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Symmetries of the Energy–Momentum Tensor for Static Plane Symmetric Spacetimes

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Abstract: This article explores matter collineations (MCs) of static plane-symmetric spacetimes, considering the stress–energy tensor in its contravariant and mixed forms. We solve the MC equations in two cases: when the energy–momentum tensor is nondegenerate and degenerate. For the case of a degenerate energy–momentum tensor, we employ a direct integration technique to solve the MC equations, which leads to an infinite-dimensional Lie algebra. On the other hand, when considering the nondegenerate energy–momentum tensor, the contravariant form results in a finite-dimensional Lie algebra with dimensions of either 4 or 10. However, in the case of the mixed form of the energy–momentum tensor, the dimension of the Lie algebra is infinite. Moreover, the obtained MCs are compared with those already found for covariant stress–energy.

Keywords: matter collineations; static plane-symmetric spacetimes; contravariant and mixed energy–momentum tensor

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1. Introduction

General Relativity (GR) is an intriguing space, time, and gravitation theory. In this theory, Einstein proposed that the presence of matter and energy induces the curvature of spacetime. Mathematically, GR is described by a set of ten interconnected nonlinear partial differential equations, known as Einstein field equations (EFE) [1]:

$$G_{ab} = R_{ab} - \frac{R}{2}\, g_{ab} = k\, T_{ab}, \tag{1}$$

where G_{ab}, R_{ab}, T_{ab}, and g_{ab} correspond to Einstein, Ricci, energy–momentum, and metric tensors, respectively. R denotes the Ricci scalar, while k defines the gravitational coupling between geometry and matter. Solving EFEs for exact solutions is a huge challenge due to their nonlinear nature. The literature contains only a limited number of physically significant exact solutions of the EFEs. Equation (1) represents the covariant form of EFEs, which can also be written in a contravariant form as:

$$G^{ab} = kT^{ab}. \tag{2}$$

Although these equations appear simple, they are highly nonlinear and pose significant challenges in finding solutions. However, some physically interesting and exact solutions of EFEs are studied in [1–3].

Spacetime symmetries are crucial for determining the exact solutions of EFEs and understanding their physical implications. For instance, spherical symmetry plays a significant role in deriving the Schwarzschild solution and explaining phenomena like the absence of gravitational radiation in a pulsating spherical star. These symmetries are characterized by specific vector fields that possess preservation properties, such as preserving spacetime geodesics, the metric tensor, the curvature tensor, or the energy–momentum tensor.

Mathematically, a smooth vector field X on a spacetime manifold M is termed as a matter collineation if it satisfies the condition:

$$\mathcal{L}_X T^{ab} = 0, \tag{3}$$

where $\mathcal{L}$ stands for the Lie derivative operator, X is a collineation vector, and T^{ab} is the contravariant form of stress–energy tensor. In its explicit form, the preceding equation can be written as:

$$T^{ab}{}_{,c}\, X^c - T^{ac} X^b{}_{,c} - T^{bc} X^a{}_{,c} = 0. \tag{4}$$

Studying the symmetries of the energy–momentum tensor serves a dual purpose. First, there is mathematical interest in exploring the invariance properties of the Einstein tensor, a geometric entity central to general relativity. This tensor relates to spacetime's matter content, T_{ab}, through Einstein's field equations (EFEs). Second, when seeking exact solutions to EFEs, simplifying assumptions about symmetries of spacetime metric are made. This gives rise to the "symmetry inheritance problem", which essentially involves understanding how physical fields reflect the metric's symmetries. A key question emerges: how does the existence of symmetry in the physical fields impact the metric tensor of spacetime? To address this, we delve into the solutions of equation $L_X T_{ab} = 0$ and explore the effects of these vector fields on the spacetime metric. This study intersects with matter and Ricci collinearity.

