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Problems Concerning Coefficients of Symmetric Starlike Functions Connected with the Sigmoid Function

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Abstract: In numerous geometric and physical applications of complex analysis, estimating the sharp bounds of coefficient-related problems of univalent functions is very important due to the fact that these coefficients describe the core inherent properties of conformal maps. The primary goal of this paper was to calculate the sharp estimates of the initial coefficients and some of their combinations (the Hankel determinants, Zalcman's functional, etc.) for the class of symmetric starlike functions linked with the sigmoid function. Moreover, we also determined the bounds of second-order Hankel determinants containing coefficients of logarithmic and inverse functions of the same class.

Keywords: starlike functions; sigmoid function; logarithmic and inverse functions; Krushkal and Zalcman functionals; Hankel determinant problems

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1. Introduction and Definitions

In order to study the basic terminology that is used in our primary outcomes, here, we have to provide some elementary ideas. Let us start with the set that indicates that the region of open unit disc $\mathbb{U}_d = \{z \in \mathbb{C} : |z| < 1\}$ as a domain and that the notation $\mathcal{A}$ stands for the family of analytic (or holomorphic) functions normalized by $g(0) = g'(0) - 1 = 0$. This shows that, if $g \in \mathcal{A}$, then it has the Taylor's series expansion:

$$g(z) = z + \sum_{n=1}^{+\infty} d_n z^n, \qquad (z \in \mathbb{U}_d). \tag{1}$$

In addition, remember that, by notation $\mathcal{S}$, we denote the family of univalent functions with series expansion (1). This family was first taken into account by Köebe in 1907. In 2012, Aleman and Constantin [1] established an astonishing connection between fluid dynamics and univalent function theory. They actually provided a simple method that shows how to use a univalent harmonic map for finding explicit solutions to incompressible two-dimensional Euler equations. It has several implications in a variety of applied scientific disciplines, including modern mathematical physics, fluid dynamics, nonlinear integrable system theory, and the theory of partial differential equations.

In 1916, Bieberbach [2] worked on the coefficients of family $\mathcal{S}$ and stated the most well-known coefficient conjecture of function theory, known as the "Bieberbach conjecture". According to this conjecture, if $g \in \mathcal{S}$, then $|d_n| \leq n$ for all $n \geq 2$. He also proved this problem for $n = 2$. It is evident that several renowned researchers have adopted a variety of approaches to address this problem, and a few of them [3–6] have succeeded in proving it for $n = 3, \ldots, 6$. Numerous scholars attempted to prove this hypothesis for $n \geq 7$ for a very long time, but no one was successful. Finally, in 1985, de-Branges [7] proved this conjecture for all $n \geq 2$ by using hypergeometric functions. Lawrence Zalcman proposed the inequality $\left|a_n^2 - a_{2n-1}\right| \leq (n-1)^2$ with $n \geq 2$ for $g \in \mathcal{S}$ in 1960 as a way of establishing the Bieberbach conjecture. Due to this, a number of articles [8–10] have been published on the Zalcman hypothesis and its generalised form $\left|\lambda a_n^2 - a_{2n-1}\right| \leq \lambda n^2 - 2n + 1$ $(\lambda \geq 0)$ for different subclasses of the set $\mathcal{S}$, however this conjecture has remained unsolved for a long time. Last but not least, Krushkal established this hypothesis in [11] for $n \leq 6$ and subsequently resolved it in an unpublished article [12] for $n \geq 2$ by using the holomorphic homotopy of univalent functions. A broader Zalcman hypothesis for $g \in \mathcal{S}$ was proposed by Ma [13] later, in 1999, and is given by

$$|a_n a_m - a_{n+m-1}| \leq (n-1)(m-1) \text{ for } n \geq 2, m \geq 2.$$

He proved it for one of the $\mathcal{S}$ subfamilies, however the challenge is still open for $\mathcal{S}$ family.

The estimates of the nth coefficient bounds for numerous subfamilies of the univalent function class, including starlike S^*, convex C, and many more, were determined between 1916 and 1985 in an effort to solve the aforementioned problem. In fact, because of that problem, this field of study has become heavily researched, which explains why this topic has gained so much popularity so rapidly. The notation S^* contains the functions that map the unit disk $\mathbb{U}_d$ onto a region that is star-shaped with respect to the origin. This idea was generalized by Sakaguchi [14] in 1959 by introducing the class S_s^* of starlike functions with respect to symmetric points. In 1977, Das and Singh [15] developed the family C_s of symmetric convex functions using this idea. In each of the above cited articles, the authors provided the analytical formulations of these classes as follows:

$$\begin{aligned} S_s^* &= \left\{ g \in \mathcal{S} : \Re\mathfrak{e} \frac{2zg'(z)}{g(z) - g(-z)} > 0, \ (z \in \mathbb{U}_d) \right\}, \\ C_s &= \left\{ g \in \mathcal{S} : \Re\mathfrak{e} \frac{2(zg'(z))'}{(g(z) - g(-z))'} > 0, \ (z \in \mathbb{U}_d) \right\}. \end{aligned}$$

