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Deviation of Geodesics, Particle Trajectories and the Propagation of Radiation in Gravitational Waves in Shapovalov Type III Wave Spacetimes

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Abstract: A class of exact (non-perturbative) models of strong gravitational waves based on Shapovalov type III spacetimes and Einstein's vacuum equations is obtained. Exact solutions are found for the trajectories of particles and radiation in a gravitational wave in privileged coordinate systems. Exact solutions are obtained for the equations of geodesic deviation and tidal acceleration of particles in a gravitational wave in privileged coordinate systems. An explicit analytical law of transition from a privileged coordinate system to a synchronous reference system associated with a freely falling observer with an explicit selection of time and spatial coordinates is obtained. An explicit form of the metric of a gravitational wave in a synchronous frame of reference is obtained. For a synchronous frame of reference, the trajectories of particles and radiation, the deviation of geodesics, and tidal accelerations in a gravitational wave are obtained. The presented methods and approaches are applicable both to Einstein's general theory of relativity and to modified theories of gravity.

Keywords: gravitational waves; Hamilton–Jacobi equation; Shapovalov spacetimes; deviation of geodesics; tidal acceleration

MSC: 83C10; 83C35



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1. Introduction

Recently, new approaches have been proposed in the theoretical study of primordial gravitational waves in Bianchi universes [1–5]. These approaches make it possible to obtain exact models of primordial gravitational waves, obtain exact solutions to the equations of test particles, obtain exact solutions to geodesic deviation equations, and exactly calculate tidal accelerations using various mathematical methods, including symmetry theory and the Hamilton–Jacobi formalism. Note that gravitational waves in Bianchi type VII universes were previously studied by Vladimir Lukash [6].

In this paper, we consider more general exact models of plane gravitational waves, which include as a special case models with spatially homogeneous symmetry, which we considered earlier [3–5]. These models are applicable to a wider class of gravitational waves and allow exact calculation of the effects created by waves, taking into account the deviation of geodesic and tidal accelerations both in general relativity and in modified theories of gravity [7–10]. These possibilities allow one to carry out an analytical comparison of the models and offer observational checks of the realism of the obtained models.

The considered exact models of strong gravitational waves make it possible to calculate both the direct impact of a wave on test particles in order to detect their motion and to

calculate the secondary physical effects, including for waves with long wavelengths, the direct detection of which is difficult. These models of gravitational waves make it possible, among other things, to estimate the plasma radiation in a gravitational wave, calculate the effects of gravitational lensing, calculate the effect of a passing wave on the change in the periods of pulsars, calculate the possibility of physical objects being captured by a gravitational wave, and make it possible to estimate the effect of a wave on the observed microwave background and the effect on the stochastic background of gravitational waves.

The data of direct detection of gravitational waves are currently analyzed using approximate and numerical methods, followed by the formation of computer databases of numerically obtained approximate models. The exact models of the gravitational waves obtained can also serve, in this case, for testing and debugging approximate and numerical methods in this area.

Shapovalov spacetimes [11] are the basis for obtaining exact models of gravitational waves in this paper. These spaces allow the existence of “privileged” coordinate systems, where the Hamilton–Jacobi equation of test particles allows exact integration through the method of separation of variables, and one of the variables on which the space metric depends in privileged coordinate systems is the wave variable, along which the spacetime interval goes to zero. At present, from the observational data of gravitational wave detection, it has been established with high accuracy that gravitational waves propagate at the speed of light [12].

Shapovalov spacetimes allow the construction of the complete integral of the Hamilton–Jacobi equation of test particles as a function of the coordinates and a set of independent constant parameters determined by the initial or boundary conditions for the motion of the test particles. The possibility of constructing such a complete integral leads to many useful consequences, such as obtaining the exact form of the test particle trajectories (i.e., the ability to find the explicit form of the geodesics) and the possibility of exactly integrating the geodesic deviation equation and calculating the tidal accelerations in a gravitational wave. All these possibilities make it feasible to analytically calculate the secondary physical effects of a gravitational wave when interacting with other physical objects and fields [13–19], as well as in theories of gravity with quantum and other modifications [20–23].

In addition, we recall that an important feature of the Shapovalov wave models under consideration is the possibility of making an explicit transition from privileged coordinate systems with wave variables to synchronous laboratory coordinate systems, where the time and spatial coordinates are separated, and a freely falling observer is at rest, while time synchronization is also possible in different points of space [24].

2. Shapovalov Spacetimes and Geodesic Deviation

Let us recall the necessary information from the Hamilton–Jacobi formalism and from the theory of Shapovalov wave spacetimes for completeness.

The Hamilton–Jacobi equation of a test particle in a gravitational field has the form (see [24])

$$g^{\alpha\beta} \frac{\partial S}{\partial x^\alpha} \frac{\partial S}{\partial x^\beta} = m^2 c^2, \quad \alpha, \beta, \gamma, \delta = 0, \dots, (n-1), \quad (1)$$

where m is the mass of the test particle, the capital letter S denotes the action function of the test particle, n is the dimension of space, and c is the speed of light, which we further set equal to unity.

The Hamilton–Jacobi formalism in mechanics is well known, and its description can be found in standard textbooks, such as in [25]. The application of the Hamilton–Jacobi formalism in the theory of gravity can be found in [24]. According to the Hamilton–Jacobi formalism, the complete integral of the Hamilton–Jacobi equation for test particles must contain n independent constants, where n is the spacetime dimension. The physical meaning of these constants is related to the coordinate system used and the existing symmetries in a specific problem and is determined by imposing initial or boundary

conditions. For particles, these constants are associated with conserved physical quantities such as energy, momentum components, and angular momentum components, as well as with symmetries such as time translation invariance, space translation invariance, and rotation invariance.

Definition 1 (Shapovalov spacetimes). *If the space allows the existence of a privileged coordinate system $\{x^\alpha\}$ where the Hamilton–Jacobi equation (Equation (1)) can be integrated by the complete separation of variables method, then the complete integral S for the test particle action function can be represented in a “separated” form:*

$$S = \phi_0(x^0, \lambda_0, \dots, \lambda_{n-1}) + \phi_1(x^1, \lambda_0, \dots, \lambda_{n-1}) + \dots + \phi_{n-1}(x^{n-1}, \lambda_0, \dots, \lambda_{n-1}), \quad (2)$$

$$\lambda_0, \lambda_1, \dots, \lambda_{n-1} = \text{const}, \quad \det \left| \frac{\partial^2 S}{\partial x^\alpha \partial \lambda_\beta} \right| \neq 0,$$

Moreover, if one of the non-ignored variables (on which the metric depends) is a wave (null) (i.e., the spacetime interval along this variable vanishes), then such a space will be called the Shapovalov wave spacetime.

Recall also that spacetimes that allow complete separation of the variables in the Hamilton–Jacobi equation (Equation (1)) were first considered by Paul Stäckel [26], and the theory of these spaces acquired a complete form in the works of Vladimir Shapovalov (see [27–29]), where a complete classification of such spaces was presented and an explicit form of the metrics of all given spaces in privileged coordinate systems was given. Shapovalov’s classification also included wave spacetimes that allowed non-ignorable wave variables, which the metric depends on in privileged coordinate systems.

The ability to construct the complete integral of the Hamilton–Jacobi equation in these spaces, in turn, allows us to find the trajectories of the test particles by determining the dependence of the coordinates x^α at the proper time τ of the test particle on the geodesic line of the particle. According to the standard Hamilton–Jacobi formalism (see, for example, [24,25]), the trajectory equations can be obtained from the known test particle action function with the following relations:

$$\frac{\partial S(x^\gamma, \lambda_\alpha)}{\partial \lambda_\beta} = \sigma_\beta, \quad \tau = S(x^\beta, \lambda_\alpha) \Big|_{m=1}, \quad (3)$$

where λ_α and σ_β are additional independent constant parameters determined by the initial conditions of the test particle motion and τ is the proper time of the particle.

The resulting complete integral of the Hamilton–Jacobi equation of the test particles also allows finding solutions for the geodesic deviation equations in the considered models of gravitational waves. Recall that the deviation of geodesics is the basic manifestation of the gravitational field and the detection of gravitational waves.

The geodesic deviation equation has the following form (see, for example, [24]):

$$\frac{D^2 \eta^\alpha}{d\tau^2} = R^\alpha{}_{\beta\gamma\delta} u^\beta u^\gamma \eta^\delta, \quad (4)$$

where $R^\alpha{}_{\beta\gamma\delta}$ is the Riemann curvature tensor of the spacetime, u^α is the four-velocity of the test particle on the base geodesic line, η^α is the geodesic deviation vector, and D is the covariant derivative. The coordinate x^α is parametrized by the proper time τ along the base geodesic line.

The four-velocity of a particle is known to satisfy the condition [24]

$$g^{\alpha\beta} u_\alpha u_\beta = 1. \quad (5)$$

The four-velocity of a particle is related to its action function S by the relation

$$u_\alpha = \frac{\partial S}{\partial x^\alpha} \Big|_{m=1}. \quad (6)$$

Thus, for Shapovalov wave spacetimes, the four-velocity of a test particle u^α depends on $(n-1)$ parameters $\lambda_1, \dots, \lambda_{(n-1)}$:

$$u_\alpha = u_\alpha(x^\beta, \lambda_1, \dots, \lambda_{(n-1)}). \quad (7)$$

The proper time of a test particle can be represented in the following form:

$$\tau = S(x^\alpha, \lambda_1, \dots, \lambda_{(n-1)}) \Big|_{m=1}. \quad (8)$$

In [30], an action was obtained whose variation gave both the Hamilton–Jacobi equation for a test particle and the geodesic variation equation as a result. The paper shows that in this case, knowing the complete integral for the Hamilton–Jacobi equation of the test particle allows us to reduce the geodesic deviation equation to an equation containing the test particle action function of the following form:

$$\eta^\alpha \frac{\partial u_\alpha(x^\beta, \lambda_i)}{\partial \lambda_k} + \rho_i \frac{\partial^2 S(x^\alpha, \lambda_j)}{\partial \lambda_i \partial \lambda_k} = \vartheta_k, \quad (9)$$

$$u_\beta(x^\alpha, \lambda_k) \eta^\beta = 0, \quad \alpha, \beta, \gamma = 0 \dots 3; \quad i, j, k = 1 \dots 3, \quad (10)$$

where λ_k , ρ_k , and ϑ_k are independent constant parameters.

The constant λ_k is given by the initial data for the test particle velocity on the base geodesic, and the constants ρ_k and ϑ_k are given by the initial data on the adjacent geodesic. The equations of motion (Equation (3)) define the dependence of the coordinates on the proper time τ .

3. Type-III Shapovalov Wave Spacetimes

Shapovalov's wave models of spacetime in a privileged coordinate system, where the Hamilton–Jacobi equation admits separation of variables, have a metric that necessarily depends on the wave (null) variable along which the interval vanishes. In addition, the metric may additionally depend on other variables (up to three). Therefore, the metric of the type I Shapovalov spacetime depends on three variables, type II spaces depend on two variables, and type III spaces depend on only one wave variable. In this paper, we will consider Shapovalov spaces of the third type.

Type-III Shapovalov wave spaces allow the existence of three commuting Killing vectors, and thus the metric of these spaces in the privileged coordinate system can generally be written in the following form [31,32]:

$$g^{\alpha\beta}(x^0) = \begin{pmatrix} 0 & 1 & g^{02}(x^0) & g^{03}(x^0) \\ 1 & 0 & 0 & 0 \\ g^{02}(x^0) & 0 & g^{22}(x^0) & g^{23}(x^0) \\ g^{03}(x^0) & 0 & g^{23}(x^0) & g^{33}(x^0) \end{pmatrix}, \quad (11)$$

where the indices α and β run through the values zero, one, two, and three. The variable x^0 is a wave (null) variable along which the spacetime interval vanishes. Thus, the metric is defined in the general case by five functions of the wave variable x^0 .