In [4], Sharif classified cylindrically symmetric static spacetimes using MCs, considering both degenerate and nondegenerate stress–energy tensors. In the case of a nondegenerate stress–energy tensor, the dimension of the Lie algebra turns out to be three, four, five, six, seven, or ten. In the case of a degenerate stress–energy tensor, a three-, four-, five- or ten-dimensional Lie algebra was obtained. In 2007, Sharif [5] studied MCs for plane-symmetric spacetimes. In this research, the author specifically examined the degenerate case of the stress–energy tensor and identified three exciting issues. The results of these cases led to a finite-dimensional group of MCs with dimensions four, six, and ten. Four obtained MCs corresponded to isometries, while the rest were proper MCs. In [6], Camci and Sahin classified Bianchi type-II spacetimes according to their MCs. In the nondegenerate stress–energy tensor, they derived a finite-dimensional Lie algebra of MCs of dimensions three, four, or five. For the case when the stress–energy tensor was degenerate, they mainly obtained an infinite-dimensional Lie algebra. However, it was concluded that, in certain cases, the dimension of the Lie algebra could be three, four, or five.

In 2003, Sharif [7] classified static plane-symmetric spacetimes based on the covariant form of the stress–energy tensor. For the nondegenerate case of the stress–energy tensor, the author determined four, five, six, seven, or ten independent MCs. Among these, four were isometries, while the rest were proper MCs. In the case of a degenerate stress–energy tensor, three interesting cases were discussed, revealing a finite-dimensional group of MCs. In these instances, the dimension of the Lie algebra was either four, six, or ten—four of them being isometries and the remaining representing proper matter collineations.

Camci and Sharif [8] examined MCs for Bianchi types I, III and Kantowski–Sachs spacetimes in both cases where the stress–energy tensor is degenerate and nondegenerate. In the case of a nondegenerate energy–momentum tensor, the dimension of Lie algebra turned out to be 4, 6, or 10, while the degenerate case of T_{ab} gave an infinite-dimensional Lie algebra. Camci [9] studied a complete classification of Bianchi type V spacetimes based on their matter collineations. In the majority of cases, the author obtained an infinite number of MCs for these spacetimes when the energy–momentum tensor is supposed to be degenerate, whereas the dimension of the Lie algebra of MCs was acquired to be four, five, six, or seven when the energy–momentum tensor is nondegenerate. The classification of some other spacetimes via matter collineations can be viewed in [10–18].

In 2007, Sharif and Ismaeel [19] classified spherically symmetric spacetimes via MCs in three different ways by considering the energy–momentum tensor in the covariant (T_{ab}), contravariant (T^{ab}) and mixed (T^a_b) forms. The authors compared their results of these three cases and proved that they are not equivalent in general. An interesting observation of the authors was that unlike the case of the covariant form of the energy–momentum tensor, where the dimension of the algebra of MCs for the nondegenerate energy–momentum tensor is always finite, this algebra was found to be infinite-dimensional for the mixed form of the energy–momentum tensor. In the literature, such a comparison is not established for any other spacetime. The present research aims to employ the same idea to investigate MCs for static plane-symmetric spacetimes by considering the energy–momentum tensor in contravariant and mixed forms. The MCs for the same spacetimes for covariant the energy–momentum tensor have already been investigated in [7] .

In Section 2, we find MCs for the mentioned spacetimes by considering the energy–momentum tensor in its contravariant form, while Section 3 presents MCs of these spacetimes for mixed forms of energy–momentum tensor. The same section also presents a comparison of the obtained results with those of the contravariant form of the energy–momentum tensor. The Secion 4 shows a summary of the present work.

2. MCs for Contravariant Energy–Momentum Tensor

The metric of static plane-symmetric spacetimes is expressed as [7]:

$$ds^2 = e^{A(x)}dt^2 - dx^2 - e^{B(x)}[dy^2 + dz^2], \tag{5}$$

where A and B are the arbitrary functions of x only. The motivation behind studying symmetries of these geometries is that they hold significance within the realm of physics as they contribute to the creation of Kasner's spatially homogeneous solutions in the field equations [20], as well as the renowned Taub's solution for the universe [21]. These solution classes have also found utility in the examination of the motion and behavior of non-rotating rigid bodies [21,22]. The metric (5) exhibits a minimum of four Killing vectors, given by:

$$X_{(1)} = \partial_t,\ X_{(2)} = \partial_y,\ X_{(3)} = \partial_z,\ X_{(4)} = z\partial_y - y\partial_z. \tag{6}$$