Furthermore, the authors asserted [14] that the class S_s^* is a subclass of the set K of close-to-convex functions and that, with reference to the origin, it encompasses the families of convex and odd starlike functions. Following that, other mathematicians developed a large number of new symmetrical point-relative univalent function subfamilies. Utilizing this idea, Goel and Kumar [16] introduced a subfamily of starlike functions defined by

$$S_{SG}^* = \left\{ g \in \mathcal{S} : \frac{zg'(z)}{g(z)} \prec \frac{2}{1 + e^{-z}}, \qquad (z \in \mathbb{U}_d) \right\}.$$

Some intriguing aspects of the function $f \in S_{SG}^*$ were investigated in the article [17]. Motivated by the last definition, we now introduce a family SS_{SG}^* of symmetric starlike functions linked to the sigmoid function, which is given by

$$SS_{SG}^* = \left\{ g \in \mathcal{S} : \frac{2zg'(z)}{g(z) - g(-z)} \prec \frac{2}{1 + e^{-z}}, \qquad (z \in \mathbb{U}_d) \right\}.$$

The determinant $\mathcal{H}_{\lambda,k}(g)$ with $\lambda, k \in \mathbb{N} = \{1, 2, \ldots\}$, given below is known as the Hankel determinant, and was investigated by Pommerenke [18,19] for the function $g \in \mathcal{S}$

$$\mathcal{H}_{\lambda,k}(g) = \begin{vmatrix} d_k & d_{k+1} & \dots & d_{k+\lambda-1} \\ d_{k+1} & d_{k+2} & \dots & d_{k+\lambda} \\ \vdots & \vdots & \dots & \vdots \\ d_{k+\lambda-1} & d_{k+\lambda} & \dots & d_{k+2\lambda-2} \end{vmatrix}.$$

This determinant is very important in several studies, including those on power series with integral coefficients by Polya [20] (p. 323) and Cantor [21], as well as singularities by Hadamard [20] (p. 329) and Edrei [22].

There are not many publications in the literature that examine the bounds of the Hankel determinant for the function g that belongs to the $\mathcal{S}$ family. The best estimate for $g \in \mathcal{S}$ was determined by Hayman in [23] and is $|\mathcal{H}_{2,n}(g)| \leq |\eta|$, where η is a constant. Additionally, for $g \in \mathcal{S}$, it was shown in [24] that the second-order Hankel determinant $|\mathcal{H}_{2,2}(g)| \leq \eta$ for $0 \leq \eta \leq 11/3$. After these findings, it was and is a difficult task for scholars to determine the exact bounds of the Hankel determinants for a particular class of functions. The first paper [25] that used the ideas of the Caratheodory functions to accurately determine the sharp estimates of $|\mathcal{H}_{2,2}(g)|$ for the two fundamental subclasses of the set $\mathcal{S}$ of univalent functions appeared in 2007. The two determinants $\mathcal{H}_{2,1}(g)$ and $\mathcal{H}_{2,2}(g)$ have been thoroughly investigated in the literature [26–35] for various subfamilies of univalent functions; however, only a small number of works have been published [36–43] in which the authors established the determinant's sharp bounds. The objective of this particular article was to compute the sharp estimates of initial coefficients, Fekete-Szegö, Krushkal, and Zalcman inequalities, as well as the second-order Hankel determinant $|\mathcal{H}_{2,2}(g)|$ for the family SS^*_{SG} of analytic functions by using a technique of subordination, which proves the result in an easier way compared to the other methodology. Furthermore, we established the sharp bounds for second-order Hankel determinants with coefficients of logarithmic and inverse functions of the same class.

2. A Set of Lemmas

Let B_0 be the family of Schwarz functions. Then, the function $w \in B_0$ may be expressed as a power series

$$w(z) = \sum_{n=1}^{+\infty} \xi_n z^n. \tag{2}$$

The subsequent Schwarz function lemmas are required for proving our primary findings.

Lemma 1 ([44])**.** *Let the Schwarz function w have the form (2). Then, for any real numbers α and β such that*

$$(\alpha, \beta) = \left\{ |\alpha| \leq \frac{1}{2}, \ -1 \leq \beta \leq 1 \right\} \cup \left\{ \frac{1}{2} \leq |\alpha| \leq 2, \ \frac{4}{27}(1+|\alpha|)^3 - (1+|\alpha|) \leq \beta \leq 1 \right\},$$

the following sharp estimate holds:

$$\left| \xi_3 + \alpha \xi_1 \xi_2 + \beta \xi_1^3 \right| \leq 1.$$

Lemma 2 ([45])**.** *If $w \in B_0$ is in the form of (2), then*

$$|\xi_2| \leq 1 - |\xi_1|^2, \tag{3}$$
$$|\xi_n| \leq 1, \quad n \geq 1. \tag{4}$$

Furthermore, the inequality of (3) can be improved in the manner of

$$\left| \xi_2 + \eta \xi_1^2 \right| \leq \max\{1, |\eta|\}, \ \textit{for} \ \ \eta \in \mathbb{C}. \tag{5}$$

Lemma 3 ([46])**.** *Let $w \in B_0$ be the series expansion* (2)*. Then,*

$$|\xi_3| \leq 1 - |\xi_1|^2 - \frac{|\xi_2|^2}{1 + |\xi_1|}, \tag{6}$$

$$|\xi_4| \leq 1 - |\xi_1|^2 - |\xi_2|^2. \tag{7}$$

Lemma 4 ([47])**.** *Let $w \in B_0$ be the series expansion* (2)*. Then,*

$$\left|\xi_1\xi_3 - \xi_2^2\right| \leq 1 - |\xi_1|^2.$$

3. Coefficient Bounds

We start with the coefficient bounds.

