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Bayes Estimation for the Rayleigh–Weibull Distribution Based on Progressive Type-II Censored Samples for Cancer Data in Medicine

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Abstract: The aim of this study is to obtain the Bayes estimators and the maximum likelihood estimators (MLEs) for the unknown parameters of the Rayleigh–Weibull (RW) distribution based on progressive type-II censored samples. The approximate Bayes estimators are calculated using the idea of Lindley, Tierney–Kadane approximations, and also the Markov Chain Monte Carlo (MCMC) method under the squared-error loss function when the Bayes estimators are not handed in explicit forms. In this study, the approximate Bayes estimates are compared with the maximum likelihood estimates in the aspect of the estimated risks (ERs) using Monte Carlo simulation. The asymptotic confidence intervals for the unknown parameters are obtained using the MLEs of parameters. In addition, the coverage probabilities the parametric bootstrap estimates are computed. Real lifetime datasets related to bladder cancer, head and neck cancer, and leukemia are used to illustrate the empirical results belonging to the approximate Bayes estimates, the maximum likelihood estimates, and the parametric bootstrap intervals.

Keywords: Rayleigh–Weibull distribution; progressive type-II censored sample; Bayes estimator; asymptotic and bootstrap confidence intervals; Lindley and Tierney–Kadane approximation; Markov Chain Monte Carlo method; squared-error loss function; Monte Carlo simulation

MSC: 62F15; 62F40; 62N01; 62P10; 26A33; 33B20

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1. Introduction

Probability distributions are often used to model real data, especially in the fields of medicine, engineering, biological studies, etc. In general, medical data such as lifetime data have a (right) skewed distribution. Therefore, statistical analysis depends on the assumed probability distribution of the skewed medical data. The Rayleigh distribution, proposed by Lord Rayleigh [1] in 1880, which is a special form of the Weibull distribution, is one of the most popular distributions in the analysis of skewed data. The Rayleigh distribution plays an important role in real life applications. It has wide applications in life and reliability analysis, especially in modeling real lifetime data in clinical research. Extensions and generalizations of the known probability distributions have been suggested in order to obtain the best model that fits the data. Likewise, Rayleigh probability distribution extensions and generalizations have been derived because of its great importance in modeling life phenomena. In the literature, statistical inferences have been made for different forms of the Rayleigh distribution, both in the case of complete and censored samples. Additionally, goodness-of-fit tests have been developed for the Rayleigh distribution (Sindhua et al. [2], Dey and Dey [3], EL-Sagheer et al. [4], Dey et al. [5], Fan and Gui [6], Shen et al. [7], Zamanzade and Mahdızadeh [8]).

The Rayleigh–Weibull distribution, introduced by Smadi and Alrefaei [9] as a new probability distribution model, provides flexibility enabling the Rayleigh distribution to obtain the best model fit with parameters (α, β) denoted by $RW(\alpha, \beta)$, where $\alpha > 0$ and

$\beta > 0$. The probability density function (pdf), cumulative distribution function (cdf), survival function, and hazard function of the random variable X has a Rayleigh–Weibull distribution with parameters (α, β) and can be given as follows:

$$f(x; \alpha, \beta) = 2\alpha\beta^2 x^{2\alpha-1} \exp\left(-\beta^2 x^{2\alpha}\right), \quad x \geq 0, \ \alpha > 0, \ \beta > 0 \tag{1}$$

$$F(x; \alpha, \beta) = 1 - \exp\left(-\beta^2 x^{2\alpha}\right) \tag{2}$$

$$\overline{F}(x; \alpha, \beta) = \exp\left(-\beta^2 x^{2\alpha}\right) \tag{3}$$

$$h(x; \alpha, \beta) = 2\alpha\beta^2 x^{2\alpha-1} \tag{4}$$

In this study, Bayesian estimators for parameters of the Rayleigh–Weibull distribution proposed to give flexibility to the Rayleigh distribution are investigated in detail. Since the prior distribution of the parameters is used in the Bayesian estimation method, it is more convenient to use Bayesian estimators of the parameters of the (right) skewed distributions in the decision-making process in medical studies. In many studies, Bayesian estimation has been investigated based on complete and censored samples for different distributions, including by Kundu and Gupta [10], Almogy et al. [11], Xie and Gui [12], Cai and Gui [13], Jiang and Gui [14].

