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Generalized Polynomials and Their Unification and Extension to Discrete Calculus

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Abstract: In this paper, we introduce a comprehensive and expanded framework for generalized calculus and generalized polynomials in discrete calculus. Our focus is on $(q;h)$-time scales. Our proposed approach encompasses both difference and quantum problems, making it highly adoptable. Our framework employs forward and backward jump operators to create a unique approach. We use a weighted jump operator α that combines both jump operators in a convex manner. This allows us to generate a time scale α, which provides a new approach to discrete calculus. This beneficial approach enables us to define a general symmetric derivative on time scale α, which produces various types of discrete derivatives and forms a basis for new discrete calculus. Moreover, we create some polynomials on α-time scales using the α-operator. These polynomials have similar properties to regular polynomials and expand upon the existing research on discrete polynomials. Additionally, we establish the α-version of the Taylor formula. Finally, we discuss related binomial coefficients and their properties in discrete cases. We demonstrate how the symmetrical nature of the derivative definition allows for the incorporation of various concepts and the introduction of fresh ideas to discrete calculus.

Keywords: (q,h)-time scale; α-time scale; α-operator; symmetric α-derivative; α-polynomial; α-binomial coefficient

MSC: 05A30; 26A24; 26E70; 26C05; 39A12; 26E05; 34N05

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1. Introduction

There are various ideas on how to develop a combined calculus for both continuous and discrete case scenarios. Among these, the theory of time scales proposed by Hilger [1] and Aulbach [2] appears to be a particularly promising approach. However, one of the remaining challenges in this theory is to devise practical formulas that can express its key concepts in a coherent manner and align with our intuition from traditional calculus.

The study of differential calculus is especially important due to its applications and its ability to approximate continuous cases (see ref. [3]). It is important to note that differential calculus is being referred to in a general sense, meaning that the grids do not necessarily have to be uniform and that quantum calculus should also be included in the study. This approach to the topic surpasses the conventional discrete calculus (cf. [4]), and the objective of this study is to acquire findings that can unify and expand existing results, which are collected in [5], for instance. It is worth mentioning at this point, however, that the need to unify the resulting calculus (separately for delta and nabla derivatives) was one of the main research concerns from the outset. It is crucial to create a symmetric derivative that is jointly defined for each discrete time scale. This should be followed by constructing the corresponding Taylor polynomials. It is worth noting that the goals form the basis of a complete theory. To look deeper, the study of transforms [6] will be the focus of upcoming research.

Applications of q-calculus and more generally discrete calculus can be found in [7,8]. For additional information on the theory's applications in physics, we recommend readers refer to Chapter 12 in [9] and to [10,11], in economics [12,13], or in biology [14,15].

The goal of this paper is to create new time scales that result in symmetric derivatives, which addresses the shortcomings of utilizing delta and nabla operators. This symmetrization process leads to exciting and new results. Comparing our definitions with previously studied ones, such as delta calculus (Section 2), nabla calculus (Section 3), quantum calculus (Section 4), and mixed time scale calculus (Section 5), in the book [5], highlights the usefulness of symmetry in defining terms. However, our take on the subject is still not fully covered by these earlier studies. In this paper, we focus on the use of time scales as a mathematical tool that allows calculus without limits, that is, the study of discrete time scales.

The concept of *time scale* $\mathbb{T}$ [1], which is a nonempty closed subset of $\mathbb{R}$, unifies and extends the discrete sets

$$h\mathbb{Z} := \{hx : x \in \mathbb{Z}\}, \quad h > 0$$

and

$$\mathbb{K}_q := \{q^n : n \in \mathbb{Z}\} \cup \{0\}, \quad q \neq 1,\ q > 0,$$

it allows one to investigate any type of continuous or discrete problem. Classical cases of differential and quantum equations are covered as special cases. However, our paper concentrates on newer efforts to construct a theory for the discrete case instead of highlighting this fact. The unification and extension approach on time scales is not unique: the Δ-derivative approach and ∇-derivative approach have been widely investigated separately [14]. To combine Δ- and ∇-derivatives, the diamond α-dynamic derivative was presented, allowing more balanced approximations not only for functions but also for solutions of differential equations [16].

When conducting studies, utilizing a general time scale can result in discrepancies and may not be relevant to certain underlying topics. To address this issue, researchers often choose to conduct studies on special time scales. For instance, researchers in [17] focused on a specific time scale called (q,h), which incorporates the analysis of both h and q. This approach helps to overcome the limitations associated with using a general time scale. In the paper in [17], a (q,h)-time scale is introduced. It combines h- and q-analysis, defined as

$$\mathbb{T}^{x_0}_{(q,h)} := \{q^n x_0 + h[n]_q :\ n \in \mathbb{Z}\} \cup \left\{\frac{h}{1-q}\right\}, \quad h \geq 0,\ q \geq 1,\ q+h > 1,\ x_0 \in \mathbb{R}, \tag{1}$$

where $[n]_q := \frac{q^n - 1}{q-1}$. It is noteworthy that this type of time scale unifies previous approaches and is well suited for studying fractional calculus discretization. It means that the results obtained in this work may also be of interest in discrete methods for problems of noninteger order. It is also necessary to mention about the concept of (q,w)-Hahn difference operators [18,19], in the study of which the unification of q-calculus and h-calculus is also the basis and which is also a special case of our operators. Again, efforts are being made to define symmetric derivatives [20].

To gain a better understanding of the paper's presented outcomes, we will gather its main goals and achieved results. Furthermore, we will briefly explain the research's motivations and how they relate to the current results.

The primary goal is to introduce a new framework that unifies and extends (q,h)-time scales (1). We have utilized the forward and backward jump operators as opposed to the delta and nabla (q,h)-derivatives to achieve this approach.

In [21], a generalized time scale is introduced as a pair $(\mathbb{T}, \alpha)$, depending on an arbitrary operator α on $\mathbb{T}$. This approach has been proven to be highly beneficial in the study of Hamiltonian systems. Inspired by [21], we introduce a weighted jump operator α, which is expressed as a convex combination of forward and backward jump operators. We introduce the α-time scale $\mathbb{T}_{\alpha}^{x_0}$, generated by the operator α. We present the α-derivative and its generalization $\alpha^{k,m}$-derivative, which is valid for $k, m \in \mathbb{Z}$. The $\alpha^{k,m}$-derivative is a generic derivative that unifies various kinds of discrete derivatives: (q,h)-derivative generator, which covers symmetric (q,h)-derivatives, delta (q,h)-derivatives [22], and nabla (q,h)-derivatives [23]; q-derivative generator, which produces symmetric/delta/nabla q-derivatives [7]; and h-derivative generator, which covers symmetric/delta/nabla h-derivatives [14].

More importantly, depending on the convex combination parameter, the $\alpha^{k,m}$-derivative provides new extensions for the (q,h)-derivatives and traditional discrete derivatives. For instance, it produces extended symmetric/delta/nabla (q,h)-derivatives, (p,q)-derivative [24,25], extended symmetric/delta/nabla q-derivatives with step size q^r, and extended symmetric/delta/nabla h-derivatives with step size hr, for $|r| < 1$.

Polynomials are crucial not only in the field of analysis and differential/difference equations but also in control theory, where they aid in approximating optimal control and offer efficient computing techniques. But is it feasible to amalgamate and expand ordinary polynomials, h- and q-polynomials? The creation of time scales offered a partial solution to this question.

Although the study on time scales generalizes and establishes many concepts well, the polynomials could have been constructed only in implicit and recursive forms by integrals as Δ-polynomials [26], ∇-polynomials [27], or diamond α-polynomials [28]. In order to present an explicit answer for this question, we constructed delta (q,h)-polynomials based on the delta (q,h)-derivative operator in [22] and nabla (q,h)-polynomials based on the nabla (q,h)-derivative operator in [23].

The key question is whether it is possible to present a structure that is symmetric and thus consolidates both delta and nabla (q,h)-polynomials and also offers their extensions. The primary goal is to uncover the form of polynomials on $\mathbb{T}_{\alpha}^{x_0}$ that reveals the characteristics of α time scales. Ordinary polynomials do not preserve their qualities on $\mathbb{T}_{\alpha}^{x_0}$.

Based on the idea of the α-operator, we introduce the α-polynomial and its generalization to the $\alpha^{k,m}$-polynomial. The α-polynomial is a unifying and extending polynomial that recovers discrete polynomials such as delta/nabla (q,h)-polynomials, delta/nabla q-polynomials, and delta/nabla h-polynomials and produces their extensions. We demonstrate that both the α-polynomial and $\alpha^{k,m}$-polynomial possess essential attributes of polynomials, including additivity and adherence to the power rule for derivative rules. Furthermore, we formulate and prove α-analogues of Taylor's formula.

This paper is organized as follows. In Section 2, we introduce the concept of the α-integer and present its basic properties. We additionally define α-analogues of factorials, permutation coefficients, and binomial coefficients. Section 3 is devoted to presenting the calculus on α-time scales. In Section 4, we construct the α- and $\alpha^{k,m}$-polynomials and present their key features. Consequently, the concept of generalized derivative is useful in defining generalized Taylor polynomials. This is an important step towards enabling further applications. Finally, in Section 5, we investigate the properties of α-analogues of binomial coefficients, such as Pascal's rule, from which we conclude that α-analogues of binomial coefficients can be represented as polynomials with symmetric coefficients.

2. α-Integers

Definition 1. *For $n \in \mathbb{Z}$, $t \in [0,1]$ and $q \in \mathbb{R}^+$, we introduce the α-integer*

$$[n]_\alpha := \begin{cases} \dfrac{(\frac{t}{q} + (1-t)q)^n - 1}{(\frac{t}{q} + (1-t)q) - 1} & \text{if} \quad \frac{t}{q} + (1-t)q \neq 1, \\ n & \text{otherwise.} \end{cases} \tag{2}$$

Because of the dependence of t and q in the definition, we can call (2) a (t, q)-integer. Instead, for simplicity, throughout this paper we prefer to call (2) the α-integer with a given notation.

When $n \in \mathbb{N}_0$, we observe that the α-integer (2) is a polynomial of $\frac{t}{q} + (1-t)q$ with degree $n-1$, namely

$$[n]_\alpha = \frac{(\frac{t}{q} + (1-t)q)^n - 1}{(\frac{t}{q} + (1-t)q) - 1} = 1 + \left(\frac{t}{q} + (1-t)q\right)^1 + \cdots + \left(\frac{t}{q} + (1-t)q\right)^{n-1}.$$

Moreover, the α-integer (2) recovers the q-integer [7] if $t = 0$

$$[n]_q = \frac{q^n - 1}{q - 1} \quad \text{if} \quad q \neq 1; \quad [n]_q = n \quad \text{if} \quad q = 1,$$

and the $\frac{1}{q}$-integer [23] if $t = 1$

$$[n]_{\frac{1}{q}} = \frac{(\frac{1}{q})^n - 1}{\frac{1}{q} - 1} \quad \text{if} \quad q \neq 1; \quad [n]_{\frac{1}{q}} = n \quad \text{if} \quad q = 1.$$

On the other hand if $t = \frac{1}{2}$, (2) turns out to be

$$\frac{\left(\frac{1}{2q} + \frac{q}{2}\right)^n - 1}{\left(\frac{1}{2q} + \frac{q}{2}\right) - 1}.$$

These reduction examples imply that the α-integer (2) not only comprises $[n]_q$ and $[n]_{\frac{1}{q}}$ for $t \in \{0, 1\}$, but it also provides extensions for any $t \in (0, 1)$. It is clear that both $[n]_q$ and $[n]_{\frac{1}{q}}$ recover the integer n when $q = 1$. According to Definition 1, if $q = 1$, then $\frac{t}{q} + (1-t)q = 1$. Therefore, $[n]_\alpha = n$ if $q = 1$ or $t = \frac{q}{q+1}$. Hence, we can conclude that the α-integer plays exactly the same role as the ordinary integer n.