Theorem 1. *If $g \in SS^*_{SG}$ has the series representation* (1)*, then*

$$|d_2| \leq \frac{1}{4},$$

$$|d_3| \leq \frac{1}{4},$$

$$|d_4| \leq \frac{1}{8},$$

$$|d_5| \leq \frac{1}{8}.$$

All these bounds are sharp.

Proof. Assume that $g \in SS^*_{SG}$. From the definition, it follows that there exists a Schwarz function w such that

$$\frac{2zg'(z)}{g(z) - g(-z)} = \frac{2}{1 + e^{-w(z)}}. \tag{8}$$

From Equation (2), we have

$$w(z) = \xi_1 z + \xi_2 z^2 + \xi_3 z^3 + \xi_4 z^4 + \cdots. \tag{9}$$

Using (1), we achieve

$$\frac{2zg'(z)}{g(z) - g(-z)} = 1 + 2d_2 z + 2d_3 z^2 + (-2d_2 d_3 + 4d_4)z^3 + \left(-2d_3^2 + 4d_5\right)z^4 + \cdots. \tag{10}$$

Using a quick computation and the series expansion of (9), we arrive at

$$\frac{2}{1 + e^{-w(z)}} = 1 + \frac{1}{2}\xi_1 z + \frac{1}{2}\xi_2 z^2 + \left(-\frac{1}{24}\xi_1^3 + \frac{1}{2}\xi_3\right)z^3 + \left(-\frac{1}{8}\xi_1^2\xi_2 + \frac{1}{2}\xi_4\right)z^4 + \cdots. \tag{11}$$

Now, by comparing (10) and (11), we obtain

$$d_2 = \frac{1}{4}\xi_1, \tag{12}$$

$$d_3 = \frac{1}{4}\xi_2, \tag{13}$$

$$d_4 = -\frac{1}{96}\xi_1^3 + \frac{1}{8}\xi_3 + \frac{1}{32}\xi_1\xi_2, \tag{14}$$

$$d_5 = \frac{1}{8}\xi_4 + \frac{1}{32}\xi_2^2 - \frac{1}{32}\xi_1^2\xi_2. \tag{15}$$

Implementing (4) in (12), we achieve

$$|d_2| \le \frac{1}{4}.$$

Applying (4) in (13), we achieve

$$|d_3| \le \frac{1}{4}.$$

From (14), we deduce that

$$|d_4| = \frac{1}{8}\left|\xi_3 + \frac{1}{4}\xi_1\xi_2 + \left(-\frac{1}{12}\right)\xi_1^3\right|.$$

From Lemma 1, let

$$\alpha = \frac{1}{4} \text{ and } \beta = -\frac{1}{12}.$$

It is clear that

$$|\alpha| \le \frac{1}{2} \text{ and } -1 \le \beta \le 1.$$

Thus, all the conditions of Lemma 1 are satisfied. Hence, we have

$$|d_4| \le \frac{1}{8}.$$

To prove the last inequality, we can write(15), as

$$\begin{aligned} |d_5| &= \frac{1}{8}\left|\xi_4 + \frac{1}{4}\xi_2^2 - \frac{1}{4}\xi_1^2\xi_2\right| \\ &\le \frac{1}{8}\left[|\xi_4| + \frac{1}{4}|\xi_2|^2 + \frac{1}{4}|\xi_1|^2|\xi_2|\right]. \end{aligned}$$

By using (3) and (7) plus the triangle inequality, we achieve

$$|d_5| \le \frac{1}{8}.$$

If $\xi_1 = 1$, then $d_2 = \frac{1}{4}$. If $\xi_2 = 1$, then $d_3 = \frac{1}{4}$. If $\xi_3 = 1$ and $\xi_k = 0$ for $k \ne 3$, then $d_4 = \frac{1}{8}$. Similarly, if $\xi_4 = 1$ and $\xi_k = 0$ for $k \ne 4$, then $d_5 = \frac{1}{8}$. This indicates that, for the functions provided by (8) with $w(z) = z$, $w(z) = z^2$, $w(z) = z^3$ and $w(z) = z^4$, respectively, the equality conditions in the theorem's statement are true. □

Theorem 2. *If $g \in SS^*_{SG}$ is of the form (1), then, for $\eta \in \mathbb{C}$,*

$$\left|d_3 - \eta d_2^2\right| \le \max\left\{\frac{1}{4}, \left|\frac{\eta}{16}\right|\right\}.$$

The Fekete–Szegö functional is sharp.

Proof. Using (12) and (13), we have

$$\begin{aligned} \left|d_3 - \eta d_2^2\right| &= \left|\frac{1}{4}\xi_2 - \frac{\eta}{16}\xi_1^2\right|. \\ &= \frac{1}{4}\left|\xi_2 + \left(-\frac{\eta}{4}\right)\xi_1^2\right|. \end{aligned}$$

The application of (5) leads us to

$$\left|d_3 - \eta d_2^2\right| \leq \max\left\{\frac{1}{4}, \left|\frac{\eta}{16}\right|\right\}.$$

The obtained bound of the Fekete–Szegö functional is sharp by considering that $w(z) = z^2$. □

After putting $\eta = 1$ in Theorem 2, we arrive at the following consequence.

