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Exploring Quantum Simpson-Type Inequalities for Convex Functions: A Novel Investigation

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Abstract: This study seeks to derive novel quantum variations of Simpson's inequality by primarily utilizing the convexity characteristics of functions. Additionally, the study examines the credibility of the obtained results through the presentation of relevant numerical examples and graphs.

Keywords: Simpson's integral inequality; convex functions; quantum calculus; integral inequalities

MSC: 05A30; 26A51; 26D10; 26D15

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1. Introduction

Inequalities are fundamental concepts in mathematics that play an important role in establishing relationships between various quantities. Inequalities play a critical role in applied mathematics, where they are used to model and analyze real-world phenomena. In many applications, the quantities of interest are subject to constraints, and inequalities are used to model and analyze these constraints. Inequalities are essential tools in mathematical analysis, providing a means to compare and analyze functions, establish bounds on integrals, and model real-world constraints. Integral inequalities are widely applied in modern mathematical analysis, making them an important component in this area of study. A broad spectrum of integral inequalities, each with multiple applications in both pure and applied sciences, is known to us. The Simpson's integral inequality is one of the most renowned integral inequalities among them. Simpson's integral inequality is a result of mathematical analysis that provides an upper bound on the error in approximating the integral of a function using Simpson's rule. Simpson's rule is a numerical integration technique that uses quadratic polynomials to approximate the value of an integral. Simpson's integral inequality is an important result in numerical analysis, and it has applications in various fields of science and engineering, where numerical integration techniques are commonly used to solve problems that involve complex functions.

Let $\Xi : I = [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a four times continuous differentiable on I°, where I° is the interior of I and $\|\Xi^{(4)}\|_\infty < \infty$. Then, the following inequality is known as Simpson's inequality:

$$\left| \frac{1}{3}\left[\frac{\Xi(a)+\Xi(b)}{2} + 2\Xi\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b \Xi(x)dx \right| \leq \frac{1}{2880}\|\Xi^{(4)}\|_\infty (b-a)^4.$$

For more details, see [1–21]. The theory of convexity plays a significant role in mathematics, especially in optimization theory, functional analysis, and geometry. Convexity refers to the property of a set or a function that any line segment between two points in the set or on the graph of the function lies entirely within the set or above the graph. One of the most important applications of convexity is in the optimization theory, where it provides a powerful tool for solving optimization problems. The classical concept of convexity also has a knit connection with the concept of symmetry. In the literature, there exist numerous properties of symmetric convex sets. One fascinating aspect of this relationship is that we work on one and apply it to the other. The theory of convexity also played a significant role in the development of the theory of integral inequalities. In particular, the theory of convex functions is closely related to the theory of integral inequalities. Many important integral inequalities can be derived using convex functions. In 2020, Chu et al. [22] introduced the notion of higher-order n-ploynomial strongly convex functions involving Katugampola fractional operators. In [23], Iqbal and colleagues derived new Simpson-type inequalities involving generalized fractional operators together with convexity characteristics. In [24], the authors visualized some upper error estimations involving generalized strongly convexity named F convex functions. For details, see [25].

New versions of classical integral inequalities have been developed using a range of modern techniques in recent years. One of the key approaches involves the use of quantum calculus instead of classical calculus, which provides a significant tool for deriving quantum (q-)analogues of classical integral inequalities. In quantum calculus, the traditional derivatives and integrals are replaced by q-derivatives and q-integrals, which depend on a parameter q. The q-derivatives and q-integrals have different algebraic properties than their classical counterparts. For more details, see [26].

The aim of this study is to derive new quantum versions of Simpson's inequality by mainly exploiting the convexity property of functions. Furthermore, the study assesses the validity of the obtained outcomes by providing relevant numerical examples and also graphical analysis.

2. Preliminaries

Some fundamental concepts and definitions of q-calculus are presented in this section. Throughout this paper, let $0 < q < 1$ and $[k_1, k_2] \subseteq \mathbb{R}$ be an interval with $k_1 < k_2$. The q-number is expressed as follows:

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}, \quad n \in \mathbb{N}.$$

The left q-derivative and integral established in [27] are presented as follows:

Definition 1 ([27]). *For a continuous function $\Xi : [k_1, k_2] \to \mathbb{R}$, the left q-derivative on $[k_1, k_2]$ is defined as:*

$$ {}_{k_1}D_q \Xi(x) = \begin{cases} \dfrac{\Xi(x) - \Xi(qx + (1-q)k_1)}{(1-q)(x-k_1)}, & if \;\; x \neq k_1; \\ \lim\limits_{x \to k_1} {}_{k_1}D_q \Xi(x), & if \;\; x = k_1. \end{cases} \tag{1}$$

The function Ξ is called a left q-differentiable function if ${}_{k_1}D_q \Xi(x)$ exists.

In Definition 1, if $k_1 = 0$, then (1) is recaptured as follows:

$$D_q \Xi(x) = \frac{\Xi(x) - \Xi(qx)}{(1-q)(x)},$$

which is the q-Jackson derivative; see [28] for more details.

Definition 2 ([27]). *For a continuous function $\Xi : [k_1, k_2] \to \mathbb{R}$, the left q-integral on $[k_1, k_2]$ is defined as:*

$$\int_{k_1}^{x} \Xi(\varrho) \, {}_{k_1}d_q\varrho = (1-q)(x-k_1) \sum_{n=0}^{\infty} q^n \Xi(q^n x + (1-q^n)k_1) \tag{2}$$

for $x \in [k_1, k_2]$. The function Ξ is called a left q-integrable function if $\int_{k_1}^{x} \Xi(\varrho) \, d_q\varrho$ for all $x \in [k_1, k_2]$ exists.