Proposition 1. *For $m, n \in \mathbb{Z}$, the following properties hold.*

(i) $[m]_\alpha - [n]_\alpha = \left(\frac{t}{q} + (1-t)q\right)^n [m-n]_\alpha$.

(ii) $\left(\frac{t}{q} + (1-t)q\right)^m [n]_\alpha + [m]_\alpha = [m+n]_\alpha$.

Proof. (i) The statement is obvious when $\frac{t}{q} + (1-t)q = 1$; otherwise, using (2), we compute

$$\begin{aligned}
[m]_\alpha - [n]_\alpha &= \frac{\left(\frac{t}{q} + (1-t)q\right)^m - 1}{\left(\frac{t}{q} + (1-t)q\right) - 1} - \frac{\left(\frac{t}{q} + (1-t)q\right)^n - 1}{\left(\frac{t}{q} + (1-t)q\right) - 1} \\
&= \frac{\left(\frac{t}{q} + (1-t)q\right)^m - \left(\frac{t}{q} + (1-t)q\right)^n}{\left(\frac{t}{q} + (1-t)q\right) - 1} \\
&= \left(\frac{t}{q} + (1-t)q\right)^n \frac{\left(\frac{t}{q} + (1-t)q\right)^{m-n} - 1}{\left(\frac{t}{q} + (1-t)q\right) - 1} = \left(\frac{t}{q} + (1-t)q\right)^n [m-n]_\alpha.
\end{aligned}$$

(ii) Similar to the previous case, one can obtain

$$\left(\frac{t}{q}+(1-t)q\right)^m[n]_\alpha+[m]_\alpha=\left(\frac{t}{q}+(1-t)q\right)^m\frac{\left(\frac{t}{q}+(1-t)q\right)^n-1}{\left(\frac{t}{q}+(1-t)q\right)-1}+\frac{\left(\frac{t}{q}+(1-t)q\right)^m-1}{\left(\frac{t}{q}+(1-t)q\right)-1}$$
$$=\frac{\left(\frac{t}{q}+(1-t)q\right)^{m+n}-1}{\left(\frac{t}{q}+(1-t)q\right)-1}=[m+n]_\alpha.$$

□

Proposition 2. *There are certain limitations that apply to the α-integer* (2)

$$\lim_{n\to\infty}[n]_\alpha=\begin{cases}\dfrac{1}{1-\left(\frac{t}{q}+(1-t)q\right)} & \text{if } \; 0<q<1, \quad 0\le t<\frac{q}{q+1},\\ \infty & \text{if } \; 0<q<1, \quad \frac{q}{q+1}\le t\le 1,\\ \dfrac{1}{1-\left(\frac{t}{q}+(1-t)q\right)} & \text{if } \; q>1, \quad \frac{q}{q+1}<t\le 1,\\ \infty & \text{if } \; q>1, \quad 0\le t\le\frac{q}{q+1},\end{cases} \tag{3}$$

and

$$\lim_{n\to-\infty}[n]_\alpha=\begin{cases}-\infty & \text{if } \; 0<q<1, \quad 0\le t\le\frac{q}{q+1},\\ \dfrac{1}{1-\left(\frac{t}{q}+(1-t)q\right)} & \text{if } \; 0<q<1, \quad \frac{q}{q+1}<t\le 1,\\ -\infty & \text{if } \; q>1, \quad \frac{q}{q+1}\le t\le 1,\\ \dfrac{1}{1-\left(\frac{t}{q}+(1-t)q\right)} & \text{if } \; q>1, \quad 0\le t<\frac{q}{q+1}.\end{cases} \tag{4}$$

Proof. The proof of (3) is discussed in two cases.

Case I: Let $0<q<1$. If $0\le t<\frac{q}{q+1}$, then $0<\frac{t}{q}+(1-t)q<1$, and therefore we have

$$\lim_{n\to\infty}[n]_\alpha=\lim_{n\to\infty}\frac{\left(\frac{t}{q}+(1-t)q\right)^n-1}{\left(\frac{t}{q}+(1-t)q\right)-1}=\frac{1}{1-\left(\frac{t}{q}+(1-t)q\right)}.$$

Otherwise, $\frac{q}{q+1}<t\le 1$ implies that $\frac{t}{q}+(1-t)q>1$ and $\lim\limits_{n\to\infty}[n]_\alpha=\infty$. Also, if $t=\frac{q}{q+1}$, then $\frac{t}{q}+(1-t)q=1$ and $\lim\limits_{n\to\infty}[n]_\alpha=\lim\limits_{n\to\infty}n=\infty$.

Case II: Let $q>1$. Similar to Case I, if $\frac{q}{q+1}<t\le 1$, we have $0<\frac{t}{q}+(1-t)q<1$ and

$$\lim_{n\to\infty}[n]_\alpha=\frac{1}{1-\left(\frac{t}{q}+(1-t)q\right)}.$$

Finally, if $0<t<\frac{q}{q+1}$, then $\frac{t}{q}+(1-t)q>1$ and $\lim\limits_{n\to\infty}[n]_\alpha=\infty$. Also, if $t=\frac{q}{q+1}$, then $\frac{t}{q}+(1-t)q=1$ and $\lim\limits_{n\to\infty}[n]_\alpha=\lim\limits_{n\to\infty}n=\infty$. The proof of (4) proceeds by a similar fashion. □

The α-integer can be generalized in the following manner:

Definition 2. *For $n, k, m \in \mathbb{Z}$ and $k \neq m$, we define the $\alpha^{k,m}$-integer*

$$[n]_{\alpha^{k,m}} := \begin{cases} \frac{(\frac{t}{q}+(1-t)q)^{kn}-(\frac{t}{q}+(1-t)q)^{mn}}{(\frac{t}{q}+(1-t)q)^{k}-(\frac{t}{q}+(1-t)q)^{m}} & \text{if} \quad \frac{t}{q}+(1-t)q \neq 1, \\ n & \text{otherwise}. \end{cases}$$

Within this concept, we conclude this section by introducing the $\alpha^{k,m}$-analogue of factorial, permutation, and binomial coefficients.

Definition 3. *For $j \in \mathbb{N}_0$, the $\alpha^{k,m}$-factorial is introduced as*

$$[j]_{\alpha^{k,m}}! := [j]_{\alpha^{k,m}}[j-1]_{\alpha^{k,m}}[j-2]_{\alpha^{k,m}} \ldots [2]_{\alpha^{k,m}}[1]_{\alpha^{k,m}}$$

with convention $[0]_{\alpha^{k,m}}! = 1$. For $n \in \mathbb{N}$, the $\alpha^{k,m}$-permutation coefficient is defined by

$$P_{\alpha^{k,m}}[n,j] := \begin{cases} [n]_{\alpha^{k,m}}[n-1]_{\alpha^{k,m}} \cdots [n-j+1]_{\alpha^{k,m}} & \text{if} \quad j \in \mathbb{N}, \\ 1 & \text{if} \quad j = 0, \end{cases}$$

and the $\alpha^{k,m}$-binomial coefficient is defined as follows:

$$\begin{bmatrix} n \\ j \end{bmatrix}_{\alpha^{k,m}} := \begin{cases} \dfrac{P_{\alpha^{k,m}}[n,j]}{[j]_{\alpha^{k,m}}!} & \text{if} \quad j \in \mathbb{N}, \\ 1 & \text{if} \quad j = 0. \end{cases} \tag{5}$$

For $k = 1$ and $m = 0$, the notions corresponding to Definition 3 are called *α-factorial, α-permutation and α-binomial coefficients*, respectively. Similarly, for $m = 0$, they are called *α^k-factorial, α^k-permutation*, and *α^k-binomial coefficients*, respectively.

3. α-Time Scales

On any time scale $\mathbb{T}$, the forward jump operator σ and the backward jump operator ρ [14] are defined by

$$\sigma(x) := \inf\{s \in \mathbb{T} : s > x\}, \qquad \rho(x) := \sup\{s \in \mathbb{T} : s < x\}.$$

Currently, we are restricted to examining each case individually (cf. [5], for instance). However, we aim to overcome this limitation by implementing symmetrizing functions to unify both approaches. For instance, in $\mathbb{T}^{x_0}_{(q,h)}$ defined by (1), for $x \in \mathbb{T}^{x_0}_{(q,h)} \setminus \{\frac{h}{1-q}\}$ if we assume $0 < q < 1,\ x_0 < \frac{h}{1-q}$ or $q > 1,\ x_0 > \frac{h}{1-q}$, then we have

$$\sigma(x) = qx + h \quad \text{and} \quad \rho(x) = \frac{x-h}{q}, \tag{6}$$

and if we assume $0 < q < 1,\ x_0 > \frac{h}{1-q}$ or $q > 1,\ x_0 < \frac{h}{1-q}$, then we derive

$$\sigma(x) = \frac{x-h}{q} \quad \text{and} \quad \rho(x) = qx + h.$$

The nabla analysis of $\mathbb{T}^{x_0}_{(q,h)}$ when $q > 1,\ x_0 > \frac{h}{1-q}$ is studied in [23,29,30] and the delta analysis of $\mathbb{T}^{x_0}_{(q,h)}$ when $0 < q < 1,\ x_0 < \frac{h}{1-q}$ is studied in [22,31].

In [21], a generalized time scale is introduced as $(\mathbb{T}, \alpha)$, depending on an arbitrary operator $\alpha : \mathbb{T} \to \mathbb{T}$. Motivated by this paper, our primary goal is to unify and extend the (q, h)-time scale (1) using the power of symmetry of introduced concepts. For this purpose, we offer a weighted jump operator α.

Definition 4. *We introduce the operator α as the convex combination of the forward and backward jump operators* (6) *as*

$$\alpha(x) := t\left(\frac{x-h}{q}\right) + (1-t)(qx+h), \tag{7}$$

where $x \in \mathbb{R}$, $h \in \mathbb{R}_0^+$, $q \in \mathbb{R}^+$, and the parameter $t \in [0,1]$.

Throughout this work, unless otherwise stated, we assume that $h, q \in \mathbb{R}^+$, $q \neq 1$ and the parameter $t \in [0,1]$. The inverse of (7), denoted by α^{-1}, can be derived from the relation $(\alpha \circ \alpha^{-1})(x) = x$, as follows

$$\alpha^{-1}(x) = \frac{x + h\left(\frac{t}{q} + (t-1)\right)}{\frac{t}{q} + (1-t)q} = \frac{\frac{t}{q}}{\frac{t}{q} + (1-t)q}(qx+h) + \frac{(1-t)q}{\frac{t}{q} + (1-t)q}\left(\frac{x-h}{q}\right), \tag{8}$$

where the convex combination structure of the forward and backward jump operators is preserved since $\frac{\frac{t}{q}}{\frac{t}{q}+(1-t)q} \in [0,1]$ and $\dfrac{\frac{t}{q}}{\frac{t}{q} + (1-t)q} + \dfrac{(1-t)q}{\frac{t}{q} + (1-t)q} = 1$.

Remark 1. *It is evident that both the operator $\alpha(x)$ and its inverse $\alpha^{-1}(x)$ unify the forward and backward jump operators* (6).