Corollary 1. *If $g \in SS^*_{SG}$ and has the form (1), then the following sharp bound holds:*

$$\left|d_3 - d_2^2\right| \leq \frac{1}{4}.$$

Theorem 3. *If $g \in SS^*_{SG}$ has series representation (1), then the sharp bounds are*

$$|d_4 - d_2 d_3| \leq \frac{1}{8} \text{ and } \left|d_5 - d_3^2\right| \leq \frac{1}{8}.$$

Proof. Utilizing (12), (13) and (14), we easily achieve

$$|d_4 - d_2 d_3| = \frac{1}{8}\left|\xi_3 + \left(-\frac{1}{4}\right)\xi_1\xi_2 + \left(-\frac{1}{12}\right)\xi_1^3\right|.$$

From Lemma 1, let

$$\alpha = -\frac{1}{4} \text{ and } \beta = -\frac{1}{12}.$$

It is clear that

$$|\alpha| \leq \frac{1}{2} \text{ and } -1 \leq \beta \leq 1.$$

Thus, all the conditions of Lemma 1 are satisfied. Hence,

$$|d_4 - d_2 d_3| \leq \frac{1}{8}.$$

To estimate $d_5 - d_3^2$, we write this expression as follows:

$$\begin{aligned} \left|d_5 - d_3^2\right| &= \frac{1}{8}\left|\xi_4 - \frac{1}{4}\xi_2^2 - \frac{1}{4}\xi_1^2\xi_2\right|. \\ &= \frac{1}{8}\left[|\xi_4| + \frac{1}{4}|\xi_2|^2 + \frac{1}{4}|\xi_1|^2|\xi_2|\right]. \end{aligned}$$

By using (3) and (7) plus the triangle inequality, we achieve

$$\begin{aligned} \left|d_5 - d_3^2\right| &\leq \frac{1}{8}\left[1 - |\xi_1|^2 - |\xi_2|^2 + \frac{1}{4}|\xi_2|^2 + \frac{1}{4}|\xi_1|^2\left(1 - |\xi_1|^2\right)\right]. \\ &\leq \frac{1}{8}\left[1 - \frac{3}{4}|\xi_1|^2 - \frac{3}{4}|\xi_2|^2 - \frac{1}{4}|\xi_1|^4\right], \end{aligned}$$

which is clearly less than or equal to $\frac{1}{8}$.

By considering $w(z) = z^3$ and $w(z) = z^4$, the above-stated Zalcman functional cases are sharp. □

Theorem 4. *If $g \in SS^*_{SG}$ and is given by (1), then*

$$\left|d_4 - d_2^3\right| \leq \frac{1}{8} \text{ and } \left|d_5 - d_2^4\right| \leq \frac{1}{8}.$$

These outcomes are sharp.

Proof. From (12) and (14), we easily obtain

$$\left|d_4 - d_2^3\right| = \frac{1}{8}\left|\xi_3 + \left(\frac{1}{4}\right)\xi_1\xi_2 + \left(-\frac{5}{4}\right)\xi_1^3\right|.$$

From Lemma 1, let

$$\alpha = \frac{1}{4} \text{ and } \beta = -\frac{5}{4}.$$

It is clear that

$$|\alpha| \leq \frac{1}{2} \text{ and } -1 \leq \beta \leq 1.$$

Thus, all the conditions of Lemma 1 are satisfied. Hence,

$$\left|d_4 - d_2^3\right| \leq \frac{1}{8}.$$

To estimate $d_5 - d_2^4$, we write this expression as follows:

$$\begin{aligned}\left|d_5 - d_2^4\right| &= \frac{1}{8}\left|\xi_4 + \frac{1}{4}\xi_2^2 - \frac{1}{4}\xi_1^2\xi_2 - \frac{1}{32}\xi_1^4\right|. \\ &= \frac{1}{8}\left[|\xi_4| + \frac{1}{4}|\xi_2|^2 + \frac{1}{4}|\xi_1|^2|\xi_2| + \frac{1}{32}|\xi_1|^4\right].\end{aligned}$$

By using (3) and (7) plus the triangle inequality, we achieve

$$\begin{aligned}\left|d_5 - d_2^4\right| &\leq \frac{1}{8}\left[1 - |\xi_1|^2 - |\xi_2|^2 + \frac{1}{4}|\xi_2|^2 + \frac{1}{4}|\xi_1|^2\left(1 - |\xi_1|^2\right) + \frac{1}{32}|\xi_1|^4\right]. \\ &\leq \frac{1}{8}\left[1 - \frac{3}{4}|\xi_1|^2 - \frac{3}{4}|\xi_2|^2 - \frac{7}{32}|\xi_1|^4\right],\end{aligned}$$

which is clearly less than or equal to $\frac{1}{8}$.

Considering $w(z) = z^3$ and $w(z) = z^4$, we can observe that the estimates of the Krushkal functionals are sharp. □

Now, let us discuss the Hankel determinants for the SS^*_{SG} class.

Theorem 5. *If $g \in SS^*_{SG}$ has the series representation form (1), then*

$$|\mathcal{H}_{2,2}(g)| \leq \frac{1}{16}.$$

The obtained inequality is sharp.