In Definition 2, if $k_1 = 0$, then (2) is recaptured as follows:

$$\int_0^x \Xi(\varrho)\, d_q\varrho = (1-q)x \sum_{n=0}^{\infty} q^n \Xi(q^n x), \tag{3}$$

which is the q-Jackson integral; see [28] for more details. Moreover, Jackson [28] gave the q-Jackson integral on the interval $[k_1, k_2]$ as follows:

$$\int_{k_1}^{k_2} \Xi(\varrho)\, d_q\varrho = \int_0^{k_2} \Xi(\varrho)\, d_q\varrho - \int_0^{k_1} \Xi(\varrho)\, d_q\varrho.$$

The right q-derivative and integral established in [29] are presented as follows:

Definition 3 ([29])**.** *For a continuous function $\Xi : [k_1, k_2] \to \mathbb{R}$, the right q-derivative on $[k_1, k_2]$ is defined as:*

$${}^{k_2}D_q\Xi(x) = \begin{cases} \dfrac{\Xi(qx + (1-q)k_2) - \Xi(x)}{(1-q)(k_2 - x)}, & if \quad x \neq k_2; \\ \lim\limits_{x \to k_2} {}^{k_2}D_q\Xi(x), & if \quad x = k_2, \end{cases} \tag{4}$$

The function Ξ is called a right q-differentiable function if ${}^{k_2}D_q\Xi(x)$ exists.

Definition 4 ([29])**.** *For a continuous function $\Xi : [k_1, k_2] \to \mathbb{R}$, the right q-integral on $[k_1, k_2]$ is defined as:*

$$\int_x^{k_2} \Xi(\varrho)\ {}^{k_2}d_q\varrho = (1-q)(k_2 - x) \sum_{n=0}^{\infty} q^n \Xi(q^n x + (1-q^n)k_2), \tag{5}$$

The function Ξ is called a right q-integrable function if $\int_x^{k_2} \Xi(\varrho)\ {}^{k_2}d_q\varrho$ for all $\varrho \in [k_1, k_2]$ exists.

Now, we provide Lemmas which can help us to prove our main findings.

Lemma 1 ([30])**.** *For continuous functions Ξ, $\Psi \to \mathbb{R}$, the following expression holds:*

$$\begin{aligned} &\int_0^c \Psi(\varrho)\ {}_{k_1}D_q\Xi(\varrho k_2 + (1-\varrho)k_1)\, d_q\varrho \\ &= \frac{\Psi(\varrho)\Xi(\varrho k_2 + (1-\varrho)k_1)}{k_2 - k_1}\bigg|_0^c - \frac{1}{k_2 - k_1}\int_0^c D_q\Psi(\varrho)\Xi(q\varrho k_2 + (1-q\varrho)k_1)\, d_q\varrho. \end{aligned} \tag{6}$$

Lemma 2 ([31])**.** *For continuous functions Ξ, $\Psi \to \mathbb{R}$, the following expression holds:*

$$\begin{aligned} &\int_0^c \Psi(\varrho)\ {}^{k_2}D_q\Xi(\varrho k_1 + (1-\varrho)k_2)\, d_q\varrho \\ &= \frac{1}{k_2 - k_1}\int_0^c D_q\Psi(\varrho)\Xi(q\varrho k_1 + (1-q\varrho)k_2)\, d_q\varrho - \frac{\Psi(\varrho)\Xi(\varrho k_1 + (1-\varrho)k_2)}{k_2 - k_1}\bigg|_0^c. \end{aligned} \tag{7}$$

The following result will be useful in calculating q-integrals.

Lemma 3 ([32])**.** *The following expression holds:*

$$\int_{k_1}^{k_2} (x - k_1)^{\alpha}\ {}_{k_1}d_q x = \frac{(k_2 - k_1)^{\alpha+1}}{[\alpha+1]_q},$$

where $\alpha \in \mathbb{R} - \{-1\}$.

3. Main Results

Within this segment, we will create quantum inequalities connected to Simpson's integral inequalities for smooth, convex functions. To demonstrate the primary outcomes, it is vital to first introduce the subsequent significant lemma.

Lemma 4. *Suppose that $\Xi : [k_1, k_2] \to \mathbb{R}$ is a q-differentiable function on (k_1, k_2) such that ${}_{k_1}D_q\Xi$ and ${}^{k_2}D_q\Xi$ are continuous and integrable functions on $[k_1, k_2]$; then,*

$$\begin{aligned}
&\frac{1}{6}\left[\Xi(k_1) + 4\Xi\left(\frac{k_1+k_2}{2}\right) + \Xi(k_2)\right] - \frac{1}{(k_2-k_1)}\left[\int_{k_1}^{\frac{k_1+k_2}{2}} \Xi(x)_{k_1} d_q x + \int_{\frac{k_1+k_2}{2}}^{k_2} \Xi(x)^{k_2} d_q x\right] \\
&= \frac{k_2-k_1}{2}\int_0^1 \left[\left(\frac{q\varrho}{2} - \frac{1}{6}\right) {}_{k_1}D_q\Xi\left(\frac{2-\varrho}{2}k_1 + \frac{\varrho}{2}k_2\right)\right. \\
&\left.+\left(\frac{1}{6} - \frac{q\varrho}{2}\right) {}^{k_2}D_q\Xi\left(\frac{\varrho}{2}k_1 + \frac{2-\varrho}{2}k_2\right)\right] d_q\varrho. \qquad (8)
\end{aligned}$$