(i) *If $t = 0$, then $\alpha(x) = qx + h$ and $\alpha^{-1}(x) = \frac{x-h}{q}$. If $0 < q < 1$, $x_0 < \frac{h}{1-q}$ or $q > 1$, $x_0 > \frac{h}{1-q}$, we have*

$$\alpha(x) = \sigma(x), \qquad \alpha^{-1}(x) = \rho(x),$$

while if $0 < q < 1$, $x_0 > \frac{h}{1-q}$ or $q > 1$, $x_0 < \frac{h}{1-q}$, we have

$$\alpha(x) = \rho(x), \qquad \alpha^{-1}(x) = \sigma(x).$$

(ii) *If $t = 1$, then $\alpha(x) = \frac{x-h}{q}$ and $\alpha^{-1}(x) = qx + h$. If $0 < q < 1$, $x_0 < \frac{h}{1-q}$ or $q > 1$, $x_0 > \frac{h}{1-q}$, we obtain*

$$\alpha(x) = \rho(x), \qquad \alpha^{-1}(x) = \sigma(x),$$

while if $0 < q < 1$, $x_0 > \frac{h}{1-q}$ or $q > 1$, $x_0 < \frac{h}{1-q}$, we derive

$$\alpha(x) = \sigma(x), \qquad \alpha^{-1}(x) = \rho(x).$$

The combined operator $\alpha(x)$ not only offers a unification for $t \in \{0,1\}$ but it also provides an extension for $t \in (0,1)$. For instance, if $t = \frac{q^{3/2}}{(1+q)(1+q^{1/2})}$, then $t \in (0,1)$ and α acts as a half jump operator

$$\alpha(x) = q^{1/2}x + \frac{1}{1+q^{1/2}}h. \tag{9}$$

If $q = 1$, Equation (9) *produces $\alpha(x) = x + \frac{h}{2}$, while for $h = 0$, it produces $\alpha(x) = q^{1/2}x$. There are also extensions that can be obtained by selecting different values of t.*

Now we are ready to introduce the generalization of the time scale $\mathbb{T}^{x_0}_{(q,h)}$ through the utilization of the α-operator.

Definition 5. *For any $x_0 \in \mathbb{R}$ with $x_0 \neq \frac{h}{1-q}$ and $\frac{t}{q} + (1-t)q \neq 1$, we introduce the α-time scale by*

$$\mathbb{T}^{x_0}_{\alpha} := \{\alpha^n(x_0) : \quad n \in \mathbb{Z}\} \cup \left\{\frac{h}{1-q}\right\}, \tag{10}$$

where α^n is the n-times composition of (7) *for $n \in \mathbb{N}_0$ and it stands for $-n$-times composition of* (8) *for $n \in \mathbb{Z}^-$.*

The time scale (10) allows us to unify results that are obtained for the time scale $\mathbb{T}_{(q,h)}^{x_0}$ [22,23,29–31] for $t \in \{0,1\}$ and provides extensions for $t \in (0,1)$. The reductions to $\mathbb{T}_{(q,h)}^{x_0}$, $h\mathbb{Z}$, $\mathbb{K}_q$, and $\mathbb{R}$ will be mentioned throughout this paper. The reason why we assume $x_0 \neq \frac{h}{1-q}$ and $\frac{t}{q} + (1-t)q \neq 1$ is explained in Remark 2 and why $\mathbb{T}_{\alpha}^{x_0}$ contains $\frac{h}{1-q}$ as the accumulation point is clarified in Proposition 4. Prior to that, we must define α^n for all $n \in \mathbb{Z}$.

Proposition 3. *The α-operator (7) admits the following form:*

$$\alpha^n(x) = \left(\frac{t}{q} + (1-t)q\right)^n x - h\left(\frac{t}{q} - (1-t)\right)[n]_\alpha, \; n \in \mathbb{Z}, \; x \in \mathbb{R}. \tag{11}$$

Proof. We prove (11) by induction.

Case I: Let $n \in \mathbb{N}_0$. For $n = 0$, we have

$$\alpha^0(x) = \left(\frac{t}{q} + (1-t)q\right)^0 x - h\left(\frac{t}{q} - (1-t)\right)[0]_\alpha = x.$$

Assume (11) holds for $n \in \mathbb{N}$. Using Proposition 1/(ii) for $m = 1$, we obtain

$$\begin{aligned}
\alpha^{n+1}(x) = \alpha(\alpha^n(x)) &= \alpha\left(\left(\frac{t}{q} + (1-t)q\right)^n x - h\left(\frac{t}{q} - (1-t)\right)[n]_\alpha\right) \\
&= t\frac{\left(\frac{t}{q} + (1-t)q\right)^n x - h\left(\frac{t}{q} - (1-t)\right)[n]_\alpha - h}{q} \\
&+ (1-t)\left(q\left(\left(\frac{t}{q} + (1-t)q\right)^n x - h\left(\frac{t}{q} - (1-t)\right)[n]_\alpha\right) + h\right) \\
&= \left(\frac{t}{q} + (1-t)q\right)^{n+1} x - h\left(\frac{t}{q} - (1-t)\right)\left(\left(\frac{t}{q} + (1-t)q\right)[n]_\alpha + 1\right) \\
&= \left(\frac{t}{q} + (1-t)q\right)^{n+1} x - h\left(\frac{t}{q} - (1-t)\right)[n+1]_\alpha.
\end{aligned}$$

Case II: Let $n \in \mathbb{Z}^-$. Rewriting (8), we derive

$$\alpha^{-1}(x) = \left(\frac{t}{q} + (1-t)q\right)^{-1} x - h\left(\frac{t}{q} - (1-t)\right)[-1]_\alpha$$

since $[-1]_\alpha = -\frac{1}{\frac{t}{q} + (1-t)q}$. Now suppose (11) holds for $-n$ and for $n \in \mathbb{N}$ and we prove it for $-(n+1)$. For any $n \in \mathbb{N}$, using Proposition 1/(ii) with $-n$ instead of n and putting $m = -1$, we obtain

$$\begin{aligned}
\alpha^{-(n+1)} = \alpha^{-1}(\alpha^{-n}(x)) &= \left(\frac{t}{q} + (1-t)q\right)^{-1} \alpha^{-n}(x) - h\left(\frac{t}{q} - (1-t)\right)[-1]_\alpha \\
&= \left(\frac{t}{q} + (1-t)q\right)^{-1}\left(\left(\frac{t}{q} + (1-t)q\right)^{-n} x - h\left(\frac{t}{q} - (1-t)\right)[-n]_\alpha\right) \\
&- h\left(\frac{t}{q} - (1-t)\right)[-1]_\alpha \\
&= \left(\frac{t}{q} + (1-t)q\right)^{-(n+1)} x - h\left(\frac{t}{q} - (1-t)\right)\left(\left(\frac{t}{q} + (1-t)q\right)^{-1}[-n]_\alpha + [-1]_\alpha\right) \\
&= \left(\frac{t}{q} + (1-t)q\right)^{-(n+1)} x - h\left(\frac{t}{q} - (1-t)\right)[-(n+1)]_\alpha.
\end{aligned}$$

□

Proposition 4. *The α-operator (7) admits the following limits:*

$$\lim_{n\to\infty} \alpha^n(x) = \begin{cases} \infty & \text{if } \frac{t}{q}+(1-t)q>1 \text{ and } x>\frac{h}{1-q}, \\ -\infty & \text{if } \frac{t}{q}+(1-t)q>1 \text{ and } x<\frac{h}{1-q}, \\ \frac{h}{1-q} & \text{if } \frac{t}{q}+(1-t)q<1 \text{ and } x>\frac{h}{1-q}, \\ \frac{h}{1-q} & \text{if } \frac{t}{q}+(1-t)q<1 \text{ and } x<\frac{h}{1-q}, \end{cases}$$

and

$$\lim_{n\to-\infty} \alpha^n(x) = \begin{cases} \frac{h}{1-q} & \text{if } \frac{t}{q}+(1-t)q>1 \text{ and } x>\frac{h}{1-q}, \\ \frac{h}{1-q} & \text{if } \frac{t}{q}+(1-t)q>1 \text{ and } x<\frac{h}{1-q}, \\ \infty & \text{if } \frac{t}{q}+(1-t)q<1 \text{ and } x>\frac{h}{1-q}, \\ -\infty & \text{if } \frac{t}{q}+(1-t)q<1 \text{ and } x<\frac{h}{1-q}. \end{cases}$$

Proof. Reconsidering (11)

$$\begin{aligned} \alpha^n(x) &= \left(\frac{t}{q}+(1-t)q\right)^n x - h\left(\frac{t}{q}-(1-t)\right)[n]_\alpha \\ &= \left(\frac{t}{q}+(1-t)q\right)^n x - h\left(\frac{t}{q}-(1-t)\right)\frac{\left(\frac{t}{q}+(1-t)q\right)^n-1}{\frac{t}{q}+(1-t)q-1} \end{aligned}$$

and using $(1-q)\left(\frac{t}{q}-(1-t)\right) = \frac{t}{q}+(1-t)q-1$, we obtain another form equivalent to (11)

$$\alpha^n(x) = \left(\frac{t}{q}+(1-t)q\right)^n\left(x-\frac{h}{1-q}\right)+\frac{h}{1-q}, \tag{12}$$

from which the proof follows. □

Remark 2. *Based on Equation (12), it can be concluded that*

$$\alpha(x)-x = \left(\frac{t}{q}+(1-t)q\right)\left(x-\frac{h}{1-q}\right)+\frac{h}{1-q}-x = \left(\frac{t}{q}+(1-t)q-1\right)\left(x-\frac{h}{1-q}\right).$$

Hence, $\alpha(x)=x$ if and only if either $x=\frac{h}{1-q}$ or $\frac{t}{q}+(1-t)q=1$. The second case happens when $q=1$ or $t=\frac{q}{q+1}$. When $x_0=\frac{h}{1-q}$ is used in (10), we obtain a trivial time scale $\mathbb{T}_\alpha^{\frac{h}{1-q}}=\{\frac{h}{1-q}\}$, and if we allow $\frac{t}{q}+(1-t)q=1$ in (10) we obtain another trivial time scale $\mathbb{T}_\alpha^{x_0}=\{x_0\}\cup\left\{\frac{h}{1-q}\right\}$.

The seed term of the time scale $\mathbb{T}_\alpha^{x_0}$ plays an important role in the location of the points of $\mathbb{T}_\alpha^{x_0}$. This is investigated and analyzed as follows.

Proposition 5. *For $x\in\mathbb{T}_\alpha^{x_0}\setminus\{\frac{h}{1-q}\}$, the following properties hold.*

(i) $x>\frac{h}{1-q}$ *if and only if* $x_0>\frac{h}{1-q}$.

(ii) $\alpha^{n+1}(x)-\alpha^n(x)>0$ *if* $x<\frac{h}{1-q}$, $\frac{t}{q}+(1-t)q<1$ *or* $x>\frac{h}{1-q}$, $\frac{t}{q}+(1-t)q>1$. *That is, in this case the α-operator acts as a forward jump operator on $\mathbb{T}_\alpha^{x_0}$.*

(iii) $\alpha^{n+1}(x)-\alpha^n(x)<0$ *if* $x<\frac{h}{1-q}$, $\frac{t}{q}+(1-t)q>1$ *or* $x>\frac{h}{1-q}$, $\frac{t}{q}+(1-t)q<1$. *That is, in this case the α-operator acts as a backward jump operator on $\mathbb{T}_\alpha^{x_0}$.*

Proof. The proof is based on the representation (12). If $x \in \mathbb{T}_\alpha^{x_0} \setminus \{\frac{h}{1-q}\}$, then $x = \alpha^k(x_0)$ for some $k \in \mathbb{Z}$. For the proof of (i), we rewrite (12),

$$x - \frac{h}{1-q} = \alpha^k(x_0) - \frac{h}{1-q} = \left(\frac{t}{q} + (1-t)q\right)^k \left(x_0 - \frac{h}{1-q}\right),$$

which implies that $x - \frac{h}{1-q}$ and $x_0 - \frac{h}{1-q}$ are of the same sign, since $\frac{t}{q} + (1-t)q > 0$. For the proof of (ii) and (iii), we consider

$$\begin{aligned}
\alpha^{n+1}(x) - \alpha^n(x) &= \alpha^{n+1}(\alpha^k(x_0)) - \alpha^n(\alpha^k(x_0)) = \alpha^{n+k+1}(x_0) - \alpha^{n+k}(x_0) \\
&= \left(\frac{t}{q} + (1-t)q\right)^{n+k+1}\left(x_0 - \frac{h}{1-q}\right) - \left(\frac{t}{q} + (1-t)q\right)^{n+k}\left(x_0 - \frac{h}{1-q}\right) \\
&= \left(\frac{t}{q} + (1-t)q\right)^{n+k}\left(\frac{t}{q} + (1-t)q - 1\right)\left(x_0 - \frac{h}{1-q}\right).
\end{aligned}$$

The sign of $\alpha^{n+1}(x) - \alpha^n(x)$ depends on the sign of the product $\left(\frac{t}{q} + (1-t)q - 1\right)$ $\left(x_0 - \frac{h}{1-q}\right)$. By (i), $x - \frac{h}{1-q}$ and $x_0 - \frac{h}{1-q}$ are of the same sign, hence the results follow. □

For an arbitrary operator $\alpha : \mathbb{T} \to \mathbb{T}$, the notion of the α-derivative is introduced in [21]. We adapt it for the time scale $\mathbb{T}_\alpha^{x_0}$ (10) as follows.

