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Evaluation of the Poly-Jindalrae and Poly-Gaenari Polynomials in Terms of Degenerate Functions

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Abstract: The fundamental aim of this paper is to introduce the concept of poly-Jindalrae and poly-Gaenari numbers and polynomials within the context of degenerate functions. Furthermore, we give explicit expressions for these polynomial sequences and establish combinatorial identities that incorporate these polynomials. This includes the derivation of Dobinski-like formulas, recurrence relations, and other related aspects. Additionally, we present novel explicit expressions and identities of unipoly polynomials that are closely linked to some special numbers and polynomials.

Keywords: modified degenerate polyexponential functions; degenerate poly-Jindalrae polynomials; degenerate poly-Gaenari polynomials degenerate unipoly functions

MSC: 11B73; 11B83; 05A19

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1. Introduction and Definitions

Focusing on the theory of special polynomials, several mathematicians have extensively studied the works and various generalizations of Bernoulli polynomials, Euler polynomials, Genocchi polynomials, and Cauchy polynomials (see [1–6] for more information). The importance of generalization of the special polynomials encompass a range of specialized polynomial families, offering a unified methodology for addressing a wide array of mathematical questions. They prove valuable not only in theoretical realms, but also in practical applications, enhancing our grasp of fundamental mathematical concepts and furnishing sophisticated resolutions to complex problems in disciplines like calculus, number theory, and physics. Moreover, recent years have witnessed a surge in research on various degenerate versions of special polynomials and numbers, reigniting the interest of mathematicians in diverse categories of special polynomials and numbers [2,7–10]. Notably, Kim and Kim [11] as well as Dolgy and Khan [12] revisited the polyexponential functions in connection with polylogarithm functions, building upon the foundational work initiated by Hardy [13].

The objective of this paper is to investigate the poly-Jindalrae and poly-Gaenari polynomials and numbers in relation to the Jindalrae–Stirling numbers of the first and second kinds, and to derive arithmetic and combinatorial findings concerning these polynomials and numbers. Initially, we define the Jindalrae–Stirling numbers of the first and second kinds as extensions of the degenerate Stirling numbers, and establish several polynomial relationships involving these special numbers. Subsequently, we introduce the Jindalrae and poly-Gaenari numbers and polynomials, providing explicit expressions and identities associated with them.

Let n be nonnegative integer. The Stirling numbers of the first kind can be characterized (see [14,15]) as

$$(x)_n = \sum_{l=0}^{n} S_1(l,n)x^l \ , \tag{1}$$

where $(x)_0 = 1, (x)_n = x(x-1)\cdots(x-n+1), (n \geq 1)$.

The Stirling numbers of the second kind can also be characterized (see [4,16]) by

$$x^n = \sum_{l=0}^{n} S_2(l,n)(x)_l. \tag{2}$$

By (1) and (2), we obtain

$$\frac{1}{k!}(e^t-1)^k = \sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!} \ \ (n \geq 0) \tag{3}$$

and

$$\frac{1}{k!}(\log(1+t))^k = \sum_{l=n}^{\infty} S_1(l,n)\frac{t^l}{l!} \ \ (n \geq 0). \tag{4}$$

The generating function of the Bell polynomials are given (see [5]) by

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}. \tag{5}$$

When $x = 1$, $Bel_n = Bel_n(1)$ are called the Bell numbers.

The degenerate exponential function is defined (see [4,5,9,14–18]) by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} \ \ e_\lambda(t) = e_\lambda^1(t) = (1+\lambda t)^{\frac{1}{\lambda}} \ \ \lambda \in \mathbb{R}. \tag{6}$$

Here we note that

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda}\frac{t^n}{n!}, \tag{7}$$

where $(x)_{0,\lambda} = 1, (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), (n \geq 1)$.

In [1,2], Carlitz introduced the Euler polynomials in their degenerate form, which can be represented as

$$\frac{2}{e_\lambda(t)+1}e_\lambda^x(t) = \sum_{n=0}^{\infty} E_{n,\lambda}(x)\frac{t^n}{n!}. \tag{8}$$

On setting $x = 0$, $E_{n,\lambda} = E_{n,\lambda}(0)$ are called degenerate Euler numbers.

The degenerate Genocchi polynomials are defined (see [4,14]) by

$$\frac{2t}{e_\lambda(t)+1}e_\lambda^x(t) = \sum_{n=0}^{\infty} G_{n,\lambda}(x)\frac{t^n}{n!}. \tag{9}$$

where $x = 0$, $G_{n,\lambda} = G_{n,\lambda}(0)$ are called the degenerate Genocchi numbers.

For $k \in \mathbb{Z}$, the modified degenerate polyexponential function [4] is defined by Kim-Kim to be

$$\mathrm{Ei}_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}x^n}{(n-1)!n^k} \ \ (|\ x\ |< 1). \tag{10}$$

Note that

$$\mathrm{Ei}_{1,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}x^n}{n!} = e_\lambda(x) - 1. \tag{11}$$

The degenerate poly-Bernoulli polynomials are defined (see [16]) by

$$\frac{\mathrm{Ei}_{k,\lambda}\big(\log_\lambda(1+t)\big)}{e_\lambda(t)-1}e_\lambda^x(t)=\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}\ \ (k\in\mathbb{Z}). \tag{12}$$

In the case when $x=0$, $\beta_{n,\lambda}^{(k)}=\beta_{n,\lambda}^{(k)}(0)$ are called the degenerate poly-Bernoulli numbers.

The degenerate poly-Genocchi polynomials are defined (see [4]) by

$$\frac{2\mathrm{Ei}_{k,\lambda}\big(\log_\lambda(1+t)\big)}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^{\infty}G_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}\ \ (k\in\mathbb{Z}). \tag{13}$$

In the case when $x=0$, $G_{n,\lambda}^{(k)}=G_{n,\lambda}^{(k)}(0)$ are called the degenerate poly-Genocchi numbers.

Kim-Kim [15] introduced the degenerate poly-Bell polynomials and numbers as follows

$$1+\mathrm{Ei}_{k,\lambda}(x(e_\lambda(t)-1))=\sum_{n=0}^{\infty}bel_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}. \tag{14}$$

When $x=1$, $bel_{n,\lambda}^{(k)}=bel_{n,\lambda}^{(k)}(0)$ are called the degenerate poly-Bell numbers.

Let $\log_\lambda(t)$ be the compositional inverse of $e_\lambda(t)$, called the degenerate logarithm function, such that $\log_\lambda(e_\lambda(t))=e_\lambda(\log_\lambda t)=t$. Then, we note that (see [3])

$$\log_\lambda(1+t)=\frac{1}{\lambda}((1+t)^\lambda-1)=\sum_{n=1}^{\infty}\lambda^{n-1}(1)_{n,\frac{1}{\lambda}}\frac{t^n}{n!}. \tag{15}$$

From (15), we obtain $\lim_{\lambda\to 0}\log_\lambda(1+t)=\log(1+t)$.

In [10], the degenerate Stirling numbers of the first kind are defined by

$$(x)_n=\sum_{l=0}^{n}S_{1,\lambda}(n,l)(x)_{l,\lambda}. \tag{16}$$

As an inversion formula of (16), the degenerate Stirling numbers of the second kind are defined (see [10]) by

$$(x)_{n,\lambda}=\sum_{k=0}^{n}S_{2,\lambda}(n,k)(x)_k,\ \ (n\geq 0). \tag{17}$$

By (16) and (17), we obtain (see [16])

$$\frac{1}{k!}(\log_\lambda(1+z))^k=\sum_{j=k}^{\infty}S_{1,\lambda}(j,k)\frac{z^j}{j!},\ \ (k\geq 0) \tag{18}$$

and

$$\frac{1}{k!}(e_\lambda(z)-1)^k=\sum_{j=k}^{\infty}S_{2,\lambda}(j,k)\frac{z^j}{j!}\ \ (k\geq 0). \tag{19}$$

The degenerate Bell polynomials $B_{n,\lambda}$ are defined (see [5]) by

$$e_\lambda^x(e_\lambda(t)-1)=\sum_{n=0}^{\infty}B_{n,\lambda}(x)\frac{t^n}{n!}, \tag{20}$$

so that

$$B_{n,\lambda}=\sum_{k=0}^{n}(x)_{k,\lambda}S_{2,\lambda}(n,k)\ \ (n\geq 0).$$

When $x=1$, $B_{n,\lambda}=B_{n,\lambda}(1)$ are called the degenerate Bell numbers.