Proof. Using Lemmas 1 and 2 and Definitions 2 and 4, we have

$$\begin{aligned}
I_1 &= \int_0^1 \left(\frac{q\varrho}{2} - \frac{1}{6}\right) {}_{k_1}D_q\Xi\left(\frac{2-\varrho}{2}k_1 + \frac{\varrho}{2}k_2\right) d_q\varrho \\
&= \frac{2}{k_2-k_1}\left(\frac{q\varrho}{2} - \frac{1}{6}\right)\Xi\left(\frac{\varrho}{2}k_1 + \frac{2-\varrho}{2}k_2\right)\Big|_0^1 - \frac{2}{k_2-k_1}\int_0^1 \frac{q}{2}\Xi\left(\frac{2-q\varrho}{2}k_1 + \frac{q\varrho}{2}k_2\right) d_q\varrho \\
&= \frac{1}{3(k_2-k_1)}\Xi(k_1) + \frac{2}{k_2-k_1}\left(\frac{q}{2} - \frac{1}{6}\right)\Xi\left(\frac{k_1+k_2}{2}\right) \\
&- \frac{1-q}{k_2-k_1}\sum_{n=0}^{\infty} q^{n+1}\Xi\left(\frac{2-q^{n+1}}{2}k_1 + \frac{q^{n+1}}{2}k_2\right) \\
&= \frac{1}{3(k_2-k_1)}\Xi(k_1) + \frac{2}{k_2-k_1}\left(\frac{q}{2} - \frac{1}{6}\right)\Xi\left(\frac{k_1+k_2}{2}\right) - \frac{1-q}{k_2-k_1}\sum_{n=1}^{\infty} q^n\Xi\left(\frac{2-q^n}{2}k_1 + \frac{q^n}{2}k_2\right) \\
&= \frac{1}{3(k_2-k_1)}\Xi(k_1) + \frac{2}{k_2-k_1}\left(\frac{q}{2} - \frac{1}{6}\right)\Xi\left(\frac{k_1+k_2}{2}\right) \\
&\quad - \frac{1-q}{k_2-k_1}\left[\sum_{n=0}^{\infty} q^n\Xi\left((1-q^n)k_1 + q^n f\left(\frac{k_1+k_2}{2}\right)\right) - \Xi\left(\frac{k_1+k_2}{2}\right)\right] \\
&= \frac{1}{3(k_2-k_1)}\Xi(k_1) + \frac{1-q}{k_2-k_1}\Xi\left(\frac{k_1+k_2}{2}\right) + \frac{2}{k_2-k_1}\left(\frac{q}{2} - \frac{1}{6}\right)\Xi\left(\frac{k_1+k_2}{2}\right) \\
&\quad - \frac{2}{(k_2-k_1)^2}\int_{k_1}^{\frac{k_1+k_2}{2}} \Xi(x)\ {}_{k_1}\ d_q x. \qquad (9)
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \int_0^1 \left(\frac{1}{6} - \frac{q\varrho}{2}\right) {}_{k_1}D_q\Xi\left(\frac{\varrho}{2}k_1 + \frac{2-\varrho}{2}k_2\right) d_q\varrho \\
&= \frac{2}{k_2-k_1}\left(\frac{q\varrho}{2} - \frac{1}{6}\right)\Xi\left(\frac{\varrho}{2}k_1 + \frac{2-\varrho}{2}k_2\right)\Big|_0^1 - \frac{2}{k_2-k_1}\int_0^1 \frac{q}{2}\Xi\left(\frac{q\varrho}{2}k_1 + \frac{2-q\varrho}{2}k_2\right) d_q\varrho \\
&= \frac{1}{3(k_2-k_1)}\Xi(k_2) + \frac{2}{k_2-k_1}\left(\frac{q}{2} - \frac{1}{6}\right)\Xi\left(\frac{k_1+k_2}{2}\right) \\
&\quad - \frac{1-q}{k_2-k_1}\sum_{n=0}^{\infty} q^{n+1}\Xi\left(\frac{q^{n+1}}{2}k_1 + \frac{2-q^{n+1}}{2}k_2\right) \\
&= \frac{1}{3(k_2-k_1)}\Xi(k_2) + \frac{2}{k_2-k_1}\left(\frac{q}{2} - \frac{1}{6}\right)\Xi\left(\frac{k_1+k_2}{2}\right) \\
&\quad - \frac{1-q}{k_2-k_1}\sum_{n=1}^{\infty} q^n\Xi\left(\frac{q^n}{2}k_1 + \frac{2-q^n}{2}k_2\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{3(k_2-k_1)}\Xi(k_2) + \frac{2}{k_2-k_1}\left(\frac{q}{2}-\frac{1}{6}\right)\Xi\left(\frac{k_1+k_2}{2}\right) \\
&- \frac{1-q}{k_2-k_1}\left[\sum_{n=0}^{\infty} q^n \Xi\left(q^n\frac{k_1+k_2}{2}+(1-q^n)k_2\right) - \Xi\left(\frac{k_1+k_2}{2}\right)\right] \\
&= \frac{1}{3(k_2-k_1)}\Xi(k_2) + \frac{1-q}{k_2-k_1}\Xi\left(\frac{k_1+k_2}{2}\right) + \frac{2}{k_2-k_1}\left(\frac{q}{2}-\frac{1}{6}\right)\Xi\left(\frac{k_1+k_2}{2}\right) \\
&- \frac{2}{(k_2-k_1)^2}\int_{\frac{k_1+k_2}{2}}^{k_2} \Xi(x)^{k_2}\, d_qx, \qquad (10)
\end{aligned}
$$

Thus, from inequalities (10) and (9), we have

$$
\begin{aligned}
&\frac{k_2-k_1}{2}(I_1+I_2) \\
&= \frac{1}{6}\left[\Xi(k_1)+4\Xi\left(\frac{k_1+k_2}{2}\right)+\Xi(k_2)\right] - \frac{1}{(k_2-k_1)}\left[\int_{k_1}^{\frac{k_1+k_2}{2}} \Xi(x)_{k_1} d_qx + \int_{\frac{k_1+k_2}{2}}^{k_2} \Xi(x)^{k_2} d_qx\right],
\end{aligned}
$$

which completes the proof. □

Remark 1. *If we take the limit as $q \to 1^-$ in Lemma 4, then the equality (8) becomes*

$$
\begin{aligned}
&\frac{1}{6}\left[\Xi(k_1)+4\Xi\left(\frac{k_1+k_2}{2}\right)+\Xi(k_2)\right] - \frac{1}{(k_2-k_1)}\int_{k_1}^{k_2}\Xi(x)dx \\
&= \frac{k_2-k_1}{2}\int_0^1 \left[\left(\frac{\varrho}{2}-\frac{1}{6}\right)\Xi'\left(\frac{2-\varrho}{2}k_1+\frac{\varrho}{2}k_2\right) + \left(\frac{1}{6}-\frac{\varrho}{2}\right)\Xi'\left(\frac{\varrho}{2}k_1+\frac{2-\varrho}{2}k_2\right)\right] d_q\varrho,
\end{aligned}
$$

which is found in [33].