Definition 6. *Let $f : \mathbb{T}_\alpha^{x_0} \to \mathbb{R}$ be any function. For $k, m \in \mathbb{Z}$ and $k \neq m$, the $\alpha^{k,m}$-derivative of f is defined by*

$$D_{\alpha^{k,m}} f(x) := \begin{cases} \dfrac{f(\alpha^k(x)) - f(\alpha^m(x))}{\alpha^k(x) - \alpha^m(x)} & \text{if} \quad x \neq \frac{h}{1-q}, \\[2ex] \displaystyle\lim_{\substack{s \to \frac{h}{1-q} \\ s \in \mathbb{T}_\alpha^{x_0}}} \frac{f(s) - f\left(\frac{h}{1-q}\right)}{s - \frac{h}{1-q}} = f'\left(\frac{h}{1-q}\right) & \text{if} \quad x = \frac{h}{1-q}, \end{cases} \tag{13}$$

if the limit exists.

It is evident that $D_{\alpha^{k,m}}$ is a linear operator. Moreover, when $m = 0$, the $\alpha^{k,m}$-derivative (13) results in the α^k-*derivative*

$$D_{\alpha^k} f(x) := \frac{f(\alpha^k(x)) - f(x)}{\alpha^k(x) - x} \tag{14}$$

and if additionally $k = 1$, (14) recovers the α-*derivative*

$$D_\alpha f(x) := \frac{f(\alpha(x)) - f(x)}{\alpha(x) - x}. \tag{15}$$

By (13), it is clear that $D_{\alpha^{k,m}} f = D_{\alpha^{m,k}} f$. Rewriting (13), we observe that the $\alpha^{k,m}$-derivative can be understood as the α^{k-m}-derivative at $\alpha^m(x)$ (or α^{m-k}-derivative at $\alpha^k(x)$) by (14). Indeed,

$$D_{\alpha^{k,m}} f(x) = \frac{f(\alpha^k(x)) - f(\alpha^m(x))}{\alpha^k(x) - \alpha^m(x)} = \frac{f(\alpha^{k-m}(\alpha^m(x))) - f(\alpha^m(x))}{\alpha^{k-m}(\alpha^m(x)) - \alpha^m(x)} = (D_{\alpha^{k-m}} f)(\alpha^m(x)).$$

If $k+m=0$, then (13) turns out to be the $\alpha^{k,-k}$-derivative

$$D_{\alpha^{k,-k}}f(x)=\frac{f(\alpha^k(x))-f(\alpha^{-k}(x))}{\alpha^k(x)-\alpha^{-k}(x)}$$

which produces the *symmetric (centralized) derivatives* widely investigated in the literature. Although the α-derivative (15) unifies and extends the discrete derivatives, in order to cover the symmetric discrete derivatives as well, we introduced the $\alpha^{k,m}$-derivative (13), whose reductions are analyzed as clearly as possible in the upcoming remark.

Remark 3. *We emphasize that the $\alpha^{k,m}$-derivative* (13) *is a* generic *derivative. For $t\in\{0,1\}$, it produces*

(a) The (q,h)-derivative generator, which covers symmetric/delta/nabla (q,h)-derivatives [22,23];
(b) The q-derivative generator, which produces symmetric/delta/nabla q-derivatives [7];
(c) The h-derivative generator, which unifies symmetric/delta/nabla h-derivatives [14].

It is important to understand that for $t\in(0,1)$, the $\alpha^{k,m}$-derivative provides extensions for the (q,h)-derivative generator, the q-derivative generator, and the h-derivative generator. This unification and extension analysis can be comprehended in the following manner.

1. *If $t=0$, then $\alpha^k(x)=q^kx+h[k]_q$ and if $t=1$, then $\alpha^k(x)=q^{-k}x-\frac{h}{q}[k]_{\frac{1}{q}}=\frac{x-h[k]_q}{q^k}=q^{-k}x+h[-k]_q$ by* (11)*. Therefore, if $t\in\{0,1\}$, we obtain*

$$D_{\alpha^{k,m}}f(x)=\frac{f(q^kx+h[k]_q)-f(q^mx+h[m]_q)}{(q^k-q^m)x+h([k]_q-[m]_q)},\quad k,m\in\mathbb{Z},\tag{16}$$

which can be understood as the (q,h)-derivative generator.

   i. *When $k+m=0$,* (16) *provides the symmetric (q,h)-derivative*

$$D_{\alpha^{k,-k}}f(x)=\frac{f(q^kx+h[k]_q)-f(q^{-k}x+h[-k]_q)}{(q^k-q^{-k})x+h([k]_q-[-k]_q)},\quad k\in\mathbb{Z}.$$

   ii. *When $k=1$, $m=0$ (or $k=0$, $m=1$),* (16) *recovers the delta (q,h)-derivative [22]*

$$D_{(q,h)}f(x)=\frac{f(qx+h)-f(x)}{(q-1)x+h},\tag{17}$$

   for $q>1,\ x_0>\frac{h}{1-q}$ or $0<q<1,\ x_0<\frac{h}{1-q}$.

   iii. *When $k=-1$, $m=0$ (or $k=0$, $m=-1$),* (16) *recovers the nabla (q,h)-derivative [23]*

$$\widetilde{D}_{(q,h)}f(x)=\frac{f(\frac{x-h}{q})-f(x)}{\frac{x-h}{q}-x},\tag{18}$$

   for $q>1,\ x_0>\frac{h}{1-q}$ or $0<q<1,\ x_0<\frac{h}{1-q}$.

2. *If $h=0$, by* (11) *we have $\alpha^k(x)=\left(\frac{t}{q}+(1-t)q\right)^kx$ and the $\alpha^{k,m}$-derivative* (13) *produces*

$$D_{\alpha^{k,m}}f(x)=\frac{f\left(\left(\frac{t}{q}+(1-t)q\right)^kx\right)-f\left(\left(\frac{t}{q}+(1-t)q\right)^mx\right)}{\left(\frac{t}{q}+(1-t)q\right)^kx-\left(\frac{t}{q}+(1-t)q\right)^mx},\quad k,m\in\mathbb{Z},\tag{19}$$

which can be regarded as the q-derivative generator . When $t\in\{0,1\}$, (19) *produces*

$$D_{\alpha^{k,m}}f(x)=\frac{f(q^kx)-f(q^mx)}{(q^k-q^m)x},\quad k,m\in\mathbb{Z},$$

whose reductions can be listed as follows.

i. *The symmetric q-derivative with k-shifts for $k+m=0$*

$$D_{\alpha^{k,-k}} f(x) = \frac{f(q^k x) - f(q^{-k} x)}{(q^k - q^{-k})x}, \quad k \in \mathbb{Z}.$$

ii. *The symmetric q-derivative for $k=1$, $m=-1$ [7]*

$$D_{\alpha^{1,-1}} f(x) = \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x}.$$

iii. *The delta q-derivative [7] for $k=1$, $m=0$ (or $k=0$, $m=1$), $q>1$*

$$\Delta_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

iv. *The nabla q-derivative [7] for $k=0$, $m=-1$ (or $k=-1$, $m=0$), $q>1$*

$$\nabla_q f(x) = \frac{f(x) - f(\frac{x}{q})}{(1 - \frac{1}{q})x}.$$

3. *If $q=1$, (11) implies that $\alpha^k(x) = x - hk(2t-1)$ and the $\alpha^{k,m}$-derivative (13) produces*

$$D_{\alpha^{k,m}} f(x) = \frac{f(x - hk(2t-1)) - f(x - hm(2t-1))}{h(2t-1)(m-k)}, \quad k,m \in \mathbb{Z}, \tag{20}$$

which can be seen as the h-derivative generator. When $t \in \{0,1\}$, by (20) we have $\alpha^k(x) = x + hk$

$$D_{\alpha^{k,m}} f(x) = \frac{f(x+hk) - f(x+hm)}{h(k-m)}, \quad k,m \in \mathbb{Z},$$

whose reductions can be listed as follows.

i. *The symmetric h-derivative with k-shifts for $k+m=0$*

$$D_{\alpha^{k,-k}} f(x) = \frac{f(x+hk) - f(x-hk)}{2hk}, \quad k \in \mathbb{Z}.$$

ii. *The symmetric h-derivative [14] for $k=1$, $m=-1$*

$$D_{\alpha^{k,-1}} f(x) = \frac{f(x+h) - f(x-h)}{2h}.$$

iii. *The delta h-derivative [14] for $k=1$, $m=0$ (or $k=0$, $m=1$)*

$$\Delta_h f(x) = \frac{f(x+h) - f(x)}{h}.$$

iv. *The nabla h-derivative [14] for $k=0$, $m=-1$ (or $k=-1$, $m=0$)*

$$\nabla_h f(x) = \frac{f(x) - f(x-h)}{h}.$$

4. *If $t \in (0,1)$, we obtain a new extension for discrete derivatives. We observe that if $|r|<1$ and $t = \frac{q^2 - q^{r+1}}{q^2 - 1}$, then $t \in (0,1)$. For such t, by (11) we obtain*

$$\alpha^k(x) = q^{rk} x + h[r]_q [k]_\alpha, \quad k \in \mathbb{Z}, \quad |r| < 1, \tag{21}$$

and new extensions for the (q,h)-derivative generator (16)

$$D_{\alpha^{k,m}} f(x) = \frac{f(q^{rk}x + h[r]_q[k]_\alpha) - f(q^{rm}x + h[r]_q[m]_\alpha)}{(q^{rk} - q^{rm})x + h[r]_q([k]_\alpha - [m]_\alpha)}, \quad k,m \in \mathbb{Z}, \quad |r| < 1, \tag{22}$$

which recovers the following extensions of discrete derivatives.

i. *The extended symmetric (q,h)-derivative for $k = 1$, $m = -1$*

$$D_{\alpha^{1,-1}} f(x) = \frac{f(q^r x + h[r]_q) - f(q^{-r}x + h[-r]_q)}{(q^r - q^{-r})x + h([r]_q - [-r]_q)},$$

the extended delta (q,h)-derivative for $k = 1$, $m = 0$, and the extended nabla (q,h)-derivative for $k = 0$, $m = -1$. Here, we assume $q > 1$, $x_0 > \frac{h}{1-q}$ or $0 < q < 1$, $x_0 < \frac{h}{1-q}$.

ii. *If $h = 0$, (22) turns out to be an extension of the q-derivative*

$$D_{\alpha^{k,m}} f(x) = \frac{f(q^{rk}x) - f(q^{rm}x)}{(q^{rk} - q^{rm})x},$$

from which we obtain the (p,q)-derivative [24] by substituting p instead of q^{rk} and q instead of q^{rm}

$$D_{\alpha^{k,m}} f(x) = \frac{f(px) - f(qx)}{(p-q)x},$$

the extended symmetric q-derivative with step size q^r for $k = 1$ and $m = -1$,

$$D_{\alpha^{1,-1}} f(x) = \frac{f(q^r x) - f(q^{-r}x)}{(q^r - q^{-r})x}, \tag{23}$$

the extended delta q-derivative for $k = 1$, $m = 0$, $q > 1$, and the extended nabla q-derivative for $k = 0$, $m = -1$, $q > 1$.

iii. *If $q = 1$, (22) allows us to derive an extension of the h-derivative with uniform step size hr*

$$D_{\alpha^{k,m}} f(x) = \frac{f(x + hrk) - f(x + hrm)}{hr(k-m)}, \quad k,m \in \mathbb{Z}$$

from which we have the (h_1,h_2)-derivative for the choices $h_1 = hrk$ and $h_2 = hrm$

$$D_{h_1,h_2} f(x) = \frac{f(x + h_1) - f(x + h_2)}{h_1 - h_2},$$

the extended symmetric h-derivative with step size hr for $k = 1$ and $m = -1$,

$$D_{\alpha^{1,-1}} f(x) = \frac{f(x + hr) - f(x - hr)}{2hr}, \tag{24}$$

the extended delta h-derivative for $k = 1$, $m = 0$, and the extended nabla h-derivative for $k = 0$, $m = -1$.