The spacetime for a gravitational wave (Equation (11)) is type N, according to Petrov's classification.

Einstein's equations with a cosmological constant Λ in a vacuum

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta} \quad (12)$$

for the metric in Equation (11) leads to the following necessary condition:

$$\Lambda = g^{02} = g^{03} = 0. \quad (13)$$

Thus, the metric of Shapovalov spacetimes of type III, taking into account the restrictions of Einstein's vacuum equations, will take the following form in the privileged coordinate system:

$$g^{\alpha\beta}(x^0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{A}(x^0) & \mathcal{B}(x^0) \\ 0 & 0 & \mathcal{B}(x^0) & \mathcal{C}(x^0) \end{pmatrix}, \quad \mathcal{A}\mathcal{C} - \mathcal{B}^2 > 0. \quad (14)$$

The space under consideration is a plane wave spacetime and admits a covariantly constant vector that specifies the direction of propagation of a gravitational wave:

$$\nabla_\alpha K_\beta = 0 \quad \rightarrow \quad K^\alpha = (0, 1, 0, 0). \quad (15)$$

In addition to the relations in Equation (13), the vacuum in Einstein's equations contains the equation $R_{00} = 0$, which allows one to express the second derivative of one of the functions in the metric. By expressing $\mathcal{C}''(x^0)$ from the remaining equation, we obtain a relation of the following form:

$$\begin{aligned} \mathcal{C}'' = & \frac{6\mathcal{B}^2(\mathcal{A}'\mathcal{C}' + (\mathcal{B}')^2) - 12\mathcal{A}\mathcal{B}\mathcal{B}'\mathcal{C}' + 3\mathcal{A}^2(\mathcal{C}')^2}{2\mathcal{A}(\mathcal{A}\mathcal{C} - \mathcal{B}^2)} \\ & + \mathcal{C} \frac{\mathcal{A}''\mathcal{B}^2 + 3\mathcal{A}(\mathcal{B}')^2 + \mathcal{B}(2\mathcal{B}''\mathcal{A} - 6\mathcal{A}'\mathcal{B}')}{\mathcal{A}(\mathcal{A}\mathcal{C} - \mathcal{B}^2)} \\ & + \frac{\mathcal{C}^2(3(\mathcal{A}')^2 - 2\mathcal{A}''\mathcal{A}) - 4\mathcal{B}''\mathcal{B}^3}{2\mathcal{A}(\mathcal{A}\mathcal{C} - \mathcal{B}^2)}, \end{aligned} \quad (16)$$

where the top prime denotes the derivative of the function with respect to the variable x^0 . Thus, the metric has three functions— $\mathcal{A}(x^0)$, $\mathcal{B}(x^0)$, and $\mathcal{C}(x^0)$ —connected by one ordinary differential equation (Equation (16)), which arises from the system of Einstein's field equations in a vacuum (where the cosmological constant Λ vanishes).

The Einstein equations for the considered wave metric are reduced to one differential relation for three initially arbitrary functions in the metric. The relations for the trajectories of the test particles include integrals of these functions. Therefore, the relationship following from Einstein's equations is not "rigid" enough to strongly influence the form of the expressions obtained for the particle trajectories, the deviation vector, and the tidal acceleration in the wave. Nevertheless, the use of this equation leads to some reduction in the "volume" of the resulting expressions.

The test particle action function in the privileged coordinate system takes the "separated" form:

$$S = \phi_0(x^0) + \sum_{k=1}^3 \lambda_k x^k, \quad (17)$$

where λ_k is the constant parameter determined by the initial or boundary values of the velocities (momentum values) of the test particle.

The equations of motion of the test particles in the Hamilton–Jacobi formalism can be integrated into the privileged coordinate system using the following notation for integrals of metric functions (Equation (14)):

$$\Theta_{ab}(x^0) = \int g^{ab}(x^0) dx^0, \quad a, b, c = 2, 3. \quad (18)$$

$$\Theta_{22}(x^0) = \int \mathcal{A}(x^0) dx^0, \quad \Theta_{23}(x^0) = \int \mathcal{B}(x^0) dx^0, \quad \Theta_{33}(x^0) = \int \mathcal{C}(x^0) dx^0. \quad (19)$$

The Hamilton–Jacobi equation (Equation (1)) can be integrated, and we find the following form of the function $\phi_0(x^0)$ (where the test particle mass m is set equal to unity):

$$\phi_0(x^0) = -\frac{\lambda_2^2 \Theta_{22} + 2\lambda_2 \lambda_3 \Theta_{23} + \lambda_3^2 \Theta_{33} - x^0}{2\lambda_1}. \quad (20)$$

Thus, we have obtained the explicit form of the complete integral $S(x^\alpha, \lambda_k)$ for the Hamilton–Jacobi equation for a test particle.

Then, in the Hamilton–Jacobi formalism, the trajectories of the test particles (Equation (3)) in the privileged coordinate system is found in the following form:

$$x^0(\tau) = \lambda_1 \tau, \quad (21)$$

$$x^1(\tau) = \frac{\tau}{2\lambda_1} - \frac{\lambda_2^2 \Theta_{22} + 2\lambda_2 \lambda_3 \Theta_{23} + \lambda_3^2 \Theta_{33}}{2\lambda_1^2} \Big|_{x^0=\lambda_1 \tau}, \quad (22)$$

$$x^2(\tau) = \frac{\lambda_2 \Theta_{22} + \lambda_3 \Theta_{23}}{\lambda_1} \Big|_{x^0=\lambda_1 \tau}, \quad (23)$$

$$x^3(\tau) = \frac{\lambda_2 \Theta_{23} + \lambda_3 \Theta_{33}}{\lambda_1} \Big|_{x^0=\lambda_1 \tau}, \quad (24)$$

where τ is the proper time of the test particle and the parameters λ_k are determined by the initial (boundary) values of the velocities (momentum values) of the test particle. The constant σ_k is set equal to zero by choosing the origin of the coordinates.

For the components of the four-velocity of the test particle u^α in the privileged coordinate system, from the equations of the trajectory, we obtain the following expressions:

$$u^0 = \lambda_1, \quad (25)$$

$$u^1(\tau) = -\frac{\lambda_2^2 \mathcal{A} + 2\lambda_2 \lambda_3 \mathcal{B} + \lambda_3^2 \mathcal{C} - 1}{2\lambda_1} \Big|_{x^0=\lambda_1 \tau}, \quad (26)$$

$$u^2(\tau) = (\lambda_2 \mathcal{A} + \lambda_3 \mathcal{B}) \Big|_{x^0=\lambda_1 \tau}, \quad (27)$$

$$u^3(\tau) = (\lambda_2 \mathcal{B} + \lambda_3 \mathcal{C}) \Big|_{x^0=\lambda_1 \tau}. \quad (28)$$

Now, we have all the necessary relations in order to write down and solve equations for the geodesic deviation vector (Equations (9) and (10)) for the metric in Equation (14). While omitting obvious calculations, we present the solution of the system of equations for the geodesic deviation vector $\eta^\alpha(\tau)$ in the privileged coordinate system:

$$\eta^0(\tau) = \rho_1 \tau - \lambda_1 \Omega, \quad x^0 = \lambda_1 \tau, \quad (29)$$

$$\eta^1(\tau) = -\frac{(\rho_1\tau - \lambda_1\Omega)(\lambda_2^2\mathcal{A} + 2\lambda_2\lambda_3\mathcal{B} + \lambda_3^2\mathcal{C} + 1)}{2\lambda_1^2} - \frac{\Theta_{23}(\lambda_2\tilde{\zeta}_3 + \lambda_3\tilde{\zeta}_2) + \lambda_2\tilde{\zeta}_2\Theta_{22} + \lambda_3\tilde{\zeta}_3\Theta_{33}}{\lambda_1^3} - \frac{\Omega}{\lambda_1} + \vartheta_1, \quad (30)$$

$$\eta^2(\tau) = \frac{\lambda_1(\rho_1\tau - \lambda_1\Omega)(\lambda_2\mathcal{A} + \lambda_3\mathcal{B}) + \tilde{\zeta}_2\Theta_{22} + \tilde{\zeta}_3\Theta_{23}}{\lambda_1^2} + \vartheta_2, \quad (31)$$

$$\eta^3(\tau) = \frac{\lambda_1(\rho_1\tau - \lambda_1\Omega)(\lambda_2\mathcal{B} + \lambda_3\mathcal{C}) + \tilde{\zeta}_2\Theta_{23} + \tilde{\zeta}_3\Theta_{33}}{\lambda_1^2} + \vartheta_3, \quad (32)$$

where τ is the proper time of the test particle on the base geodesic line. Here, the functions g^{ab} and Θ_{ab} in Equation (18) are functions of one variable x^0 , which on a geodesic with the proper time τ has the form $x^0 = \lambda_1\tau$.

Equations (29)–(32) include both independent parameters (λ_k , ρ_k , and ϑ_k) as well as dependent auxiliary constants ($\tilde{\zeta}_2$, $\tilde{\zeta}_3$, and Ω), which are introduced to shorten the notation:

$$\tilde{\zeta}_2 = \lambda_1\rho_2 - \lambda_2\rho_1, \quad \tilde{\zeta}_3 = \lambda_1\rho_3 - \lambda_3\rho_1, \quad (33)$$

$$\Omega = \lambda_1\vartheta_1 + \lambda_2\vartheta_2 + \lambda_3\vartheta_3. \quad (34)$$

The constant parameters λ_k are determined by the initial or boundary values for the momentum values of a test particle on the base geodesic, and the parameters ρ_k and ϑ_k are determined by the initial or boundary values of the momentum values and the relative positions of particles on neighboring geodesics.

The resulting deviation vector now allows us to calculate the deviation velocity $D\eta^\alpha/d\tau$ and the tidal acceleration $D^2\eta^\alpha/d\tau^2$.

Let us find the deviation rate $V^\alpha(\tau) = D\eta^\alpha/d\tau$ for the resulting general solution of the deviation equations (Equations (29)–(32)) in the metric in Equation (14) in the privileged coordinate system:

$$V^0 = \rho_1 = \text{const}, \quad (35)$$

$$\begin{aligned} V^1(\tau) = & \frac{1}{2\lambda_1^2(\mathcal{B}^2 - \mathcal{A}\mathcal{C})} \left[-\mathcal{C}(\tilde{\zeta}_2\Theta_{22} + \tilde{\zeta}_3\Theta_{23})(\lambda_2\mathcal{A}' + \lambda_3\mathcal{B}') \right. \\ & \lambda_1^2\vartheta_2(\mathcal{B}(\lambda_2\mathcal{B}' + \lambda_3\mathcal{C}') - \mathcal{C}(\lambda_2\mathcal{A}' + \lambda_3\mathcal{B}')) \\ & + \lambda_1^2\vartheta_3(\mathcal{B}(\lambda_2\mathcal{A}' + \lambda_3\mathcal{B}') - \mathcal{A}(\lambda_2\mathcal{B}' + \lambda_3\mathcal{C}')) \\ & + \rho_1(\mathcal{A}\mathcal{C} - \mathcal{B}^2)(\lambda_2^2\mathcal{A} + 2\lambda_2\lambda_3\mathcal{B} + \lambda_3^2\mathcal{C} + 1) \\ & + \mathcal{B}(\Theta_{23}(\mathcal{B}'(\lambda_2\tilde{\zeta}_3 + \lambda_3\tilde{\zeta}_2) + \lambda_2\tilde{\zeta}_2\mathcal{A}' + \lambda_3\tilde{\zeta}_3\mathcal{C}')) \\ & + \tilde{\zeta}_3\Theta_{33}(\lambda_2\mathcal{A}' + \lambda_3\mathcal{B}') + \tilde{\zeta}_2\Theta_{22}(\lambda_2\mathcal{B}' + \lambda_3\mathcal{C}')) \\ & - \mathcal{A}((\tilde{\zeta}_2\Theta_{23} + \tilde{\zeta}_3\Theta_{33})(\lambda_2\mathcal{B}' + \lambda_3\mathcal{C}')) \\ & - 2\mathcal{B}\mathcal{C}(\lambda_2\tilde{\zeta}_3 + \lambda_3\tilde{\zeta}_2) + 2\lambda_2\tilde{\zeta}_2\mathcal{B}^2 - 2\lambda_3\tilde{\zeta}_3\mathcal{C}^2) \\ & \left. - 2\mathcal{B}^3(\lambda_2\tilde{\zeta}_3 + \lambda_3\tilde{\zeta}_2) + 2\lambda_2\tilde{\zeta}_2\mathcal{A}^2\mathcal{C} - 2\lambda_3\tilde{\zeta}_3\mathcal{B}^2\mathcal{C} \right], \quad (36) \end{aligned}$$