The non-vanishing Ricci tensor components for the above metric are:

$$\begin{aligned} R_{00} &= \frac{e^{A(x)}}{4}\Big[2A''(x) + A'^2(x) + 2A'(x)B'(x)\Big], \\ R_{11} &= \frac{1}{4}\Big[-2A''(x) - A'^2(x) - 4B''(x) - 2B'^2(x)\Big], \\ R_{22} &= R_{33} = \frac{e^{B(x)}}{4}\Big[A'(x)B'(x) + 2B''(x) + B'^2(x)\Big], \end{aligned} \tag{7}$$

where the primes represent the derivative of the metric functions with respect to x. The Ricci scalar R for the static plane symmetric spacetime can be calculated using $R = g^{ab}R_{ab}$, so that:

$$R = \frac{1}{4}\big[4A''(x) + 2A'^2(x) + 3A'(x)B'(x) + 6B''(x) + 4B'^2(x)\big]. \tag{8}$$

Substituting (7) and (8) in Equation (1), we obtain the covariant stress–energy tensor components as:

$$\begin{aligned} T_{00} &= \frac{-e^{A(x)}}{4}[3B'^2(x)+4B''(x)], \\ T_{11} &= \frac{1}{4}[B'^2(x)+2A'(x)B'(x)], \\ T_{22} &= T_{33}=\frac{e^{B(x)}}{4}[B'^2(x)+A'(x)B'(x)+A'^2(x)+2A''(x)+2B''(x)]. \end{aligned} \tag{9}$$

From the above covariant stress–energy tensor, the components of the contravariant stress–energy tensor can be obtained by the relation $T^{ab}=g^{ak}g^{bl}T_{kl}$, and hence:

$$\begin{aligned} T^{00} &= \frac{-1}{4e^{A(x)}}[3B'^2(x)+4B''(x)], \\ T^{11} &= \frac{1}{4}[B'^2(x)+2A'(x)B'(x)], \\ T^{\,22} &= T^{33}=\frac{1}{4e^{B(x)}}[B'^2(x)+A'(x)B'(x)+A'^2(x)+2A''(x)+2B''(x)]. \end{aligned} \tag{10}$$

From these values of the contravariant stress–energy tensor, the components of mixed energy–momentum tensor can be obtained as $T^a_b=T^{ac}g_{bc}$, so that:

$$\begin{aligned} T^0_0 &= \frac{-1}{4}[3B'^2(x)+4B''(x)], \\ T^1_1 &= \frac{-1}{4}[B'^2(x)+2A'(x)B'(x)], \\ T^2_2 &= T^3_3=\frac{-1}{4}[B'^2(x)+A'(x)B'(x)+A'^2(x)+2A''(x)+2B''(x)]. \end{aligned} \tag{11}$$

Using the values from (10) in (4), we obtain the following ten MC equations:

$$\begin{aligned} T^{00}_{,1}X^1-2T^{00}X^0_{,0}&=0, \\ T^{00}X^1_{,0}+T^{11}X^0_{,1}&=0, \\ T^{00}X^2_{,0}+T^{\,22}X^0_{,2}&=0, \\ T^{00}X^3_{,0}+T^{\,22}X^0_{,3}&=0, \\ T^{11}_{,1}X^1-2T^{11}X^1_{,1}&=0, \\ T^{11}X^2_{,1}+T^{\,22}X^1_{,2}&=0, \\ T^{11}X^3_{,1}+T^{\,22}X^1_{,3}&=0, \\ T^{\,22}_{,1}X^1-2T^{\,22}X^2_{,2}&=0, \\ T^{\,22}(X^3_{,2}+X^2_{,3})&=0, \\ T^{\,22}_{,1}X^1-2T^{\,22}X^3{}_{,3}&=0. \end{aligned} \tag{12}$$

where the components of the vector field X generating MCs are represented by X^0, X^1, X^2, X^3 and the commas in the subscripts represent partial derivatives with respect to spacetime coordinates. In the following sections, we solve these equations for degenerate and non-degenerate contravariant energy–momentum tensor, to obtain the explicit form of MCs for static plane-symmetric spacetimes. For this purpose, we use the direct integration approach. In this approach, The equations representing matter collineations are decoupled and integrated with respect to the coordinate variables to obtain the components of the vector field representing MCs in the explicit form. The process of solving MC equations ends when the components of the vector field representing MCs contains no unknown function, that is given in terms of arbitrary constants. The number of constants involved in the final form of the MC vector field gives the dimension of the algebra of MCs.

