}(\eta,\gamma,\beta,m,\mu)$, *where* $\mathcal{T}_h^{\vartheta}(\eta,\gamma,\beta,m,\mu)$ *represents the functions* $\Upsilon \in \mathbb{A}$ *that satisfies* (13) *for* $\mathcal{R}_h^{\vartheta,q}$ *replaced with* $\mathcal{I}_h^{\lambda}$ (7).

(ii) *Putting* $c_t = \dfrac{(-1)^{t-1}\Gamma(\rho+1)}{4^{t-1}(t-1)!\Gamma(t+\rho)}$, $\rho > 0$, *we get the subclass* $\mathcal{P}_\rho^{\vartheta,q}(\eta,\gamma,\beta,m,\mu)$, *that represents the functions* $\Upsilon \in \mathbb{A}$ *that satisfies* (13) *for* $\mathcal{R}_h^{\vartheta,q}$ *replaced with* $\mathcal{N}_{\rho,q}^{\vartheta}$ (8).

(iii) *Putting* $c_t = \left(\dfrac{m+1}{m+t}\right)^{\alpha}$, $\alpha > 0$, $m \geq 0$, *we have the class* $\mathcal{R}_{m,\alpha}^{\vartheta,q}(\eta,\gamma,\beta,m,\mu)$, *that represents the functions* $\Upsilon \in \mathbb{A}$ *that satisfies* (13) *for* $\mathcal{R}_h^{\vartheta,q}$ *replaced with* $\mathcal{M}_{m,q}^{\vartheta,\alpha}$ (10).

(iv) *Putting* $c_t = \dfrac{n^{t-1}}{(t-1)!}e^{-n}$, $n > 0$, *we get the class* $\mathcal{D}_n^{\vartheta,q}(\eta,\gamma,\beta,m,\mu)$, *that represents the functions* $\Upsilon \in \mathbb{A}$ *that satisfies* (13) *for* $\mathcal{R}_h^{\vartheta,q}$ *replaced with* $\mathcal{I}_q^{\vartheta,n}$ (11).

(v) *Putting* $c_t = \left[\dfrac{1+\ell+\lambda(t-1)}{1+\ell}\right]^n$, $n \in \mathbb{Z}$, $\ell \geq 0$, $\lambda \geq 0$, *we have the class* $\mathcal{J}_{n,\ell,\lambda}^{\vartheta,q}(\eta,\gamma,\beta,m,\mu)$, *that represents the functions* $\Upsilon \in \mathbb{A}$ *that satisfies* (13) *for* $\mathcal{R}_h^{\vartheta,q}$ *replaced with* $\mathcal{J}_{q,\ell,\lambda}^{\vartheta,n}$ (12).

The main object of the present investigation is to derive some coefficient bounds for functions in the subclasses $\mathcal{SC}_h^{\vartheta,q}(\eta,\gamma,\beta)$ and $\mathcal{N}_h^{\vartheta,q}(\eta,\gamma,\beta,m,\mu)$ of $\mathbb{A}$.

2. Coefficient Estimates for the Function Class $\mathcal{SC}_h^{\vartheta,q}(\eta,\gamma,\beta)$

Unless otherwise mentioned, we assume throughout this paper that:
$\eta \in \mathbb{C}^*$, $0 \leq \gamma \leq 1$, $0 \leq \beta < 1$; $m \in \mathbb{N}^*$; $\mu \in \mathbb{R}\backslash(-\infty,-1]$, $\vartheta > -1$; $0 < q < 1$, $\zeta \in \Lambda$.