$$\begin{aligned}
V^2(\tau) = & \frac{1}{2\lambda_1(\mathcal{B}^2 - \mathcal{A}\mathcal{C})} \left[\lambda_1^2 \vartheta_2 (\mathcal{A}'\mathcal{C} - \mathcal{B}\mathcal{B}') \right. \\
& + \lambda_1^2 \vartheta_3 (\mathcal{A}\mathcal{B}' - \mathcal{A}'\mathcal{B}) - 2\rho_1 (\mathcal{A}\mathcal{C} - \mathcal{B}^2) (\lambda_2\mathcal{A} + \lambda_3\mathcal{B}) \\
& - \mathcal{B} (\mathcal{A}'(\xi_2\Theta_{23} + \xi_3\Theta_{33}) + \mathcal{B}'(\xi_2\Theta_{22} + \xi_3\Theta_{23})) \\
& + \mathcal{A} (\mathcal{B}'(\xi_2\Theta_{23} + \xi_3\Theta_{33}) - 2\xi_3\mathcal{B}\mathcal{C} + 2\xi_2\mathcal{B}^2) \\
& \left. + \mathcal{A}'\mathcal{C}(\xi_2\Theta_{22} + \xi_3\Theta_{23}) - 2\xi_2\mathcal{A}^2\mathcal{C} + 2\xi_3\mathcal{B}^3 \right], \quad (37)
\end{aligned}$$

$$\begin{aligned}
V^3(\tau) = & \frac{1}{2\lambda_1(\mathcal{B}^2 - \mathcal{A}\mathcal{C})} \left[\lambda_1^2 \vartheta_3 (\mathcal{A}\mathcal{C}' - \mathcal{B}\mathcal{B}') \right. \\
& + \lambda_1^2 \vartheta_2 (\mathcal{B}'\mathcal{C} - \mathcal{B}\mathcal{C}') + 2\rho_1 (\mathcal{B}^2 - \mathcal{A}\mathcal{C}) (\lambda_2\mathcal{B} + \lambda_3\mathcal{C}) \\
& - \mathcal{B} (2\xi_2\mathcal{A}\mathcal{C} + \xi_2\Theta_{22}\mathcal{C}' + \xi_3\Theta_{33}\mathcal{B}' + \xi_2\Theta_{23}\mathcal{B}' + \xi_3\Theta_{23}\mathcal{C}') \\
& + \mathcal{A} (\mathcal{C}'(\xi_2\Theta_{23} + \xi_3\Theta_{33}) - 2\xi_3\mathcal{C}^2) \\
& \left. + \mathcal{B}'\mathcal{C}(\xi_2\Theta_{22} + \xi_3\Theta_{23}) + 2\xi_3\mathcal{B}^2\mathcal{C} + 2\xi_2\mathcal{B}^3 \right]. \quad (38)
\end{aligned}$$

Recall that the functions g^{ab} and Θ_{ab} (Equation (18)) depend on one variable x^0 , which is equal to $x^0 = \lambda_1\tau$ on the base geodesic, where τ is the proper time on the base geodesic. The constants λ_k are determined by the initial or boundary conditions for the velocity on the base geodesic, and the constants ρ_k and ϑ_k are determined by the initial or boundary conditions for the velocity and relative position on the adjacent geodesic. The constants ξ_2 , ξ_3 , and Ω are defined by the relations in Equations (33) and (34).

The expressions for the tidal acceleration $A^\alpha = D^2\eta^\alpha/d\tau^2$ have a cumbersome form, and we have moved them to Appendix A.

The trajectories of test particles $x^\alpha(\tau)$ (Equations (21)–(24)), geodesic deviation vector $\eta^\alpha(\tau)$ (Equations (29)–(32)), and geodesic deviation rate $V^\alpha(\tau)$ (Equations (35)–(38)) were found above for Shapovalov type-III gravitational waves in general form (i.e., obtained without taking into account the Einstein equation, shown in Equation (16)). Therefore, they can be applied to the metric in Equation (14) in any field equations based on it, including application in modified gravity theories (see [20–23]).

The obtained characteristics of a gravitational wave make it possible to evaluate the influence of a gravitational wave on physical objects, fields, and the physical media through which the wave passes. For example, we can now calculate the radiation of charges moving with tidal acceleration in a gravitational wave.

4. Synchronous Reference System

The use of a privileged coordinate system allowed us to integrate the equations of motion of the test particles, obtain an exact solution for the geodesic deviation vector, and obtain an exact form of the deviation velocity and tidal accelerations in a gravitational wave.

The form of the trajectories of the test particles in the field of a gravitational wave (Equations (21)–(24)) obtained using the complete integral of the Hamilton–Jacobi equation makes it possible to additionally implement a transition from the privileged coordinate system used to the laboratory synchronous reference frame $\tilde{x}^k$ associated with an observer freely falling along the base geodesic. The advantages of a synchronous reference system are related to the fact that time and the spatial variables in this reference system are separated, and time at different points in space can be synchronized.

The transition to the synchronous coordinate system $\tilde{x}^k$ can be explicitly implemented using the relations in Equations (21)–(24) according to the rules (see [24]):

$$x^\alpha \rightarrow \tilde{x}^\alpha = (\tau, \lambda_1, \lambda_2, \lambda_3). \quad (39)$$

Thus, transformations from the privileged coordinate system $\{x^a\}$ to the synchronous reference system $\{\tilde{x}^a\}$ take the following form:

$$x^0 = \tilde{x}^1 \tau, \quad (40)$$

$$x^1 = - \frac{(\tilde{x}^2)^2 \Theta_{22} + 2\tilde{x}^2 \tilde{x}^3 \Theta_{23} + (\tilde{x}^3)^2 \Theta_{33} - \tilde{x}^1 \tau}{2(\tilde{x}^1)^2} \Big|_{x^0 = \tilde{x}^1 \tau}, \quad (41)$$

$$x^2 = \frac{\tilde{x}^2 \Theta_{22} + \tilde{x}^3 \Theta_{23}}{\tilde{x}^1} \Big|_{x^0 = \tilde{x}^1 \tau}, \quad (42)$$

$$x^3 = \frac{\tilde{x}^2 \Theta_{23} + \tilde{x}^3 \Theta_{33}}{\tilde{x}^1} \Big|_{x^0 = \tilde{x}^1 \tau}. \quad (43)$$

In the resulting synchronous frame of reference, the four-velocity of the test particle will take the form $\tilde{u}^k = \{1, 0, 0, 0\}$. Thus, the observer rests on the base geodesic in the obtained synchronous frame of reference, and the observer's proper time τ becomes the time of the frame of reference. The metric in Equation (14) in the synchronous reference system will take the following form:

$$ds^2 = d\tau^2 - dl^2 = d\tau^2 + \tilde{g}_{ij}(\tau, \tilde{x}^k) d\tilde{x}^i d\tilde{x}^j, \quad (44)$$

where dl is the spatial distance element and τ is the time in the synchronous frame of reference. (The speed of light c is set equal to unity.)

When switching to a synchronous frame of reference, it is required in the functions $g^{ab}(x^0)$ and $\Theta_{ab}(x^0)$ to change $x^0 \rightarrow \tau \tilde{x}^1$. In this case, the components of the gravitational wave metric in Equation (14) will take the following form in the synchronous reference frame $\{\tilde{x}^a\}$:

$$\tilde{g}_{00} = 1, \quad \tilde{g}_{0k} = 0, \quad (45)$$

$$\begin{aligned} \tilde{g}_{11}(\tau, \tilde{x}^k) = & -\frac{\tau^2}{(\tilde{x}^1)^2} - \frac{\Theta_{23}^2 \left((\tilde{x}^2)^2 \mathcal{A} + \tilde{x}^3 (\tilde{x}^3 \mathcal{C} - 2\tilde{x}^2 \mathcal{B}) \right)}{(\tilde{x}^1)^4 (\mathcal{B}^2 - \mathcal{A}\mathcal{C})} \\ & + \frac{2\Theta_{23} (\tilde{x}^3 \Theta_{33} (\tilde{x}^3 \mathcal{B} - \tilde{x}^2 \mathcal{A}) + \tilde{x}^2 \Theta_{22} (\tilde{x}^2 \mathcal{B} - \tilde{x}^3 \mathcal{C}))}{(\tilde{x}^1)^4 (\mathcal{B}^2 - \mathcal{A}\mathcal{C})} \\ & - \frac{(\tilde{x}^2)^2 \Theta_{22}^2 \mathcal{C} - 2\tilde{x}^2 \tilde{x}^3 \Theta_{22} \Theta_{33} \mathcal{B} + (\tilde{x}^3)^2 \Theta_{33}^2 \mathcal{A}}{(\tilde{x}^1)^4 (\mathcal{B}^2 - \mathcal{A}\mathcal{C})}, \end{aligned} \quad (46)$$

$$\begin{aligned} \tilde{g}_{12}(\tau, \tilde{x}^k) = & \frac{\tilde{x}^2 (\Theta_{23}^2 \mathcal{A} - 2\Theta_{22} \Theta_{23} \mathcal{B} + \Theta_{22}^2 \mathcal{C})}{(\tilde{x}^1)^3 (\mathcal{B}^2 - \mathcal{A}\mathcal{C})} \\ & + \frac{\tilde{x}^3 (\Theta_{23} \Theta_{33} \mathcal{A} - \Theta_{22} \Theta_{33} \mathcal{B} + \Theta_{22} \Theta_{23} \mathcal{C} - \Theta_{23}^2 \mathcal{B})}{(\tilde{x}^1)^3 (\mathcal{B}^2 - \mathcal{A}\mathcal{C})}, \end{aligned} \quad (47)$$

$$\begin{aligned} \tilde{g}_{13}(\tau, \tilde{x}^k) = & \frac{\tilde{x}^2 (\Theta_{23} \Theta_{33} \mathcal{A} - \Theta_{22} \Theta_{33} \mathcal{B} + \Theta_{22} \Theta_{23} \mathcal{C} - \Theta_{23}^2 \mathcal{B})}{(\tilde{x}^1)^3 (\mathcal{B}^2 - \mathcal{A}\mathcal{C})} \\ & + \frac{\tilde{x}^3 (\Theta_{33}^2 \mathcal{A} - 2\Theta_{23} \Theta_{33} \mathcal{B} + \Theta_{23}^2 \mathcal{C})}{(\tilde{x}^1)^3 (\mathcal{B}^2 - \mathcal{A}\mathcal{C})}, \end{aligned} \quad (48)$$

$$\tilde{g}_{22}(\tau, \tilde{x}^k) = -\frac{\Theta_{23}^2 \mathcal{A} - 2\Theta_{22}\Theta_{23}\mathcal{B} + \Theta_{22}^2 \mathcal{C}}{(\tilde{x}^1)^2 (\mathcal{B}^2 - \mathcal{A}\mathcal{C})}, \quad (49)$$

$$\tilde{g}_{23}(\tau, \tilde{x}^k) = \frac{\Theta_{22}\Theta_{33}\mathcal{B} + \Theta_{23}^2 \mathcal{B} - \Theta_{23}(\Theta_{33}\mathcal{A} + \Theta_{22}\mathcal{C})}{(\tilde{x}^1)^2 (\mathcal{B}^2 - \mathcal{A}\mathcal{C})}, \quad (50)$$

$$\tilde{g}_{33}(\tau, \tilde{x}^k) = -\frac{\Theta_{33}^2 \mathcal{A} - 2\Theta_{23}\Theta_{33}\mathcal{B} + \Theta_{23}^2 \mathcal{C}}{(\tilde{x}^1)^2 (\mathcal{B}^2 - \mathcal{A}\mathcal{C})}, \quad (51)$$

where the one-variable functions $\mathcal{A}(x^0)$, $\mathcal{B}(x^0)$, $\mathcal{C}(x^0)$, and $\Theta_{ab}(x^0)$ (Equation (18)) now depend on the product $\tau\tilde{x}^1$ and the observer's proper time on the base geodesic τ is the universal time in the synchronous frame of reference.