For $k \geq 0$, the Jindalrae–Stirling numbers of the first kind and second kind are given (see [19]) by

$$\frac{1}{k!}\left(\log_\lambda(\log_\lambda(1+t)+1)\right)^k = \sum_{n=k}^{\infty} S_{J,\lambda}^{(1)}(n,k)\frac{t^n}{n!} \quad (21)$$

and

$$\frac{1}{k!}(e_\lambda(e_\lambda-1)-1)^k = \sum_{n=k}^{\infty} S_{J,\lambda}^{(2)}(n,k)\frac{t^n}{n!}. \quad (22)$$

In [19], Kim et al. introduced Jindalrae and Gaenari polynomials defined by

$$e_\lambda^x(e_\lambda(e_\lambda-1)-1)) = \sum_{n=0}^{\infty} J_{n,\lambda}(x)\frac{t^n}{n!} \quad (23)$$

and

$$e_\lambda^x(\log_\lambda(\log_\lambda(1+t)+1)) = \sum_{n=0}^{\infty} G_{n,\lambda}(x)\frac{t^n}{n!}. \quad (24)$$

When $x=1$, $J_{n,\lambda} = J_{n,\lambda}(1)$ and $G_{n,\lambda} = G_{n,\lambda}(1)$ are called the Jindalrae and Gaenari numbers.

Kim-Kim [11] defined the unipoly function attached to polynomials $p(x)$ by

$$u_k(x|p) = \sum_{n=1}^{\infty} \frac{p(n)}{n^k}x^n, (k \in \mathbb{Z}). \quad (25)$$

Moreover,

$$u_k(x|1) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} = \mathrm{Li}_k(x) \quad (26)$$

is the ordinary polylogarithm function (see [7]).

The degenerate unipoly function attached to polynomials $p(x)$ is as follows (see [3])

$$u_{k,\lambda}(x|p) = \sum_{i=1}^{\infty} p(i)\frac{(1)_{i,\lambda}}{i^k}x^i. \quad (27)$$

It is worthy to note that

$$u_{k,\lambda}(x|1/\Gamma) = \mathrm{Ei}_{k,\lambda}(x) \quad (28)$$

is the modified degenerate polyexponential function.

This paper is structured as follows. Section 1 provides an overview of essential concepts that are fundamental, including the degenerate exponential functions, degenerate logarithm function, degenerate Stirling numbers of the first and second kinds, and degenerate Bell numbers. It is important to note that the degenerate poly-Bell polynomials $bel_{n,\lambda}^{(k)}(x)$ (refer to [15]) differ from the degenerate Bell polynomials $bel_{n,\lambda}(x)$ discussed in [5], and the new type degenerate Bell polynomials $Bel_{n,\lambda}(x)$ introduced in [5].

In Section 2, we introduce poly-Jindalrae and poly-Gaenari polynomials as extensions of the Jindalrae and Gaenari polynomials. We establish connections between these special numbers, degenerate Stirling numbers of the first and second kinds, and degenerate Bell numbers and polynomials. Furthermore, we define poly-Jindalrae numbers and polynomials as extensions of the degenerate Bell numbers and polynomials. We derive explicit expressions and identities involving these numbers and polynomials, Jindalrae–Stirling numbers of the first and second kinds, degenerate Stirling numbers of the first and second kinds, and degenerate Bell polynomials.

In Section 3, we introduce the degenerate unipoly-Jindalrae and unipoly-Gaenari polynomials by utilizing the degenerate unipoly functions associated with polynomials $p(x)$. We provide explicit expressions and identities involving these polynomials.

2. Degenerate Poly-Jindalrae and Poly-Gaenari Polynomials and Numbers

In this section, we define the degenerate poly-Jindalrae and poly-Gaenari polynomials by using of the degenerate polyexponential functions and represent the Jindalrae and Gaenari numbers (more precisely, the values of ordinary degenerate Bell polynomials at 1) when $k = 1$. At the same time, we give explicit expressions and identities involving those polynomials.

Motivated and inspired by Equation (23), for $k \in \mathbb{Z}$, we consider the degenerate poly-Jindalrae polynomials by

$$1 + \mathrm{Ei}_{k,\lambda}(x(e_\lambda(e_\lambda(t) - 1) - 1)) = \sum_{n=0}^{\infty} J_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}, \tag{29}$$

and $J_{0,\lambda}^{(k)}(x) = 1$.

In the special case when $x = 1$, $J_{n,\lambda}^{(k)} = J_{n,\lambda}^{(k)}(1)$ are called the degenerate poly-Jindalrae numbers.

By $k = 1$ in (29), we note that

$$\begin{aligned} 1 + \mathrm{Ei}_{1,\lambda}(x(e_\lambda(e_\lambda(1) - 1) - 1)) &= 1 + \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}[(x(e_\lambda(e_\lambda(1) - 1) - 1))]^n}{(n-1)!n!} \\ &= \sum_{n=0}^{\infty} \frac{(1)_{n,\lambda}[(x(e_\lambda(e_\lambda(1) - 1) - 1))]^n}{(n-1)!n!} \\ &= \sum_{n=0}^{\infty} J_{n,\lambda}(x) \frac{t^n}{n!}. \end{aligned} \tag{30}$$

Combining with (29) and (30), we have

$$J_{n,\lambda}^{(1)}(x) = J_{n,\lambda}(x).$$

When $\lambda \to 0$,

$$\mathrm{Ei}_k(x(e^{(e^t-1)} - 1)) + 1 = \sum_{n=0}^{\infty} J_n^{(k)}(x) \frac{t^n}{n!} \tag{31}$$

are called the poly-Jindalrae polynomials.

When $x = 1$, $J_n^{(k)} = J_n^{(k)}(1)$ are called the poly-Jindalrae numbers.

Theorem 1. *For $k \in \mathbb{Z}$, we have*

$$J_{n,\lambda}^{(k)}(x) = \sum_{m=1}^{n} \sum_{l=1}^{m} \frac{(1)_{l,\lambda} x^l l!}{l^{k-1}} S_{2,\lambda}(m,l) S_{2,\lambda}(n,m). \tag{32}$$

Proof. From (9) and (29), we note that

$$\begin{aligned} \mathrm{Ei}_{k,\lambda}(x(e_\lambda(e_\lambda(1) - 1) - 1)) &= \sum_{l=1}^{\infty} \frac{(1)_{l,\lambda} x^l l!}{l^{k-1}} \frac{[e_\lambda(e_\lambda(t) - 1) - 1]^l}{l!} \\ &= \sum_{l=1}^{\infty} \frac{(1)_{l,\lambda} x^l l!}{l^{k-1}} \sum_{m=l}^{\infty} S_{2,\lambda}(m,l) \frac{[e_\lambda(t) - 1]^m}{m!} \\ &= \sum_{m=1}^{\infty} \sum_{l=1}^{m} \frac{(1)_{l,\lambda} x^l l!}{l^{k-1}} S_{2,\lambda}(m,l) \sum_{n=m}^{\infty} S_{2,\lambda}(n,m) \frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} \sum_{l=1}^{m} \frac{(1)_{l,\lambda} x^l l!}{l^{k-1}} S_{2,\lambda}(m,l) S_{2,\lambda}(n,m) \right) \frac{t^n}{n!}. \end{aligned} \tag{33}$$

Therefore, by Equations (29) and (33), we obtain the result. □

Theorem 2. *Let $k \in \mathbb{Z}$ and $n \geq 0$; we have*

$$J_{n,\lambda}^{(k)}(x) = \sum_{m=1}^{n} \frac{(1)_{l,\lambda} x^l l!}{l^{k-1}} S_{J,\lambda}^{(2)}(n,l). \tag{34}$$

Proof. From (22) and (29), we have

$$\begin{aligned} \mathrm{Ei}_{k,\lambda}(x(e_\lambda(e_\lambda(1)-1)-1)) &= \sum_{l=1}^{\infty} \frac{(1)_{l,\lambda} x^l l!}{l^{k-1}} \frac{[e_\lambda(e_\lambda(t)-1)-1]^l}{l!} \\ &= \sum_{l=1}^{\infty} \frac{(1)_{l,\lambda} x^l l!}{l^{k-1}} \sum_{n=l}^{\infty} S_{J,\lambda}^{(2)}(n,l) \frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} \frac{(1)_{l,\lambda} x^l l!}{l^{k-1}} S_{J,\lambda}^{(2)}(n,l) \right) \frac{t^n}{n!}. \end{aligned} \tag{35}$$