Theorem 1. *Suppose that $\Xi : [k_1,k_2] \to \mathbb{R}$ is a q-differentiable function on (k_1,k_2) such that ${}_{k_1}D_q\Xi$ and ${}^{k_2}D_q\Xi$ are continuous and integrable functions on $[k_1,k_2]$. If $\left|{}_{k_1}D_q\Xi\right|$ and $\left|{}^{k_2}D_q\Xi\right|$ are convex and integrable functions on $[k_1,k_2]$, then*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[\Xi(k_1)+4\Xi\left(\frac{k_1+k_2}{2}\right)+\Xi(k_2)\right] - \frac{1}{k_2-k_1}\left[\int_{k_1}^{\frac{k_1+k_2}{2}} \Xi(x)_{k_1} d_qx + \int_{\frac{k_1+k_2}{2}}^{k_2} \Xi(x)^{k_2} d_qx\right]\right| \\
&\le \frac{k_2-k_1}{2}\left[\left(\Phi_1(q)\left|{}_{k_1}D_q\Xi(k_1)\right| + \Phi_2(q)\left|{}_{k_1}D_q\Xi(k_2)\right|\right)\right. \\
&\left.+\left(\Phi_2(q)\left|{}^{k_2}D_q\Xi(k_1)\right| + \Phi_1(q)\left|{}^{k_2}D_q\Xi(k_2)\right|\right)\right], \qquad (11)
\end{aligned}
$$

where $\Phi_i(q), i = 1,2$ are defined by

$$
\Phi_1(q) = \int_0^1 \left|\frac{q\varrho}{2}-\frac{1}{6}\right|\frac{2-\varrho}{2}\, d_q\varrho = \begin{cases} \dfrac{-4q^3+1}{12[2]_q[3]_q}, & for\ \ 0<q<\frac{1}{3}; \\ \dfrac{36q^3+12q^2+10q+3}{108[2]_q[3]_q}, & for\ \ \frac{1}{3}\le q<1, \end{cases}
$$

$$
\Phi_2(q) = \int_0^1 \left|\frac{q\varrho}{2}-\frac{1}{6}\right|\frac{\varrho}{2}\, d_q\varrho = \begin{cases} \dfrac{1-2q-2q^2}{48[2]_q[3]_q}, & for\ \ 0<q<\frac{1}{3}; \\ \dfrac{-9q^3+18q^2+18q+2}{108[2]_q[3]_q}, & for\ \ \frac{1}{3}\le q<1, \end{cases}
$$

Proof. By taking the absolute value of both sides of inequality (8), we observe that

$$\begin{aligned}
&\left|\frac{1}{6}\left[\Xi(k_1)+4\Xi\left(\frac{k_1+k_2}{2}\right)+\Xi(k_2)\right]-\frac{1}{(k_2-k_1)}\left[\int_{k_1}^{\frac{k_1+k_2}{2}}\Xi(x)_{k_1}d_qx+\int_{\frac{k_1+k_2}{2}}^{k_2}\Xi(x)^{k_2}d_qx\right]\right| \\
&\le \frac{k_2-k_1}{2}\int_0^1\left[\left|\frac{q\varrho}{2}-\frac{1}{6}\right|\left|{}_{k_1}D_q\Xi\left(\frac{2-\varrho}{2}k_1+\frac{\varrho}{2}k_2\right)\right|+\left|\frac{1}{6}-\frac{q\varrho}{2}\right|\left|{}^{k_2}D_q\Xi\left(\frac{\varrho}{2}k_1+\frac{2-\varrho}{2}k_1\right)\right|\right]d_q\varrho \\
&\le \frac{k_2-k_1}{2}\int_0^1\left|\frac{q\varrho}{2}-\frac{1}{6}\right|\left[\left(\frac{2-\varrho}{2}|{}_{k_1}D_q\Xi(k_1)|+\frac{\varrho}{2}|{}_{k_1}D_q\Xi(k_2)|\right)\right. \\
&\quad\left.+\left(\frac{\varrho}{2}\left|{}^{k_2}D_q\Xi(k_1)\right|+\frac{2-\varrho}{2}\left|{}^{k_2}D_q\Xi(k_2)\right|\right)\right]d_q\varrho \\
&=\frac{k_2-k_1}{2}\left[\left(\Phi_1(q)|{}_{k_1}D_q\Xi(k_2)|+\Phi_2(q)|{}_{k_1}D_q\Xi(k_1)|\right)\right. \\
&\quad\left.+\left(\Phi_1(q)\left|{}^{k_2}D_q\Xi(k_1)\right|+\Phi_2(q)\left|{}^{k_2}D_q\Xi(k_2)\right|\right)\right].
\end{aligned}$$

Using Lemma 3, one can easily compute the integrals as follows:

$$\Phi_1(q)=\int_0^1\left|\frac{q\varrho}{2}-\frac{1}{6}\right|\frac{2-\varrho}{2}d_q\varrho=\begin{cases}\dfrac{-4q^3+1}{12[2]_q[3]_q}, & for\ \ 0<q<\frac{1}{3};\\[2ex] \dfrac{36q^3+12q^2+10q+3}{108[2]_q[3]_q}, & for\ \ \frac{1}{3}\le q<1,\end{cases}$$

$$\Phi_2(q)=\int_0^1\left|\frac{q\varrho}{2}-\frac{1}{6}\right|\frac{\varrho}{2}d_q\varrho=\begin{cases}\dfrac{1-2q-2q^2}{48[2]_q[3]_q}, & for\ \ 0<q<\frac{1}{3};\\[2ex] \dfrac{-9q^3+18q^2+18q+2}{108[2]_q[3]_q}, & for\ \ \frac{1}{3}\le q<1,\end{cases}$$

which completes the proof. □

Remark 2. *If we take the limit as $q\to 1^-$ in Theorem 1, then the inequality (11) becomes*

$$\left|\frac{1}{6}\left[\Xi(k_1)+4\Xi\left(\frac{k_1+k_2}{2}\right)+\Xi(k_2)\right]-\frac{1}{(k_2-k_1)}\int_{k_1}^{k_2}\Xi(x)dx\right|\le\frac{5(k_2-k_1)}{72}\left[|\Xi'(k_1)|+|\Xi'(k_2)|\right],$$

which is found in [33].