A metric with superscripts $\tilde{g}^{\alpha\beta}(\tau, \tilde{x}^k)$ has a more compact and clearer form in a synchronous frame of reference:

$$\tilde{g}^{00} = 1, \quad \tilde{g}^{0k} = 0, \quad \tilde{g}^{1k}(\tau, \tilde{x}^i) = -\frac{\tilde{x}^1 \tilde{x}^k}{\tau^2}, \quad (52)$$

$$\tilde{g}^{ab}(\tau, \tilde{x}^i) = -\frac{\tilde{x}^a \tilde{x}^b}{\tau^2} + (\tilde{x}^1)^2 F^{ab}(x) \Big|_{x=\tilde{x}^1 \tau}, \quad (53)$$

where three independent functions of one variable $F^{ab}(x)$ are related to the metric functions in the privileged coordinate system by the following relations:

$$F^{ab}(\tilde{x}^1 \tau) = \left(\sum_{c,d=2}^3 [\Theta^{-1}]^{ac} g^{cd} [\Theta^{-1}]^{db} \right) \Big|_{x^0 \rightarrow \tilde{x}^1 \tau} = -\frac{d}{dx^0} [\Theta^{-1}]^{ab} \Big|_{x^0 \rightarrow \tilde{x}^1 \tau}, \quad (54)$$

where $[\Theta^{-1}]^{ab}$ is the inverse of the matrix $\Theta_{ab}(\tau\tilde{x}^1)$ from Equation (18):

$$F^{22}(\tilde{x}^1 \tau) = \frac{\Theta_{33}^2 \mathcal{A} - 2\Theta_{23}\Theta_{33}\mathcal{B} + \Theta_{23}^2 \mathcal{C}}{(\Theta_{23}^2 - \Theta_{22}\Theta_{33})^2} \Big|_{x^0 \rightarrow \tilde{x}^1 \tau}, \quad (55)$$

$$F^{23} = F^{32} = \frac{\Theta_{22}\Theta_{33}\mathcal{B} - \Theta_{23}(\Theta_{33}\mathcal{A} + \Theta_{22}\mathcal{C} - \Theta_{23}\mathcal{B})}{(\Theta_{23}^2 - \Theta_{22}\Theta_{33})^2} \Big|_{x^0 \rightarrow \tilde{x}^1 \tau}, \quad (56)$$

$$F^{33}(\tilde{x}^1 \tau) = \frac{\Theta_{23}^2 \mathcal{A} - 2\Theta_{22}\Theta_{23}\mathcal{B} + \Theta_{22}^2 \mathcal{C}}{(\Theta_{23}^2 - \Theta_{22}\Theta_{33})^2} \Big|_{x^0 \rightarrow \tilde{x}^1 \tau}. \quad (57)$$

Thus, for calculations with a gravitational wave (Equation (14)) in a synchronous (laboratory) frame of reference associated with a freely falling observer, we can initially proceed from the general form of the metric in Equations (52) and (53) without using the relations in Equations (55)–(57), defining three independent functions of one variable $F^{ab}(\tilde{x}^1 \tau)$ in the synchronous system reference directly from the field equations of the corresponding theory of gravity.

The geodesic deviation vector $\tilde{\eta}^\alpha(\tau)$ in a gravitational wave with the metrics in Equations (52) and (53) after transformation of the coordinates (Equations (40) and (41)) in the synchronous reference system will take the following form:

$$\tilde{\eta}^0 = 0, \quad (58)$$

$$\tilde{\eta}^1(\tau) = \rho_1 - \frac{\lambda_1 \Omega}{\tau}, \quad (59)$$

$$\tilde{\eta}^2(\tau) = \rho_2 - \frac{\lambda_2 \Omega}{\tau} + \frac{\lambda_1 \vartheta_2 \Theta_{33}}{\Theta_{22} \Theta_{33} - \Theta_{23}^2} + \frac{\lambda_1 \vartheta_3 \Theta_{23}}{\Theta_{23}^2 - \Theta_{22} \Theta_{33}}, \quad (60)$$

$$\tilde{\eta}^3(\tau) = \rho_3 - \frac{\lambda_3 \Omega}{\tau} + \frac{\lambda_1 \vartheta_2 \Theta_{23}}{\Theta_{23}^2 - \Theta_{22} \Theta_{33}} + \frac{\lambda_1 \vartheta_3 \Theta_{22}}{\Theta_{22} \Theta_{33} - \Theta_{23}^2}, \quad (61)$$

where the constant parameters λ_k and ϑ_k are determined by the initial (boundary) values of the velocities on neighboring geodesics and the constants ρ_k are determined by the initial (boundary) values of the relative deviation of the geodesics. The auxiliary constant Ω is defined by the relation in Equation (34). The functions $\Theta_{ab}(x^0)$ (Equation (18)) depend on one variable equal to the product of $\lambda_1 \tau$, where the variable τ is the time in the synchronous frame of reference.

Let us find the deviation rate $\tilde{V}^\alpha(\tau) = D\tilde{\eta}^\alpha/d\tau$ in the synchronous frame of reference using the coordinate transformations in Equations (40) and (41):

$$\tilde{V}^0 = 0, \quad (62)$$

$$\tilde{V}^1 = \rho_1/\tau, \quad (63)$$

$$\begin{aligned} \tilde{V}^2 = & \frac{1}{2\lambda_1 \tau (\Theta_{23}^2 - \Theta_{22} \Theta_{33}) (\mathcal{B}^2 - \mathcal{A} \mathcal{C})} \left[\lambda_1^3 \tau \vartheta_2 \left(\Theta_{33} (\mathcal{B} \mathcal{B}' - \mathcal{A}' \mathcal{C}) \right. \right. \\ & + \Theta_{23} (\mathcal{B}' \mathcal{C} - \mathcal{B} \mathcal{C}') \Big) \\ & + \lambda_1^3 \tau \vartheta_3 (\Theta_{23} (\mathcal{A} \mathcal{C}' - \mathcal{B} \mathcal{B}') + \Theta_{33} (\mathcal{A}' \mathcal{B} - \mathcal{A} \mathcal{B}')) \\ & + \Theta_{23}^2 \left(\mathcal{A} (\lambda_1 \tau \xi_2 \mathcal{C}' - 2\lambda_2 \rho_1 \mathcal{C}) - \lambda_1 \tau \mathcal{B} (\xi_2 \mathcal{B}' + \xi_3 \mathcal{C}') \right. \\ & + 2\lambda_2 \rho_1 \mathcal{B}^2 + \lambda_1 \tau \xi_3 \mathcal{B}' \mathcal{C} \Big) \\ & + \Theta_{33} \left(\Theta_{22} \left(\mathcal{C} (2\lambda_2 \rho_1 \mathcal{A} - \lambda_1 \tau \xi_2 \mathcal{A}') - 2\lambda_2 \rho_1 \mathcal{B}^2 + \lambda_1 \tau \xi_2 \mathcal{B} \mathcal{B}' \right) \right. \\ & + \lambda_1 \tau \left(\xi_3 \Theta_{33} \mathcal{A}' \mathcal{B} - \mathcal{A} (\xi_3 \Theta_{33} \mathcal{B}' - 2\xi_3 \mathcal{B} \mathcal{C} + 2\xi_2 \mathcal{B}^2) \right. \\ & + 2\xi_2 \mathcal{A}^2 \mathcal{C} - 2\xi_3 \mathcal{B}^3 \Big) \Big) \\ & + \lambda_1 \tau \Theta_{23} \left(\mathcal{C} (\xi_2 \Theta_{22} \mathcal{B}' - \xi_3 \Theta_{33} \mathcal{A}') \right. \\ & + \mathcal{A} \left(\Theta_{33} (\xi_3 \mathcal{C}' - \xi_2 \mathcal{B}') - 2\xi_3 \mathcal{C}^2 \right) \\ & \left. \left. + \xi_2 \mathcal{B} (\Theta_{33} \mathcal{A}' - 2\mathcal{A} \mathcal{C} - \Theta_{22} \mathcal{C}') + 2\xi_3 \mathcal{B}^2 \mathcal{C} + 2\xi_2 \mathcal{B}^3 \right) \right], \quad (64) \end{aligned}$$

$$\begin{aligned}
\tilde{V}^3 = & \frac{1}{2\lambda_1\tau(\Theta_{23}^2 - \Theta_{22}\Theta_{33})(\mathcal{B}^2 - \mathcal{A}\mathcal{C})} \left[\lambda_1^3\tau\theta_2 \left(\Theta_{23}(\mathcal{A}'\mathcal{C} - \mathcal{B}\mathcal{B}') \right. \right. \\
& + \Theta_{22}(\mathcal{B}\mathcal{C}' - \mathcal{B}'\mathcal{C}) \Big) \\
& + \lambda_1^3\tau\theta_3 (\Theta_{22}(\mathcal{B}\mathcal{B}' - \mathcal{A}\mathcal{C}') + \Theta_{23}(\mathcal{A}\mathcal{B}' - \mathcal{A}'\mathcal{B})) \\
& + \Theta_{22} \left(\Theta_{33} \left(\mathcal{A}(2\lambda_3\rho_1\mathcal{C} - \lambda_1\tau\zeta_3\mathcal{C}') - 2\lambda_3\rho_1\mathcal{B}^2 + \lambda_1\tau\zeta_3\mathcal{B}\mathcal{B}') \right. \right. \\
& + \lambda_1\tau \left(\mathcal{C}(2\zeta_3\mathcal{A}\mathcal{C} - \zeta_2\Theta_{22}\mathcal{B}') + \zeta_2\mathcal{B}(2\mathcal{A}\mathcal{C} + \Theta_{22}\mathcal{C}') \right. \\
& \left. \left. - 2\zeta_3\mathcal{B}^2\mathcal{C} - 2\zeta_2\mathcal{B}^3 \right) \right) \\
& + \Theta_{23}^2 \left(\mathcal{A}(\lambda_1\tau\zeta_2\mathcal{B}' - 2\lambda_3\rho_1\mathcal{C}) - \lambda_1\tau\mathcal{B}(\zeta_2\mathcal{A}' + \zeta_3\mathcal{B}') \right. \\
& + 2\lambda_3\rho_1\mathcal{B}^2 + \lambda_1\tau\zeta_3\mathcal{A}'\mathcal{C} \Big) \\
& + \lambda_1\tau\Theta_{23} \left(\Theta_{22}\mathcal{C}(\zeta_2\mathcal{A}' - \zeta_3\mathcal{B}') + \mathcal{B}(\zeta_3\Theta_{22}\mathcal{C}' - \zeta_3\Theta_{33}\mathcal{A}') \right. \\
& + \mathcal{A} \left(-\zeta_2\Theta_{22}\mathcal{C}' + \zeta_3\Theta_{33}\mathcal{B}' - 2\zeta_3\mathcal{B}\mathcal{C} + 2\zeta_2\mathcal{B}^2 \right) \\
& \left. \left. - 2\zeta_2\mathcal{A}^2\mathcal{C} + 2\zeta_3\mathcal{B}^3 \right) \right], \tag{65}
\end{aligned}$$

where the functions g^{ab} and Θ_{ab} (Equation (18)) depend on one variable equal to the product of $\lambda_1\tau$, the variable τ is the time in the synchronous frame of reference, and the auxiliary constants ζ_2 and ζ_3 are defined by the relations in Equation (33).