Therefore, by Equations (29) and (35), we obtain the result. □

Theorem 3. *Let $k \in \mathbb{Z}$ and $n \geq 0$; we have*

$$bel_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n} J_{l,\lambda}^{(k)}(x) S_{1,\lambda}(n,m). \tag{36}$$

Proof. Replacing t with $\log_\lambda(1+t)$ in (29), we obtain

$$\begin{aligned} 1 + \mathrm{Ei}_{k,\lambda}(x((e_\lambda(t)-1)) &= \sum_{l=0}^{\infty} J_{l,\lambda}^{(k)}(x) \frac{[\log_\lambda(1+t)]^j}{j!} \\ &= \sum_{l=0}^{\infty} J_{l,\lambda}^{(k)}(x) \sum_{n=l}^{\infty} S_{1,\lambda}(n,m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^{n} J_{l,\lambda}^{(k)}(x) S_{1,\lambda}(n,m) \frac{t^n}{n!}. \end{aligned}$$

On the other hand,

$$1 + \mathrm{Ei}_{k,\lambda}(x((e_\lambda(t)-1)) = \sum_{n=0}^{\infty} bel_{n,\lambda}^{(k)}(x) \frac{t^n}{n!} \quad \text{(see [15])}. \tag{37}$$

Therefore, by comparing the coefficients of t on both sides of equations, we obtain the result. □

Theorem 4. *Let $k \in \mathbb{Z}$ and $n \geq 0$; we have*

$$J_{n,\lambda}^{k)}(x) = \sum_{l=1}^{n} bel_{n,\lambda}^{(k)}(x) S_{2,\lambda}(n,l). \tag{38}$$

Proof. Replacing t with $e_\lambda(t) - 1$ in (37), we obtain

$$\begin{aligned} 1 + \mathrm{Ei}_{k,\lambda}(x(e_\lambda(e_\lambda(1)-1)-1)) &= \sum_{l=0}^{\infty} bel_{n,\lambda}^{(k)}(x) \frac{[e_\lambda(t)-1]^l}{l!} \\ &= \sum_{l=0}^{\infty} bel_{n,\lambda}^{(k)}(x) \sum_{n=l}^{\infty} S_{2,\lambda}(n,l) \frac{t^n}{n!} \end{aligned} \tag{39}$$

$$= \sum_{n=0}^{\infty} \sum_{l=1}^{n} bel_{n,\lambda}^{(k)}(x) S_{2,\lambda}(n,l) \frac{t^n}{n!}.$$

Therefore, by (29) and (39), we obtain the result. □

Theorem 5 (Dobinski-like formulas). *For $n \geq 0$, we have*

$$J_{n,\lambda}^{(k)}(x) = \sum_{m=1}^{\infty} \sum_{l=1}^{m} \sum_{s=0}^{m} \binom{m}{s} (-1)^{s-m} \frac{(1)_{l,\lambda} x^l}{l^{k-1} m!} S_{2,\lambda}(m,l)(s)_{n,\lambda}. \tag{40}$$

Proof. From (29), we note that

$$\begin{aligned}
\sum_{n=1}^{\infty} J_{n,\lambda}^{(k)}(x) \frac{t^n}{n!} &= \sum_{l=1}^{\infty} \frac{(1)_{l,\lambda} x^l}{(l-1)! l^k} [e_\lambda(e_\lambda(t)-1)-1]^l \\
&= \sum_{l=1}^{\infty} \frac{(1)_{l,\lambda} x^l}{(l-1)! l^k} \sum_{m=l}^{\infty} S_{2,\lambda}(m,l) \frac{1}{m!} (e_\lambda(t)-1)^m \\
&= \sum_{m=1}^{\infty} \sum_{l=1}^{m} \frac{(1)_{l,\lambda} x^l}{l^{k-1} m!} S_{2,\lambda}(m,l) \sum_{s=0}^{m} \binom{m}{s} (-1)^{s-m} e_\lambda^s(t) \\
&= \sum_{m=1}^{\infty} \sum_{l=1}^{m} \frac{(1)_{l,\lambda} x^l}{l^{k-1} m!} S_{2,\lambda}(m,l) \sum_{s=0}^{m} \binom{m}{s} (-1)^{s-m} \sum_{n=0}^{\infty} (s)_{n,\lambda} \frac{t^n}{n!} \qquad (41) \\
&= \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{m} \sum_{s=0}^{m} \binom{m}{s} (-1)^{s-m} \frac{(1)_{l,\lambda} x^l}{l^{k-1} m!} S_{2,\lambda}(m,l)(s)_{n,\lambda} \frac{t^n}{n!} \\
&= \sum_{m=1}^{\infty} \sum_{l=1}^{m} \sum_{s=0}^{m} \binom{m}{s} \frac{(1)_{l,\lambda} x^l}{l^{k-1} m!} (-1)^{s-m} S_{2,\lambda}(m,l) \\
&+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{m} \sum_{s=0}^{m} \binom{m}{s} (-1)^{s-m} \frac{(1)_{l,\lambda} x^l}{l^{k-1} m!} S_{2,\lambda}(m,l)(s)_{n,\lambda} \frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients of t on both sides of Equation (41), we obtain the result.

$$\sum_{m=1}^{\infty} \sum_{l=1}^{m} \sum_{s=0}^{m} \binom{m}{s} \frac{(1)_{l,\lambda} x^l}{l^{k-1} m!} (-1)^{s-m} S_{2,\lambda}(m,l) = 0$$

and

$$J_{n,\lambda}^{(k)}(x) = \sum_{m=1}^{\infty} \sum_{l=1}^{m} \sum_{s=0}^{m} \binom{m}{s} (-1)^{s-m} \frac{(1)_{l,\lambda} x^l}{l^{k-1} m!} S_{2,\lambda}(m,l)(s)_{n,\lambda} \quad (n \geq 1).$$

□

Theorem 6. *Let $k \in \mathbb{Z}$and $n \geq 0$; we have*

$$\sum_{m=0}^{n-1} \binom{n}{m} J_{n-m,\lambda} J_{m+1,\lambda}^{(k)}(x) = \sum_{j=1}^{n} \sum_{i=0}^{j} \binom{n}{j} (1-\lambda)_{i,\lambda} S_{2,\lambda}(j,i) J_{n-j,\lambda}^{(k-1)}(x). \tag{42}$$

Proof. Differentiating with respect to t in (29), the left hand side of (29) is

$$\begin{aligned}
\frac{\partial}{\partial t}[1+\mathrm{Ei}_{k,\lambda}(x(e_\lambda(e_\lambda(t)-1)-1))] &= \frac{\partial}{\partial t}\left[1+\sum_{n=1}^{\infty} \frac{(1)_{n,\lambda} x^n (e_\lambda(e_\lambda(t)-1)-1)^n}{(n-1)! n^k}\right] \\
&= \frac{e_\lambda^{1-\lambda}(e_\lambda(t)-1)}{e_\lambda(e_\lambda(t)-1)-1} \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda} x^n (e_\lambda(e_\lambda(t)-1)-1)^n}{(n-1)! n^{k-1}} \qquad (43)
\end{aligned}$$

$$= \frac{e_\lambda^{1-\lambda}(e_\lambda(t)-1)}{e_\lambda(e_\lambda(t)-1)-1}\mathrm{Ei}_{k-1,\lambda}(x(e_\lambda(e_\lambda(t)-1)-1))$$
$$= \frac{e_\lambda^{1-\lambda}(e_\lambda(t)-1)}{e_\lambda(e_\lambda(t)-1)-1}\sum_{n=1}^{\infty} J_{n,\lambda}^{(k-1)}(x)\frac{t^n}{n!}.$$

On the other hand, we have

$$\frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty} J_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}\right) = \sum_{n=1}^{\infty} J_{n,\lambda}^{(k)}(x)\frac{t^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} J_{n+1,\lambda}^{(k)}(x)\frac{t^n}{n!}. \tag{44}$$