There are additional extensions available for various values of t within the range of $(0,1)$. These extensions can be very beneficial in several computational applications.

5. *As $(q,h) \to (1,0)$, the $\alpha^{k,m}$-derivative recovers the ordinary derivative.*

Example 1. *If $r = \frac{1}{\pi}$, then (23) implies*

$$D_{\alpha^{1,-1}} f(x) = \frac{f(q^{1/\pi}x) - f(q^{-1/\pi}x)}{q^{1/\pi}x - q^{-1/\pi}x}.$$

If $r = \frac{1}{2}$, *then* (24) *implies*

$$D_{\alpha^{1,-1}} f(x) = \frac{f(x + \frac{1}{2}h) - f(x - \frac{1}{2}h)}{h}.$$

Proposition 6. *If* $f, g : \mathbb{T}_\alpha^{x_0} \to \mathbb{R}$ *are any functions with* $x \neq \frac{h}{1-q}$, *then the product rule for the* $\alpha^{k,m}$*-derivative is given by*

$$\begin{aligned} D_{\alpha^{k,m}}(f(x)g(x)) &= f(\alpha^k(x))D_{\alpha^{k,m}}g(x) + g(\alpha^m(x))D_{\alpha^{k,m}}f(x) \\ &= g(\alpha^k(x))D_{\alpha^{k,m}}f(x) + f(\alpha^m(x))D_{\alpha^{k,m}}g(x). \end{aligned}$$

Proof. We add and subtract the term $f(\alpha^k(x))g(\alpha^m(x))$

$$\begin{aligned} D_{\alpha^{k,m}}(f(x)g(x)) =& \frac{(fg)(\alpha^k(x)) - f(\alpha^k(x))g(\alpha^m(x)) + f(\alpha^k(x))g(\alpha^m(x)) - (fg)(\alpha^m(x))}{\alpha^k(x) - \alpha^m(x)} \\ =& f(\alpha^k(x))\frac{g(\alpha^k(x)) - g(\alpha^m(x))}{\alpha^k(x) - \alpha^m(x)} + g(\alpha^m(x))\frac{f(\alpha^k(x)) - f(\alpha^m(x))}{\alpha^k(x) - \alpha^m(x)} \\ =& f(\alpha^k(x))D_{\alpha^{k,m}}g(x) + g(\alpha^m(x))D_{\alpha^{k,m}}f(x). \end{aligned}$$

Since $D_{\alpha^{k,m}} = D_{\alpha^{m,k}}$, replacing k and m in the first form we obtain the second form. □

4. α-Polynomials

The main motivation of this section is to introduce a proper polynomial obeying the nature of $\mathbb{T}_\alpha^{x_0}$. If we defined the derivative operator on $\mathbb{T}_\alpha^{x_0}$ with a convex combination of the delta (q, h)-derivative (17) and the nabla (q, h)-derivative (18), we would obtain nothing but implicit polynomials that are recursively defined by integrals on time scales [14] or in diamond α-time scales [28]. Instead, we define the α-operator (7) as a convex combination of forward and backward jump operators from which we construct polynomials. The framework considered in the current paper is more suitable for obtaining the explicit form of the polynomials given below.

Definition 7. *Let* $a \in \mathbb{R}$, $k, m \in \mathbb{Z}$, *and* $n \in \mathbb{N}_0$. *The* $\alpha^{k,m}$*-polynomial of order* n *is introduced by*

$$(x-a)^n_{\alpha^{k,m}} := \begin{cases} \prod_{j=0}^{n-1}\left(x - \alpha^{j(k-m)}(a)\right) & \textit{if} \quad n > 0, \\ 1 & \textit{if} \quad n = 0. \end{cases} \tag{25}$$

In order to clarify polynomial (25), let us set $s := k - m$. Depending on the value of the integer s, (25) is described by powers of α, for instance for $s = 1$, we have the polynomial

$$(x-a)^n_\alpha := \prod_{j=0}^{n-1}\left(x - \alpha^j(a)\right) = (x-a)(x-\alpha(a))(x-\alpha^2(a))\cdots(x-\alpha^{n-1}(a)), \tag{26}$$

which can be called the α*-polynomial*, or by powers of α^{-1}, for instance for $s = -1$

$$(x-a)^n_{\alpha^{-1}} = (x-a)(x-\alpha^{-1}(a))(x-\alpha^{-2}(a))\cdots(x-\alpha^{-(n-1)}(a)).$$

We can observe that each difference in the product (25)

$$(x-a)^n_{\alpha^s} := (x-a)(x-\alpha^s(a))(x-\alpha^{2s}(a))\cdots(x-\alpha^{(n-1)s}(a)), \tag{27}$$

can provide a wider or narrower difference, and therefore can provide more balanced approximations depending on the conditions presented in Proposition 5 (ii) and (iii) and depending on the value of s. Note also that (25) recovers the ordinary polynomial

$$\prod_{j=0}^{n-1}(x-a)=(x-a)^n,$$

for $q=1$ and $h=0$ (this case is also covered by the choice $k=m$ or $t=\frac{q}{q+1}$).

Remark 4. *The α-polynomial (26) is a generic polynomial. For $t\in\{0,1\}$, it produces*

(a) The delta (q,h)-polynomial [22] and the nabla (q,h)-polynomial [23];

(b) Theq-polynomial generator, which produces delta/nabla q-polynomials [7];

(c) Theh-polynomial generator, which unifies delta/nabla h-polynomials [14].

Moreover, for $t\in(0,1)$, the α-polynomial provides extensions for the delta/nabla (q,h)-polynomials, q-polynomial generator, and h-polynomial generator.

To understand the analysis mentioned earlier, follow these steps.

1. i. *If $t=0$, then (11) implies that $\alpha^j(a)=q^ja+h[j]_q$ and the α-polynomial (26) produces the delta (q,h)-polynomial $(x-a)^n_{q,h}$, presented in [22]*

$$\prod_{j=0}^{n-1}\left(x-q^ja-h[j]_q\right)=(x-a)^n_{q,h},\tag{28}$$

for $q>1,\ x_0>\frac{h}{1-q}$ or $0<q<1,\ x_0<\frac{h}{1-q}$.

ii. *If $t=1$, then (11) leads to $\alpha^j(a)=\frac{a-h[j]_q}{q^j}$ and (26) reduces to the nabla (q,h)-polynomial $(x-a)^n_{\widetilde{q,h}}$, studied in [23],*

$$\prod_{j=0}^{n-1}\left(x-\frac{a-h[j]_q}{q^j}\right)=(x-a)^n_{\widetilde{q,h}},\tag{29}$$

for $q>1,\ x_0>\frac{h}{1-q}$ or $0<q<1,\ x_0<\frac{h}{1-q}$.

2. *If $h=0$, then from (11) we have $\alpha^j(a)=\left(\frac{t}{q}+(1-t)q\right)^ja$ and in this case the α-polynomial (26) produces the q-polynomial generator.*

$$\prod_{j=0}^{n-1}\left(x-\left(\frac{t}{q}+(1-t)q\right)^ja\right),$$

which reduces to the following q-polynomials for $q>1$.

i. *The q-polynomial (or delta q-polnomial) [7] for $t=0$,*

$$\prod_{j=1}^{n}\left(x-q^{j-1}a\right)=(x-a)(x-qa)(x-q^2a)\cdots(x-q^{n-1}a)=(x-a)^n_q.$$

ii. *The $\frac{1}{q}$-polynomial (or nabla q-polnomial) [23] for $t=1$,*

$$\prod_{j=0}^{n-1}\left(x-q^{-j}a\right)=(x-a)(x-q^{-1}a)(x-q^{-2}a)\cdots(x-q^{-n+1}a)=(x-a)^n_{\frac{1}{q}}.$$

3. *If $q = 1$, then by (11) we have $\alpha^j(a) = a - hj(2t-1)$, from which the α-polynomial (26) turns out to be the h-polynomial generator*

$$\prod_{j=0}^{n-1}(x - (a - h(2t-1)j),$$

which reduces to the following h-polynomials.

   i. *The delta h-polynomial for $t = 0$*

$$\prod_{j=0}^{n-1}(x - (a + jh)) = (x-a)(x-(a+h))(x-(a+2h))\cdots(x-(a+(n-1)h)).$$

   ii. *The nabla h-polynomial for $t = 1$*

$$\prod_{j=0}^{n-1}(x - (a - jh)) = (x-a)(x-(a-h))(x-(a-2h))\cdots(x-(a-(n-1)h)).$$

4. *If $|r| < 1$ and $t = \frac{q^2 - q^{r+1}}{q^2-1}$, then $t \in (0,1)$. For such t, by (21) and (26), we have*

$$\prod_{j=0}^{n-1}\left(x - q^{rj}a - h[r]_q[j]_\alpha\right), \quad |r| < 1$$

which recovers the following extended polynomials.

   i. *The extension of the delta (q,h)-polynomial (28) for $0 < r < 1$ and the extension of the nabla (q,h)-polynomial (29) for $-1 < r < 0$. Here, $q > 1$, $x_0 > \frac{h}{1-q}$ or $0 < q < 1$, $x_0 < \frac{h}{1-q}$.*

   ii. *If $h = 0$ and $q > 1$, the extended delta q-polynomial for $0 < r < 1$,*

$$\prod_{j=0}^{n-1}\left(x - q^{rj}a\right) = (x-a)(x-q^r a)(x-q^{2r}a)\cdots(x-q^{2(n-1)}a),$$

   and the extended nabla q-polynomial for $-1 < r < 0$.

   iii. *If $q = 1$, the extended delta h-polynomial for $0 < r < 1$,*

$$\prod_{j=0}^{n-1}(x - a - hrj) = (x-a)(x-a-hr)(x-a-2hr)\cdots(x-a-(n-1)hr),$$

   and the extended nabla h-polynomial for $-1 < r < 0$.

Other extensions can be found for the different choices of t.