Theorem 2. *Suppose that $\Xi:[k_1,k_2]\to\mathbb{R}$ is a q-differentiable function on (k_1,k_2) such that ${}_{k_1}D_q\Xi$ and ${}^{k_2}D_q\Xi$ are continuous and integrable functions on $[k_1,k_2]$ and $r\ge 1$. If $|{}_{k_1}D_q\Xi|^r$ and $|{}^{k_2}D_q\Xi|^r$ are convex and integrable functions on $[k_1,k_2]$, then*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\Xi(k_1)+4\Xi\left(\frac{k_1+k_2}{2}\right)+\Xi(k_2)\right]-\frac{1}{(k_2-k_1)}\left[\int_{k_1}^{\frac{k_1+k_2}{2}}\Xi(x)_{k_1}d_qx+\int_{\frac{k_1+k_2}{2}}^{k_2}\Xi(x)^{k_2}d_qx\right]\right| \\
&\le\frac{(k_2-k_1)(\Phi_3(q))^{1-\frac{1}{r}}}{2}\left[\left(\Phi_1(q)|{}_{k_1}D_q\Xi(k_1)|^r+\Phi_2(q)|{}_{k_1}D_q\Xi(k_2)|^r\right)^{\frac{1}{r}}\right. \\
&\quad\left.+\left(\Phi_2(q)\left|{}^{k_2}D_q\Xi(k_1)\right|^r+\Phi_1(q)\left|{}^{k_2}D_q\Xi(k_2)\right|^r\right)^{\frac{1}{r}}\right],
\end{aligned}\tag{12}$$

where $\Phi_i(q), i=1,2$ are defined in Theorem 1 and $\Phi_3(q)$ is defined by

$$\Phi_3(q)=\begin{cases}\dfrac{1-2q}{6[2]_q}, & for\ \ 0<q<\frac{1}{3};\\[2ex] \dfrac{6q-1}{18[2]_q}, & for\ \ \frac{1}{3}\le q<1,\end{cases}$$

Proof. Taking the absolute value of both sides of inequality (8), applying the power mean inequality, and using the convexity of $|\,_{k_1}D_q\Xi|^r$ and $|\,^{k_2}D_q\Xi|^r$, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[\Xi(k_1)+4\Xi\left(\frac{k_1+k_2}{2}\right)+\Xi(k_2)\right]-\frac{1}{(k_2-k_1)}\left[\int_{k_1}^{\frac{k_1+k_2}{2}}\Xi(x)\,_{k_1}d_qx+\int_{\frac{k_1+k_2}{2}}^{k_2}\Xi(x)\,^{k_2}d_qx\right]\right| \\
&\leq \frac{k_2-k_1}{2}\int_0^1\left[\left|\frac{q\varrho}{2}-\frac{1}{6}\right|\left|\,_{k_1}D_q\Xi\left(\frac{2-\varrho}{2}k_1+\frac{\varrho}{2}k_2\right)\right|+\left|\frac{1}{6}-\frac{q\varrho}{2}\right|\left|\,^{k_2}D_q\Xi\left(\frac{\varrho}{2}k_1+\frac{2-\varrho}{2}k_1\right)\right|\right]d_q\varrho \\
&\leq \frac{k_2-k_1}{2}\left[\int_0^1\left(\left|\frac{q\varrho}{2}-\frac{1}{6}\right|d_q\varrho\right)^{1-\frac{1}{r}}\left[\int_0^1\left|\frac{q\varrho}{2}-\frac{1}{6}-\right|\left|\,_{k_1}D_q\Xi\left(\frac{2-\varrho}{2}k_1+\frac{\varrho}{2}k_2\right)\right|^r d_q\varrho\right]^{\frac{1}{r}}\right. \\
&\quad\left.+\int_0^1\left(\left|\frac{1}{6}-\frac{q\varrho}{2}\right|d_q\varrho\right)^{1-\frac{1}{r}}\left[\int_0^1\left|\frac{1}{6}-\frac{q\varrho}{2}\right|\left|\,^{k_2}D_q\Xi\left(\frac{\varrho}{2}k_1+\frac{2-\varrho}{2}k_2\right)\right|^r d_q\varrho\right]^{\frac{1}{r}}\right] \\
&\leq \frac{k_2-k_1}{2}\left[\int_0^1\left(\left|\frac{q\varrho}{2}-\frac{1}{6}\right|d_q\varrho\right)^{1-\frac{1}{r}}\left[\int_0^1\left|\frac{q\varrho}{2}-\frac{1}{6}\right|\left(\frac{2-\varrho}{2}|\,_{k_1}D_q\Xi(k_1)|^r+\frac{\varrho}{2}|\,_{k_1}D_q\Xi(k_2)|^r\right)d_q\varrho\right]^{\frac{1}{r}}\right. \\
&\quad\left.+\int_0^1\left(\left|\frac{1}{6}-\frac{q\varrho}{2}\right|d_q\varrho\right)^{1-\frac{1}{r}}\left[\int_0^1\left|\frac{1}{6}-\frac{q\varrho}{2}\right|\left(\frac{\varrho}{2}\left|\,^{k_2}D_q\Xi(k_1)\right|^r+\frac{2-\varrho}{2}\left|\,^{k_2}D_q\Xi(k_2)\right|^r\right)d_q\varrho\right]^{\frac{1}{r}}\right] \\
&=\frac{(k_2-k_1)(\Phi_3(q)^{1-\frac{1}{r}})}{2}\left[\left(\Phi_1(q)|\,_{k_1}D_q\Xi(k_1)|^r+\Phi_2(q)|\,_{k_1}D_q\Xi(k_2)|^r\right)^{\frac{1}{r}}\right. \\
&\quad\left.+\left(\Phi_2(q)\left|\,^{k_2}D_q\Xi(k_1)\right|^r+\Phi_1(q)\left|\,^{k_2}D_q\Xi(k_2)\right|^r\right)^{\frac{1}{r}}\right].
\end{aligned}$$

Using Lemma 3, one can easily compute the integral as follows:

$$\Phi_3(q)=\int_0^1\left|\frac{1}{6}-\frac{q\varrho}{2}\right|d_q\varrho=\begin{cases}\dfrac{1-2q}{6[2]_q}, & for\ \ 0<q<\frac{1}{3};\\[2mm] \dfrac{6q-1}{18[2]_q}, & for\ \ \frac{1}{3}\leq q<1,\end{cases}$$

which completes the proof. □

Remark 3. *If we take the limit as $q\to 1^-$ in Theorem 2, then the inequality (12) becomes*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\Xi(k_1)+4\Xi\left(\frac{k_1+k_2}{2}\right)+\Xi(k_2)\right]-\frac{1}{(k_2-k_1)}\int_{k_1}^{k_2}\Xi(x)dx\right| \\
&\leq \frac{k_2-k_1}{72}(5)^{1-\frac{1}{r}}\left[\left(\frac{61|\Xi'(k_1)|^r+29|\Xi'(k_2)|^r}{18}\right)^{\frac{1}{r}}+\left(\frac{61|\Xi'(k_2)|^r+29|\Xi'(k_1)|^r}{18}\right)^{\frac{1}{r}}\right],
\end{aligned}$$

which is found in [33].