Theorem 1. *Assume that the function* Υ *given by* (1) *belongs to the class* $\mathcal{SC}_h^{\vartheta,q}(\eta,\gamma,\beta)$, *then*

$$|a_t| \leq \frac{[\vartheta+1]_{q,t-1}\prod_{i=0}^{t-2}[i+2(1-\beta)|\eta|]}{(t-1)![1+\gamma(t-1)][t]_q!c_t}(t \in \mathbb{N}*). \tag{14}$$

Proof. The function $\Upsilon \in \mathbb{A}$ be given by (1) and let the function $\mathcal{F}(\zeta)$ be defined by

$$\mathcal{F}(\zeta) = (1-\gamma)\mathcal{R}_h^{\vartheta,q}\Upsilon(\zeta) + \gamma\zeta\left(\mathcal{R}_h^{\vartheta,q}\Upsilon(\zeta)\right)'.$$

Then from (13) and the definition of the function $\mathcal{F}(\zeta)$ above, it is easily seen that

$$\Re\left\{1 + \frac{1}{\eta}\left(\frac{\zeta\mathcal{F}'(\zeta)}{\mathcal{F}(\zeta)} - 1\right)\right\} > \beta$$

with

$$\mathcal{F}(\zeta) = \zeta + \sum_{t=2}^{+\infty}\Theta_t\zeta^t \ \left(\Theta_t = \tfrac{[t]_q!}{[\vartheta+1]_{q,t-1}}[1+\gamma(t-1)]a_t\,c_t;\ t \in \mathbb{N}^*\right).$$

Thus, by setting

$$\frac{1 + \frac{1}{\eta}\left(\frac{\zeta\mathcal{F}'(\zeta)}{\mathcal{F}(\zeta)} - 1\right) - \beta}{1-\beta} = g(\zeta)$$

or, equivalently,

$$\zeta\mathcal{F}'(\zeta) = [1 + \eta(1-\beta)(g(\zeta)-1)]\mathcal{F}(\zeta), \tag{15}$$

we get

$$g(\zeta) = 1 + d_1\zeta + d_2\zeta^2 + \ldots\ldots. \tag{16}$$

Since $\Re\{g(\zeta)\} > 0$, we conclude that $|d_t| \leq 2$ $(t \in \mathbb{N})$ (see [14]).
We get from (15) and (16) that

$$(t-1)\Theta_t = \eta(1-\beta)[d_1\Theta_{t-1} + d_2\Theta_{t-2} + \cdots + d_{t-1}].$$

For $t = 2, 3, 4$, we have

$$\Theta_2 = \eta(1-\beta)d_1 \quad \Rightarrow |\Theta_2| \leq 2(1-\beta)|\eta|,$$

$$2\Theta_3 = \eta(1-\beta)(d_1\Theta_2 + d_2) \quad \Rightarrow |\Theta_3| \leq \frac{2(1-\beta)|\eta|[1+2(1-\beta)|\eta|]}{2!},$$

and

$$3\Theta_4 = \eta(1-\beta)(d_1\Theta_3 + d_2\Theta_2 + d_3) \quad \Rightarrow |\Theta_4| \leq \frac{2(1-\beta)|\eta|[1+2(1-\beta)]|\eta|[2+2(1-\beta)|\eta|]}{3!},$$

respectively. Using the principle of mathematical induction, we obtain

$$|\Theta_t| \leq \frac{\prod_{i=0}^{t-2}[i+2(1-\beta)|\eta|]}{(t-1)!} \ (t \in \mathbb{N}^*). \tag{17}$$

Using the relationship between the functions $\Upsilon(\zeta)$ and $\mathcal{F}(\zeta)$, we get

$$\Theta_t = \tfrac{[t]_q!}{[\vartheta+1]_{q,t-1}}[1+\gamma(t-1)]a_t\, c_t \ (t \in \mathbb{N}^*), \tag{18}$$

and then we get

$$|a_t| \leq \frac{[\vartheta+1]_{q,t-1} \prod_{i=0}^{t-2}[i+2(1-\beta)|\eta|]}{(t-1)![1+\gamma(t-1)][t]_q!c_t} \ (t \in \mathbb{N}^*).$$

This completes the proof of Theorem 1.