In medical studies, since researchers can not observe the entire lifetime of all subjects in a life test experiment due to the time and cost constraints, censored data are needed. Since the complete data are not always available, there are censoring schemes that reduce time and cost. In life test experiments, one of the most frequently used censoring schemes is the progressive Type-II right (PTR-II) censoring scheme. Progressive censoring is useful in both industrial life testing applications and clinical settings. It allows removal of surviving experimental units before testing is terminated [3]. In the PTR-II censoring scheme, the items are removed from the experiment and then a censored sample is created, thus saving time and cost. This type of censored scheme is explained as follows. Suppose that n identical items are put to the test and m failures are to be observed. At the time of the first failure, R_1 items from the rest of the surviving $n - R_1 - 1$ items are randomly selected, and then removed. Likewise, at the time of the second failure, R_2 items of the remaining $n - R_2 - 2$ items are randomly selected, and then removed, and so the process continues. Lastly, at the time of the m^{th} failure, all the surviving items are censored. The PTR-II censoring scheme is visually demonstrated with the $\mathbf{R} = (R_1, R_2, \ldots, R_m)$ scheme. In this lifetime process, $\mathbf{X}^{\mathbf{R}} = \left(X^{R_1}_{1:m:n}, X^{R_2}_{2:m:n}, \ldots, X^{R_m}_{m:m:n}\right)$ with $X^{R_1}_{1:m:n} < X^{R_2}_{2:m:n} < \ldots < X^{R_m}_{m:m:n}$ is called the PTR-II censored sample with $\mathbf{R} = (R_1, R_2, \ldots, R_m)$. In PTR-II censoring, using $\mathbf{R} = (0, 0, \ldots, n - m)$, Type-II right censoring is obtained. The joint probability density function (pdf) of this censored sample is given by ([15,16])

$$f_{X^{\mathbf{R}}_{1:m:n}, X^{\mathbf{R}}_{2:m:n}, \ldots, X^{\mathbf{R}}_{m:m:n}}(x_1, x_2, \ldots, x_m) = c\prod_{i=1}^{m} f(x_i)[1 - F(x_i)]^{R_i}, \ -\infty < x_1 < x_2 < \cdots < x_m < \infty \tag{5}$$

where $c = n(n - R_1 - 1) \times \cdots \times (n - R_1 - R_2 - \cdots - R_{m-1} - m + 1)$.

There are a lot of studies that refer to the parameter estimation of different distributions under PTR-II censored samples (Ali Mousa [17], Balakrishnan [18], Ali Mousa and Al-Sagheer [19], Wu et al. [20], Panahi and Asadi [21], Aljuaid [22], Ahmed [23], Singh et al. [24], Liao ang Gui [25], Abbas et al. [26], Sultan et al. [27], Alshenawy [28], Mukhtar [29], Wu and Gui [30], Almongy et al. [31], Qiao and Gui [32], Wu [33], El-Morshedy et al. [34], El-Sherpieny et al. [35], Liang et al. [36], Alshenawy et al. [37], Almetwally et al. [38], Muhammed and Almetwally [39], Çiftci et al. [40]). Based on the above studies, the main purpose of this study is to obtain the approximate Bayes estimators under the square error loss functions and then to check them with maximum likelihood estimators (MLEs) in the aspect of the estimated risk (ER).

The sections of this study are organized as follows: In the first section, an introduction to the RW distribution and PTR-II censored sample is given. In Section 2, the MLEs, asymptotic and bootstrap confidence intervals for the unknown parameters are obtained. In Section 3, the approximate Bayes estimators under the squared error loss function using Lindley's approximations, Tierney–Kadane approximations, and the Markov Chain Monte Carlo (MCMC) method for the unknown parameters are acquired. In Section 4, the approximate Bayes estimations are compared with the maximum likelihood (ML) estimations in the aspect of the ER, and then the coverage probabilities of the asymptotic confidence intervals and the bootstrap confidence intervals are observed by using Monte Carlo simulation. In Section 5, the real lifetime datasets of bladder cancer, neck cancer, and leukemia are given to illustrate the empirical results of the approximate Bayes estimates, the maximum likelihood estimates, and the parametric bootstrap intervals. In Section 6, conclusions are given.