Due to the cumbersome nature of the expressions, the form of the tidal acceleration $\tilde{A}^\alpha(\tau) = D^2\tilde{\eta}^\alpha/d\tau^2$ in the synchronous reference system is given in Appendix B.

Note here that the tidal acceleration components $\tilde{A}^0$ and $\tilde{A}^1$ vanish in the used synchronous frame of reference:

$$\tilde{A}^0 = 0, \quad \tilde{A}^1 = 0, \tag{66}$$

We see that the tidal acceleration acts on particles only in the plane of the space variables x^2 and x^3 , while the gravitational wave propagates along the space variable x^1 .

Thus, for the considered models of gravitational waves, we obtained the trajectories of the test particles, the geodesic deviation vector and the deviation velocity vector and the form of the tidal accelerations both in privileged and in synchronous coordinate systems. We also found the structure of the gravitational wave metric in a synchronous frame of reference.

The characteristics of a gravitational wave obtained here completely determine its effect on the particles, fields, and the physical medium through which the wave passes. An example of application of the obtained results to the calculation of secondary physical effects for a gravitational wave in the type-7 Bianchi universe [4] is given to obtain the retarded electromagnetic potentials of the Lienard–Wiechert type when charges move due to the tidal acceleration in a gravitational wave.

5. Propagation of Radiation in a Gravitational Wave

The models of gravitational waves under consideration make it possible to analytically determine the light cone formulas for an observer on the basic geodesic.

To construct the light cone, we will use the eikonal equation, which determines the propagation of light rays in a gravitational wave:

$$g^{\alpha\beta} \frac{\partial \Psi}{\partial x^\alpha} \frac{\partial \Psi}{\partial x^\beta} = 0, \tag{67}$$

where Ψ is the eikonal function.

For the gravitational wave models we are considering, the eikonal function in the privileged coordinate system can be represented by

$$\Psi = \psi_0(x^0) + \sum_{k=1}^3 p_k x^k + \psi_c, \quad \psi_c = \text{const}, \quad (68)$$

where p_k are independent constant parameters of the propagation of a light beam, which is determined by the initial or boundary conditions.

Moreover, from the eikonal equation, we have the relation

$$\psi_0(x^0) = -\frac{1}{2p_1} \sum_{a,b=2}^3 p_a p_b \Theta_{ab}(x^0), \quad (69)$$

where the functions in the matrix $\Theta_{ab}(x^0)$ are given by the relations in Equation (18).

The trajectories of the light rays according to the Hamilton–Jacobi formalism are found from the equations

$$\frac{\partial \Psi}{\partial p_k} = q_k, \quad i, j, k = 1, 2, 3, \quad (70)$$

where q_k and p_k are independent constant parameters of the light beam trajectory determined by the initial or boundary conditions.

The trajectory of a light beam in a gravitational wave in the privileged coordinate system takes the form

$$x^1 = q_1 - \frac{1}{2p_1^2} \sum_{a,b=2}^3 p_a p_b \Theta_{ab}(x^0), \quad (71)$$

$$x^a = q_a + \frac{1}{p_1} \sum_{b=2}^3 p_b \Theta_{ab}(x^0) \quad a, b, c = 2, 3. \quad (72)$$

From the relations for light ray paths in Equations (71) and (72), in the privileged coordinate system, we obtain a linear relation for the coordinates, which is valid on the trajectories of the light beam in the considered gravitational wave:

$$2p_1 x^1 + \sum_{a=2}^3 p_a x^a = 2p_1 q_1 + \sum_{a=2}^3 p_a q_a = \text{const}. \quad (73)$$

Note also that in the considered gravitational wave along the trajectories of the light beam, the eikonal function is constant:

$$\Psi = \sum_{k=1}^3 p_k q_k + \psi_c. \quad (74)$$

Let us consider the transition to a synchronous reference system. The equations for the trajectory of a light beam in a gravitational wave (Equations (71) and (72)) in a synchronous frame of reference $\{\tilde{x}^k\}$ (Equations (40)–(43)) take the following form:

$$\frac{\tau}{\tilde{x}^1} - \sum_{a,b=2}^3 q_a q_b \left[\Theta^{-1} \right]^{ab} (\tau \tilde{x}^1) = \frac{2}{p_1} \sum_{k=1}^3 p_k q_k, \quad (75)$$

$$\tilde{x}^a = \tilde{x}^1 \left(\sum_{b=2}^3 q_b \left[\Theta^{-1} \right]^{ab} (\tau \tilde{x}^1) + \frac{p_a}{p_1} \right), \quad (76)$$

where $[\Theta^{-1}]^{ab}$ is the inverse of the matrix $\Theta_{ab}(\tau\tilde{x}^1)$ from Equation (18) and Equation (75) implicitly specifies the dependence of the coordinate $\tilde{x}^1$ at time τ on the ray trajectory. In Equation (75), the variable $\tilde{x}^1$ is the spatial coordinate of the light beam along the direction of propagation of the gravitational wave.

Then, the equation of the light cone of the observer on the basic geodesic in a gravitational wave will include all possible trajectories of light rays (Equations (75) and (76)) passing through the world point of the observer whose coordinates in the synchronous reference system have the form $\tilde{x}^\alpha = \{\tau, \lambda_1, \lambda_2, \lambda_3\}$, which gives three additional equations for determining the admissible parameters q_k or p_k .

If the charge emitted radiation at the point $\tilde{x}^\alpha = \{\tau', x^{1'}, x^{a'}\}$, then for the constant parameters q_k of the propagating radiation, we obtain

$$q_1 = \sigma_1 + \frac{m^2 x^{1'} \tau'}{2\lambda_1^2} + \frac{1}{2} \Theta^{ab}(x^{1'} \tau') \left(\frac{p_a}{p_1} + \frac{\lambda_a}{\lambda_1} \right) \left(\frac{p_b}{p_1} - \frac{\lambda_b}{\lambda_1} \right), \quad (77)$$

$$q_a = \sigma_a + \Theta^{ab}(x^{1'} \tau') \left(\frac{\lambda_b}{\lambda_1} - \frac{p_b}{p_1} \right), \quad (78)$$

where m is the mass of the charge, λ_k and σ_k are the constant parameters of the charge motion in the gravitational wave, and p_k are constant parameters of propagation of the radiation emitted by the charge.

The spatial distance l traveled by a light beam in a gravitational wave can be found from the integral along the beam path (Equations (75) and (76)):

$$l = \int \sqrt{-\tilde{g}_{ij}(\tau, \tilde{x}^k) d\tilde{x}^i d\tilde{x}^j} = \int_{\tau_1}^{\tau_2} d\tau \sqrt{-\tilde{g}_{ij}(\tau, \tilde{x}^k(\tau)) \frac{d\tilde{x}^i(\tau)}{d\tau} \frac{d\tilde{x}^j(\tau)}{d\tau}}, \quad (79)$$

where the metric functions $\tilde{g}^{ik}$ in the synchronous reference system are defined by the relations in Equations (52) and (53). The expressions for the derivatives of the coordinates on the trajectory of the light beam can be found from the relations for the trajectory using the following relation:

$$[\Theta^{-1}]^{ab'} = - \sum_{c,d} [\Theta^{-1}]^{ac} g^{cd} [\Theta^{-1}]^{db} = -F^{ab}, \quad (80)$$

Then, from Equations (75) and (76), we find the following form of the velocity components along the path of the light beam:

$$\frac{d\tilde{x}^1(\tau)}{d\tau} = \frac{\tilde{x}^1}{\tau} \left(1 - \frac{2}{1 + (\tilde{x}^1)^2 q_a q_b F^{ab}(\tau\tilde{x}^1)} \right), \quad (81)$$

$$\frac{d\tilde{x}^a(\tau)}{d\tau} = \frac{d\tilde{x}^1}{d\tau} \left(\frac{p_a}{p_1} + [\Theta^{-1}]^{ab} p_b - \tau\tilde{x}^1 F^{ab}(\tau\tilde{x}^1) q_b \right) - (\tilde{x}^1)^2 F^{ab}(\tau\tilde{x}^1) q_b, \quad (82)$$

Due to the invariance of the speed of light c , the following condition must be satisfied ($c = 1$):

$$l = \tau_2 - \tau_1. \quad (83)$$

The relations obtained are sufficient for calculating any secondary physical effects in a gravitational wave. In particular, the relations in Equations (79) and (83), together with the relations in Equations (81) and (82), allow us to set the retarded potentials of an electric charge moving under the influence of the tidal accelerations of a gravitational wave.

6. Discussion

The general results obtained in this paper for the deviation of geodesics in a gravitational wave can be used to calculate the physical effects of the influence of a gravitational wave on particles and fields. In particular, on the basis of the results obtained, one can calculate the radiation of charges moving with tidal acceleration in a gravitational wave and other secondary effects: the generation of density waves in a medium, the formation of local dense accumulations of matter, the birth of black holes in a gravitational wave [33,34], the influence of the primary gravitational waves on the microwave background of the universe and on the stochastic gravitational background, etc.

The characteristics of the secondary effects make it possible to estimate the parameters of the gravitational wave itself when direct detection of the wave is impossible. The dimensions of the “shoulders” of the satellite gravitational wave detectors planned in the future are limited by the distances between the Lagrange points in the solar system. Therefore, the lengths of gravitational waves, which are larger than these sizes, can conditionally be called large wavelengths, since their detection with the standard method is difficult.

The considered gravitational waves contain special cases of Bianchi universes of types 4, 5, and 7, on the basis of which, using the described approaches, exact models of primary gravitational waves have already been created [3–5]. For these models, it is possible to estimate the effect of radiation generated by a wave in the primordial plasma on the parameters of the cosmic microwave background [35], including the anisotropy observed by satellite telescopes.

The exact solutions for gravitational waves obtained on the basis of the approaches proposed in this work can also be used to debug more complex computer models and train artificial intelligence systems to analyze gravitational wave signals. The results obtained in this work can be used for LIGO, LISA, and the next generation of gravitational wave detectors.

7. Conclusions

For models of plane gravitational waves in Shapovalov spaces of type III, on the basis of the Hamilton–Jacobi formalism, exact solutions for the trajectories of test particles and the exact solution of geodesic deviation equations were obtained. The geodesic deviation velocities and tidal accelerations of a gravitational wave were found in a privileged coordinate system, where the spacetime metric depended on the wave variable.

Using the obtained results, an explicit analytical transformation into a synchronous (laboratory) frame of reference associated with a freely falling observer on the base geodesic was found. The synchronous reference system allows one to synchronize time at different points in space and separate time and spatial variables, which is important for the observer and makes the resulting models physically illustrative.

In the synchronous reference system, the form of the metric, the geodesic deviation vector, the deviation velocity, and the tidal accelerations in a gravitational wave are found.

The obtained exact solutions make it possible to calculate the secondary physical effects through the action of gravitational waves.

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Appendix A. Tidal Accelerations of a Gravitational Wave in a Privileged Coordinate System

We will use the prime at the top to denote the derivative of functions of one variable:

$$(g^{ab})' = \frac{d}{dx^0} g^{ab}(x^0). \quad (A1)$$

Let us recall that on the base geodesic, the variable x^0 is equal to $\lambda_1 \tau$, where τ is the proper time of the particle and λ_k represents the constants determined by the initial values of the particle velocity components on the base geodesic. The constants ϑ_k and ρ_k are determined by the initial values of the velocity and relative position of the particle on the adjacent geodesic. The constants ξ_2 and ξ_3 are auxiliary notations given by the relations in Equation (33).