From (43) and (44), we obtain

$$(e_\lambda(e_\lambda(t)-1)-1)\sum_{n=0}^{\infty} J_{n+1,\lambda}^{(k)}(x)\frac{t^n}{n!} = e_\lambda^{1-\lambda}(e_\lambda(t)-1)\sum_{n=1}^{\infty} J_{n,\lambda}^{(k-1)}(x)\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} J_{n,\lambda}\frac{t^n}{n!}\sum_{m=0}^{\infty} J_{m+1,\lambda}^{(k)}(x)\frac{t^m}{m!}$$
$$= \sum_{i=0}^{\infty}(1-\lambda)_{i,\lambda}\frac{[e_\lambda(t)-1]^i}{i!}\sum_{n=1}^{\infty} J_{n,\lambda}^{(k-1)}(x)\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}J_{n-m,\lambda}J_{m+1,\lambda}^{(k)}(x)\frac{t^n}{n!}$$
$$= \sum_{j=0}^{\infty}\sum_{i=0}^{j}(1-\lambda)_{i,\lambda}S_{2,\lambda}(j,i)\frac{t^j}{j!}\sum_{n=1}^{\infty} J_{n,\lambda}^{(k-1)}(x)\frac{t^n}{n!}$$

$$\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\binom{n}{m}J_{n-m,\lambda}J_{m+1,\lambda}^{(k)}(x)\frac{t^n}{n!}$$
$$= \sum_{n=1}^{\infty}\sum_{j=1}^{n}\sum_{i=0}^{j}\binom{n}{j}(1-\lambda)_{i,\lambda}S_{2,\lambda}(j,i)J_{n-j,\lambda}^{(k-1)}(x)\frac{t^n}{n!}.$$

By comparing the coefficients of t^n on both sides, we obtain the result. □

Theorem 7. *Let $k \in \mathbb{Z}$ and $n \geq 1$. Then*

$$J_{n,\lambda}^{(k)}(x) = \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{h=1}^{j}\sum_{m=1}^{h}\frac{(1)_{m,\lambda}}{m^{k-1}}S_{1,\lambda}(h,m)S_{J,\lambda}^{(2)}(j,h|x)S_{2,\lambda}(i,j)S_{2,\lambda}(n,i). \tag{45}$$

Proof. From (10) and (22), we observe that

$$\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t)) = \sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}(\log_\lambda(1+t))^m}{(m-1)!m^k}$$
$$= \sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}}{m^{k-1}}\frac{1}{m!}(\log_\lambda(1+t))^m \tag{46}$$
$$= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\frac{(1)_{m,\lambda}}{m^{k-1}}S_{1,\lambda}(n,m)\right)\frac{t^n}{n!}.$$

Replacing t by $e_\lambda^x(e_\lambda(e_\lambda(t)-1)-1)-1$ in (46), we obtain

$$\mathrm{Ei}_{k,\lambda}(x(e_\lambda(e_\lambda(t)-1)-1)) = \sum_{h=1}^{\infty}\sum_{m=1}^{h}\frac{(1)_{m,\lambda}}{m^{k-1}}S_{1,\lambda}(h,m)\frac{[e_\lambda^x(e_\lambda(e_\lambda(t)-1)-1)-1]^h}{h!}$$

$$
\begin{aligned}
&= \sum_{h=1}^{\infty}\sum_{m=1}^{h}\frac{(1)_{m,\lambda}}{m^{k-1}}S_{1,\lambda}(h,m)\sum_{j=h}^{\infty}S_{J,\lambda}^{(2)}(j,h|x)\frac{(e_\lambda(e_\lambda(t)-1)-1)^j}{j!} \\
&= \sum_{j=1}^{\infty}\sum_{h=1}^{j}\sum_{m=1}^{h}\frac{(1)_{m,\lambda}}{m^{k-1}}S_{1,\lambda}(h,m)S_{J,\lambda}^{(2)}(j,h|x)\sum_{i=j}^{\infty}S_{2,\lambda}(i,j)\frac{(e_\lambda(t)-1)^i}{i!} \qquad (47)\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\sum_{h=1}^{j}\sum_{m=1}^{h}\frac{(1)_{m,\lambda}}{m^{k-1}}S_{1,\lambda}(h,m)S_{J,\lambda}^{(2)}(j,h|x)S_{2,\lambda}(i,j)\sum_{n=i}^{\infty}S_{2,\lambda}(n,i)\frac{t^n}{n!} \\
&= \sum_{n=1}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{h=1}^{j}\sum_{m=1}^{h}\frac{(1)_{m,\lambda}}{m^{k-1}}S_{1,\lambda}(h,m)S_{J,\lambda}^{(2)}(j,h|x)S_{2,\lambda}(i,j)S_{2,\lambda}(n,i)\frac{t^n}{n!}.
\end{aligned}
$$

Therefore, by Equations (29) and (47), we obtain the result. □

For next theorem, we observe that (see [16])

$$
\begin{aligned}
\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k)}(x)\frac{t^n}{n!} &= \frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t)-1}e_\lambda^x(t) \\
&= \sum_{m=0}^{\infty}\beta_{m,\lambda}^{(k)}\frac{t^m}{m!}\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!} \qquad (48)\\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\beta_{m,\lambda}^{(k)}(x)_{n-m,\lambda}\right)\frac{t^n}{n!}.
\end{aligned}
$$

By comparing the coefficients on both sides of (48), we obtain

$$
\beta_{n,\lambda}^{(k)}(x) = \sum_{m=0}^{n}\binom{n}{m}\beta_{m,\lambda}^{(k)}(x)_{n-m,\lambda}. \qquad (49)
$$

Theorem 8. *Let $k \in \mathbb{Z}$ and $n \geq 1$. Then*

$$
J_{n,\lambda}^{(k)}(x) = \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{h=1}^{j}\left(\beta_{h,\lambda}^{(k)}(1)-\beta_{h,\lambda}^{(k)}\right)S_{J,\lambda}^{(2)}(j,h|x)S_{2,\lambda}(i,j)S_{2,\lambda}(n,i). \qquad (50)
$$

Proof. From (10), (12) and (49), we observe that

$$
\begin{aligned}
\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t)) &= (e_\lambda(t)-1)\sum_{j=0}^{\infty}\beta_{j,\lambda}^{(k)}\frac{t^j}{j!} \\
&= \left(\sum_{m=0}^{\infty}\frac{(1)_{m,\lambda}}{m!}t^m - 1\right)\sum_{j=0}^{\infty}\beta_{j,\lambda}^{(k)}\frac{t^j}{j!} \qquad (51)\\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}(1)_{n-m,\lambda}\beta_{m,\lambda}^{(k)}(x)-\beta_{m,\lambda}^{(k)}\right)\frac{t^n}{n!} \\
&= \sum_{n=1}^{\infty}\left(\beta_{n,\lambda}^{(k)}(1)-\beta_{n,\lambda}^{(k)}\right)\frac{t^n}{n!}.
\end{aligned}
$$

Replacing t by $e_\lambda^x(e_\lambda(e_\lambda(t)-1)-1)-1$ in (51), we obtain

$$
\mathrm{Ei}_{k,\lambda}(x(e_\lambda(e_\lambda(t)-1)-1)) = \sum_{h=1}^{\infty}\left(\beta_{h,\lambda}^{(k)}(1)-\beta_{h,\lambda}^{(k)}\right)\frac{[e_\lambda^x(e_\lambda(e_\lambda(t)-1)-1)-1)]^h}{h!}
$$

$$
\begin{aligned}
&= \sum_{h=1}^{\infty}\sum_{j=h}^{\infty}\Big(\beta_{h,\lambda}^{(k)}(1)-\beta_{h,\lambda}^{(k)}\Big)S_{J,\lambda}^{(2)}(j,h|x)\frac{(e_\lambda(e_\lambda(t)-1)-1)^j}{j!} \\
&= \sum_{j=1}^{\infty}\sum_{h=1}^{j}\Big(\beta_{h,\lambda}^{(k)}(1)-\beta_{h,\lambda}^{(k)}\Big)S_{J,\lambda}^{(2)}(j,h|x)\sum_{i=j}^{\infty}S_{2,\lambda}(i,j)\frac{(e_\lambda(t)-1)^i}{i!} \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\sum_{h=1}^{j}\Big(\beta_{h,\lambda}^{(k)}(1)-\beta_{h,\lambda}^{(k)}\Big)S_{J,\lambda}^{(2)}(j,h|x)S_{2,\lambda}(i,j)\sum_{n=i}^{\infty}S_{2,\lambda}(n,i)\frac{t^n}{n!} \\
&= \sum_{n=1}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{h=1}^{j}\Big(\beta_{h,\lambda}^{(k)}(1)-\beta_{h,\lambda}^{(k)}\Big)S_{J,\lambda}^{(2)}(j,h|x)S_{2,\lambda}(i,j)S_{2,\lambda}(n,i)\frac{t^n}{n!}.
\end{aligned} \tag{52}
$$