2. Maximum Likelihood Estimation

In terms of possessing the asymptotic normality property under mild regularity conditions, the maximum likelihood (ML) estimation is highly favored within the field of statistical inference. Recently, Wang et al. [41] studied the ML methodology for inverse Gaussian distribution based on maximum rank set sampling with unequal samples. Let $\mathbf{X}^{\mathbf{R}} = \left(X_{1:m:n}^{R_1}, X_{2:m:n}^{R_2}, \ldots, X_{m:m:n}^{R_m}\right)$, which denotes a PTR-II censored sample taken from a $RW(\alpha, \beta)$ distribution with the pdf and cdf in Equations (1) and (2). Then, the likelihood function $\ell(\alpha, \beta)$ can be written as follows:

$$\ell(\alpha, \beta) = c\prod_{i=1}^{m} 2\alpha\beta^2 x_{i:m:n}^{2\alpha-1} \exp\left(-\beta^2 x_{i:m:n}^{2\alpha}\right) \left[\exp\left(-\beta^2 x_{i:m:n}^{2\alpha}\right)\right]^{R_i}$$

$$\ell(\alpha, \beta) \propto \alpha^m \beta^{2m} \exp\left(\sum_{i=1}^{m} -\beta^2 x_{i:m:n}^{2\alpha}\right) \prod_{i=1}^{m} x_{i:m:n}^{2\alpha-1} \left[\exp\left(-\beta^2 x_{i:m:n}^{2\alpha}\right)\right]^{R_i} \tag{6}$$

The log-likelihood function, $L(\alpha, \beta) = \ln \ell(\alpha, \beta)$ can be given as follows:

$$L(\alpha, \beta) \propto m\ln(\alpha) + 2m\ln(\beta) + (2\alpha - 1)\sum_{i=1}^{m} \ln(x_{i:m:n}) - \beta^2 \sum_{i=1}^{m} (1 + R_i) x_{i:m:n}^{2\alpha} \tag{7}$$

Taking the partial derivatives of $L(\alpha, \beta)$ according to the α and β parameters, and then equalizing them to zero, the following equations can be obtained:

$$L_1 = \frac{\partial L}{\partial \alpha} = \frac{m}{\hat{\alpha}} + 2\sum_{i=1}^{m} \ln(x_{i:m:n}) - \hat{\beta}^2 \sum_{i=1}^{m} 2(1 + R_i) x_{i:m:n}^{2\hat{\alpha}} \ln(x_{i:m:n}) = 0 \tag{8}$$

$$L_2 = \frac{\partial L}{\partial \beta} = \frac{2m}{\hat{\beta}} - 2\hat{\beta}\sum_{i=1}^{m} (1 + R_i) x_{i:m:n}^{2\hat{\alpha}} = 0 \tag{9}$$

The nonlinear equations given by Equations (8) and (9) can be solved by using the Newton–Raphson (NR) iterative method in MATLAB 2016.

2.1. Asymptotic Confidence Interval (ACI)

Let $\Theta = (\alpha, \beta)$ be the Fisher information matrix of Θ parameter vector given by

$$I(\Theta) = -E\begin{bmatrix} \frac{\partial^2 lnL}{\partial \alpha^2} & \frac{\partial^2 lnL}{\partial \alpha \partial \beta} \\ \frac{\partial^2 lnL}{\partial \alpha \partial \beta} & \frac{\partial^2 lnL}{\partial \beta^2} \end{bmatrix}$$