Theorem 3. *Suppose that $\Xi:[k_1,k_2]\to\mathbb{R}$ is a q-differentiable function on (k_1,k_2) such that $_{k_1}D_q\Xi$ and $^{k_2}D_q\Xi$ are continuous and integrable functions on $[k_1,k_2]$ and $s^{-1}+r^{-1}=1$, if $|\,_{k_1}D_q\Xi|^r$ and $|\,^{k_2}D_q\Xi|^r$ are convex and integrable functions on $[k_1,k_2]$. Then,*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\Xi(k_1)+4\Xi\left(\frac{k_1+k_2}{2}\right)+\Xi(k_2)\right]-\frac{1}{(k_2-k_1)}\left[\int_{k_1}^{\frac{k_1+k_2}{2}}\Xi(x)\,_{k_1}d_qx+\int_{\frac{k_1+k_2}{2}}^{k_2}\Xi(x)\,^{k_2}d_qx\right]\right| \\
&\leq \frac{k_2-k_1}{12}\left(\frac{1+(3q-1)^{s+1}}{3q[s+1]_q}\right)^{\frac{1}{s}}\left[\left(\frac{(1+2q)|\,_{k_1}D_q\Xi(k_1)|^r+|\,_{k_1}D_q\Xi(k_2)|^r}{2[2]_q}\right)^{\frac{1}{r}}\right.
\end{aligned}$$

$$+\left(\frac{|\,^{k_2}D_q\Xi(k_1)|^r+(1+2q)|\,^{k_2}D_q\Xi(k_2)|^r}{2[2]_q}\right)^{\frac{1}{r}}\Bigg]. \tag{13}$$

Proof. Taking the absolute value of both sides of inequality (8), applying the Hölder's inequality, and using the convexity of $|\,_{k_1}D_q\Xi|^r$ and $|\,^{k_2}D_q\Xi|^r$, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[\Xi(k_1)+4\Xi\left(\frac{k_1+k_2}{2}\right)+\Xi(k_2)\right]-\frac{1}{(k_2-k_1)}\left[\int_{k_1}^{\frac{k_1+k_2}{2}}\Xi(x)_{k_1}d_qx+\int_{\frac{k_1+k_2}{2}}^{k_2}\Xi(x)^{k_2}d_qx\right]\right| \\
&\leq\frac{k_2-k_1}{2}\int_0^1\left[\left|\frac{q\varrho}{2}-\frac{1}{6}\right|\left|\,_{k_1}D_q\Xi\left(\frac{2-\varrho}{2}k_1+\frac{\varrho}{2}k_2\right)\right|+\left|\frac{1}{6}-\frac{q\varrho}{2}\right|\left|\,^{k_2}D_q\Xi\left(\frac{\varrho}{2}k_1+\frac{2-\varrho}{2}k_1\right)\right|\right]d_q\varrho \\
&\leq\frac{k_2-k_1}{2}\left[\int_0^1\left(\left|\frac{q\varrho}{2}-\frac{1}{6}\right|^s d_q\varrho\right)^{\frac{1}{s}}\left[\int_0^1\left|\,_{k_1}D_q\Xi\left(\frac{2-\varrho}{2}k_1+\frac{\varrho}{2}k_2\right)\right|^r d_q\varrho\right]^{\frac{1}{r}}\right. \\
&\quad\left.+\int_0^1\left(\left|\frac{1}{6}-\frac{q\varrho}{2}-\right|^s d_q\varrho\right)^{\frac{1}{s}}\left[\int_0^1\left|\,^{k_2}D_q\Xi\left(\frac{\varrho}{2}k_1+\frac{2-\varrho}{2}k_2\right)\right|^r d_q\varrho\right]^{\frac{1}{r}}\right] \\
&\leq\frac{k_2-k_1}{2}\left[\int_0^1\left(\left|\frac{q\varrho}{2}-\frac{1}{6}\right|^s d_q\varrho\right)^{\frac{1}{s}}\left[\int_0^1\left(\frac{2-\varrho}{2}|\,_{k_1}D_q\Xi(k_1)|^r+\frac{\varrho}{2}|\,_{k_1}D_q\Xi(k_2)|^r\right)d_q\varrho\right]^{\frac{1}{r}}\right. \\
&\quad\left.+\int_0^1\left(\left|\frac{1}{6}-\frac{q\varrho}{2}\right|^s d_q\varrho\right)^{\frac{1}{s}}\left[\int_0^1\left(\frac{\varrho}{2}\left|\,^{k_2}D_q\Xi(k_1)\right|^r+\frac{2-\varrho}{2}\left|\,^{k_2}D_q\Xi(k_2)\right|^r\right)d_q\varrho\right]^{\frac{1}{r}}\right].
\end{aligned}$$

Using Lemma 3, it is easy to see that

$$\begin{aligned}
\int_0^1\left|\frac{q\varrho}{2}-\frac{1}{6}\right|^s d_q\varrho&=\int_0^{\frac{1}{3q}}\left(\frac{1}{6}-\frac{q\varrho}{2}\right)^s d_q\varrho+\int_{\frac{1}{3q}}^1\left(\frac{q\varrho}{2}-\frac{1}{6}\right)^s d_q\varrho \\
&=(-1)^{s+1}\frac{q^s}{2^s}\int_{\frac{1}{3q}}^0\left(\varrho-\frac{1}{3q}\right)^s d_q\varrho+\frac{q^s}{2^s}\int_{\frac{1}{3q}}^1\left(\varrho-\frac{1}{3q}\right)^s d_q\varrho \\
&=(-1)^{s+1}\frac{q^s}{2^s[s+1]_q}\left(0-\frac{1}{3q}\right)^{s+1}+\frac{q^s}{2^s[s+1]_q}\left(1-\frac{1}{3q}\right)^{s+1} \\
&=\frac{2(1+(3q-1)^{s+1})}{6^{s+1}q[s+1]_q}.
\end{aligned}$$