Proof. The determinant $|\mathcal{H}_{2,2}(g)|$ can be reconsidered as follows:

$$|\mathcal{H}_{2,2}(g)| = \left|d_2d_4 - d_3^2\right|.$$

From (12)–(14), we obtain

$$\begin{aligned}\left|d_2d_4 - d_3^2\right| &= \frac{1}{8}\left|\frac{1}{2}\xi_2^2 - \frac{1}{4}\xi_1\xi_3 - \frac{1}{16}\xi_1^2\xi_2 + \frac{1}{48}\xi_1^4\right|. \\ &= \frac{1}{8}\left|\frac{1}{4}\left(\xi_2^2 - \xi_1\xi_3\right) + \frac{1}{4}\left(\xi_2^2 - \frac{1}{4}\xi_1^2\xi_2 + \frac{1}{12}\xi_1^4\right)\right|.\end{aligned}$$

Since

$$\begin{aligned}\left|\xi_2^2+\frac{1}{4}\xi_1^2\xi_2+\frac{1}{12}\xi_1^4\right| &\leq \left(1-|\xi_1|^2\right)^2+\frac{1}{4}\xi_1^2\left(1-|\xi_1|^2\right)+\frac{1}{12}\xi_1^4.\\ &= 1-\frac{7}{4}|\xi_1|^2+\frac{5}{6}|\xi_1|^4.\end{aligned}$$

and the function

$$1-\frac{7}{4}t+\frac{5}{6}t^2,$$

is decreasing in $[0,1]$. Hence, we deduce that

$$\left|\xi_2^2+\frac{1}{4}\xi_1^2\xi_2+\frac{1}{12}\xi_1^4\right| \leq 1.$$

Thus, by utilizing Lemma 4 along with the last inequality, we obtain the required stated bound.

Moreover, the determinant $|\mathcal{H}_{2,2}(g)| = \frac{1}{16}$ if $\xi_2 = 1$ and $\xi_k = 0$ with $k \neq 2$; that is, considering $w(z) = z^2$ in (8). □

4. Logarithmic Coefficient

The logarithmic coefficients μ_n are provided by the following formula:

$$G_g(z) := \log\left(\frac{g(z)}{z}\right) = 2\sum_{n=1}^{+\infty}\mu_n z^n \text{ for } z \in \mathbb{U}_d. \tag{16}$$

The theory of Schlicht functions is significantly impacted by these coefficients in various estimations. In 1985, de-Branges [7] determined that

$$\sum_{k=1}^{n} k(n-k+1)|\mu_n|^2 \leq \sum_{k=1}^{n}\frac{n-k+1}{k}, \text{ for } n \geq 1,$$

and, for the particular function $g(z) = z/\left(1-e^{i\theta}z\right)$ with $\theta \in \mathbb{R}$, equality is attained. Evidently, this inequality gives rise to the broadest formulation of the well-known Bieberbach–Robertson–Milin conjectures involving Taylor coefficients of g that belong to $\mathcal{S}$. For more information on how de-Brange's claim is explained, see [48–50]. Brennan's conjecture for conformal mappings was answered by Kayumov [51] in 2005 by taking the logarithmic coefficients into account. Several works [52–54] that have significantly advanced the study of logarithmic coefficients are included in this article.

It is easy to determine from the definition given above that the logarithmic coefficients for g belonging to $\mathcal{S}$ are given by

$$\mu_1 = \frac{1}{2}d_2, \tag{17}$$

$$\mu_2 = \frac{1}{2}\left(d_3-\frac{1}{2}d_2^2\right), \tag{18}$$

$$\mu_3 = \frac{1}{2}\left(d_4-d_2d_3+\frac{1}{3}d_2^3\right), \tag{19}$$

$$\mu_4 = \frac{1}{2}\left(d_5-d_2d_4+d_2^2d_3-\frac{1}{2}d_3^2-\frac{1}{4}d_2^4\right). \tag{20}$$

The Hankel determinant $\mathcal{H}_{q,n}(G_g/2)$ with logarithmic coefficients was initially developed by Kowalczyk and Lecko in [55,56] and is given by

$$\mathcal{H}_{q,n}(G_g/2) := \begin{vmatrix} \mu_n & \mu_{n+1} & \cdots & \mu_{n+q-1} \\ \mu_{n+1} & \mu_{n+2} & \cdots & \mu_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ \mu_{n+q-1} & \mu_{n+q} & \cdots & \mu_{n+2q-2} \end{vmatrix}. \tag{21}$$

It has been observed that

$$\begin{aligned} \mathcal{H}_{2,1}(G_g/2) &= \begin{vmatrix} \mu_1 & \mu_2 \\ \mu_2 & \mu_3 \end{vmatrix} = \mu_1\mu_3 - \mu_2^2, \\ \mathcal{H}_{2,2}(G_g/2) &= \begin{vmatrix} \mu_2 & \mu_3 \\ \mu_3 & \mu_4 \end{vmatrix} = \mu_2\mu_4 - \mu_3^2. \end{aligned}$$

For further investigations of the Hankel determinant on logarithmic coefficients, see [57–60]. In this section, we compute the sharp estimates of logarithmic coefficients up to μ_3 and Fekete–Szegö, Zalcman, and Krushkal inequalities along with the Hankel determinant $|\mathcal{H}_{2,1}(G_g/2)|$ for the class SS^*_{SG}.