Therefore, by (29) and (52), we obtain the result. □

Theorem 9. *Let $k \in \mathbb{Z}$ and $n \geq 1$. Then*

$$
J_{n,\lambda}^{(k)}(x) = \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{h=1}^{j}\sum_{m=1}^{h}\binom{h}{m}(1)_{m,\lambda}\beta_{h-m,\lambda}^{(k)}S_{J,\lambda}^{(2)}(j,h|x)S_{2,\lambda}(i,j)S_{2,\lambda}(n,i). \tag{53}
$$

Proof. From (10) and (12), we observe that

$$
\begin{aligned}
\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t)) &= (e_\lambda(t)-1)\sum_{j=0}^{\infty}\beta_{j,\lambda}^{(k)}\frac{t^j}{j!} \\
&= \left(\sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}}{m!}t^m\right)\sum_{j=0}^{\infty}\beta_{j,\lambda}^{(k)}\frac{t^j}{j!} \\
&= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\binom{n}{m}(1)_{m,\lambda}\beta_{n-m,\lambda}^{(k)}\right)\frac{t^n}{n!}.
\end{aligned} \tag{54}
$$

Replacing t with $e_\lambda^x(e_\lambda(e_\lambda(t)-1)-1)-1$ in (54), we obtain

$$
\begin{aligned}
\mathrm{Ei}_{k,\lambda}(x(e_\lambda(e_\lambda(t)-1)-1)) &= \sum_{h=1}^{\infty}\sum_{m=1}^{h}\binom{h}{m}(1)_{m,\lambda}\beta_{h-m,\lambda}^{(k)}\frac{[e_\lambda^x(e_\lambda(e_\lambda(t)-1)-1)-1)]^h}{h!} \\
&= \sum_{h=1}^{\infty}\sum_{j=h}^{\infty}\sum_{m=1}^{h}\binom{h}{m}(1)_{m,\lambda}\beta_{h-m,\lambda}^{(k)}S_{J,\lambda}^{(2)}(j,h|x)\frac{(e_\lambda(e_\lambda(t)-1)-1)^j}{j!} \\
&= \sum_{j=1}^{\infty}\sum_{h=1}^{j}\sum_{m=1}^{h}\binom{h}{m}(1)_{m,\lambda}\beta_{h-m,\lambda}^{(k)}S_{J,\lambda}^{(2)}(j,h|x)\sum_{i=j}^{\infty}S_{2,\lambda}(i,j)\frac{(e_\lambda(t)-1)^i}{i!} \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\sum_{h=1}^{j}\sum_{m=1}^{h}\binom{h}{m}(1)_{m,\lambda}\beta_{h-m,\lambda}^{(k)}S_{J,\lambda}^{(2)}(j,h|x)S_{2,\lambda}(i,j)\sum_{n=i}^{\infty}S_{2,\lambda}(n,i)\frac{t^n}{n!} \\
&= \sum_{n=1}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{h=1}^{j}\sum_{m=1}^{h}\binom{h}{m}(1)_{m,\lambda}\beta_{h-m,\lambda}^{(k)}S_{J,\lambda}^{(2)}(j,h|x)S_{2,\lambda}(i,j)S_{2,\lambda}(n,i)\frac{t^n}{n!}.
\end{aligned} \tag{55}
$$

Therefore, by (29) and (55), we obtain the result. □

Theorem 10. *Let $k \in \mathbb{Z}$ and $n \geq 1$. Then*

$$
J_{n,\lambda}^{(k)}(x) = \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{h=1}^{j}\sum_{m=1}^{h}\binom{h}{m}\Big((1)_{m,\lambda}G_{h-m,\lambda}^{(k)}+2G_{h,\lambda}^{(k)}\Big)S_{J,\lambda}^{(2)}(j,h|x)S_{2,\lambda}(i,j)S_{2,\lambda}(n,i). \tag{56}
$$

Proof. From (10) and (13), we observe that

$$
\begin{aligned}
2\mathrm{Ei}_{k,\lambda}(\log_{\lambda}(1+t)) &= (e_{\lambda}(t)+1)\sum_{j=0}^{\infty} G_{j,\lambda}^{(k)}\frac{t^j}{j!} \\
&= \left(\sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}}{m!}t^m\right)\sum_{j=0}^{\infty}\beta_{j,\lambda}^{(k)}\frac{t^j}{j!}+2\sum_{j=0}^{\infty}\beta_{j,\lambda}^{(k)}\frac{t^j}{j!} \\
&= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\binom{n}{m}(1)_{m,\lambda}G_{n-m,\lambda}^{(k)}+2G_{n-m,\lambda}^{(k)}\right)\frac{t^n}{n!}.
\end{aligned} \tag{57}
$$

Replacing t with $e_{\lambda}^{x}(e_{\lambda}(e_{\lambda}(t)-1)-1)-1$ in (57), we obtain

$$
\begin{aligned}
\mathrm{Ei}_{k,\lambda}(x(e_{\lambda}(e_{\lambda}(t)-1)-1)) &= \sum_{h=1}^{\infty}\sum_{m=1}^{h}\binom{h}{m}\Big((1)_{m,\lambda}G_{h-m,\lambda}^{(k)}+2G_{h,\lambda}^{(k)}\Big)\frac{[e_{\lambda}^{x}(e_{\lambda}(e_{\lambda}(t)-1)-1)-1)]^h}{h!} \\
&= \sum_{h=1}^{\infty}\sum_{j=h}^{\infty}\sum_{m=1}^{h}\binom{h}{m}\Big((1)_{m,\lambda}G_{h-m,\lambda}^{(k)}+2G_{h,\lambda}^{(k)}\Big)S_{J,\lambda}^{(2)}(j,h|x)\frac{(e_{\lambda}(e_{\lambda}(t)-1)-1)^j}{j!} \\
&= \sum_{j=1}^{\infty}\sum_{h=1}^{j}\sum_{m=1}^{h}\binom{h}{m}(1)_{m,\lambda}\beta_{h-m,\lambda}^{(k)}S_{J,\lambda}^{(2)}(j,h|x)\sum_{i=j}^{\infty}S_{2,\lambda}(i,j)\frac{(e_{\lambda}(t)-1)^i}{i!} \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\sum_{h=1}^{j}\sum_{m=1}^{h}\binom{h}{m}\Big((1)_{m,\lambda}G_{h-m,\lambda}^{(k)}+2G_{h,\lambda}^{(k)}\Big)S_{J,\lambda}^{(2)}(j,h|x)S_{2,\lambda}(i,j)\sum_{n=i}^{\infty}S_{2,\lambda}(n,i)\frac{t^n}{n!} \\
&= \sum_{n=1}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{h=1}^{j}\sum_{m=1}^{h}\binom{h}{m}\Big((1)_{m,\lambda}G_{h-m,\lambda}^{(k)}+2G_{h,\lambda}^{(k)}\Big)S_{J,\lambda}^{(2)}(j,h|x)S_{2,\lambda}(i,j)S_{2,\lambda}(n,i)\frac{t^n}{n!}.
\end{aligned} \tag{58}
$$

Therefore, by (29) and (58), we obtain the result. □

Motivated and inspired by Equation (24), we define the degenerate poly-Gaenari polynomials given by

$$
1+\mathrm{Ei}_{k,\lambda}\big(x(\log_{\lambda}(\log_{\lambda}(1+t))+1)\big)=\sum_{n=0}^{\infty}G_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}\quad (k\in\mathbb{Z}), \tag{59}
$$

and $G_{0,\lambda}^{(k)}(x)=1$, $G_{n,\lambda}^{(k)}=G_{n,\lambda}^{(k)}(1)$ are called the poly-Gaenari numbers.