2.1. MCs for Non-Degenerate T^{ab}

In the case where the contravariant stress–energy tensor is non-degenerate, that is $det(T^{ab}) \neq 0$, it follows that $T^{00} \neq 0$, $T^{11} \neq 0$, and $T^{22} \neq 0$, implying that none of the contravariant stress–energy tensor components can equal zero. Integrating the system of Equation (12), we have obtained the following solution of the MC equations in the form of unknown functions, dependent solely on t and x:

$$\begin{aligned} X^0 &= -\frac{T^{00}}{T^{22}}\left[\frac{1}{2}(y^2+z^2)G^1_t(t,x) + zG^2_t(t,x) + yG^3_t(t,x)\right] + G^4(t,x), \\ X^1 &= -\frac{T^{11}}{T^{22}}\left[\frac{1}{2}(y^2+z^2)G^1_x(t,x) + zG^2_x(t,x) + yG^3_x(t,x)\right] + G^5(t,x), \\ X^2 &= c_1\left[\frac{yz^2}{2} - \frac{y^3}{6}\right] + c_2 yz + c_3\left[\frac{z^2}{2} - \frac{y^2}{2}\right] + c_4 z + yG^1(t,x) + G^3(t,x), \\ X^3 &= c_1\left[\frac{z^3}{6} - \frac{y^2 z}{2}\right] + c_2\left[\frac{z^2}{2} - \frac{y^2}{2}\right] - c_3 yz - c_4 z + zG^1(t,x) + G^2(t,x). \end{aligned} \tag{13}$$

Here, c_1,c_2,c_3 and c_4 represent arbitrary constants. Upon substituting the above values of X^a , for $a = 0,1,2,3$ into the system of Equation (12), it is observed that six out of ten equations are satisfied identically, resulting in the vanishing of the constant c_1. However, the remaining four equations yield the following integrability conditions that enforce particular constraints on T^{ab}:

$$\begin{aligned} T^{00}_{,1}G^1_x(t,x) - \frac{2T^{00}T^{00}}{T^{11}}G^1_{tt}(t,x) &= 0, \\ T^{00}_{,1}G^3_x(t,x) - \frac{2T^{00}T^{00}}{T^{11}}G^3_{tt}(t,x) &= 0, \\ T^{00}_{,1}G^2_x(t,x) - \frac{2T^{0}T^{0}}{T^{11}}G^2_{tt}(t,x) &= 0, \\ T^{00}_{,1}G^5(t,x) - 2T^{00}G^4_t(t,x) &= 0, \\ G^1_{tx}(t,x) + \frac{T^{22}}{2T^{00}}\left(\frac{T^{00}}{T^{22}}\right)_{,1}G^1_t(t,x) &= 0, \\ G^3_{tx}(t,x) + \frac{T^{22}}{2T^{00}}\left(\frac{T^{00}}{T^{22}}\right)_{,1}G^3_t(t,x) &= 0, \\ G^2_{tx}(t,x) + \frac{T^{22}}{2T^{00}}\left(\frac{T^{00}}{T^{22}}\right)_{,1}G^2_t(t,x) &= 0, \\ T^{00}G^5_t(t,x) + T^{11}G^4_x(t,x) &= 0, \\ -T^{11}_{,1}\frac{T^{11}}{2T^{22}}G^1_x(t,x) + T^{11}\left(\frac{T^{11}}{T^{22}}\right)_{,1}G^1_x(t,x) + \frac{T^{11}T^{11}}{T^{22}}G^1_{xx}(t,x) &= 0, \\ -T^{11}_{,1}\frac{T^{11}}{2T^{22}}G^3_x(t,x) + T^{11}\left(\frac{T^{11}}{T^{22}}\right)_{,1}G^3_x(t,x) + \frac{T^{11}T^{11}}{T^{22}}G^3_{xx}(t,x) &= 0, \\ -T^{11}_{,1}\frac{T^{1}}{2T^{22}}G^2_x(t,x) + T^{11}\left(\frac{T^{11}}{T^{22}}\right)_{,1}G^2_x(t,x) + \frac{T^{11}T^{1}}{T^{22}}G^2_{xx}(t,x) &= 0, \\ T^{11}_{,1}G^5(t,x) - 2T^{11}G^5_x(t,x) &= 0, \\ T^{22}_{,1}\frac{T^{11}}{2T^{22}}G^1_x(t,x) &= 0, \\ T^{22}_{,1}\frac{T^{1}}{2T^{22}}G^3_x(t,x) - c_3T^{22} &= 0, \\ T^{22}_{,1}\frac{T^{1}}{2T^{22}}G^2_x(t,x) + c_2T^{22} &= 0, \\ T^{22}_{,1}G^5(t,x) - 2T^{22}G^1(t,x) &= 0. \end{aligned} \tag{14}$$