Theorem 6. *If $g \in SS^*_{SG}$ is of the form (1), then*

$$\begin{aligned} |\mu_1| &\leq \frac{1}{8}, \\ |\mu_2| &\leq \frac{1}{8}, \\ |\mu_3| &\leq \frac{1}{16}. \end{aligned}$$

These bounds are sharp.

Proof. Applying (12)–(15) in (17)–(20), we obtain

$$\mu_1 = \frac{1}{8}\xi_1, \tag{22}$$

$$\mu_2 = \frac{1}{8}\xi_2 - \frac{1}{64}\xi_1^2, \tag{23}$$

$$\mu_3 = -\frac{1}{384}\xi_1^3 + \frac{1}{16}\xi_3 - \frac{1}{64}\xi_1\xi_2, \tag{24}$$

$$\mu_4 = \frac{1}{16}\xi_4 - \frac{3}{256}\xi_1^2\xi_2 + \frac{5}{6144}\xi_1^4 - \frac{1}{64}\xi_1\xi_3. \tag{25}$$

Using (4) in (22), we achieve

$$|\mu_2| \leq \frac{1}{8}.$$

For the second inequality, we may write (23) by

$$|\mu_2| = \frac{1}{8}\left|\xi_2 + \left(-\frac{1}{2}\right)\xi_1^2\right|.$$

Applying (5), we achieve

$$|\mu_2| \leq \frac{1}{8}.$$

For (24), we deduce that

$$|\mu_3| = \frac{1}{16}\left|\xi_3 + \left(-\frac{1}{4}\right)\xi_1\xi_2 + \left(-\frac{1}{24}\right)\xi_1^3\right|.$$

From Lemma 1, let
$$\alpha = -\frac{1}{4} \quad \text{and} \quad \beta = -\frac{1}{24}.$$
It is clear that
$$|\alpha| \leq \frac{1}{2} \text{ and } -1 \leq \beta \leq 1.$$

Thus, Lemma 1's assumptions are all satisfied, and, when we apply it, we obtain the below desired outcome:
$$|\mu_3| \leq \frac{1}{16}.$$

If $\xi_1 = 1$, then $\mu_1 = \frac{1}{8}$. If $\xi_2 = 1$ and $\xi_k = 0$ for $k \neq 2$, then $\mu_2 = \frac{1}{8}$. Similarly, if $\xi_3 = 1$ and $\xi_k = 0$ for $k \neq 3$, then $\mu_3 = \frac{1}{16}$. This indicates that, for the functions provided by (8) with $w(z) = z$, $w(z) = z^2$, $w(z) = z^3$ and $w(z) = z^4$, respectively, the equality conditions in the theorem's statement are true. □

Theorem 7. *If $g \in SS^*_{SG}$ and has the expansion (1), then, for $\eta \in \mathbb{C}$,*
$$\left|\mu_2 - \eta\mu_1^2\right| \leq \max\left\{\frac{1}{8}, \left|\frac{\eta+1}{64}\right|\right\}.$$

The Fekete–Szegö functional is sharp.

Proof. Using (22) and (23), we easily obtain
$$\begin{aligned} \left|\mu_2 - \eta\mu_1^2\right| &= \left|\frac{1}{8}\xi_2 - \frac{\eta}{64}\xi_1^2 - \frac{1}{64}\xi_1^2\right|. \\ &= \frac{1}{8}\left|\xi_2 + \left(\frac{-\eta-1}{8}\right)\xi_1^2\right|. \end{aligned}$$

The application of (5) leads us to
$$\left|\mu_2 - \eta\mu_1^2\right| \leq \max\left\{\frac{1}{8}, \left|\frac{\eta+1}{64}\right|\right\}.$$

The obtained bound of the Fekete–Szegö functional is sharp by considering $w(z) = z^2$. □

The below result is obtained by putting $\eta = 1$ in Theorem 7.

Corollary 2. *If $g \in SS^*_{SG}$ has the expansion representation (1), then*
$$\left|\mu_2 - \mu_1^2\right| \leq \frac{1}{8}.$$

This result is the best possible.

Theorem 8. *If $g \in SS^*_{SG}$ has the series form (1), then sharp bounds are*
$$|\mu_3 - \mu_1\mu_2| \leq \frac{1}{16} \text{ and } \left|\mu_3 - \mu_1^3\right| \leq \frac{1}{16}.$$

Proof. Using (22)–(24), we achieve
$$|\mu_3 - \mu_1\mu_2| = \frac{1}{16}\left|\xi_3 + \left(-\frac{1}{2}\right)\xi_1\xi_2 + \left(-\frac{1}{96}\right)\xi_1^3\right|.$$