We find that

$$\begin{aligned}
&\int_0^1\left(\frac{2-\varrho}{2}|\,_{k_1}D_q\Xi(k_1)|^r+\frac{\varrho}{2}|\,_{k_1}D_q\Xi(k_2)|^r\right)d_q\varrho \\
&=\frac{(1+2q)|\,_{k_1}D_q\Xi(k_1)|^r+|\,_{k_1}D_q\Xi(k_2)|^r}{2[2]_q},
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\left(\frac{\varrho}{2}\left|\,^{k_2}D_q\Xi(k_1)\right|^r+\frac{2-\varrho}{2}\left|\,^{k_2}D_q\Xi(k_2)\right|^r\right)d_q\varrho \\
&=\frac{|\,^{k_2}D_q\Xi(k_1)|^r+(1+2q)|\,^{k_2}D_q\Xi(k_2)|^r}{2[2]_q}.
\end{aligned}$$

Thus, the proof is completed. □

Remark 4. *If we take the limit as $q\to 1^-$ in Theorem 3, then the inequality (13) becomes*

$$\left|\frac{1}{6}\left[\Xi(k_1)+4\Xi\left(\frac{k_1+k_2}{2}\right)+\Xi(k_2)\right]-\frac{1}{(k_2-k_1)}\int_{k_1}^{k_2}\Xi(x)dx\right|$$

$$\leq \frac{k_2-k_1}{12}\left(\frac{1+2^{s+1}}{3(s+1)}\right)^{\frac{1}{s}}\left[\left(\frac{3|\Xi'(k_1)|^r+|\Xi'(k_2)|^r}{4}\right)^{\frac{1}{r}}+\left(\frac{3|\Xi'(k_2)|^r+|\Xi'(k_1)|^r}{4}\right)^{\frac{1}{r}}\right],$$

which is found in [33].

4. Numerical Examples

Within this section, we provide illustrations that bolster the recently established inequalities outlined in the previous section.

Example 1. *Define function $\Xi : [0,1] \to \mathbb{R}$ by $\Xi(x) = x^2$. From Theorem 1 with $q = \frac{1}{2}$, the left side of the inequality (11) becomes*

$$\left|\frac{1}{6}\left[\Xi(k_1)+4\Xi\left(\frac{k_1+k_2}{2}\right)+\Xi(k_2)\right]-\frac{1}{(k_2-k_1)}\left[\int_{k_1}^{\frac{k_1+k_2}{2}}\Xi(x)_{k_1}d_qx+\int_{\frac{k_1+k_2}{2}}^{k_2}\Xi(x)^{k_2}d_qx\right]\right|$$
$$=\left|\frac{1}{6}\left[\Xi(0)+4\Xi\left(\frac{1}{2}\right)+\Xi(1)\right]-\left[\int_0^{\frac{1}{2}}x^2{}_0d_{\frac{1}{2}}x+\int_{\frac{1}{2}}^{1}x^{2\,1}d_{\frac{1}{2}}x\right]\right|\approx 0.02380,$$

and the right side of the inequality (11) becomes

$$\frac{k_2-k_1}{2}\left[\left(\Phi_1(q)\left|{}_{k_1}D_q\Xi(k_2)\right|+\Phi_2(q)\left|{}_{k_1}D_q\Xi(k_1)\right|\right)+\left(\Phi_1(q)\left|{}^{k_2}D_q\Xi(k_1)\right|+\Phi_2(q)\left|{}^{k_2}D_q\Xi(k_2)\right|\right)\right]$$
$$\frac{1}{2}\left[\left(\Phi_1\left(\frac{1}{2}\right)\left|{}_0D_{\frac{1}{2}}\Xi(0)\right|+\Phi_2\left(\frac{1}{2}\right)\left|{}_0D_{\frac{1}{2}}\Xi(1)\right|\right)+\left(\Phi_2\left(\frac{1}{2}\right)\left|{}^1D_{\frac{1}{2}}\Xi(0)\right|+\Phi_1\left(\frac{1}{2}\right)\left|{}^1D_{\frac{1}{2}}\Xi(1)\right|\right)\right]$$
$$\approx 0.0694.$$

It is clear that

$$0.02380 \leq 0.0694,$$

which shows that the inequality (11) is valid.

In Figure 1 green and red lines indicate the right and left hand sides of Theorem 1 respectively.

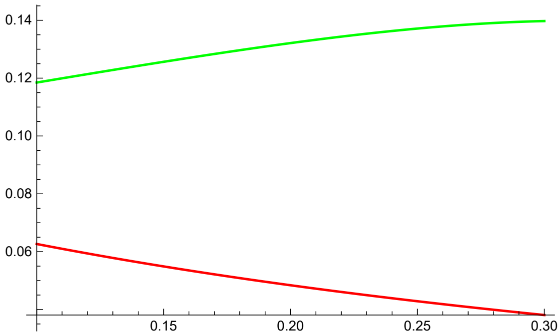

Figure 1. Graphical analysis of left and right sides of Theorem 1.

Example 2. *Define function $\Xi : [0,1] \to \mathbb{R}$ by $\Xi(x) = x^2$. From Theorem 2 with $q = \frac{1}{2}$ and $r = 2$, the left side of the inequality (12) becomes*

$$\left|\frac{1}{6}\left[\Xi(k_1)+4\Xi\left(\frac{k_1+k_2}{2}\right)+\Xi(k_2)\right]-\frac{1}{(k_2-k_1)}\left[\int_{k_1}^{\frac{k_1+k_2}{2}}\Xi(x)_{k_1}d_qx+\int_{\frac{k_1+k_2}{2}}^{k_2}\Xi(x)^{k_2}d_qx\right]\right|$$
$$=\left|\frac{1}{6}\left[\Xi(0)+4\Xi\left(\frac{1}{2}\right)+\Xi(1)\right]-\left[\int_0^{\frac{1}{2}}x^2{}_0d_{\frac{1}{2}}x+\int_{\frac{1}{2}}^{1}x^{2\,1}d_{\frac{1}{2}}x\right]\right|\approx 0.02380,$$