Solving these integrability conditions allows us to derive the final expression for the vector field X that generates MCs. However, due to the highly nonlinear nature of these equations, it is impossible to solve them in a general manner. Hence, we consider different conditions on the derivatives of the contravariant stress–energy tensor components, leading to eight distinct cases as follows:

(I) $T^{00'} = 0$, $T^{11'} = 0$, and $T^{22'} = 0$ **(II)** $T^{00'} = 0$, $T^{11'} = 0$, and $T^{22'} \neq 0$
(III) $T^{00'} = 0$, $T^{22'} = 0$, and $T^{11'} \neq 0$ **(IV)** $T^{22'} = 0$, $T^{11'} = 0$, and $T^{00'} \neq 0$
(V) $T^{00'} \neq 0$, $T^{11'} \neq 0$, and $T^{22'} = 0$ **(VI)** $T^{00'} \neq 0$, $T^{22'} \neq 0$, and $T^{11'} = 0$
(VII) $T^{11'} \neq 0$, $T^{22'} \neq 0$, and $T^{00'} = 0$ **(VIII)** $T^{00'} \neq 0$, $T^{11'} = 0$, and $T^{22'} \neq 0$

We have solved Equation (14) for each of the aforementioned cases, giving the final form of MCs. We omit to write the basic calculations and present the results obtained for these eight cases in Table 1. In some cases, we have obtained a maximum (of ten) matter collineations, consisting of four basic Killing vectors and six additional matter collineations. In the remaining cases, the number of matter collineations turned out to be four, where all of these matter collineations are the same as the minimum Killing vectors of the spacetime under consideration.

Table 1. MCs for non-degenerate T^{ab}.