Since $I(\Theta)$ is difficult to compute, the observed Fisher information $I(\hat{\Theta})$ is used as an approximate to expect Fisher information matrix. Let $\hat{\Theta} = (\hat{\alpha}, \hat{\beta})$ be the MLEs of the parameters $\Theta = (\alpha, \beta)$. The observed Fisher information matrix is given by

$$I(\hat{\Theta}) = \begin{bmatrix} -\frac{\partial^2 lnL}{\partial\alpha^2} & -\frac{\partial^2 lnL}{\partial\alpha\partial\beta} \\ -\frac{\partial^2 lnL}{\partial\alpha\partial\beta} & -\frac{\partial^2 lnL}{\partial\beta^2} \end{bmatrix}_{\Theta=\hat{\Theta}}$$

Therefore, the observed variance–covariance matrix for the MLEs $(\hat{\alpha}, \hat{\beta})$ is the inverse of the observed information matrix given by [42]

$$I^{-1}(\hat{\Theta}) = \begin{bmatrix} \hat{V}ar(\hat{\alpha}) & \hat{C}ov(\alpha, \beta) \\ \hat{C}ov(\alpha, \beta) & \hat{V}ar(\hat{\beta}) \end{bmatrix}$$

Under some regularity conditions, $\hat{\Theta}$ is approximately bivariately normally distributed with mean Θ and variance–covariance matrix $I(\hat{\Theta})$ as $\left[\hat{\Theta} \sim N(\Theta, I^{-1}(\hat{\Theta}))\right]$ [43]. Thus, the 100 $(1-\delta)$% confidence interval for α and β can be constructed as $\left(\hat{\alpha} \mp z_{\frac{\delta}{2}} \times \sqrt{Var(\hat{\alpha})}\right)$ and $\left(\hat{\beta} \mp z_{\frac{\delta}{2}} \times \sqrt{Var(\hat{\beta})}\right)$, where z_δ denotes the upper $\delta - th$ quantile of the standard normal distribution.

2.2. Bootstrap Confidence Interval

Confidence intervals for the unknown $\Theta = (\alpha, \beta)$ parameters are obtained by using the percentile bootstrap confidence interval (P-BCI) method proposed by Efron [44]. The steps for estimating the bootstrap parametric confidence intervals of the parameters $\Theta = (\alpha, \beta)$ by using the P-BCI method are given as follows [45].

1. Generate the PTR-II censored samples $\mathbf{X^R} = \left(X_{1:m:n}^{R_1}, X_{2:m:n}^{R_2}, \ldots, X_{m:m:n}^{R_m}\right)$ taken from the RW distribution with the $\Theta = (\alpha, \beta)$ parameters.

2. Let ML estimates of the $\Theta = (\alpha, \beta)$ parameters be $\hat{\Theta} = (\hat{\alpha}_{MLE}, \hat{\beta}_{MLE})$.

3. Generate the bootstrap samples $\mathbf{X^R} = \left(X_{1:m:n}^{R_1}, X_{2:m:n}^{R_2}, \ldots, X_{m:m:n}^{R_m}\right)$ with the $R_1, R_2, \ldots, R_m$ scheme, using the $\hat{\Theta} = (\hat{\alpha}_{MLE}, \hat{\beta}_{MLE})$. Find the bootstrap estimate of the $\Theta = (\alpha, \beta)$ parameters as $\hat{\Theta}^* = (\hat{\alpha}^*_{MLE}, \hat{\beta}^*_{MLE})$.

4. Repeat Step 3 NBoot times.

5. Let $F^*(x) = P\left(\hat{\Theta}^* \leq x\right)$ as the cumulative distribution function of $\hat{\Theta}^*$. Define $\hat{\Theta}^* = F^{*-1}(x)$ for a given x. The approximate bootstrap 100 $(1-\gamma)$% confidence interval for Θ is given as $\left(\hat{\Theta}^*_{\frac{\gamma}{2}}, \hat{\Theta}^*_{1-\frac{\gamma}{2}}\right)$.