Theorem 1. *For any non-negative integers n, ℓ, the $\alpha^{k,m}$-polynomial defined by (25) satisfies the following additive property*

$$(x-a)^{n+\ell}_{\alpha^{k,m}} = (x-a)^n_{\alpha^{k,m}} \cdot \left(x - \alpha^{n(k-m)}(a)\right)^\ell_{\alpha^{k,m}}. \tag{30}$$

Proof. The cases $n = 0$ or $\ell = 0$ or both bring out trivial cases. To continue, assume that both n and ℓ are positive. With this assumption, we can derive the following result

$$\begin{aligned}(x-a)^{n+\ell}_{\alpha^{k,m}} &= \prod_{j=0}^{n+\ell-1}\Big(x-\alpha^{j(k-m)}(a)\Big) = \prod_{j=0}^{n-1}\Big(x-\alpha^{j(k-m)}(a)\Big)\prod_{j=n}^{n+\ell-1}\Big(x-\alpha^{j(k-m)}(a)\Big)\\ &= (x-a)^{n}_{\alpha^{k,m}}\Big(x-\alpha^{n(k-m)}(a)\Big)\Big(x-\alpha^{(n+1)(k-m)}(a)\Big)\cdots\Big(x-\alpha^{(n+\ell-1)(k-m)}(a)\Big).\end{aligned}$$

Since

$$\begin{aligned}\prod_{j=n}^{n+\ell-1}\Big(x-\alpha^{j(k-m)}(a)\Big) &= \prod_{j=1}^{\ell-1}\Big(x-\alpha^{(j+n)(k-m)}(a)\Big) = \prod_{j=1}^{\ell-1}\Big(x-\alpha^{j(k-m)}(\alpha^{n(k-m)}(a))\Big)\\ &= \Big(x-\alpha^{n(k-m)}(a)\Big)^{\ell}_{\alpha^{k,m}},\end{aligned}$$

we obtain (30). □

Lemma 1. *For $n, m, k \in \mathbb{Z}$, the α-operator (7) satisfies the following shift identity*

$$\alpha^k(x) - \alpha^m(a) = \left(\frac{t}{q} + (1-t)q\right)^n (\alpha^{k-n}(x) - \alpha^{m-n}(a)).$$

Proof. It follows from (12) that

$$\begin{aligned}\alpha^k(x) - \alpha^m(a) &= \left(\frac{t}{q}+(1-t)q\right)^k\left(x-\frac{h}{1-q}\right)+\frac{h}{1-q}-\left(\frac{t}{q}+(1-t)q\right)^m\left(a-\frac{h}{1-q}\right)\\ &\qquad -\frac{h}{1-q}\\ &= \left(\frac{t}{q}+(1-t)q\right)^n\Bigg(\left(\frac{t}{q}+(1-t)q\right)^{k-n}\left(x-\frac{h}{1-q}\right)+\frac{h}{1-q}\\ &\qquad -\left(\frac{t}{q}+(1-t)q\right)^{m-n}\left(a-\frac{h}{1-q}\right)-\frac{h}{1-q}\Bigg)\\ &= \left(\frac{t}{q}+(1-t)q\right)^n\Big(\alpha^{k-n}(x)-\alpha^{m-n}(a)\Big).\end{aligned}$$

□

Theorem 2. *The $\alpha^{k,m}$-derivative of the $\alpha^{k,m}$-polynomial (25) is determined by*

$$D_{\alpha^{k,m}}(x-a)^n_{\alpha^{k,m}} = [n]_{\alpha^{k,m}}(x-\alpha^{-m}(a))^{n-1}_{\alpha^{k,m}}, \qquad n \in \mathbb{N}. \tag{31}$$

Proof. For $x \neq \frac{h}{1-q}$, we prove via induction on n. Using the definition of the $\alpha^{k,m}$-*derivative* (13) on $(x-a)^1_{\alpha^{k,m}}$, it is straightforward that

$$D_{\alpha^{k,m}}(x-a)^1_{\alpha^{k,m}} = D_{\alpha^{k,m}}(x-a) = \frac{\alpha^k(x)-a-(\alpha^m(x)-a)}{\alpha^k(x)-\alpha^m(x)} = 1 = [1]_{\alpha^{k,m}}.$$

Assume the induction hypothesis (31) holds. By Theorem 1, we have the identity

$$(x-a)^{n+1}_{\alpha^{k,m}} = (x-a)^1_{\alpha^{k,m}}(x-\alpha^{k-m}(a))^n_{\alpha^{k,m}} = (x-a)(x-\alpha^{k-m}(a))^n_{\alpha^{k,m}}. \tag{32}$$

The proof follows by using Lemma 1 and Proposition 6 on (32)

$$\begin{aligned}
D_{\alpha^{k,m}}(x-a)^{n+1}_{\alpha^{k,m}} &= D_{\alpha^{k,m}}\Big((x-a)(x-\alpha^{k-m}(a))^{n}_{\alpha^{k,m}}\Big)\\
&= (\alpha^{k}(x)-\alpha^{k-m}(a))^{n}_{\alpha^{k,m}} + [n]_{\alpha^{k,m}}(x-\alpha^{k-2m}(a))^{n-1}_{\alpha^{k,m}}(\alpha^{m}(x)-a)\\
&= \left(\frac{t}{q}+(1-t)q\right)^{kn}(x-\alpha^{-m}(a))^{n}_{\alpha^{k,m}}\\
&\qquad + [n]_{\alpha^{k,m}}(x-\alpha^{k-2m}(a))^{n-1}_{\alpha^{k,m}}\left(\frac{t}{q}+(1-t)q\right)^{m}(x-\alpha^{-m}(a)).
\end{aligned}$$

Using Theorem 1 once more, we obtain

$$\begin{aligned}
(x-\alpha^{k-2m}(a))^{n-1}_{\alpha^{k,m}}(x-\alpha^{-m}(a)) &= (x-\alpha^{k-2m}(a))^{n-1}_{\alpha^{k,m}}(x-\alpha^{-m}(a))^{1}_{\alpha^{k,m}}\\
&= (x-\alpha^{-m}(a))^{n}_{\alpha^{k,m}},
\end{aligned}$$

from which we end up with

$$\begin{aligned}
D_{\alpha^{k,m}}(x-a)^{n+1}_{\alpha^{k,m}} &= \left(\left(\frac{t}{q}+(1-t)q\right)^{kn} + [n]_{\alpha^{k,m}}\left(\frac{t}{q}+(1-t)q\right)^{m}\right)(x-\alpha^{-m}(a))^{n}_{\alpha^{k,m}}\\
&= [n+1]_{\alpha^{k,m}}(x-\alpha^{-m}(a))^{n}_{\alpha^{k,m}}.
\end{aligned}$$

We also note that the derivative formula still holds at the accumulation point $\frac{h}{1-q}$. One can derive this result by applying L'Hôspital's rule and the product rule for ordinary derivatives. □

Corollary 1. *The α^k-derivative (14) of the α^k-polynomial (27) is given by*

$$D_{\alpha^k}(x-a)^{n}_{\alpha^k} = [n]_{\alpha^k}(x-a)^{n-1}_{\alpha^k}.$$

Corollary 2. *The α-derivative (15) of the α-polynomial (26) is given by*

$$D_{\alpha}(x-a)^{n}_{\alpha} = [n]_{\alpha}(x-a)^{n-1}_{\alpha}, \qquad n\in\mathbb{N}.$$

Corollary 3. *Let $k,m\in\mathbb{Z}$, $n,j\in\mathbb{N}_0$, and $0\le j\le n$, then the following Leibniz rules hold.*

(i) $D^{j}_{\alpha^{k,m}}(x-a)^{n}_{\alpha^{k,m}} = P_{\alpha^{k,m}}[n,j](x-\alpha^{-jm}(a))^{n-j}_{\alpha^{k,m}}$.

(ii) $D^{j}_{\alpha^{k,m}}\left(\dfrac{(x-a)^{n}_{\alpha^{k,m}}}{[n]_{\alpha^{k,m}}!}\right) = \dfrac{(x-\alpha^{-jm}(a))^{n-j}_{\alpha^{k,m}}}{[n-j]_{\alpha^{k,m}}!}$.

Proof. Using Theorem 2 j-times, we compute

$$\begin{aligned}
D^{j}_{\alpha^{k,m}}(x-a)^{n}_{\alpha^{k,m}} &= [n]_{\alpha^{k,m}}[n-1]_{\alpha^{k,m}}\cdots[n-j+1]_{\alpha^{k,m}}(x-\alpha^{-jm}(a))^{n-j}_{\alpha^{k,m}}\\
&= P_{\alpha^{k,m}}[n,j](x-\alpha^{-jm}(a))^{n-j}_{\alpha^{k,m}}.
\end{aligned}$$

Similarly,

$$D^{j}_{\alpha^{k,m}}\left(\frac{(x-a)^{n}_{\alpha^{k,m}}}{[n]_{\alpha^{k,m}}!}\right) = \frac{P_{\alpha^{k,m}}[n,j]}{[n]_{\alpha^{k,m}}!}(x-\alpha^{-jm}(a))^{n-j}_{\alpha^{k,m}} = \frac{(x-\alpha^{-jm}(a))^{n-j}_{\alpha^{k,m}}}{[n-j]_{\alpha^{k,m}}!}.$$

□

Theorem 3. *The set $\{Q_0, Q_1, \cdots Q_{n-1}\}$ is a basis for the vector space of polynomials of degree at most $n-1$, where*

$$Q_j(x) = \frac{(x-a)^j_{\alpha^{k,m}}}{[j]_{\alpha^{k,m}}!}, \qquad 0 \le j \le n-1. \tag{33}$$

Moreover, if $m = 0$, the generalized Taylor's formula can be presented for any polynomial Q of order at most $n-1$

$$Q(x) = \sum_{j=0}^{n-1} (D^j_{\alpha^{k,m}} Q)(a) \frac{(x-a)^j_{\alpha^{k,m}}}{[j]_{\alpha^{k,m}}!}. \tag{34}$$

If $m \neq 0$, then the generalized Taylor's formula (34) holds for $a \in \mathbb{R}$ satisfying $\alpha(a) = a$.

Remark 5. *We stress that if $m = 0$, then the generalized Taylor's formula (34) is satisfied without any requirement. Otherwise, for $m \neq 0$, the condition given by $\alpha(a) = a$ implies that $a = \frac{h}{1-q}$ by Remark 2. For such $a = \frac{h}{1-q}$, Theorem 3 can be considered as an $\alpha^{k,m}$-analogue of the Maclaurin polynomial, which is basically written as*

$$Q(x) = \sum_{j=0}^{n-1} (D^j_{\alpha^{k,m}} Q)\left(\frac{h}{1-q}\right) \frac{(x - \frac{h}{1-q})^j_{\alpha^{k,m}}}{[j]_{\alpha^{k,m}}!}.$$

Nevertheless, we can omit the condition $\alpha(a) = a$ on the α^k-polynomial case because of Corollary 1. To be more precise, the following theorem holds for any $a \in \mathbb{R}$.