Case	Constraints	MCs
I	$T^{00'} = 0$ $T^{11'} = 0$ $T^{22'} = 0$	$X^0 = -c_5 z - c_6 y + c_7 x + c_9$, $X^1 = -c_{12} z - c_8 y - c_7 t + c_{10}$, $X^2 = c_4 z + c_6 t + c_8 x + c_{11}$, $X^3 = -c_4 y + c_5 t + c_{12} x + c_{13}$.
II	$T^{00'} = 0$ $T^{11'} = 0$ and $T^{22'} \neq 0$	$X^0 = c_5$, $X^1 = 0$, $X^2 = c_4 z + c_6$, $X^3 = -c_4 y + c_7$.
III	$T^{00'} = 0$ $T^{22'} = 0$ and $T^{11'} \neq 0$	$X^0 = -\left[c_4 z + c_6\right] + c_7 y \int \frac{dx}{\sqrt{T^{11}}} + c_8$, $X^1 = -T^{11}\left[\frac{c_9 z}{\sqrt{T^{11}}} + \frac{c_{10} y}{\sqrt{T^{11}}}\right] - \frac{c_7 t}{\sqrt{T^{11}}} + \sqrt{T^{11}} c_{11}$, $X^2 = c_4 z + c_6 t + c_{10} \int \frac{dx}{\sqrt{T^{11}}} + c_{12}$, $X^3 = -c_4 y + c_5 t + c_9 \int \frac{dx}{\sqrt{T^{11}}} + c_{13}$.
IV	$T^{22'} = 0$ $T^{11'} = 0$ and $T^{00'} \neq 0$	$X^0 = -\frac{1}{x^2}[xz\{-c_5 \sin t + c_6 \cos t\} + xy\{-c_7 \sin t + c_8 \cos t\}] + \frac{1}{x}\{-c_9 \sin t + c_{10} \cos t\} + c_{11}$, $X^1 = -z\{c_5 \cos t + c_6 \sin t\} + y\{c_7 \cos t + c_8 \sin t\} + c_{11} \cos t + c_{10} \sin t$, $X^2 = c_4 z + x\{c_7 \cos t + c_8 \sin t\} + c_{12}$, $X^3 = -c_4 y + x\{c_5 \cos t + c_6 \sin t\} + c_{13}$.
V	$T^{00'} \neq 0$ $T^{11'} \neq 0$ and $T^{22'} = 0$	$X^0 = c_5$, $X^1 = 0$, $X^2 = c_4 z + c_6$, $X^3 = -c_4 y + c_7$.
VI	$T^{00'} \neq 0$ $T^{22'} \neq 0$ and $T^{11'} = 0$	$X^0 = c_5$, $X^1 = 0$, $X^2 = c_4 z + c_6$, $X^3 = -c_4 y + c_7$.
VII	$T^{11'} \neq 0$ $T^{22'} \neq 0$ and $T^{00'} = 0$	$X^0 = c_5$, $X^1 = 0$, $X^2 = c_4 z + c_6$, $X^3 = -c_4 y + c_7$.
VIII	$T^{00'} \neq 0$ $T^{11'} \neq 0$ $T^{22'} \neq 0$	$X^0 = c_5$, $X^1 = 0$, $X^2 = c_4 z + c_6$, $X^3 = -c_4 y + c_7$.

2.2. MCs for Degenerate T^{ab}

When the contravariant form of stress–energy tensor is degenerate, that is *det* $T^{ab} = T^{00}T^{11}T^{\,22} = 0$, then one of the components T^{00}, T^{11} and $T^{\,22}$ must be zero, and hence we have the following six possibilities:

(D1) $T^{00} = 0,\ T^{11} = 0,$ and $T^{\,22} \neq 0$ **(D2)** $T^{11} = 0, T^{\,22} = 0,$ and $T^{00} \neq 0$
(D3) $T^{00} = 0,\ T^{\,22} = 0,$ and $T^{11} \neq 0$ **(D4)** $T^{00} = 0, T^{11} \neq 0, T^{\,22} \neq 0$
(D5) $T^{00} \neq 0,\ T^{11} \neq 0,$ and $T^{\,22} = 0$ **(D6)** $T^{\,22} \neq 0, T^{00} \neq 0,$ and $T^{11} = 0$

We have solved the set of MC equations, Equation (12), for all the above six cases, and as a result, we have obtained the components of the vector field X involving arbitrary functions, showing that the dimension of the group of MCs is infinite in all these cases. Table 2 presents the MCs obtained for the six cases.

Table 2. MCs for degenerate T^{ab}.