Below is a general form of the tidal accelerations $A^\alpha(\tau) = D^2 \eta^\alpha / d\tau^2$ acting on test particles in a gravitational wave with the metric $g^{\alpha\beta}(x^0)$ (Equation (14)) and the deviation vector $\eta^\alpha(\tau)$ (Equations (29)–(32)) in the privileged coordinate system in a general case without taking into account the field equations:

$$A^0 = 0, \quad (A2)$$

$$\begin{aligned} A^1(\tau) = & \frac{\lambda_1 \vartheta_3}{4(\mathcal{B}^2 - \mathcal{A}\mathcal{C})^2} \left[2(\lambda_2 \mathcal{A}'' + \lambda_3 \mathcal{B}'') \mathcal{B}^3 \right. \\ & - \left(3\lambda_3 \mathcal{B}'^2 + 3\mathcal{A}'(2\lambda_2 \mathcal{B}' + \lambda_3 \mathcal{C}') + 2\mathcal{A}(\lambda_2 \mathcal{B}'' + \lambda_3 \mathcal{C}'') \right) \mathcal{B}^2 \\ & + \mathcal{B} \left(3\mathcal{A} \left(2\lambda_2 \mathcal{B}'^2 + 3\lambda_3 \mathcal{C}' \mathcal{B}' + \lambda_2 \mathcal{A}' \mathcal{C}' \right) \right. \\ & \left. \left. + \mathcal{C} \left(3\lambda_2 \mathcal{B}'^2 + 3\lambda_3 \mathcal{B}' \mathcal{A}' - 2\mathcal{A}(\lambda_2 \mathcal{A}'' + \lambda_3 \mathcal{B}'') \right) \right) \right. \\ & \left. + \mathcal{A} \left(\mathcal{C} \left(-3\lambda_3 \mathcal{B}'^2 - 3\lambda_2 \mathcal{A}' \mathcal{B}' + 2\mathcal{A}(\lambda_2 \mathcal{B}'' + \lambda_3 \mathcal{C}'') \right) \right. \right. \\ & \left. \left. - 3\mathcal{A}\mathcal{C}'(\lambda_2 \mathcal{B}' + \lambda_3 \mathcal{C}') \right) \right] \\ & - \frac{\lambda_1 \vartheta_2}{4(\mathcal{B}^2 - \mathcal{A}\mathcal{C})^2} \left[\left(3\lambda_2 \mathcal{B}'^2 + 3\lambda_3 \mathcal{B}' \mathcal{A}' - 2\mathcal{A}(\lambda_2 \mathcal{A}'' + \lambda_3 \mathcal{B}'') \right) \mathcal{C}^2 \right. \\ & + \mathcal{C} \left(2(\lambda_2 \mathcal{A}'' + \lambda_3 \mathcal{B}'') \mathcal{B}^2 + 2\mathcal{A}(\lambda_2 \mathcal{B}'' + \lambda_3 \mathcal{C}'') \right. \\ & \left. \left. - \left(6\lambda_3 \mathcal{B}'^2 + 3\mathcal{A}'(3\lambda_2 \mathcal{B}' + \lambda_3 \mathcal{C}') \right) \mathcal{B} + 3\mathcal{A}\mathcal{B}'(\lambda_2 \mathcal{B}' + \lambda_3 \mathcal{C}') \right) \right. \\ & \left. + \mathcal{B} \left(-2(\lambda_2 \mathcal{B}'' + \lambda_3 \mathcal{C}'') \mathcal{B}^2 - 3\mathcal{A}\mathcal{C}'(\lambda_2 \mathcal{B}' + \lambda_3 \mathcal{C}') \right) \right. \\ & \left. \left. + 3 \left(\lambda_2 \mathcal{B}'^2 + 2\lambda_3 \mathcal{C}' \mathcal{B}' + \lambda_2 \mathcal{A}' \mathcal{C}' \right) \mathcal{B} \right) \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4\lambda_1(\mathcal{B}^2 - \mathcal{A}\mathcal{C})^2} \left[\xi_3 \Theta_{33} \left(2(\lambda_2 \mathcal{A}'' + \lambda_3 \mathcal{B}'') \mathcal{B}^3 \right. \right. \\
& - \left(3\lambda_3 \mathcal{B}'^2 + 3\mathcal{A}'(2\lambda_2 \mathcal{B}' + \lambda_3 \mathcal{C}') + 2\mathcal{A}(\lambda_2 \mathcal{B}'' + \lambda_3 \mathcal{C}'') \right) \mathcal{B}^2 \\
& + \mathcal{B} \left(3\mathcal{A} \left(2\lambda_2 \mathcal{B}'^2 + 3\lambda_3 \mathcal{C}' \mathcal{B}' + \lambda_2 \mathcal{A}' \mathcal{C}' \right) \right. \\
& + \left. \left. \mathcal{C} \left(3\lambda_2 \mathcal{B}'^2 + 3\lambda_3 \mathcal{B}' \mathcal{A}' - 2\mathcal{A}(\lambda_2 \mathcal{A}'' + \lambda_3 \mathcal{B}'') \right) \right) \right. \\
& + \left. \mathcal{A} \left(\mathcal{C} \left(-3\lambda_3 \mathcal{B}'^2 - 3\lambda_2 \mathcal{A}' \mathcal{B}' + 2\mathcal{A}(\lambda_2 \mathcal{B}'' + \lambda_3 \mathcal{C}'') \right) \right. \right. \\
& - \left. \left. 3\mathcal{A}\mathcal{C}'(\lambda_2 \mathcal{B}' + \lambda_3 \mathcal{C}') \right) \right) \\
& - \xi_2 \Theta_{22} \left(\left(3\lambda_2 \mathcal{B}'^2 + 3\lambda_3 \mathcal{B}' \mathcal{A}' - 2\mathcal{A}(\lambda_2 \mathcal{A}'' + \lambda_3 \mathcal{B}'') \right) \mathcal{C}^2 \right. \\
& + \left. \mathcal{C} \left(2(\lambda_2 \mathcal{A}'' + \lambda_3 \mathcal{B}'') \mathcal{B}^2 \right. \right. \\
& - \left(6\lambda_3 \mathcal{B}'^2 + 3\mathcal{A}'(3\lambda_2 \mathcal{B}' + \lambda_3 \mathcal{C}') - 2\mathcal{A}(\lambda_2 \mathcal{B}'' + \lambda_3 \mathcal{C}'') \right) \mathcal{B} \\
& + \left. 3\mathcal{A}\mathcal{B}'(\lambda_2 \mathcal{B}' + \lambda_3 \mathcal{C}') \right) \\
& + \left. \mathcal{B} \left(-2(\lambda_2 \mathcal{B}'' + \lambda_3 \mathcal{C}'') \mathcal{B}^2 - 3\mathcal{A}\mathcal{C}'(\lambda_2 \mathcal{B}' + \lambda_3 \mathcal{C}') \right. \right. \\
& + \left. \left. 3 \left(\lambda_2 \mathcal{B}'^2 + 2\lambda_3 \mathcal{C}' \mathcal{B}' + \lambda_2 \mathcal{A}' \mathcal{C}' \right) \mathcal{B} \right) \right) \\
& + \Theta_{23} \left(2(\lambda_2 \xi_2 \mathcal{A}'' + (\lambda_3 \xi_2 + \lambda_2 \xi_3) \mathcal{B}'' + \lambda_3 \xi_3 \mathcal{C}'') \mathcal{B}^3 \right. \\
& - \mathcal{B}^2 \left[3(\lambda_3 \xi_2 + \lambda_2 \xi_3) \mathcal{B}'^2 + 6\lambda_3 \xi_3 \mathcal{C}' \mathcal{B}' + 2\xi_2 \mathcal{A}(\lambda_2 \mathcal{B}'' + \lambda_3 \mathcal{C}'') \right. \\
& + \left. \left. 3\mathcal{A}'(2\lambda_2 \xi_2 \mathcal{B}' + (\lambda_3 \xi_2 + \lambda_2 \xi_3) \mathcal{C}') + 2\xi_3 \mathcal{C}(\lambda_2 \mathcal{A}'' + \lambda_3 \mathcal{B}'') \right] \right. \\
& + \left. \mathcal{B} \left(3\mathcal{A} \left(2\lambda_2 \xi_2 \mathcal{B}'^2 + (3\lambda_3 \xi_2 + \lambda_2 \xi_3) \mathcal{C}' \mathcal{B}' + \mathcal{C}'(\lambda_2 \xi_2 \mathcal{A}' + \lambda_3 \xi_3 \mathcal{C}') \right) \right. \right. \\
& + \left. \left. \mathcal{C} \left(3\lambda_2 \xi_2 \mathcal{B}'^2 + 3((\lambda_3 \xi_2 + 3\lambda_2 \xi_3) \mathcal{B}' + \lambda_3 \xi_3 \mathcal{C}') \mathcal{A}' + 6\lambda_3 \xi_3 \mathcal{B}'^2 \right. \right. \right. \\
& - \left. \left. 2\mathcal{A}(\lambda_2 \xi_2 \mathcal{A}'' + (\lambda_3 \xi_2 + \lambda_2 \xi_3) \mathcal{B}'' + \lambda_3 \xi_3 \mathcal{C}'') \right) \right) \\
& - \left. 3\xi_2 \mathcal{A}^2 \mathcal{C}'(\lambda_2 \mathcal{B}' + \lambda_3 \mathcal{C}') \right. \\
& + \left. \xi_3 \mathcal{C}^2 \left(-3\lambda_2 \mathcal{B}'^2 - 3\lambda_3 \mathcal{B}' \mathcal{A}' + 2\mathcal{A}(\lambda_2 \mathcal{A}'' + \lambda_3 \mathcal{B}'') \right) \right. \\
& + \left. \mathcal{A}\mathcal{C} \left(-3(\lambda_3 \xi_2 + \lambda_2 \xi_3) \mathcal{B}'^2 - 3\lambda_2 \xi_2 \mathcal{A}' \mathcal{B}' \right. \right. \\
& - \left. \left. 3\lambda_3 \xi_3 \mathcal{C}' \mathcal{B}' + 2\xi_2 \mathcal{A}(\lambda_2 \mathcal{B}'' + \lambda_3 \mathcal{C}'') \right) \right) \Bigg], \tag{A3}
\end{aligned}$$