For $k=1$ in (59), we note that

$$
\begin{aligned}
1+\mathrm{Ei}_{1,\lambda}\big(x(\log_{\lambda}(\log_{\lambda}(1+t))+1)\big) &= 1+\sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}[x(\log_{\lambda}(\log_{\lambda}(1+t))+1]^n}{(n-1)!n} \\
&= \sum_{n=0}^{\infty}G_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}.
\end{aligned} \tag{60}
$$

Combining with (59) and (60), we have

$$
G_{n,\lambda}^{(k)}(x)=G_{n,\lambda}(x)\quad (n\geq 0).
$$

Theorem 11. *Let $k \in \mathbb{Z}$ and $n \geq 1$. Then*

$$G_{n,\lambda}^{(k)}(x) = \sum_{l=1}^{j} \sum_{j=1}^{n} \frac{(1)_{l,\lambda} x^l}{l^{k-1}} S_{1,\lambda}(j,l) S_{1,\lambda}(n,j). \tag{61}$$

Proof. From (18) and (60), we observe that

$$\begin{aligned} \mathrm{Ei}_{k,\lambda}\big(x(\log_\lambda(\log_\lambda(1+t))+1)\big) &= \sum_{l=1}^{\infty} \frac{(1)_{l,\lambda} x^l}{l^{k-1}} \frac{1}{l!} [\log_\lambda(\log_\lambda(1+t))+1)]^l \\ &= \sum_{l=1}^{\infty} \frac{(1)_{l,\lambda} x^l}{l^{k-1}} \sum_{j=l}^{\infty} S_{1,\lambda}(j,l) \frac{1}{j!} [\log_\lambda(1+t)]^j \\ &= \sum_{l=1}^{\infty} \frac{(1)_{l,\lambda} x^l}{l^{k-1}} \sum_{j=l}^{\infty} S_{1,\lambda}(j,l) \sum_{n=j}^{\infty} S_{1,\lambda}(n,j) \frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty} \sum_{l=1}^{j} \sum_{j=1}^{n} \frac{(1)_{l,\lambda} x^l}{l^{k-1}} S_{1,\lambda}(j,l) S_{1,\lambda}(n,j) \frac{t^n}{n!}. \end{aligned} \tag{62}$$

In view of (59) and (62), we obtain the desired result. □

Theorem 12. *Let $k \in \mathbb{Z}$ and $n \geq 1$ Then*

$$G_{n,\lambda}^{(k)}(x) = \sum_{l=1}^{n} \frac{(1)_{l,\lambda} x^l}{l^{k-1}} S_{J,\lambda}^{(1)}(n,l). \tag{63}$$

Proof. From (21) and (59), we observe that

$$\begin{aligned} \mathrm{Ei}_{k,\lambda}\big(x(\log_\lambda(\log_\lambda(1+t))+1)\big) &= \sum_{l=1}^{\infty} \frac{(1)_{l,\lambda} x^l}{l^{k-1}} \frac{1}{l!} [\log_\lambda(\log_\lambda(1+t))+1)]^l \\ &= \sum_{l=1}^{\infty} \frac{(1)_{l,\lambda} x^l}{l^{k-1}} \sum_{n=l}^{\infty} S_{J,\lambda}^{(1)}(n,l) \\ &= \sum_{n=1}^{\infty} \sum_{l=1}^{n} \frac{(1)_{l,\lambda} x^l}{l^{k-1}} S_{J,\lambda}^{(1)}(n,l) \frac{t^n}{n!}. \end{aligned} \tag{64}$$

In view of (59) and (64), we obtain the desired result. □

Theorem 13. *Let $k \in \mathbb{Z}$ and $n \geq 1$. Then*

$$\sum_{m=0}^{n} G_{m,\lambda}^{(k)}(x) S_{2,\lambda}(n,m) = \sum_{l=0}^{n} \frac{(1)_{l,\lambda} x^l}{l^{k-1}} S_{1,\lambda}(n,l). \tag{65}$$

Proof. By replacing t with $e_\lambda(t) - 1$ in (59), we obtain

$$\begin{aligned} 1 + \mathrm{Ei}_{k,\lambda}\big(x(\log_\lambda(1+t))\big) &= \sum_{m=0}^{\infty} G_{m,\lambda}^{(k)}(x) \frac{(e_\lambda(t)-1)^m}{m!} \\ &= \sum_{m=0}^{\infty} G_{m,\lambda}^{(k)}(x) \sum_{n=m}^{\infty} S_{2,\lambda}(n,m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} G_{m,\lambda}^{(k)}(x) S_{2,\lambda}(n,m) \right) \frac{t^n}{n!}. \end{aligned} \tag{66}$$

On the other hand, we have

$$
\begin{aligned}
1+\mathrm{Ei}_{k,\lambda}\big(x(\log_{\lambda}(1+t))\big) &= 1+\sum_{l=1}^{\infty}\frac{(1)_{l,\lambda}x^{l}}{(l-1)!l^{k}}(\log_{\lambda}(1+t))^{l} \\
&= 1+\sum_{l=1}^{\infty}\frac{(1)_{l,\lambda}x^{l}}{l^{k-1}}\sum_{n=l}^{\infty}S_{1,\lambda}(n,l)\frac{t^{n}}{n!} \\
&= 1+\sum_{n=1}^{\infty}\sum_{l=1}^{n}\frac{(1)_{l,\lambda}x^{l}}{l^{k-1}}S_{1,\lambda}(n,l)\frac{t^{n}}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{(1)_{l,\lambda}x^{l}}{l^{k-1}}S_{1,\lambda}(n,l)\frac{t^{n}}{n!}.
\end{aligned} \tag{67}
$$

In view of (66) and (67), we obtain the result. □

Theorem 14. *Let $k \in \mathbb{Z}$ and $n \geq 1$. Then*

$$
\sum_{m=0}^{n} G_{m,\lambda}^{(k)}(x)S_{J,\lambda}^{(2)}(n,m) = \frac{(1)_{n,\lambda}x^{n}}{n^{k-1}}. \tag{68}
$$

Proof. By replacing t with $e_{\lambda}(e_{\lambda}(t)-1)-1$ in (59), we obtain

$$
\begin{aligned}
1+\mathrm{Ei}_{k,\lambda}(xt)) &= \sum_{m=0}^{\infty} G_{m,\lambda}^{(k)}(x)\frac{(e_{\lambda}(e_{\lambda}(t)-1)-1)^{m}}{m!} \\
&= \sum_{m=0}^{\infty} G_{m,\lambda}^{(k)}(x)\sum_{n=m}^{\infty}S_{J,\lambda}^{(2)}(n,m)\frac{t^{n}}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} G_{m,\lambda}^{(k)}(x)S_{J,\lambda}^{(2)}(n,m)\right)\frac{t^{n}}{n!}.
\end{aligned} \tag{69}
$$

On the other hand, we have

$$
\begin{aligned}
1+\mathrm{Ei}_{k,\lambda}(xt)) &= 1+\sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}x^{n}}{(n-1)!n^{k}}t^{l} \\
&= 1+\sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}x^{n}}{n^{k-1}}\frac{t^{n}}{n!} \\
&= \sum_{n=0}^{\infty}\frac{(1)_{n,\lambda}x^{n}}{n^{k-1}}\frac{t^{n}}{n!}.
\end{aligned} \tag{70}
$$

In view of (69) and (70), we obtain the result. □

Theorem 15. *Let $k \in \mathbb{Z}$ and $n \geq 1$. Then*

$$
\sum_{m=0}^{n} J_{m,\lambda}^{(k)}(x)S_{J,\lambda}^{(1)}(n,m) = \frac{(1)_{n,\lambda}x^{n}}{n^{k-1}}. \tag{71}
$$

Proof. By replacing t with $\log_{\lambda}\big(\log_{\lambda}(1+t)+1\big)$ in (29) and using (21), we obtain

$$
\begin{aligned}
1+\mathrm{Ei}_{k,\lambda}(xt)) &= \sum_{m=0}^{\infty} J_{m,\lambda}^{(k)}(x)\frac{(\log_{\lambda}(\log_{\lambda}(1+t)+1))^{m}}{m!} \\
&= \sum_{m=0}^{\infty} J_{m,\lambda}^{(k)}(x)\sum_{n=m}^{\infty}S_{J,\lambda}^{(1)}(n,m)\frac{t^{n}}{n!}
\end{aligned} \tag{72}
$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} J_{m,\lambda}^{(k)}(x) S_{J,\lambda}^{(1)}(n,m) \right) \frac{t^n}{n!}.$$

In view of (70) and (72), we obtain the result. □

Theorem 16. *Let $k \in \mathbb{Z}$ and $n \geq 1$. Then*

$$\sum_{m=0}^{n} J_{m,\lambda}^{(k)}(x) S_{J,\lambda}^{(1)}(n,m) = \sum_{m=0}^{n} G_{m,\lambda}^{(k)}(x) S_{J,\lambda}^{(2)}(n,m). \tag{73}$$

Proof. By using (70) and (72), the complete proof the theorem. □

3. Degenerate Unipoly-Jindalrae and Unipoly-Gaenari Polynomials

In this section, we define the degenerate unipoly-Jindalrae and unipoly-Gaenari polynomials by using of the degenerate unipoly functions attached to polynomials $p(x)$ and we give explicit expressions and identities involving those polynomials.