Case	Constraints	MCs
I	$T^{00} = 0\ \ T^{11} = 0$ and $T^{\,22} \neq 0$	$X^0 = G^1(t,x),\ X^1 = G^2(t,x),$ $X^2 = \frac{y}{2}\frac{T^2_{,1}}{T^{\,22}}G^2(t,x) + F^1(t,x,z),$ $X^3 = \frac{z}{2}\frac{T^{\,22}_{,1}}{T^{\,22}}G^2(t,x) + F^2(t,x,z).$
II	$T^{11} = 0\ \ T^{\,22} = 0$ and $T^{00} \neq 0$	$X^0 = \frac{T^{00}_{,1}}{2T^{00}}F^1(x,y,z) + F^4(x,y,z),\ X^1 = F^1(x,y,z),$ $X^2 = F^2(x,y,z),\ X^3 = F^3(x,y,z).$
III	$T^{00} = 0\ \ T^{\,22} = 0$ and $T^{11} \neq 0$	$X^0 = F^1(t,y,z),\ X^1 = \sqrt{T^1}F^4(t,y,x),\ X^2 = F^2(t,y,z),$ $X^3 = F^3(t,y,z).$
IV	$T^{00} = 0$ and $T^{11} \neq 0, T^{\,22} \neq 0$	$X^0 = H^1(t),\ X^1 = 0,$ $X^2 = -zH^2(t) + H^4(t),\ X^3 = yH^2(t) + H^3(t).$
V	$T^{00} \neq 0\ \ T^{11} \neq 0\ T^{\,22} = 0$	$X^0 = \frac{1}{2}\frac{T^{00}_{,1}\sqrt{T^{11}}}{T^{00}}G^3(y,z) + F^1(x,y,z), X^1 = 0,$ $X^2 = G^1(y,z),\ X^3 = G^2(y,z).$
VI	$T^{\,22} \neq 0, T^{00} \neq 0$ and $T^{11} = 0$	$X^0 = \frac{T^{00}_{,1}}{2T^{00}}tH^1(x) - \frac{T^{00}}{T^{\,22}}yH^2(x) + G^1(x,z), X^1 = H^1(x),$ $X^2 = \frac{T^{\,22}_{,1}}{2T^{\,22}}yH^1(x) + tH^2(x) - zH^3(x) + H^5(x),$ $X^3 = \frac{T^{\,22}_{,1}}{2T^{\,22}}zH^1(x) - \frac{T^{\,22}}{T^{00}}tG^1_z(x,z) + yH^3(x) + H^4(x).$

3. MCs for Mixed Energy–Momentum Tensor

In this section, we explore MCs for static plane-symmetric spacetimes by considering the energy–momentum tensor in its mixed form, that is T^a_b. Like the case of the contravariant energy–momentum tensor, we use the definition of MCs as:

$$\mathcal{L}_X T^a_b = 0, \tag{15}$$

where $\mathcal{L}$ represents the Lie derivative operator, X is a collineation vector and T^a_b is the energy–momentum tensor in its mixed form. The preceding equation can be written in its explicit form:

$$T^a_{b\,,c}\, X^c - T^c_b X^a{}_{,c} + T^a_c X^c{}_{,b} = 0. \tag{16}$$

Using the components of T^a_b, given in (11), in Equation (16), we obtain the following thirteen MC equations:

$$
\begin{aligned}
(T^0_0)'X^1 &= 0\\
(T^0_0 - T^1_1)X^0{}_{,1} &= 0\\
(T^0_0 - T^1_1)X^1{}_{,0} &= 0\\
(T^0_0 - T^2_2)X^0{}_{,2} &= 0\\
(T^0_0 - T^2_2)X^2{}_{,0} &= 0\\
(T^0_0 - T^2_2)X^0{}_{,3} &= 0\\
(T^0_0 - T^2_2)X^3{}_{,0} &= 0\\
(T^1_1)'X^1 &= 0\\
(T^1_1 - T^2_2)X^1{}_{,2} &= 0\\
(T^1_1 - T^2_2)X^2{}_{,1} &= 0\\
(T^1_1 - T^2_2)X^1{}_{,3} &= 0\\
(T^1_1 - T^2_2)X^3{}_{,1} &= 0\\
(T^2_2)'X^1 &= 0.
\end{aligned} \tag{17}
$$

To compare the obtained MCs for contravariant and mixed forms of energy–momentum tensor, we have solved the above equations for the same cases of degenerate and nondegenerate energy–momentum tensor, as discussed in the previous section. However, we have obtained infinite MCs for all the cases considered here. The following tables present the comparisons of the obtained MCs for the contravariant and mixed form of the energy–momentum tensor and with those obtained in ref. [7], where the energy–momentum tensor was considered in its covariant form. These comparisons are presented for both the nondegenerate and degenerate cases of the energy–momentum tensor in Tables 3 and 4, respectively.

Table 3. Comparison in non-degenerate case.