$$\begin{aligned}
A^2(\tau) = & \frac{\lambda_1^2 \vartheta_3}{4(\mathcal{B}^2 - \mathcal{A}\mathcal{C})^2} \left[-2\mathcal{A}''\mathcal{B}^3 + 2(3\mathcal{A}'\mathcal{B}' + \mathcal{A}\mathcal{B}'')\mathcal{B}^2 \right. \\
& - \left(3\mathcal{A}(2\mathcal{B}'^2 + \mathcal{A}'\mathcal{C}') + \mathcal{C}(3\mathcal{B}'^2 - 2\mathcal{A}\mathcal{A}'') \right) \mathcal{B} \\
& \left. + \mathcal{A}(3\mathcal{A}\mathcal{B}'\mathcal{C}' + \mathcal{C}(3\mathcal{A}'\mathcal{B}' - 2\mathcal{A}\mathcal{B}'')) \right] \\
& + \frac{\lambda_1^2 \vartheta_2}{4(\mathcal{B}^2 - \mathcal{A}\mathcal{C})^2} \left[(3\mathcal{B}'^2 - 2\mathcal{A}\mathcal{A}'')\mathcal{C}^2 \right. \\
& + (2\mathcal{A}''\mathcal{B}^2 + (2\mathcal{A}\mathcal{B}'' - 9\mathcal{A}'\mathcal{B}')\mathcal{B} + 3\mathcal{A}\mathcal{B}'^2)\mathcal{C} \\
& \left. + \mathcal{B}(-2\mathcal{B}''\mathcal{B}^2 + 3(\mathcal{B}'^2 + \mathcal{A}'\mathcal{C}')\mathcal{B} - 3\mathcal{A}\mathcal{B}'\mathcal{C}') \right] \\
& + \frac{1}{4(\mathcal{B}^2 - \mathcal{A}\mathcal{C})^2} \left[\xi_3 \Theta_{33} \left(-2\mathcal{A}''\mathcal{B}^3 + 2(3\mathcal{A}'\mathcal{B}' + \mathcal{A}\mathcal{B}'')\mathcal{B}^2 \right. \right. \\
& - \left(3\mathcal{A}(2\mathcal{B}'^2 + \mathcal{A}'\mathcal{C}') + \mathcal{C}(3\mathcal{B}'^2 - 2\mathcal{A}\mathcal{A}'') \right) \mathcal{B} \\
& \left. \left. + \mathcal{A}(3\mathcal{A}\mathcal{B}'\mathcal{C}' + \mathcal{C}(3\mathcal{A}'\mathcal{B}' - 2\mathcal{A}\mathcal{B}'')) \right) \right. \\
& + \xi_2 \Theta_{22} \left((3\mathcal{B}'^2 - 2\mathcal{A}\mathcal{A}'')\mathcal{C}^2 \right. \\
& + (2\mathcal{A}''\mathcal{B}^2 + (2\mathcal{A}\mathcal{B}'' - 9\mathcal{A}'\mathcal{B}')\mathcal{B} + 3\mathcal{A}\mathcal{B}'^2)\mathcal{C} \\
& \left. \left. + \mathcal{B}(-2\mathcal{B}''\mathcal{B}^2 + 3(\mathcal{B}'^2 + \mathcal{A}'\mathcal{C}')\mathcal{B} - 3\mathcal{A}\mathcal{B}'\mathcal{C}') \right) \right. \\
& + \Theta_{23} \left(-2(\xi_2\mathcal{A}'' + \xi_3\mathcal{B}'')\mathcal{B}^3 \right. \\
& + (3\xi_3\mathcal{B}'^2 + 3\mathcal{A}'(2\xi_2\mathcal{B}' + \xi_3\mathcal{C}') + 2\xi_3\mathcal{C}\mathcal{A}'' + 2\xi_2\mathcal{A}\mathcal{B}'')\mathcal{B}^2 \\
& - \mathcal{B} \left[3\mathcal{A}(2\xi_2\mathcal{B}'^2 + \xi_3\mathcal{C}'\mathcal{B}' + \xi_2\mathcal{A}'\mathcal{C}') \right. \\
& \left. \left. + \mathcal{C}(3\xi_2\mathcal{B}'^2 + 9\xi_3\mathcal{B}'\mathcal{A}' - 2\mathcal{A}(\xi_2\mathcal{A}'' + \xi_3\mathcal{B}'')) \right) \right. \\
& + 3\xi_2\mathcal{A}^2\mathcal{B}'\mathcal{C}' + \xi_3\mathcal{C}^2(3\mathcal{B}'^2 - 2\mathcal{A}\mathcal{A}'') \\
& \left. \left. + \mathcal{A}\mathcal{C}(3\xi_3\mathcal{B}'^2 + 3\xi_2\mathcal{A}'\mathcal{B}' - 2\xi_2\mathcal{A}\mathcal{B}'') \right) \right] \Bigg], \tag{A4}
\end{aligned}$$

$$\begin{aligned}
A^3(\tau) = & \frac{\lambda_1^2 \vartheta_3}{4(\mathcal{B}^2 - \mathcal{A}\mathcal{C})^2} \left[-2\mathcal{B}''\mathcal{B}^3 + (3\mathcal{B}'^2 + 3\mathcal{A}'\mathcal{C}' + 2\mathcal{A}\mathcal{C}'')\mathcal{B}^2 \right. \\
& + (\mathcal{C}(2\mathcal{A}\mathcal{B}'' - 3\mathcal{A}'\mathcal{B}') - 9\mathcal{A}\mathcal{B}'\mathcal{C}')\mathcal{B} \\
& \left. + \mathcal{A}(3\mathcal{A}\mathcal{C}'^2 + \mathcal{C}(3\mathcal{B}'^2 - 2\mathcal{A}\mathcal{C}'')) \right] \\
& + \frac{\lambda_1^2 \vartheta_2}{4(\mathcal{B}^2 - \mathcal{A}\mathcal{C})^2} \left[(3\mathcal{A}'\mathcal{B}' - 2\mathcal{A}\mathcal{B}'')\mathcal{C}^2 \right. \\
& + (2\mathcal{B}''\mathcal{B}^2 + (-6\mathcal{B}'^2 - 3\mathcal{A}'\mathcal{C}' + 2\mathcal{A}\mathcal{C}'')\mathcal{B} + 3\mathcal{A}\mathcal{B}'\mathcal{C}')\mathcal{C} \\
& \left. + \mathcal{B}(-2\mathcal{C}''\mathcal{B}^2 + 6\mathcal{B}'\mathcal{C}'\mathcal{B} - 3\mathcal{A}\mathcal{C}'^2) \right] \\
& + \frac{1}{4(\mathcal{B}^2 - \mathcal{A}\mathcal{C})^2} \left[\xi_3 \Theta_{33} \left(-2\mathcal{B}''\mathcal{B}^3 + (3\mathcal{B}'^2 + 3\mathcal{A}'\mathcal{C}' + 2\mathcal{A}\mathcal{C}'')\mathcal{B}^2 \right. \right. \\
& + (\mathcal{C}(2\mathcal{A}\mathcal{B}'' - 3\mathcal{A}'\mathcal{B}') - 9\mathcal{A}\mathcal{B}'\mathcal{C}')\mathcal{B} \\
& + \mathcal{A}(3\mathcal{A}\mathcal{C}'^2 + \mathcal{C}(3\mathcal{B}'^2 - 2\mathcal{A}\mathcal{C}'')) \left. \right) \\
& + \xi_2 \Theta_{22} \left((3\mathcal{A}'\mathcal{B}' - 2\mathcal{A}\mathcal{B}'')\mathcal{C}^2 \right. \\
& + (2\mathcal{B}''\mathcal{B}^2 + (-6\mathcal{B}'^2 - 3\mathcal{A}'\mathcal{C}' + 2\mathcal{A}\mathcal{C}'')\mathcal{B} + 3\mathcal{A}\mathcal{B}'\mathcal{C}')\mathcal{C} \\
& + \mathcal{B}(-2\mathcal{C}''\mathcal{B}^2 + 6\mathcal{B}'\mathcal{C}'\mathcal{B} - 3\mathcal{A}\mathcal{C}'^2) \left. \right) \\
& + \Theta_{23} \left(-2\mathcal{B}^3(\xi_2\mathcal{B}'' + \xi_3\mathcal{C}'') \right. \\
& + (3\xi_2\mathcal{B}'^2 + 6\xi_3\mathcal{C}'\mathcal{B}' + 3\xi_2\mathcal{A}'\mathcal{C}' + 2\xi_3\mathcal{C}\mathcal{B}'' + 2\xi_2\mathcal{A}\mathcal{C}'')\mathcal{B}^2 \\
& - (3\mathcal{A}\mathcal{C}'(3\xi_2\mathcal{B}' + \xi_3\mathcal{C}')) \\
& + \mathcal{C}(6\xi_3\mathcal{B}'^2 + 3\mathcal{A}'(\xi_2\mathcal{B}' + \xi_3\mathcal{C}') - 2\mathcal{A}(\xi_2\mathcal{B}'' + \xi_3\mathcal{C}'')) \left. \right) \mathcal{B} \\
& + 3\xi_2\mathcal{A}^2\mathcal{C}'^2 + \xi_3\mathcal{C}^2(3\mathcal{A}'\mathcal{B}' - 2\mathcal{A}\mathcal{B}'') \\
& \left. + \mathcal{A}\mathcal{C}(3\xi_2\mathcal{B}'^2 + 3\xi_3\mathcal{C}'\mathcal{B}' - 2\xi_2\mathcal{A}\mathcal{C}'') \right) \left. \right]. \tag{A5}
\end{aligned}$$

Here, we use the notation for the metric components $g^{\alpha\beta}$ and their integrals introduced earlier:

$$\mathcal{A} = g^{22}(x^0), \quad \mathcal{B} = g^{23}(x^0), \quad \mathcal{C} = g^{33}(x^0), \quad \Theta_{ab}(x^0) = \int g^{ab}(x^0) dx^0.$$

The quantities $\lambda_1, \lambda_2, \lambda_3, \vartheta_1, \vartheta_2, \vartheta_3, \xi_2$, and ξ_3 are constant parameters determined by the initial conditions.

Appendix B. Tidal Accelerations of a Gravitational Wave in a Synchronous Frame of Reference

Below is the form of the tidal accelerations $\tilde{A}^\alpha(\tau) = D^2\tilde{\eta}^\alpha/d\tau^2$ acting on test particles in a gravitational wave with the metric $\tilde{g}^{\alpha\beta}(\tau, \tilde{x}^k)$ (Equations (52) and (53)) and deviation vector $\tilde{\eta}^\alpha(\tau)$, taking into account the Einstein vacuum in Equation (16) in the synchronous reference system:

$$\tilde{A}^0 = 0, \quad (\text{A6})$$

$$\tilde{A}^1 = 0, \quad (\text{A7})$$

$$\begin{aligned} \tilde{A}^2(\tau) = & \frac{\lambda_1^3 \vartheta_3}{4(\Theta_{23}^2 - \Theta_{22}\Theta_{33})(\mathcal{B}^2 - \mathcal{A}\mathcal{C})^2} \left[\Theta_{33} \left(2\mathcal{A}''\mathcal{B}^3 \right. \right. \\ & + \left(3\mathcal{A}(2\mathcal{B}'^2 + \mathcal{A}'\mathcal{C}') + \mathcal{C}(3\mathcal{B}'^2 - 2\mathcal{A}\mathcal{A}'') \right) \mathcal{B} \\ & + \mathcal{A}(\mathcal{C}(2\mathcal{A}\mathcal{B}'' - 3\mathcal{A}'\mathcal{B}') - 3\mathcal{A}\mathcal{B}'\mathcal{C}') \\ & \left. \left. - 2(3\mathcal{A}'\mathcal{B}' + \mathcal{A}\mathcal{B}'')\mathcal{B}^2 \right) \right. \\ & + \Theta_{23} \left(-2\mathcal{B}''\mathcal{B}^3 + (3\mathcal{B}'^2 + 3\mathcal{A}'\mathcal{C}' + 2\mathcal{A}\mathcal{C}'')\mathcal{B}^2 \right. \\ & + (\mathcal{C}(2\mathcal{A}\mathcal{B}'' - 3\mathcal{A}'\mathcal{B}') - 9\mathcal{A}\mathcal{B}'\mathcal{C}')\mathcal{B} \\ & \left. \left. + \mathcal{A}(3\mathcal{A}\mathcal{C}'^2 + \mathcal{C}(3\mathcal{B}'^2 - 2\mathcal{A}\mathcal{C}'')) \right) \right] \\ & - \frac{\lambda_1^3 \vartheta_2}{4(\Theta_{23}^2 - \Theta_{22}\Theta_{33})(\mathcal{B}^2 - \mathcal{A}\mathcal{C})^2} \left[\Theta_{33} \left((3\mathcal{B}'^2 - 2\mathcal{A}\mathcal{A}'')\mathcal{C}^2 \right. \right. \\ & + (2\mathcal{A}''\mathcal{B}^2 + (2\mathcal{A}\mathcal{B}'' - 9\mathcal{A}'\mathcal{B}')\mathcal{B} + 3\mathcal{A}\mathcal{B}'^2)\mathcal{C} \\ & + \mathcal{B}(-2\mathcal{B}''\mathcal{B}^2 + 3(\mathcal{B}'^2 + \mathcal{A}'\mathcal{C}')\mathcal{B} - 3\mathcal{A}\mathcal{B}'\mathcal{C}') \Big) \\ & + \Theta_{23} \left((2\mathcal{A}\mathcal{B}'' - 3\mathcal{A}'\mathcal{B}')\mathcal{C}^2 \right. \\ & + \mathcal{C}(\mathcal{B}(6\mathcal{B}'^2 + 3\mathcal{A}'\mathcal{C}' - 2\mathcal{A}\mathcal{C}'')) \\ & \left. \left. - 2\mathcal{B}''\mathcal{B}^2 - 3\mathcal{A}\mathcal{B}'\mathcal{C}' \right) \right. \\ & \left. \left. + \mathcal{B}(2\mathcal{C}''\mathcal{B}^2 - 6\mathcal{B}'\mathcal{C}'\mathcal{B} + 3\mathcal{A}\mathcal{C}'^2) \right) \right] \\ & + \frac{\lambda_1 \xi_3}{4(\Theta_{23}^2 - \Theta_{22}\Theta_{33})(\mathcal{B}^2 - \mathcal{A}\mathcal{C})^2} \left[\Theta_{23}^2 \left((3\mathcal{A}'\mathcal{B}' - 2\mathcal{A}\mathcal{B}'')\mathcal{C}^2 \right. \right. \\ & + (2\mathcal{B}''\mathcal{B}^2 + (-6\mathcal{B}'^2 - 3\mathcal{A}'\mathcal{C}' + 2\mathcal{A}\mathcal{C}'')\mathcal{B} + 3\mathcal{A}\mathcal{B}'\mathcal{C}')\mathcal{C} \\ & + \mathcal{B}(-2\mathcal{C}''\mathcal{B}^2 + 6\mathcal{B}'\mathcal{C}'\mathcal{B} - 3\mathcal{A}\mathcal{C}'^2) \Big) \\ & + \Theta_{23}\Theta_{33} (2\mathcal{A}\mathcal{A}'' - 3\mathcal{A}'\mathcal{B} + 3\mathcal{A}\mathcal{C}'^2) \\ & + \Theta_{33}^2 (2\mathcal{A}''\mathcal{B}^3 - 2(3\mathcal{A}'\mathcal{B}' + \mathcal{A}\mathcal{B}'')\mathcal{B}^2 \\ & + (3\mathcal{A}(2\mathcal{B}'^2 + \mathcal{A}'\mathcal{C}') + \mathcal{C}(3\mathcal{B}'^2 - 2\mathcal{A}\mathcal{A}''))\mathcal{B} \\ & \left. \left. + \mathcal{A}(\mathcal{C}(2\mathcal{A}\mathcal{B}'' - 3\mathcal{A}'\mathcal{B}') - 3\mathcal{A}\mathcal{B}'\mathcal{C}') \right) \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{\lambda_1 \xi_2}{4(\Theta_{23}^2 - \Theta_{22}\Theta_{33})(B^2 - AC)^2} \left[\Theta_{23}^2 \left(-2B''B^3 \right. \right. \\
& + \left(3B'^2 + 3A'C' + 2AC'' \right) B^2 \\
& + \left(C(2AB'' - 3A'B') - 9AB'C' \right) B \\
& + A \left(3AC'^2 + C \left(3B'^2 - 2AC'' \right) \right) \Big) \\
& + \Theta_{23} \left(\Theta_{33} \left(2A''B^3 - 2(3A'B' + AB'')B^2 \right. \right. \\
& + \left(3A(2B'^2 + A'C') + C(3B'^2 - 2AA'') \right) B \\
& + A(C(2AB'' - 3A'B') - 3AB'C') \Big) \\
& + \Theta_{22} \left((3A'B' - 2AB'')C^2 \right. \\
& + C(2B''B^2 + (3AB'C' - 6B'^2 - 3A'C' + 2AC'')B) \\
& + B(-2C''B^2 + 6B'C'B - 3AC'^2) \Big) \Big) \\
& - \Theta_{22}\Theta_{33} \left((3B'^2 - 2AA'')C^2 \right. \\
& + (2A''B^2 + (2AB'' - 9A'B')B + 3AB'^2)C \\
& + B(-2B''B^2 + 3(B'^2 + A'C')B - 3AB'C') \Big) \Big], \tag{A8} \\
\tilde{A}^3(\tau) = & - \frac{\lambda_1^3 \vartheta_3}{4(\Theta_{23}^2 - \Theta_{22}\Theta_{33})(B^2 - AC)^2} \left[\Theta_{23} \left(2A''B^3 \right. \right. \\
& - 2(3A'B' + AB'')B^2 \\
& + (3A(2B'^2 + A'C') + C(3B'^2 - 2AA''))B \\
& + A(C(2AB'' - 3A'B') - 3AB'C') \Big) \\
& + \Theta_{22} \left(-2B''B^3 + (3B'^2 + 3A'C' + 2AC'')B^2 \right. \\
& + (C(2AB'' - 3A'B') - 9AB'C')B \\
& + A(3AC'^2 + C(3B'^2 - 2AC'')) \Big) \Big]
\end{aligned}$$

$$\begin{aligned}
& + \frac{\lambda_1^3 \vartheta_2}{4(\Theta_{23}^2 - \Theta_{22}\Theta_{33})(\mathcal{B}^2 - \mathcal{A}\mathcal{C})^2} \left[\Theta_{23} \left((3\mathcal{B}'^2 - 2\mathcal{A}\mathcal{A}'')\mathcal{C}^2 \right. \right. \\
& + (2\mathcal{A}''\mathcal{B}^2 + (2\mathcal{A}\mathcal{B}'' - 9\mathcal{A}'\mathcal{B}')\mathcal{B} + 3\mathcal{A}\mathcal{B}'^2)\mathcal{C} \\
& + \mathcal{B}(-2\mathcal{B}''\mathcal{B}^2 + 3(\mathcal{B}'^2 + \mathcal{A}'\mathcal{C}')\mathcal{B} - 3\mathcal{A}\mathcal{B}'\mathcal{C}')) \\
& + \Theta_{22} \left((2\mathcal{A}\mathcal{B}'' - 3\mathcal{A}'\mathcal{B}')\mathcal{C}^2 \right. \\
& + (-2\mathcal{B}''\mathcal{B}^2 + (6\mathcal{B}'^2 + 3\mathcal{A}'\mathcal{C}' - 2\mathcal{A}\mathcal{C}'')\mathcal{B} - 3\mathcal{A}\mathcal{B}'\mathcal{C}')\mathcal{C} \\
& + \mathcal{B}(2\mathcal{C}''\mathcal{B}^2 - 6\mathcal{B}'\mathcal{C}'\mathcal{B} + 3\mathcal{A}\mathcal{C}'^2)) \left. \right] \\
& + \frac{\lambda_1 \xi_2}{4(\Theta_{23}^2 - \Theta_{22}\Theta_{33})(\mathcal{B}^2 - \mathcal{A}\mathcal{C})^2} \left[\Theta_{22}^2 \left((2\mathcal{A}\mathcal{B}'' - 3\mathcal{A}'\mathcal{B}')\mathcal{C}^2 \right. \right. \\
& + (-2\mathcal{B}''\mathcal{B}^2 + (6\mathcal{B}'^2 + 3\mathcal{A}'\mathcal{C}' - 2\mathcal{A}\mathcal{C}'')\mathcal{B} - 3\mathcal{A}\mathcal{B}'\mathcal{C}')\mathcal{C} \\
& + \mathcal{B}(2\mathcal{C}''\mathcal{B}^2 - 6\mathcal{B}'\mathcal{C}'\mathcal{B} + 3\mathcal{A}\mathcal{C}'^2)) \\
& + \Theta_{22}\Theta_{23} \left((3\mathcal{B}'^2 - 2\mathcal{A}\mathcal{A}'')\mathcal{C}^2 \right. \\
& + 2(\mathcal{C}''\mathcal{A}^2 - 3\mathcal{B}\mathcal{A}'\mathcal{B}' + \mathcal{B}^2\mathcal{A}'')\mathcal{C} \\
& + \mathcal{A}(-2\mathcal{C}''\mathcal{B}^2 + 6\mathcal{B}'\mathcal{C}'\mathcal{B} - 3\mathcal{A}\mathcal{C}'^2)) \\
& + \Theta_{23}^2 \left(-2\mathcal{A}''\mathcal{B}^3 + 2(3\mathcal{A}'\mathcal{B}' + \mathcal{A}\mathcal{B}'')\mathcal{B}^2 \right. \\
& - \mathcal{B}(3\mathcal{A}(2\mathcal{B}'^2 + \mathcal{A}'\mathcal{C}') + \mathcal{C}(3\mathcal{B}'^2 - 2\mathcal{A}\mathcal{A}'')) \\
& + \mathcal{A}(3\mathcal{A}\mathcal{B}'\mathcal{C}' + \mathcal{C}(3\mathcal{A}'\mathcal{B}' - 2\mathcal{A}\mathcal{B}'')) \left. \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{\lambda_1 \xi_3}{4(\Theta_{23}^2 - \Theta_{22}\Theta_{33})(\mathcal{B}^2 - \mathcal{A}\mathcal{C})^2} \left[\Theta_{23}^2 \left((3\mathcal{B}'^2 - 2\mathcal{A}\mathcal{A}'')\mathcal{C}^2 \right. \right. \\
& + (2\mathcal{A}''\mathcal{B}^2 + (2\mathcal{A}\mathcal{B}'' - 9\mathcal{A}'\mathcal{B}')\mathcal{B} + 3\mathcal{A}\mathcal{B}'^2)\mathcal{C} \\
& + \mathcal{B}(-2\mathcal{B}''\mathcal{B}^2 + 3(\mathcal{B}'^2 + \mathcal{A}'\mathcal{C}')\mathcal{B} - 3\mathcal{A}\mathcal{B}'\mathcal{C}') \Big) \\
& + \Theta_{23}\Theta_{33} \left(-2\mathcal{A}''\mathcal{B}^3 + 2(3\mathcal{A}'\mathcal{B}' + \mathcal{A}\mathcal{B}'')\mathcal{B}^2 \right. \\
& - (3\mathcal{A}(2\mathcal{B}'^2 + \mathcal{A}'\mathcal{C}') + \mathcal{C}(3\mathcal{B}'^2 - 2\mathcal{A}\mathcal{A}''))\mathcal{B} \\
& + \mathcal{A}(3\mathcal{A}\mathcal{B}'\mathcal{C}' + \mathcal{C}(3\mathcal{A}'\mathcal{B}' - 2\mathcal{A}\mathcal{B}'')) \Big) \\
& + \Theta_{22}\Theta_{23} \left((2\mathcal{A}\mathcal{B}'' - 3\mathcal{A}'\mathcal{B}')\mathcal{C}^2 \right. \\
& - (2\mathcal{B}''\mathcal{B}^2 - (6\mathcal{B}'^2 + 3\mathcal{A}'\mathcal{C}' - 2\mathcal{A}\mathcal{C}'')\mathcal{B} + 3\mathcal{A}\mathcal{B}'\mathcal{C}')\mathcal{C} \\
& + \mathcal{B}(2\mathcal{C}''\mathcal{B}^2 - 6\mathcal{B}'\mathcal{C}'\mathcal{B} + 3\mathcal{A}\mathcal{C}'^2) \Big) \\
& + \Theta_{22}\Theta_{33} \left(2\mathcal{B}''\mathcal{B}^3 - (3\mathcal{B}'^2 + 3\mathcal{A}'\mathcal{C}' + 2\mathcal{A}\mathcal{C}'')\mathcal{B}^2 \right. \\
& + (9\mathcal{A}\mathcal{B}'\mathcal{C}' + \mathcal{C}(3\mathcal{A}'\mathcal{B}' - 2\mathcal{A}\mathcal{B}''))\mathcal{B} \\
& \left. \left. + \mathcal{A}(\mathcal{C}(2\mathcal{A}\mathcal{C}'' - 3\mathcal{B}'^2) - 3\mathcal{A}\mathcal{C}'^2) \right) \right]. \tag{A9}
\end{aligned}$$

Here the one-variable functions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and Θ_{ab} depend on the base geodesic line on the product $\lambda_1 \tau$, where τ is the time in the synchronous reference frame. The quantities λ_1 , λ_2 , λ_3 , ϑ_1 , ϑ_2 , ϑ_3 , ξ_2 , and ξ_3 are constant parameters determined by the initial conditions.

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