Here, we define the degenerate unipoly-Jindalrae polynomials attached to polynomials $p(x)$ by

$$1 + u_{k,\lambda}(x(e_\lambda(e_\lambda(t)-1)-1)|p) = \sum_{n=0}^{\infty} J_{n,\lambda,p}^{(k)}(x) \frac{t^n}{n!}. \tag{74}$$

In the case when $x = 1$, $J_{n,\lambda,p}^{(k)} = J_{n,\lambda,p}^{(k)}(1)$ are called the degenerate unipoly-Jindalrae numbers attached to p.

From (74), we see

$$\sum_{n=0}^{\infty} J_{n,\lambda,1/\Gamma}^{(k)} \frac{t^n}{n!} = 1 + u_{k,\lambda}(x(e_\lambda(e_\lambda(t)-1)-1)|1/\Gamma) = \sum_{r=1}^{\infty} \frac{(1)_{r,\lambda}(e_\lambda(e_\lambda(t)-1)-1)^r}{r^k (r-1)!} \tag{75}$$

$$= 1 + \mathrm{Ei}_{k,\lambda}((e_\lambda(e_\lambda(t)-1)-1)) = \sum_{n=0}^{\infty} J_{n,\lambda}^{(k)} \frac{t^n}{n!}.$$

Thus, by (75), we have

$$J_{n,\lambda,1/\Gamma}^{(k)} = J_{n,\lambda}^{(k)} \ \ (n \geq 0).$$

Theorem 17. *For $k \in \mathbb{Z}$, we have*

$$J_{n,\lambda,p}^{(k)}(x) = \sum_{m=1}^{n} \sum_{l=1}^{m} \frac{p(l)(1)_{l,\lambda} x^l l!}{l^k} S_{2,\lambda}(m,l) S_{2,\lambda}(n,m). \tag{76}$$

Proof. From (19) and (74), we note that

$$\begin{aligned}
\mathrm{u}_{k,\lambda}(x(e_\lambda(e_\lambda(1)-1)-1)|p) &= \sum_{l=1}^{\infty} \frac{p(l)(1)_{l,\lambda} x^l l!}{l^k} \frac{[e_\lambda(e_\lambda(t)-1)-1]^l}{l!} \\
&= \sum_{l=1}^{\infty} \frac{p(l)(1)_{l,\lambda} x^l l!}{l^k} \sum_{m=l}^{\infty} S_{2,\lambda}(m,l) \frac{[e_\lambda(t)-1]^m}{m!} \\
&= \sum_{m=1}^{\infty} \sum_{l=1}^{m} \frac{p(l)(1)_{l,\lambda} x^l l!}{l^k} S_{2,\lambda}(m,l) \sum_{n=m}^{\infty} S_{2,\lambda}(n,m) \frac{t^n}{n!} \\
&= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} \sum_{l=1}^{m} \frac{p(l)(1)_{l,\lambda} x^l l!}{l^k} S_{2,\lambda}(m,l) S_{2,\lambda}(n,m) \right) \frac{t^n}{n!}.
\end{aligned} \tag{77}$$

Therefore, by Equations (74) and (77), we obtain the result. □

Theorem 18. *Let $k \in \mathbb{Z}$ and $n \geq 0$; we have*

$$J^{(k)}_{n,\lambda,p}(x) = \sum_{m=1}^{n} \frac{p(l)(1)_{l,\lambda} x^l l!}{l^k} S^{(2)}_{J,\lambda}(n,l). \tag{78}$$

Proof. From (22) and (74), we have

$$\begin{aligned} \mathrm{u}_{k,\lambda}(x(e_\lambda(e_\lambda(1)-1)-1)|p) &= \sum_{l=1}^{\infty} \frac{p(l)(1)_{l,\lambda} x^l l!}{l^k} \frac{[e_\lambda(e_\lambda(t)-1)-1]^l}{l!} \\ &= \sum_{l=1}^{\infty} \frac{p(l)(1)_{l,\lambda} x^l l!}{l^k} \sum_{n=l}^{\infty} S^{(2)}_{J,\lambda}(n,l) \frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} \frac{p(l)(1)_{l,\lambda} x^l l!}{l^k} S^{(2)}_{J,\lambda}(n,l) \right) \frac{t^n}{n!}. \end{aligned} \tag{79}$$

Therefore, by Equations (74) and (79), we obtain the result. □

Theorem 19. *Let $k \in \mathbb{Z}$ and $n \geq 0$; we have*

$$\sum_{l=0}^{n} J^{(k)}_{l,\lambda,p}(x) S_{1,\lambda}(n,l) = \sum_{l=0}^{n} \frac{p(l)(1)_{l,\lambda} x^l l!}{l^k} S_{2,\lambda}(n,l). \tag{80}$$

Proof. Replacing t with $\log_\lambda(1+t)$ in (74), we obtain

$$\begin{aligned} 1 + \mathrm{u}_{k,\lambda}(x((e_\lambda(t)-1))|p) &= \sum_{l=0}^{\infty} J^{(k)}_{l,\lambda,p}(x) \frac{[\log_\lambda(1+t)]^j}{j!} \\ &= \sum_{l=0}^{\infty} J^{(k)}_{l,\lambda,p}(x) \sum_{n=l}^{\infty} S_{1,\lambda}(n,l) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^{n} J^{(k)}_{l,\lambda,p}(x) S_{1,\lambda}(n,l) \frac{t^n}{n!}. \end{aligned} \tag{81}$$

On the other hand,

$$\begin{aligned} 1 + \mathrm{u}_{k,\lambda}(x((e_\lambda(t)-1))|p) &= 1 + \sum_{l=1}^{\infty} \frac{p(l)(1)_{l,\lambda} x^l l!}{l^k} \frac{[e_\lambda(t)-1]^l}{l!} \\ &= 1 + \sum_{l=1}^{\infty} \frac{p(l)(1)_{l,\lambda} x^l l!}{l^k} \sum_{n=l}^{\infty} S_{2,\lambda}(n,l) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \frac{p(l)(1)_{l,\lambda} x^l l!}{l^k} S_{2,\lambda}(n,l) \right) \frac{t^n}{n!}. \end{aligned} \tag{82}$$

Therefore, by comparing the coefficients of t on both sides of Equations (81) and (82), we obtain the result. □

Theorem 20. *Let $k \in \mathbb{Z}$ and $n \geq 0$; we have*

$$J^{k)}_{n,\lambda,p}(x) = \sum_{j=}^{n} \sum_{l=0}^{j} \frac{p(l)(1)_{l,\lambda} x^l l!}{l^k} S_{2,\lambda}(j,l) S_{2,\lambda}(n,j). \tag{83}$$

Proof. Replacing t with $e_\lambda(t)-1$ in (74), we obtain

$$\begin{aligned}1+u_{k,\lambda}(x(e_\lambda(e_\lambda(1)-1)-1)|p) &= \sum_{j=0}^{\infty}\sum_{l=0}^{j}\frac{p(l)(1)_{l,\lambda}x^l l!}{l^k}S_{2,\lambda}(j,l)\frac{(e_\lambda(t)-1)^j}{j!} \\ &= \sum_{j=0}^{\infty}\sum_{l=0}^{j}\frac{p(l)(1)_{l,\lambda}x^l l!}{l^k}S_{2,\lambda}(j,l)\sum_{n=j}^{\infty}S_{2,\lambda}(n,j)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{j=}^{n}\sum_{l=0}^{j}\frac{p(l)(1)_{l,\lambda}x^l l!}{l^k}S_{2,\lambda}(j,l)S_{2,\lambda}(n,j)\frac{t^n}{n!}.\end{aligned} \tag{84}$$

Therefore, by (74) and (84), we obtain the result. □

Now, we define the degenerate unipoly-Gaenari polynomials given by

$$1+u_{k,\lambda}\big(x(\log_\lambda(\log_\lambda(1+t))+1)|p\big) = \sum_{n=0}^{\infty}G_{n,\lambda,p}^{(k)}(x)\frac{t^n}{n!} \quad (k\in\mathbb{Z}). \tag{85}$$

When $x=1$, $G_{n,\lambda,p}^{(k)}=G_{n,\lambda,p}^{(k)}(1)$ are called the unipoly-Gaenari numbers attached to p.