Cases	$\mathcal{L}_X T_{ab} = 0$ [7]	$\mathcal{L}_X T^{ab} = 0$	$\mathcal{L}_X T^a_b = 0$
I	6	10	Infinite Dimensional
II	5	4	Infinite Dimensional
III	4	10	Infinite Dimensional
IV	7	10	Infinite Dimensional
V	10	4	Infinite Dimensional
VI-a	10	4	Infinite Dimensional
VI-b	10	4	Infinite Dimensional
VII	6	4	Infinite Dimensional
VIII	–	4	Infinite Dimensional

Table 4. Comparison in degenerate case.

Cases	$\mathcal{L}_X T_{ab} = 0$ [7]	$\mathcal{L}_X T^{ab} = 0$	$\mathcal{L}_X T^a_b = 0$
I	Infinite Dimensional	Infinite Dimensional	Infinite Dimensional
II	Infinite Dimensional	Infinite Dimensional	Infinite Dimensional
III	Infinite Dimensional	Infinite Dimensional	Infinite Dimensional
IV	Infinite Dimensional	Infinite Dimensional	Infinite Dimensional
V	Infinite Dimensional	Infinite Dimensional	Infinite Dimensional
VI	4,6,10	Infinite Dimensional	Infinite Dimensional

4. Conclusions

In this article, we have investigated matter collineations of static plane-symmetric spacetimes. We specifically examined MCs for contravariant and mixed forms of the stress–

energy tensor, while MCs for the same spacetimes for the covariant energy–momentum tensor were already explored in ref. [7]. The MC equations are solved for both degenerate and non-degenerate cases. Our findings indicate that for the degenerate stress–energy tensor in its contravariant and mixed form, the Lie algebra of MCs exhibits an infinite dimension due to unknown functions in the components of the vector field X. However, the author of ref. [7] obtained finite and infinite MCs for a degenerate energy–momentum tensor in its covariant form.

On the other hand, for nondegenerate stress–energy tensor in contravariant form, we have observed that the Lie algebra has finite dimensions, typically four or ten. However, in ref. [7], the author obtained the 4-, 5-, 6-, 7- and 10-dimensional algebra of MCs for the nondegenerate stress–energy tensor in its covariant form. Moreover, when considering the nondegenerate mixed form of the stress–energy tensor, the dimension of the Lie algebra is found to be infinite, while for the nondegenerate contravariant form of the stress–energy tensor the author of [7] obtained a 4-, 5-, 6-, 7-, and 10-dimensional algebra of MCs.

Summarizing the results, we can say that like in the case of static spherically symmetric spacetimes, the algebras of MCs for static plane-symmetric spacetimes are also of different dimensions for different forms of energy–momentum tensor. A similar comparative study of MCs with three different forms of energy–momentum tensor for some other spacetimes is under consideration. Another open direction to follow is to extend the same work for the case of homothetic and conformal matter collineations with three mentioned forms of energy–momentum tensor.

To add some physical implications, we find some particular plane symmetric metrics satisfying the conditions under which the spacetime under consideration admits the obtained MCs. Considering $A(x) = 0$ and $B(x)$, we obtain the following metric:

$$ds^2 = dt^2 - dx^2 - e^x[dy^2 + dz^2]. \tag{18}$$

For this metric, we have found that $T^{00} = -\frac{3}{4}$, $T^{11} = \frac{1}{4}$ and $T^{22} = T^{33} = \frac{1}{4e^x}$, which satisfy the constraints of case II in Section 2.1. This metric represents a perfect fluid as its energy–momentum tensor is of the form $T^{ab} = (p + \rho)u^a u^b - pg^{ab}$ with $p = \frac{1}{4}$ as pressure, $\rho = -\frac{3}{4}$ as density and $u^a = e^{-A(x)}$ as the four velocity vector. Similarly, if we take $A(x) = B(x) = x$, the static plane symmetric metric becomes:

$$ds^2 = e^x dt^2 - dx^2 - e^x[dy^2 + dz^2]. \tag{19}$$

For this metric, we have $T^{00} = T^{22} = -\frac{3}{4}e^{-x}$ and $T^{11} = \frac{3}{4}$, which satisfy the conditions of case VIII of Section 2.1. Like the above metric, this metric also gives a perfect fluid with $p = \rho = -\frac{3}{4}$. Similarly, one may find metrics corresponding to the remaining cases and find their energy–momentum tensor components to see the nature of the corresponding matter.

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