□

Putting $q \to 1^-$ in Theorem 1, we obtain the following corollary:

Corollary 1. *If the function* Υ *given by* (1) *belongs to the class* $\mathcal{G}_h^{\vartheta}(\eta,\gamma,\beta)$*, then*

$$|a_t| \leq \frac{(\vartheta+1)_{t-1} \prod_{i=0}^{t-2}[i+2(1-\beta)|\eta|]}{t[(t-1)!]^2[1+\gamma(t-1)]c_t} \ (t \in \mathbb{N}^*). \tag{19}$$

Taking $c_t = \dfrac{(-1)^{t-1}\Gamma(\rho+1)}{4^{t-1}(t-1)!\Gamma(t+\rho)}$, $\rho > 0$ in Theorem 1, we obtain the following special case:

Example 1. *If the function* Υ *given by* (1) *belongs to the class* $\mathcal{B}_\rho^{\vartheta,q}(\eta,\gamma,\beta)$*, then*

$$|a_t| \leq \frac{4^{t-1}\Gamma(t+\rho)[\vartheta+1]_{q,t-1}\prod_{i=0}^{t-2}[i+2(1-\beta)|\eta|]}{(-1)^{t-1}\Gamma(\rho+1)[1+\gamma(t-1)][t]_q!} \quad (t \in \mathbb{N}^*).$$

Considering $c_t = \left(\frac{m+1}{m+t}\right)^{\alpha}$, $\alpha > 0$, $m \geq 0$ in Theorem 1, we obtain the following result:

Example 2. *If the function* Υ *given by* (1) *belongs to the class* $\mathcal{M}_{m,\alpha}^{\vartheta,q}(\eta,\gamma,\beta)$*, then*

$$|a_t| \leq \frac{(m+t)^{\alpha}[\vartheta+1]_{q,t-1}\prod_{i=0}^{t-2}[i+2(1-\beta)|\eta|]}{(t-1)![1+\gamma(t-1)][t]_q!(m+1)^{\alpha}} \quad (t \in \mathbb{N}^*).$$

Putting $c_t = \frac{n^{t-1}}{(t-1)!}e^{-n}$, $n > 0$ in Theorem 1, we obtain the following special case:

Example 3. *If the function* Υ *given by* (1) *belongs to the class* $\mathcal{I}_n^{\vartheta,q}(\eta,\gamma,\beta)$*, then*

$$|a_t| \leq \frac{[\vartheta+1]_{q,t-1}\prod_{i=0}^{t-2}[i+2(1-\beta)|\eta|]}{n^{t-1}[1+\gamma(t-1)][t]_q!e^{-n}} \quad (t \in \mathbb{N}^*).$$

Putting $c_t = \left[\frac{1+\ell+\lambda(t-1)}{1+\ell}\right]^n$, $n \in \mathbb{Z}$, $\ell \geq 0$, $\lambda \geq 0$ in Theorem 1, we obtain the following special case:

Example 4. *If the function* Υ *given by* (1) *belongs to the class* $\mathcal{J}_{n,\ell,\lambda}^{\vartheta,q}(\eta,\gamma,\beta,m,\mu)$*, then*

$$|a_t| \leq \frac{(1+\ell)^n[\vartheta+1]_{q,t-1}\prod_{i=0}^{t-2}[i+2(1-\beta)|\eta|]}{(t-1)![1+\gamma(t-1)][t]_q!(1+\ell+\lambda(t-1))^n} \quad (t \in \mathbb{N}^*).$$

Putting $c_t = 1$ and $\vartheta = 1$ in Corollary 1, we obtain the following special case:

Example 5. *If the function* Υ *given by* (1) *belongs to the class* $\mathcal{G}^1_{\frac{\zeta}{1-\zeta}}(\eta,\gamma,\beta)$*, then*

$$|a_t| \leq \frac{\prod_{i=0}^{t-2}[i+2(1-\beta)|\eta|]}{(t-1)![1+\gamma(t-1)]} \quad (t \in \mathbb{N}^*).$$

3. Coefficient Estimates for the Function Class $\mathcal{N}_h^{\vartheta,q}(\eta,\gamma,\beta,m,\mu)$

Our main coefficient bounds for function in the class $\mathcal{N}_h^{\vartheta,q}(\eta,\gamma,\beta,m,\mu)$ are given by Theorem 2 below.