Theorem 4. *The set $\{Q_0, Q_1, \cdots Q_{n-1}\}$ is a basis for n-dimensional vector space of polynomials of degree at most $n-1$ where*

$$Q_j(x) = \frac{(x-a)^j_{\alpha^k}}{[j]_{\alpha^k}!}, \qquad 0 \le j \le n-1.$$

Moreover, for any polynomial $Q(x)$ of order at most $n-1$, the generalized Taylor's formula can be presented by

$$Q(x) = \sum_{j=0}^{n-1} (D^j_{\alpha^k} Q)(a) \frac{(x-a)^j_{\alpha^k}}{[j]_{\alpha^k}!}.$$

Proof of Theorem 3. For each $0 \le j \le n-1$, $\deg Q_j = j$, which assures that the set $\{Q_0, Q_1, \cdots Q_{n-1}\}$ is a linearly independent set of polynomials. We can also conclude that it spans the n-dimensional vector space of polynomials since $|\{Q_0, Q_1, \cdots Q_{n-1}\}| = n$ and $\{Q_0, Q_1, \cdots Q_{n-1}\}$ forms a basis. Therefore, we can present any polynomial $Q(x)$ of degree $n-1$ as a linear combination of polynomials in $\{Q_0, Q_1, \cdots Q_{n-1}\}$, i.e.,

$$Q(x) = \sum_{j=0}^{n-1} c_j Q_j(x). \tag{35}$$

We aim to determine the constants c_j. We first observe that the polynomials (33) obey the conditions

$$Q_0(a) = 1, \qquad Q_j(a) = 0, \quad 0 < j \le n-1, \tag{36}$$

from which we obtain the first constant c_0 uniquely as

$$Q(a) = \sum_{j=0}^{m-1} c_j Q_j(a) = c_0 Q_0(a) = c_0.$$

Using the linearity of the operator $D_{\alpha^{k,m}}$, one can apply $D_{\alpha^{k,m}}$ on (35) and use Corollary 3(ii) to obtain

$$D_{\alpha^{k,m}}Q(x) = \sum_{j=0}^{n-1} c_j D_{\alpha^{k,m}}Q_j(x) = \sum_{j=1}^{n-1} c_j \frac{(x-\alpha^{-m}(a))_{\alpha^{k,m}}^{j-1}}{[j-1]_{\alpha^{k,m}}!} = \sum_{j=1}^{n-1} c_j \frac{(x-a)_{\alpha^{k,m}}^{j-1}}{[j-1]_{\alpha^{k,m}}!} = \sum_{j=1}^{n-1} c_j Q_{j-1}(x).$$

The above equation holds if $m = 0$ or if $\alpha(a) = a$, which implies that $\alpha^{-m}(a) = a$ for any $m \in \mathbb{Z}$. Thus, by the use of (36), we provide the next constant

$$D_{\alpha^{k,m}}Q(a) = \sum_{j=1}^{n-1} c_j Q_{j-1}(a) = c_1 Q_0(a) = c_1.$$

We successively apply $D^i_{\alpha^{k,m}}$ to (35), and use Corollary 3(ii) and the assumption $\alpha(a) = a$ to derive

$$D^i_{\alpha^{k,m}}Q(x) = \sum_{j=0}^{n-1} c_j D^i_{\alpha^{k,m}}Q_j(x) = \sum_{j=i}^{n-1} c_j \frac{(x-\alpha^{-im}(a))_{\alpha^{k,m}}^{j-i}}{[j-i]_{\alpha^{k,m}}!} = \sum_{j=i}^{n-1} c_j \frac{(x-a)_{\alpha^{k,m}}^{j-i}}{[j-i]_{\alpha^{k,m}}!} = \sum_{j=i}^{n-1} c_j Q_{j-i}(x).$$

Imposing (36), we compute all the constants

$$D^i_{\alpha^{k,m}}Q(a) = c_i, \qquad 0 \le i \le n-1. \tag{37}$$

As a conclusion, we rewrite (35) using (37)

$$Q(x) = \sum_{j=0}^{n-1} c_j Q_j(x) = \sum_{j=0}^{n-1} (D^j_{\alpha^{k,m}}Q)(a) Q_j(x) = \sum_{j=0}^{m-1} (D^j_{\alpha^{k,m}}Q)(a) \frac{(x-a)_{\alpha^{k,m}}^{j}}{[j]_{\alpha^{k,m}}!}.$$

□

The proof of Theorem 4 proceeds similarly.

Lemma 2. *For $m, k \in \mathbb{Z}$, the α-operator (7) satisfies the following identity*

$$\alpha^k(x) - \alpha^m(x) = \left(\frac{t}{q} + (1-t)q\right)^m [k-m]_\alpha (\alpha(x) - x).$$

Proof. By (12) and Remark 2, we derive

$$\begin{aligned}
\alpha^k(x) - \alpha^m(x) &= \left(\frac{t}{q} + (1-t)q\right)^k \left(x - \frac{h}{1-q}\right) - \left(\frac{t}{q} + (1-t)q\right)^m \left(x - \frac{h}{1-q}\right) \\
&= \left(\frac{t}{q} + (1-t)q\right)^m \left(\left(\frac{t}{q} + (1-t)q\right)^{k-m} - 1\right)\left(x - \frac{h}{1-q}\right) \\
&= \left(\frac{t}{q} + (1-t)q\right)^m [k-m]_\alpha \left(\frac{t}{q} + (1-t)q - 1\right)\left(x - \frac{h}{1-q}\right) \\
&= \left(\frac{t}{q} + (1-t)q\right)^m [k-m]_\alpha (\alpha(x) - x).
\end{aligned}$$

□

Proposition 7. *The α^k-derivative (14) of the α^{-k} polynomial (27) is determined by*

$$D_{\alpha^k}(x-a)^n_{\alpha^{-k}} = [n]_{\alpha^k}(x - \alpha^{-k}(a))^{n-1}_{\alpha^{-k}}.$$

Proof. It follows from Theorem 1 that

$$\begin{aligned}
D_{\alpha^k}(x-a)^n_{\alpha^{-k}} &= \frac{(\alpha^k(x)-a)^n_{\alpha^{-k}} - (x-a)^n_{\alpha^{-k}}}{\alpha^k(x)-x} \\
&= \frac{(\alpha^k(x)-a)^{n-1}_{\alpha^{-k}}(\alpha^k(x)-\alpha^{-(n-1)k}(a)) - (x-a)(x-\alpha^{-k}(a))^{n-1}_{\alpha^{-k}}}{\alpha^k(x)-x}.
\end{aligned}$$

Applying Lemma 1, we obtain

$$(\alpha^k(x)-a)^{n-1}_{\alpha^{-k}} = \left(\frac{t}{q}+(1-t)q\right)^{(n-1)k}(x-\alpha^{-k}(a))^{n-1}_{\alpha^{-k}}.$$

and

$$(\alpha^k(x)-\alpha^{-(n-1)k}(a)) = \left(\frac{t}{q}+(1-t)q\right)^{-(n-1)k}(\alpha^{nk}(x)-a),$$

We finally use Lemma 2 to end up with

$$\begin{aligned}
D_{\alpha^k}(x-a)^n_{\alpha^{-k}} &= \frac{(x-\alpha^{-k}(a))^{n-1}_{\alpha^{-k}}(\alpha^{nk}(x)-a) - (x-a)(x-\alpha^{-k}(a))^{n-1}_{\alpha^{-k}}}{\alpha^k(x)-x} \\
&= \frac{(x-\alpha^{-k}(a))^{n-1}_{\alpha^{-k}}((\alpha^{nk}(x)-a)-(x-a))}{\alpha^k(x)-x} \\
&= (x-\alpha^{-k}(a))^{n-1}_{\alpha^{-k}}\frac{[nk]_\alpha}{[k]_\alpha} = [n]_{\alpha^k}(x-\alpha^{-k}(a))^{n-1}_{\alpha^{-k}}.
\end{aligned}$$

□

Corollary 4. $D_\alpha(x-a)^n_{\alpha^{-1}} = [n]_\alpha(x-\alpha^{-1}(a))^{n-1}_{\alpha^{-1}}$.

The proof is a direct consequence of Proposition 7 by substituting $k=1$.

Corollary 5. $D^j_\alpha(x-a)^n_{\alpha^{-1}} = P_\alpha[n,j](x-\alpha^{-j}(a))^{n-j}_{\alpha^{-1}}$.

The proof follows by Corollary 4 and induction.

Corollary 6. $D_{\alpha^{-1}}(x-a)^n_\alpha = [n]_{\alpha^{-1}}(x-\alpha(a))^{n-1}_\alpha$.

The proof is a direct consequence of Proposition 7 by substituting $k=-1$.

Corollary 7. $D^j_{\alpha^{-1}}(x-a)^n_\alpha = P_{\alpha^{-1}}[n,j](x-\alpha^j(a))^{n-j}_\alpha$.

The proof follows by Corollary 6 and the induction method.

Remark 6. *Once the $\alpha^{k,m}$-polynomial (25) admits the additivity rule stated in Theorem 1 and the power rule stated in Theorem 2, we can confirm that the $\alpha^{k,m}$-polynomial deserves its name. Instead of shifts on a, it is also possible to define a polynomial concept on $\mathbb{T}^{x_0}_\alpha$ using shifts on x and a as well. One may define the polynomial by using m-shifts on x and k-shifts on a as*

$$(x-a)^n_{\widetilde{\alpha^{k,m}}} := \begin{cases} \prod_{j=0}^{n-1}\left(\alpha^{mj}(x)-\alpha^{kj}(a)\right) & \textit{if} \quad n>0, \\ 1 & \textit{if} \quad n=0, \end{cases}$$

which admits a similar derivative rule as follows

$$D_{\alpha^{k,m}}(x-a)^n_{\widetilde{\alpha^{k,m}}} = [n]_{\alpha^{k,m}}(\alpha^m(x)-a)^{n-1}_{\widetilde{\alpha^{k,m}}}, \qquad n\in\mathbb{N}. \tag{38}$$

Indeed, using Lemma 1, it is easy to see that $(x-a)^n_{\alpha^{k,m}}$ *and* $(x-a)^n_{\widetilde{\alpha^{k,m}}}$ *are the same up to a constant, namely*

$$(x-a)^n_{\widetilde{\alpha^{k,m}}} = \left(\frac{t}{q}+(1-t)q\right)^{\frac{m(n-1)n}{2}} (x-a)^n_{\alpha^{k,m}}.$$

Hence, the analogues of Theorem 1 and (38) remain valid.

5. Properties of α-Binomial Coefficients

In this section, we aim to present the properties of the $\alpha^{k,m}$-binomial coefficient (5). We first introduce the $\alpha^{k,m}$-analogue of the Pascal rule, which allows us to demonstrate that the $\alpha^{k,m}$-binomial coefficient serves as a polynomial of $\dfrac{t}{q}+(1-t)q$.

Theorem 5. *Let* $n \in \mathbb{N}_0$ *and* $1 \le j \le n-1$*. Then, the* $\alpha^{k,m}$*-analogue of the Pascal rule can be presented by*

$$\begin{bmatrix} n \\ j \end{bmatrix}_{\alpha^{k,m}} = \left(\frac{t}{q}+(1-t)q\right)^{(n-j)m} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_{\alpha^{k,m}} + \left(\frac{t}{q}+(1-t)q\right)^{jk} \begin{bmatrix} n-1 \\ j \end{bmatrix}_{\alpha^{k,m}}. \tag{39}$$

Proof. Using Definition 3, we rewrite

$$\begin{aligned}
\begin{bmatrix} n \\ j \end{bmatrix}_{\alpha^{k,m}} &= \frac{P_{\alpha^{k,m}}[n,j]}{[j]_{\alpha^{k,m}}!} = \frac{[n]_{\alpha^{k,m}}!}{[j]_{\alpha^{k,m}}![n-j]_{\alpha^{k,m}}!} = \frac{[n]_{\alpha^{k,m}}[n-1]_{\alpha^{k,m}}!}{[j]_{\alpha^{k,m}}![n-j]_{\alpha^{k,m}}!} \\
&= \frac{[n-1]_{\alpha^{k,m}}!}{[j]_{\alpha^{k,m}}![n-j]_{\alpha^{k,m}}!}\left(\left(\frac{t}{q}+(1-t)q\right)^{(n-j)m}[j]_{\alpha^{k,m}} + \left(\frac{t}{q}+(1-t)q\right)^{jk}[n-j]_{\alpha^{k,m}}\right) \\
&= \left(\frac{t}{q}+(1-t)q\right)^{(n-j)m} \frac{[n-1]_{\alpha^{k,m}}!}{[j-1]_{\alpha^{k,m}}![n-1-(j-1)]_{\alpha^{k,m}}!} \\
&\qquad + \left(\frac{t}{q}+(1-t)q\right)^{jk} \frac{[n-1]_{\alpha^{k,m}}!}{[j]_{\alpha^{k,m}}![n-1-j]_{\alpha^{k,m}}!} \\
&= \left(\frac{t}{q}+(1-t)q\right)^{(n-j)m} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_{\alpha^{k,m}} + \left(\frac{t}{q}+(1-t)q\right)^{jk} \begin{bmatrix} n-1 \\ j \end{bmatrix}_{\alpha^{k,m}}.
\end{aligned}$$