From Lemma 1, let
$$\alpha = -\frac{1}{2} \text{ and } \beta = -\frac{1}{96}.$$
It is clear that
$$|\alpha| \leq \frac{1}{2} \text{ and } -1 \leq \beta \leq 1.$$
Thus, all the conditions of Lemma 1 are satisfied. Hence,
$$|\mu_3 - \mu_1\mu_2| \leq \frac{1}{16}.$$
To estimate $\mu_3 - \mu_1^3$, we write this expression as follows:
$$\left|\mu_3 - \mu_1^3\right| = \frac{1}{16}\left|\xi_3 + \left(-\frac{1}{4}\right)\xi_1\xi_2 + \left(-\frac{7}{96}\right)\xi_1^3\right|.$$
From Lemma 1, let
$$\alpha = -\frac{1}{4} \quad \text{and} \quad \beta = -\frac{7}{96}.$$
It is clear that
$$|\alpha| \leq \frac{1}{2} \text{ and } -1 \leq \beta \leq 1.$$
Thus, all the conditions of Lemma 1 are satisfied. Hence,
$$\left|\mu_3 - \mu_1^3\right| \leq \frac{1}{16}.$$
Considering $w(z) = z^3$, we can observe that the above two estimates are sharp. □

Theorem 9. *If $g \in SS^*_{SG}$ has the expansion form (1), then*
$$|\mathcal{H}_{2,1}(G_g/2)| \leq \frac{1}{64}.$$
The above stated inequality is sharp.

Proof. The determinant $|\mathcal{H}_{2,1}(G_g/2)|$ might be reconsidered as follows:
$$|\mathcal{H}_{2,1}(G_g/2)| = \left|\mu_1\mu_3 - \mu_2^2\right|.$$
From (22)–(24), we obtain
$$\begin{aligned} |\mathcal{H}_{2,1}(G_g/2)| &= \frac{1}{64}\left|\frac{7}{192}\xi_1^4 - \frac{1}{2}\xi_1\xi_3 - \frac{1}{8}\xi_1^2\xi_2 + \xi_2^2\right|. \\ &= \frac{1}{64}\left|\frac{1}{2}\left(\xi_2^2 - \xi_1\xi_3\right) + \frac{1}{2}\left(\frac{7}{96}\xi_1^4 - \frac{1}{4}\xi_1^2\xi_2 + \xi_2^2\right)\right|. \end{aligned}$$
Since
$$\begin{aligned} \left|\frac{7}{96}\xi_1^4 + \frac{1}{4}\xi_1^2\xi_2 + \xi_2^2\right| &\leq \frac{7}{96}|\xi_1|^4 + \frac{1}{4}|\xi_1|^2\left(1 - |\xi_1|^2\right) + \left(1 - |\xi_1|^2\right)^2. \\ &= \frac{79}{96}|\xi_1|^4 + 1 - \frac{7}{4}|\xi_1|^2. \end{aligned}$$
and so the function
$$\frac{79}{96}t^4 + 1 - \frac{7}{4}t^2,$$

is decreasing in $[0,1]$. Thus, we conclude that

$$\left|\frac{7}{96}\xi_1^4 + \frac{1}{4}\xi_1^2\xi_2 + \xi_2^2\right| \leq 1,$$

and hence by utilizing Lemma 4 along with the last inequality, we obtain the required stated bound.

Moreover, the determinant $\left|\mathcal{H}_{2,1}(G_g/2)\right| = \frac{1}{64}$ if $\xi_2 = 1$ and $\xi_k = 0$ with $k \neq 2$; that is; considering $w(z) = z^2$ in (8). □

5. Inverse Coefficient

For each univalent function g defined in $\mathbb{U}_d$, the well-known Köebe 1/4-theorem guarantees that its inverse g^{-1} exists at least on a disc of radius 1/4 with the following Taylor's series expansion:

$$g^{-1}(w) = w + \sum_{n=2}^{+\infty} \zeta_n w^n, \quad \left(|w| < \frac{1}{4}\right). \tag{26}$$

By the virtue of $g\left(g^{-1}(w)\right) = w$, we obtain

$$\zeta_2 = -d_2, \tag{27}$$
$$\zeta_3 = -d_3 + 2d_2^2, \tag{28}$$
$$\zeta_4 = -d_4 + 5d_2d_3 - 5d_2^3, \tag{29}$$
$$\zeta_5 = -d_5 + 3d_3^2 + 14d_2^4 + 6d_2d_4 - 21d_2^2d_3. \tag{30}$$

Many authors studied Hankel determinants for the inverse functions; see [61–64]. Here, in this portion, we study the sharp estimates of some initial coefficient, Fekete–Szegö functional, and Hankel determinant $\left|\mathcal{H}_{2,2}\left(g^{-1}\right)\right|$ for the inverse functions of the class SS^*_{SG}.

Theorem 10. *If $g \in SS^*_{SG}$ has the series form (1), then*

$$|\zeta_2| \leq \frac{1}{4},$$
$$|\zeta_3| \leq \frac{1}{4}.$$

These bounds are sharp.

Proof. Applying (12)–(15) in (27)–(30), we obtain

$$\zeta_2 = -\frac{1}{4}\xi_1, \tag{31}$$
$$\zeta_3 = -\frac{1}{4}\xi_2 + \frac{1}{8}\xi_1^2, \tag{32}$$
$$\zeta_4 = -\frac{13}{192}\xi_1^3 - \frac{1}{8}\xi_3 + \frac{9}{32}\xi_1\xi_2, \tag{33}$$
$$\zeta_5 = \frac{5}{32}\xi_2^2 - \frac{1}{4}\xi_1^2\xi_2 + \frac{5}{128}\xi_1^4 - \frac{1}{8}\xi_4 + \frac{3}{16}\xi_1\xi_3. \tag{34}$$