Theorem 2. *If the function* Υ *given by* (1) *belongs to the class* $\mathcal{N}_h^{\vartheta,q}(\eta,\gamma,\beta,m,\mu)$*, then*

$$|a_t| \leq \frac{[\vartheta+1]_{q,t-1}\prod_{i=0}^{t-2}[i+2(1-\beta)|\eta|]\prod_{i=0}^{m-1}(\mu+i+1)}{(t-1)![1+\gamma(t-1)][t]_q!\prod_{i=0}^{m-1}(\mu+i+t)c_t} \quad (t \in \mathbb{N}^*). \tag{20}$$

Proof. Let the function $\Upsilon \in \mathbb{A}$ be given by (1) and let the function g define as follows

$$g(\zeta) = \zeta + \sum_{t=2}^{+\infty} d_t \zeta^t \in \mathcal{SC}_h^{\vartheta,q}(\eta,\gamma,\beta), \tag{21}$$

so that

$$a_t = \frac{\prod_{i=0}^{m-1}(\mu+i+1)}{\prod_{i=0}^{m-1}(\mu+i+t)} d_t \;\; (t,m \in \mathbb{N}^*;\; \mu \in \mathbb{R}\backslash(-\infty,-1]). \tag{22}$$

$$|a_t| \leq \frac{[\vartheta+1]_{q,t-1} \prod_{r=0}^{t-2}[r+2(1-\beta)|\eta|] \prod_{i=0}^{m-1}(\mu+i+1)}{(t-1)![1+\gamma(t-1)][t]_q! \prod_{i=0}^{m-1}(\mu+i+t)c_t} \;(j \in \mathbb{N}^*).$$

Thus, by using Theorem 1, we readily complete the proof of Theorem 2. □

Putting $q \to 1^-$ in Theorem 1, we obtain the following corollary:

Corollary 2. *If the function* Υ *given by* (1) *belongs to the class* $\mathcal{T}_h^{\vartheta}(\eta,\gamma,\beta,m,\mu)$*, then*

$$|a_t| \leq \frac{(\vartheta+1)_{t-1} \prod_{r=0}^{t-2}[r+2(1-\beta)|\eta|] \prod_{i=0}^{m-1}(\mu+i+1)}{t[(t-1)!]^2[1+\gamma(t-1)] \prod_{i=0}^{m-1}(\mu+i+t)c_t} \;(t \in \mathbb{N}^*).$$

Putting $c_t = 1$ and $\vartheta = 1$ in Corollary 2, we obtain the following example:

Example 6. *If the function* Υ *given by* (1) *belongs to the class* $\mathcal{T}^1_{\frac{\zeta}{1-\zeta}}(\eta,\gamma,\beta,m,\mu)$*, then*

$$|a_j| \leq \frac{\prod_{r=0}^{t-2}[r+2(1-\beta)|\eta|] \prod_{i=0}^{m-1}(\mu+i+1)}{(t-1)![1+\gamma(t-1)] \prod_{i=0}^{m-1}(\mu+i+t)} \;(t \in \mathbb{N}^*).$$

4. Conclusions

We investigated certain subclasses of analytic functions of complex order combined with the linear q-convolution operator. For the functions in this new class, we obtained the coefficient bounds and introduced here by means of a certain non-homogeneous Cauchy–Euler-type differential equation of order m. There was also consideration of several interesting corollaries and applications of the results by suitably fixing the parameters, as illustrated in Remark 1.

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