Theorem 21. *Let $k\in\mathbb{Z}$ and $n\geq 1$. Then*

$$G_{n,\lambda,p}^{(k)}(x) = \sum_{l=1}^{j}\sum_{j=1}^{n}\frac{p(l)(1)_{l,\lambda}x^l l!}{l^k}S_{1,\lambda}(j,l)S_{1,\lambda}(n,j). \tag{86}$$

Proof. From (18) and (85), we observe that

$$\begin{aligned}u_{k,\lambda}\big(x(\log_\lambda(\log_\lambda(1+t))+1)|p\big) &= \sum_{l=1}^{\infty}\frac{p(l)(1)_{l,\lambda}x^l l!}{l^k}\frac{1}{l!}[\log_\lambda(\log_\lambda(1+t))+1]^l \\ &= \sum_{l=1}^{\infty}\frac{p(l)(1)_{l,\lambda}x^l l!}{l^k}\sum_{j=l}^{\infty}S_{1,\lambda}(j,l)\frac{1}{j!}[\log_\lambda(1+t)]^j \\ &= \sum_{l=1}^{\infty}\frac{p(l)(1)_{l,\lambda}x^l l!}{l^k}\sum_{j=l}^{\infty}S_{1,\lambda}(j,l)\sum_{n=j}^{\infty}S_{1,\lambda}(n,j)\frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty}\sum_{l=1}^{j}\sum_{j=1}^{n}\frac{p(l)(1)_{l,\lambda}x^l l!}{l^k}S_{1,\lambda}(j,l)S_{1,\lambda}(n,j)\frac{t^n}{n!}.\end{aligned} \tag{87}$$

In view of (85) and (87), we obtain the desired result. □

Theorem 22. *Let $k\in\mathbb{Z}$ and $n\geq 1$. Then*

$$G_{n,\lambda,p}^{(k)}(x) = \sum_{l=1}^{n}\frac{p(l)(1)_{l,\lambda}x^l l!}{l^k}S_{J,\lambda}^{(1)}(n,l). \tag{88}$$

Proof. From (21) and (85), we observe that

$$\begin{aligned}u_{k,\lambda}\big(x(\log_\lambda(\log_\lambda(1+t))+1)|p\big) &= \sum_{l=1}^{\infty}\frac{p(l)(1)_{l,\lambda}x^l l!}{l^k}\frac{1}{l!}[\log_\lambda(\log_\lambda(1+t))+1]^l \\ &= \sum_{l=1}^{\infty}\frac{p(l)(1)_{l,\lambda}x^l l!}{l^k}\sum_{n=l}^{\infty}S_{J,\lambda}^{(1)}(n,l)\end{aligned} \tag{89}$$

$$= \sum_{n=1}^{\infty} \sum_{l=1}^{n} \frac{p(l)(1)_{l,\lambda} x^l l!}{l^k} S_{J,\lambda}^{(1)}(n,l) \frac{t^n}{n!}.$$

In view of (85) and (89), we obtain the desired result. □

Theorem 23. *Let $k \in \mathbb{Z}$ and $n \geq 1$. Then*

$$\sum_{m=0}^{n} G_{m,\lambda,p}^{(k)}(x) S_{2,\lambda}(n,m) = \sum_{l=0}^{n} \frac{p(l)(1)_{l,\lambda} x^l l!}{l^k} S_{1,\lambda}(n,l). \tag{90}$$

Proof. By replacing t with $e_\lambda(t) - 1$ in (85), we obtain

$$\begin{aligned} 1 + u_{k,\lambda}\big(x(\log_\lambda(1+t))|p\big) &= \sum_{m=0}^{\infty} G_{m,\lambda,p}^{(k)}(x) \frac{(e_\lambda(t)-1)^m}{m!} \\ &= \sum_{m=0}^{\infty} G_{m,\lambda,p}^{(k)}(x) \sum_{n=m}^{\infty} S_{2,\lambda}(n,m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} G_{m,\lambda,p}^{(k)}(x) S_{2,\lambda}(n,m) \right) \frac{t^n}{n!}. \end{aligned} \tag{91}$$

On the other hand, we have

$$\begin{aligned} 1 + u_{k,\lambda,p}\big(x(\log_\lambda(1+t))|p\big) &= 1 + \sum_{l=1}^{\infty} \frac{p(l)(1)_{l,\lambda} x^l l!}{l^k} (\log_\lambda(1+t))^l \\ &= 1 + \sum_{l=1}^{\infty} \frac{p(l)(1)_{l,\lambda} x^l l!}{l^k} \sum_{n=l}^{\infty} S_{1,\lambda}(n,l) \frac{t^n}{n!} \\ &= 1 + \sum_{n=1}^{\infty} \sum_{l=1}^{n} \frac{p(l)(1)_{l,\lambda} x^l l!}{l^k} S_{1,\lambda}(n,l) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \frac{p(l)(1)_{l,\lambda} x^l l!}{l^k} S_{1,\lambda}(n,l) \frac{t^n}{n!}. \end{aligned} \tag{92}$$

In view of (91) and (92), we obtain the result. □

Theorem 24. *Let $k \in \mathbb{Z}$ and $n \geq 1$. Then*

$$\sum_{m=0}^{n} G_{m,\lambda,p}^{(k)}(x) S_{J,\lambda}^{(2)}(n,m) = \frac{p(n)(1)_{n,\lambda} x^n n!}{n^k}. \tag{93}$$

Proof. By replacing t with $e_\lambda(e_\lambda(t)-1) - 1$ in (85), we obtain

$$\begin{aligned} 1 + u_{k,\lambda}(xt|p)) &= \sum_{m=0}^{\infty} G_{m,\lambda,p}^{(k)}(x) \frac{(e_\lambda(e_\lambda(t)-1)-1)^m}{m!} \\ &= \sum_{m=0}^{\infty} G_{m,\lambda,p}^{(k)}(x) \sum_{n=m}^{\infty} S_{J,\lambda}^{(2)}(n,m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} G_{m,\lambda,p}^{(k)}(x) S_{J,\lambda}^{(2)}(n,m) \right) \frac{t^n}{n!}. \end{aligned} \tag{94}$$

On the other hand, we have

$$1 + u_{k,\lambda}(xt|p)) = 1 + \sum_{n=1}^{\infty} \frac{p(n)(1)_{n,\lambda} x^n n!!}{n^k} \frac{t^n}{n!}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{p(n)(1)_{n,\lambda} x^n n!}{n^k} \frac{t^n}{n!} \tag{95}$$
$$= \sum_{n=0}^{\infty} \frac{p(n)(1)_{n,\lambda} x^n n!}{n^k} \frac{t^n}{n!}.$$

In view of (94) and (95), we obtain the result. □

Theorem 25. *Let $k \in \mathbb{Z}$ and $n \geq 1$. Then*

$$\sum_{m=0}^{n} J_{m,\lambda,p}^{(k)}(x) S_{J,\lambda}^{(1)}(n,m) = \frac{p(n)(1)_{n,\lambda} x^n n!}{n^k}. \tag{96}$$

Proof. By replacing t with $\log_\lambda\big(\log_\lambda(1+t)+1\big)$ in (74) and using (21), we obtain

$$1 + u_{k,\lambda}(xt|p)) = \sum_{m=0}^{\infty} J_{m,\lambda,p}^{(k)}(x) \frac{\big(\log_\lambda\big(\log_\lambda(1+t)+1\big)\big)^m}{m!}$$
$$= \sum_{m=0}^{\infty} J_{m,\lambda,p}^{(k)}(x) \sum_{n=m}^{\infty} S_{J,\lambda}^{(1)}(n,m) \frac{t^n}{n!} \tag{97}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} J_{m,\lambda,p}^{(k)}(x) S_{J,\lambda}^{(1)}(n,m) \right) \frac{t^n}{n!}.$$

In view of (95) and (97), we obtain the result. □

Theorem 26. *Let $k \in \mathbb{Z}$ and $n \geq 1$. Then*

$$\sum_{m=0}^{n} J_{m,\lambda,p}^{(k)}(x) S_{J,\lambda}^{(1)}(n,m) = \sum_{m=0}^{n} G_{m,\lambda,p}^{(k)}(x) S_{J,\lambda}^{(2)}(n,m).$$

Proof. By using (94) and (97), we complete the proof of the theorem. □

4. Conclusions

Inspired by the contributions of Kim et al., as shown in [19], we have introduced the poly-Jindalrae and poly-Gaenari polynomials via an innovative utilization of the polyexponential function. Subsequently, we have meticulously derived explicit identities, encompassing the Jindalrae–Stirling numbers of the first and second categories, the degenerate Stirling numbers of both kinds, and the degenerate Bell polynomials. Moreover, we have extended our investigation to the realm of degenerate unipoly functions associated with the polynomial $p(x)$, resulting in the derivation of unipoly-Jindalrae and unipoly-Gaenari polynomials, replete with explicit expressions.

As we conclude this work, we believe that this paper will have a potential applications of our results in the realms of science, engineering, and other mathematical disciplines in near future such as statistics, probability, differential equations, etc.

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