Implementing (4) in (31), we achieve

$$|\zeta_2| \leq \frac{1}{4}.$$

For the proof of the second inequality, we consider (32) as:

$$|\zeta_3| = \frac{1}{4}\left|\xi_2 + \left(-\frac{1}{2}\right)\xi_1^2\right|.$$

Applying (5), we achieve

$$|\zeta_3| \leq \frac{1}{4}.$$

If $\xi_1 = 1$, then $\zeta_2 = \frac{1}{4}$. Similarly, if $\xi_2 = 1$ and $\xi_k = 0$ for $k \neq 2$, then $\zeta_3 = \frac{1}{4}$. This demonstrates that the functions produced by (8) with $w(z) = z$ and $w(z) = z^2$, respectively, satisfy the equalities in the statement of this theorem. □

Theorem 11. *If $g \in SS^*_{SG}$ has the representation (1), then*

$$\left|\zeta_3 - \eta\zeta_2^2\right| \leq \max\left\{\frac{1}{4}, \left|\frac{\eta - 2}{16}\right|\right\}.$$

The Fekete–Szegö functional is sharp.

Proof. Using (31) and (32), we easily obtain

$$\begin{aligned}\left|\zeta_3 - \eta\zeta_2^2\right| &= \left|-\frac{1}{4}\xi_2 - \frac{\eta}{16}\xi_1^2 + \frac{1}{8}\xi_1^2\right|.\\ &= \frac{1}{4}\left|\xi_2 + \left(\frac{\eta}{4} - \frac{1}{2}\right)\xi_1^2\right|.\end{aligned}$$

The application of (5) leads us to

$$\left|\zeta_3 - \eta\zeta_2^2\right| \leq \max\left\{\frac{1}{4}, \left|\frac{\eta - 2}{16}\right|\right\}.$$

Considering that $w(z) = z^2$, we can observe that the estimate of the Fekete–Szegö functional is sharp. □

Putting $\eta = 1$ in Theorem 11, we deduce the following corollary.

Corollary 3. *If $g \in SS^*_{SG}$ and has the form (1), then the sharp bound is*

$$\left|\zeta_3 - \zeta_2^2\right| \leq \frac{1}{4}.$$

Theorem 12. *If $g \in SS^*_{SG}$ and has the series form (1), then*

$$\left|\mathcal{H}_{2,2}\left(g^{-1}\right)\right| \leq \frac{1}{16}.$$

The stated inequality is sharp.

Proof. The determinant $\left|\mathcal{H}_{2,2}\left(g^{-1}\right)\right|$ might be rearranged as:

$$\left|\mathcal{H}_{2,2}\left(g^{-1}\right)\right| = \left|\zeta_2\zeta_4 - \zeta_3^2\right|.$$

From (31)–(33), we achieve

$$\begin{aligned}\left|\mathcal{H}_{2,2}\left(g^{-1}\right)\right| &= \frac{1}{16}\left|-\frac{1}{48}\xi_1^4-\frac{1}{2}\xi_1\xi_3+\frac{1}{8}\xi_1^2\xi_2+\xi_2^2\right|.\\ &= \frac{1}{16}\left|\frac{1}{2}\left(\xi_2^2-\xi_1\xi_3\right)+\frac{1}{2}\left(-\frac{1}{24}\xi_1^4+\frac{1}{4}\xi_1^2\xi_2+\xi_2^2\right)\right|.\end{aligned}$$

Since

$$\begin{aligned}\left|\frac{1}{24}\xi_1^4+\frac{1}{4}\xi_1^2\xi_2+\xi_2^2\right| &\leq \frac{1}{24}|\xi_1|^4+\frac{1}{4}|\xi_1|^2\left(1-|\xi_1|^2\right)+\left(1-|\xi_1|^2\right)^2.\\ &= \frac{19}{24}|\xi_1|^4+1-\frac{7}{4}|\xi_1|^2.\end{aligned}$$

and, for $t\in[0,1]$, the below function

$$\frac{19}{24}t^4+1-\frac{7}{4}t^2,$$

is decreasing, it thus follows that

$$\left|\frac{1}{24}\xi_1^4+\frac{1}{4}\xi_1^2\xi_2+\xi_2^2\right|\leq 1.$$

We may thus obtain the necessary stated inequality by applying Lemma 4 and the final inequality.

Moreover, the determinant $\left|\mathcal{H}_{2,1}\left(g^{-1}\right)\right|=\frac{1}{16}$ if $\xi_2=1$ and $\xi_k=0$ with $k\neq 2$; that is, considering that $w(z)=z^2$ in (8). □

6. Conclusions

One of the most challenging tasks in geometric function theory is to determine how to acquire the sharp estimates of the functionals consisting of coefficients that appear in the Taylor–Maclaurin series of analytic or univalent functions. One of such coefficient-related problems is to determine the sharp bounds of the Hankel determinants. Our goal in this study was to compute the sharp bounds of the second-order Hankel determinant, Zalcman's functional, and Fekete–Szegö inequalities for the class of symmetric starlike functions connected with the sigmoid function. Furthermore, we calculated the sharp bounds of second-order Hankel determinants, which consist of the coefficients of logarithmic and inverse functions of the same defined class. This work may be extended to obtain the sharp bounds of third, fourth, and fifth-order Hankel determinants for the same class as well as for some novel classes of analytic functions. The obtained results can also be studied for classes of meromophic functions.

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