and the right side of the inequality (12) *becomes*

$$
\begin{aligned}
&\frac{(k_2-k_1)(\Phi_3(q))^{1-\frac{1}{r}}}{2} \\
&\times\left[\left(\Phi_1(q)\left|\,{}_{k_1}D_q\Xi(k_1)\right|^r+\Phi_2(q)\left|\,{}_{k_1}D_q\Xi(k_2)\right|^r\right)^{\frac{1}{r}}+\left(\Phi_2(q)\left|\,{}^{k_2}D_q\Xi(k_1)\right|^r+\Phi_1(q)\left|\,{}^{k_2}D_q\Xi(k_2)\right|^r\right)^{\frac{1}{r}}\right] \\
&=\frac{(\Phi_3\left(\frac{1}{2}\right))^{1-\frac{1}{2}}}{2}\left[\left(\Phi_1\left(\frac{1}{2}\right)\left|\,{}_0D_{\frac{1}{2}}\Xi(0)\right|^2+\Phi_2\left(\frac{1}{2}\right)\left|\,{}_0D_{\frac{1}{2}}\Xi(1)\right|^2\right)^{\frac{1}{2}}\right. \\
&\quad\left.+\left(\Phi_2\left(\frac{1}{2}\right)\left|\,{}^1D_{\frac{1}{2}}\Xi(0)\right|^2+\Phi_1\left(\frac{1}{2}\right)\left|\,{}^1D_{\frac{1}{2}}\Xi(1)\right|^2\right)^{\frac{1}{2}}\right] \\
&\approx 0.08535.
\end{aligned}
$$

It is clear that

$$0.02380 \leq 0.08535,$$

which shows that the inequality (12) *is valid.*

In Figure 2 green and red lines indicate the right and left hand sides of Theorem 2 respectively.

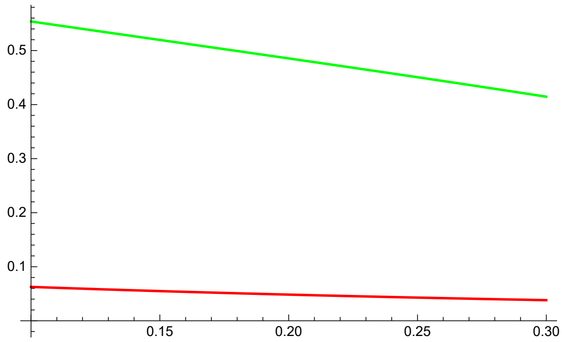

Figure 2. Graphical analysis of left and right sides of Theorem 2.

Example 3. *Define function* $\Xi : [0,1] \to \mathbb{R}$ *by* $\Xi(x) = x^2$. *From Theorem* 3 *with* $q = \frac{1}{2}$, $r = 2$ *and* $s = 2$, *the left side of the inequality* (13) *becomes*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[\Xi(k_1)+4\Xi\left(\frac{k_1+k_2}{2}\right)+\Xi(k_2)\right]-\frac{1}{(k_2-k_1)}\left[\int_{k_1}^{\frac{k_1+k_2}{2}}\Xi(x)_{k_1}d_qx+\int_{\frac{k_1+k_2}{2}}^{k_2}\Xi(x)^{k_2}d_qx\right]\right| \\
=&\left|\frac{1}{6}\left[\Xi(0)+4\Xi\left(\frac{1}{2}\right)+\Xi(1)\right]-\left[\int_0^{\frac{1}{2}}x^2{}_0d_{\frac{1}{2}}x+\int_{\frac{1}{2}}^1 x^2{}^1d_{\frac{1}{2}}x\right]\right| \approx 0.02380,
\end{aligned}
$$

and the right side of the inequality (13) *becomes*

$$
\begin{aligned}
&\frac{k_2-k_1}{12}\left(\frac{1+(3q-1)^{s+1}}{3q[s+1]_q}\right)^{\frac{1}{s}} \\
&\times\left[\left(\frac{(1+2q)|\,{}_{k_1}D_q\Xi(k_1)|^r+|\,{}_{k_1}D_q\Xi(k_2)|^r}{2[2]_q}\right)^{\frac{1}{r}}+\left(\frac{|\,{}^{k_2}D_q\Xi(k_1)|^r+(1+2q)|\,{}^{k_2}D_q\Xi(k_2)|^r}{2[2]_q}\right)^{\frac{1}{r}}\right] \\
&=\frac{1}{12}\left(\frac{1+\left(\frac{1}{2}\right)^3}{\frac{3}{2}[3]_q}\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\times \left[\left(\frac{3|\,{}_0D_{\frac{1}{2}}\Xi(0)|^2 + \,{}_0D_{\frac{1}{2}}\Xi(1)|^r}{2[2]_q} \right)^{\frac{1}{2}} + \left(\frac{|\,{}^1D_{\frac{1}{2}}\Xi(0)|^2 + 3|\,{}^1D_{\frac{1}{2}}\Xi(1)|^2}{2[2]_q} \right)^{\frac{1}{2}} \right]$$
$$\approx 0.13638.$$

It is clear that

$$0.02380 \leq 0.13638,$$

which shows that the inequality (13) *is valid.*

In Figure 3 green and red lines indicate the right and left hand sides of Theorem 3 respectively.

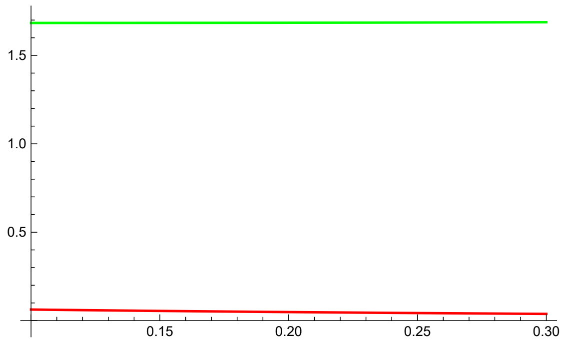

Figure 3. Graphical analysis of left and right sides of Theorem 3.

5. Conclusions

This paper has introduced various fresh quantum versions of Simpson's inequality, relying on the convexity of the functions. To verify our findings, we have provided numerical illustrations and graphical analysis. Our intention is that the concepts and methods presented in this research will inspire additional exploration in this area. In the future, we plan to extend these findings to additional classes of convexity and expand them to post-quantum calculus.

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