□

The $\alpha^{k,m}$-Pascal rule (39) reduces to the α-*Pascal rules* for $k=1$, $m=0$,

$$\begin{bmatrix} n \\ j \end{bmatrix}_{\alpha} = \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_{\alpha} + \left(\frac{t}{q}+(1-t)q\right)^{j} \begin{bmatrix} n-1 \\ j \end{bmatrix}_{\alpha}, \tag{40}$$

and for $k=0$, $m=1$,

$$\begin{bmatrix} n \\ j \end{bmatrix}_{\alpha} = \left(\frac{t}{q}+(1-t)q\right)^{n-j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_{\alpha} + \begin{bmatrix} n-1 \\ j \end{bmatrix}_{\alpha}. \tag{41}$$

When $t=0$, Equations (40) and (41) reduce, respectively, to q-Pascal rules [7]

$$\begin{bmatrix} n \\ j \end{bmatrix}_{q} = q^j \begin{bmatrix} n-1 \\ j \end{bmatrix}_{q} + \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_{q}, \qquad \begin{bmatrix} n \\ j \end{bmatrix}_{q} = \begin{bmatrix} n-1 \\ j \end{bmatrix}_{q} + q^{n-j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_{q}.$$

A natural question arises of whether the α-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_{\alpha}$ is a polynomial in $\frac{t}{q}+(1-t)q$ or not. An answer can be given for special values of j. Indeed,

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_{\alpha} = \begin{bmatrix} n \\ n \end{bmatrix}_{\alpha} = 1 = \left(\frac{t}{q}+(1-t)q\right)^{0}$$

is a polynomial of $\left(\frac{t}{q}+(1-t)q\right)$ with degree 0 and

$$\begin{bmatrix} n \\ 1 \end{bmatrix}_\alpha = \begin{bmatrix} n \\ n-1 \end{bmatrix}_\alpha = [n]_\alpha = 1 + \left(\frac{t}{q}+(1-t)q\right) + \cdots + \left(\frac{t}{q}+(1-t)q\right)^{n-1}$$

is a polynomial of $\left(\frac{t}{q}+(1-t)q\right)$ with degree $n-1$. For an arbitrary j, we state the answer in the forthcoming theorem, whose proof is based on the α-Pascal rule (40).

Theorem 6. *For $n \in \mathbb{N}_0$, the α-binomial coefficient (5) is a polynomial of $\frac{t}{q}+(1-t)q$ with degree $j(n-j)$. Additionally, the coefficients, c_i, of this polynomial are symmetric, i.e.,*

$$c_i = c_{j(n-j)-i}, \qquad 0 \le i \le j(n-j). \tag{42}$$

Proof. To prove the statement, we will use induction on the value of n. Let us start with the base case where $n=1$. It is easy to see that the statement holds true in this case:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}_\alpha = 1 = \left(\frac{t}{q}+(1-t)q\right)^0$$

is a polynomial with degree $0 = 1\cdot(1-1)$, which is $j(n-j)$ for $n=j=1$, and

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_\alpha = 1 = \left(\frac{t}{q}+(1-t)q\right)^0$$

is a polynomial with degree $0 = 0\cdot(1-0)$, which is $j(n-j)$ for $n=1,\ j=0$. Assume that $\begin{bmatrix} n \\ j \end{bmatrix}_\alpha$ is a polynomial of $\frac{t}{q}+(1-t)q$ with degree $j(n-j)$. Then, utilizing the α-Pascal rule (40),

$$\begin{bmatrix} n+1 \\ j \end{bmatrix}_\alpha = \begin{bmatrix} n \\ j-1 \end{bmatrix}_\alpha + \left(\frac{t}{q}+(1-t)q\right)^j \begin{bmatrix} n \\ j \end{bmatrix}_\alpha$$

is also a polynomial of $\left(\frac{t}{q}+(1-t)q\right)$ as the sum of two polynomials. Let us compute its degree.

$$\begin{aligned}
\deg \begin{bmatrix} n+1 \\ j \end{bmatrix}_\alpha &= \max\left\{\deg \begin{bmatrix} n \\ j-1 \end{bmatrix}_\alpha, \deg\left(\left(\frac{t}{q}+(1-t)q\right)^j \begin{bmatrix} n \\ j \end{bmatrix}_\alpha\right)\right\} \\
&= \max\{(j-1)(n-(j-1)), j+j(n-j)\} \\
&= \max\{(j-1)(n+1-j), j(n+1-j)\} = j(n+1-j).
\end{aligned}$$

Since the α-binomial coefficient (5) is a polynomial of $\left(\frac{t}{q}+(1-t)q\right)$ of degree $j(n-j)$, we can represent it by

$$Q\left(\frac{t}{q}+(1-t)q\right) = \begin{bmatrix} n \\ j \end{bmatrix}_\alpha = \sum_{i=0}^{j(n-j)} c_i \left(\frac{t}{q}+(1-t)q\right)^i. \tag{43}$$

On the other hand, by definition of the α-binomial coefficient (5), we have

$$Q\left(\frac{t}{q}+(1-t)q\right) = \begin{bmatrix} n \\ j \end{bmatrix}_\alpha = \prod_{i=1}^{j} \frac{\left(\frac{t}{q}+(1-t)q\right)^{n-i+1}-1}{\left(\frac{t}{q}+(1-t)q\right)^{i}-1}. \tag{44}$$

Hence, using (44), we acquire

$$Q\left(\frac{1}{\frac{t}{q}+(1-t)q}\right)=\prod_{i=1}^{j}\frac{\frac{1}{\left(\frac{t}{q}+(1-t)q\right)^{n-i+1}}-1}{\frac{1}{\left(\frac{t}{q}+(1-t)q\right)^{i}}-1}$$
$$=\frac{\left(\frac{t}{q}+(1-t)q\right)^{1+2+\cdots+j}}{\left(\frac{t}{q}+(1-t)q\right)^{n+(n-1)+\cdots(n-j+1)}}\prod_{i=1}^{j}\left(\frac{1-\left(\frac{t}{q}+(1-t)q\right)^{n-i+1}}{1-\left(\frac{t}{q}+(1-t)q\right)^{i}}\right)$$
$$=\left(\frac{t}{q}+(1-t)q\right)^{-j(n-j)}Q\left(\frac{t}{q}+(1-t)q\right),$$

which leads us to have the beneficial identity

$$\left(\frac{t}{q}+(1-t)q\right)^{j(n-j)}Q\left(\frac{1}{\frac{t}{q}+(1-t)q}\right)=Q\left(\frac{t}{q}+(1-t)q\right). \tag{45}$$

Using (43), we obtain

$$\left(\frac{t}{q}+(1-t)q\right)^{j(n-j)}Q\left(\frac{1}{\frac{t}{q}+(1-t)q}\right)=\left(\frac{t}{q}+(1-t)q\right)^{j(n-j)}\cdot\sum_{i=0}^{j(n-j)}c_i\left(\frac{t}{q}+(1-t)q\right)^{-i}$$
$$=\sum_{i=0}^{j(n-j)}c_i\left(\frac{t}{q}+(1-t)q\right)^{j(n-j)-i}. \tag{46}$$

Making use of the identity (45), and analyzing the coefficients of both (43) and (46), we conclude that the coefficients are symmetric, i.e., the relation (42) holds. □

Corollary 8. *For $n,k,m\in\mathbb{N}_0$, the $\alpha^{k,m}$-binomial coefficient (5) is a polynomial in $\frac{t}{q}+(1-t)q$ with degree $\max\{k,m\}j(n-j)$ in the form*

$$\begin{bmatrix}n\\j\end{bmatrix}_{\alpha^{k,m}}=\sum_{i=0}^{j(n-j)}c_i\left(\frac{t}{q}+(1-t)q\right)^{j(n-j)\min\{k,m\}+|k-m|i},$$

where the coefficient c_i has the symmetry relation

$$c_i=c_{j(n-j)-i},\qquad 0\le i\le j(n-j).$$

Proof. We prove the case $k>m$. The proof of the case $m>k$ is the same. Noting that

$$[n]_{\alpha^{k,m}}=\frac{\left(\frac{t}{q}+(1-t)q\right)^{nk}-\left(\frac{t}{q}+(1-t)q\right)^{nm}}{\left(\frac{t}{q}+(1-t)q\right)^{k}-\left(\frac{t}{q}+(1-t)q\right)^{m}}$$
$$=\left(\frac{t}{q}+(1-t)q\right)^{(n-1)m}\frac{\left(\frac{t}{q}+(1-t)q\right)^{n(k-m)}-1}{\left(\frac{t}{q}+(1-t)q\right)^{k-m}-1}$$
$$=\left(\frac{t}{q}+(1-t)q\right)^{(n-1)m}[n]_{\alpha^{k-m}},$$

we obtain

$$\begin{bmatrix}n\\j\end{bmatrix}_{\alpha^{k,m}}=\prod_{i=1}^{j}\frac{[n-i+1]_{\alpha^{k,m}}}{[j-i+1]_{\alpha^{k,m}}}=\prod_{i=1}^{j}\left(\frac{t}{q}+(1-t)q\right)^{(n-j)m}\frac{[n-i+1]_{\alpha^{k-m}}}{[j-i+1]_{\alpha^{k-m}}}$$
$$=\left(\frac{t}{q}+(1-t)q\right)^{j(n-j)m}\begin{bmatrix}n\\j\end{bmatrix}_{\alpha^{k-m}}.$$

Since

$$\begin{bmatrix} n \\ j \end{bmatrix}_{\alpha^{k-m}} = \prod_{i=1}^{j} \frac{[n-i+1]_{\alpha^{k-m}}}{[i]_{\alpha^{k-m}}} = \prod_{i=1}^{j} \frac{\left(\frac{t}{q}+(1-t)q\right)^{(n-i+1)(k-m)}-1}{\left(\frac{t}{q}+(1-t)q\right)^{i(k-m)}-1}$$

$$= \prod_{i=1}^{j} \frac{\left(\left(\frac{t}{q}+(1-t)q\right)^{k-m}\right)^{n-i+1}-1}{\left(\left(\frac{t}{q}+(1-t)q\right)^{k-m}\right)^{i}-1},$$

it follows from (44) and (43) that

$$\begin{bmatrix} n \\ j \end{bmatrix}_{\alpha^{k,m}} = \left(\frac{t}{q}+(1-t)q\right)^{j(n-j)m} \begin{bmatrix} n \\ j \end{bmatrix}_{\alpha^{k-m}} = \left(\frac{t}{q}+(1-t)q\right)^{j(n-j)m} Q\left(\left(\frac{t}{q}+(1-t)q\right)^{k-m}\right)$$

$$= \sum_{i=0}^{j(n-j)} c_i \left(\frac{t}{q}+(1-t)q\right)^{j(n-j)m+(k-m)i}.$$

□

6. Conclusions

This paper presents a framework for discrete calculus that aims to unify and extend upon previous research. The primary goal is to introduce a new class of discrete time scales $\mathbb{T}_{\alpha}^{x_0}$ that unifies and extends (q,h)-time scales (1). We utilize the forward and backward jump operators as opposed to the delta and nabla (q,h)-derivatives to achieve this approach. We develop a symmetric definition for the α derivative. We also offer a generic polynomial, namely the α-polynomial. Consequently, we state a generalized version of Taylor polynomials. Moreover, we investigate the properties of α-analogues of binomial coefficients such as Pascal's rule, from which we conclude that α-analogues of binomial coefficients can be represented as polynomials with symmetric coefficients.

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