3. Bayes Estimation

For Bayesian estimation, it is assumed that the α and β parameters of the $RW(\alpha, \beta)$ distribution have the following independent prior $Gamma(a_1, b_1)$, and $Gamma(a_2, b_2)$ densities, respectively:

$$\pi_1(\alpha) = \frac{\alpha^{a_1-1} \exp(-b_1\alpha) b_1^{a_1}}{\Gamma(a_1)} \quad a_1, b_1, \alpha > 0 \tag{10}$$

$$\pi_2(\beta) = \frac{\beta^{a_2-1} \exp(-b_2\beta) b_2^{a_2}}{\Gamma(a_2)} \quad a_2, b_2, \beta > 0 \tag{11}$$

In this case, the joint prior distribution of the α and β parameters can be written as follows:

$$\pi(\alpha, \beta) = \frac{\alpha^{a_1-1} b_1^{a_1} \beta^{a_2-1} b_2^{a_2}}{\Gamma(a_1)\Gamma(a_2)} \exp(-b_1\alpha) \exp(-b_2\beta) \quad a_i, b_i, \alpha, \beta > 0\,, i = 1,2 \tag{12}$$

From Equation (12), the log of the prior density function is given as follows:

$$\begin{aligned}\rho(\alpha,\beta) = (a_1-1)\ln\alpha + (a_2-1)\ln\beta - b_1\alpha - b_2\beta + \\ a_1\ln(b_1) + a_2\ln(b_2) - \ln[\Gamma(a_1)] - \ln[\Gamma(a_2)]\end{aligned} \tag{13}$$

By using the $L(\alpha,\beta)$, and $\pi(\alpha,\beta)$, the joint posterior density function of the α and β parameters can be written as follows:

$$P(\alpha,\beta) = \frac{k(\alpha,\beta)\exp\left(\sum_{i=1}^{m} -\beta^2 x_{i:m:n}^{2\alpha}\right)\prod_{i=1}^{m} x_{i:m:n}^{2\alpha-1}\left[\exp\left(-\beta^2 x_{i:m:n}^{2\alpha}\right)\right]^{R_i}}{\int_0^\infty\int_0^\infty k(\alpha,\beta)\exp\left(\sum_{i=1}^{m} -\beta^2 x_{i:m:n}^{2\alpha}\right)\prod_{i=1}^{m} x_{i:m:n}^{2\alpha-1}\left[\exp\left(-\beta^2 x_{i:m:n}^{2\alpha}\right)\right]^{R_i} d\alpha d\beta} \tag{14}$$

where $k(\alpha,\beta) = \alpha^{m+a_1-1}\beta^{2m+a_2-1}\exp(-b_1\alpha)\exp(-b_2\beta)$.

Thus, the Bayes estimate of any function of α and β, say $u(\alpha,\beta)$, under the squared error loss function can be written as follows:

$$\hat{u}_B(\alpha,\beta) = E[u(\alpha,\beta)] = \frac{\int_0^\infty\int_0^\infty u(\alpha,\beta)e^{[L(\alpha,\beta)+\rho(\alpha,\beta)]}\,d\alpha d\beta}{\int_0^\infty\int_0^\infty e^{[L(\alpha,\beta)+\rho(\alpha,\beta)]}d\alpha d\beta} \tag{15}$$

The Bayes estimate of any function of α and β given in Equation (15), which consists of the ratio of two integrals, can not be obtained in closed form, and then the Bayes estimators of these parameters using the Lindley's approximation, and Tierney–Kadane approximation under the squared error loss (quadratic loss) function are computed.

3.1. Lindley's Approximation

Lindley's approximation, suggested by Lindley [46], is an approximate Bayes method used to approximate the ratio of two integrals, such as those given in Equation (15), that cannot be solved analytically. This method uses third derivatives of the log-likelihood function, and has an error of order $O(n^{-1})$. Lindley's approximation has been used by many authors, such as Ahmad and Jaheen [47], Kundu and Gupta [48], and Preda et al. [49], to compute the approximate Bayes estimators of different lifetime distributions based on the censored samples. For the two-parameter case, where θ_1 and θ_2 notations are used for the α and β parameters, the formula with the Lindley's approximation can be written as follows:

$$u_{BLindley}(\hat{\theta}_1,\hat{\theta}_2) = E[u(\theta_1,\theta_2)/X] \approx$$

$$\approx \left[u_{MLE}(\hat{\theta}_1,\hat{\theta}_2) + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}(u_{ij}+2u_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2}L_{ijk}\sigma_{ij}\sigma_{kl}u_l\right]$$

$$= u_{MLE}(\hat{\theta}_1,\hat{\theta}_2) + \frac{1}{2}[a_{11}+a_{12}+a_{21}+a_{22}] + \frac{1}{2}[(u_1\sigma_{11}+u_2\sigma_{12})d + (u_1\sigma_{21}+u_2\sigma_{22})e] \tag{16}$$

where $\hat{\theta}_1$ and $\hat{\theta}_2$ are the MLE of the θ_1 and θ_2 parameters, respectively, and let

$$a_{1i} = (u_{1i}+2u_1\rho_i)\sigma_{1i};\ a_{2i} = (u_{2i}+2u_2\rho_i)\sigma_{2i} \quad i=1,2$$

$$d = L_{111}\sigma_{11} + L_{121}\sigma_{12} + L_{211}\sigma_{21} + L_{221}\sigma_{22}$$

$$e = L_{112}\sigma_{11} + L_{122}\sigma_{12} + L_{212}\sigma_{21} + L_{222}\sigma_{22}$$

and

$$\rho_i = \frac{\partial \rho(\theta_1, \theta_2)}{\partial \theta_i}, \ i = 1,2 \ u_i = \frac{\partial u(\theta_1, \theta_2)}{\partial \theta_i}, \ i = 1,2 \ u_{ij} = \frac{\partial^2 u(\theta_1, \theta_2)}{\partial \theta_i \partial \theta_j}, \ i,j = 1,2$$

$$L_{ij} = \frac{\partial^2 L(\theta_1, \theta_2)}{\partial \theta_i \partial \theta_j}, \ i,j = 1,2 \ L_{ijk} = \frac{\partial^3 L(\theta_1, \theta_2)}{\partial \theta_i \partial \theta_j \partial \theta_k}, \ i,j,k = 1,2$$

$$\left[-L_{ij}\right]^{-1} = \left[\sigma_{ij}\right], \ i,j = 1,2$$

σ_{ij} is the (i, j)-th element of the matrix $\left[\sigma_{ij}\right]$.

From Equation (13), we get

$$\rho_1 = \frac{a_1 - 1}{\alpha} - b_1, \ \rho_2 = \frac{a_2 - 1}{\beta} - b_2$$

and then, the following values of L_{ij} for i, j = 1, 2 and L_{ijk} for i, j, k = 1, 2 are handed as follows:

$$L_{11} = -\frac{m}{\alpha^2} - \beta^2 \sum_{i=1}^{m} 4(1 + R_i) x_{i:m:n}^{2\alpha} \ln(x_{i:m:n})^2$$

$$L_{12} = L_{21} = -2\beta \sum_{i=1}^{m} 2(1 + R_i) x_{i:m:n}^{2\alpha} \ln(x_{i:m:n})$$

$$L_{22} = -\frac{2m}{\beta^2} - 2\sum_{i=1}^{m} (1 + R_i) x_{i:m:n}^{2\alpha}$$

$$L_{111} = \frac{2m}{\alpha^3} - \beta^2 \sum_{i=1}^{m} 8(1 + R_i) x_{i:m:n}^{2\alpha} \ln(x_{i:m:n})^3$$

$$L_{112} = L_{112} = L_{121} = L_{211} = -2\beta \sum_{i=1}^{m} 4(1 + R_i) x_{i:m:n}^{2\alpha} \ln(x_{i:m:n})^2$$

$$L_{122} = L_{122} = L_{221} = L_{212} = -2\sum_{i=1}^{m} 2(1 + R_i) x_{i:m:n}^{2\alpha} \ln(x_{i:m:n})$$

$$L_{222} = \frac{4m}{\beta^3}$$

Finally, the approximate Bayes estimators for the α and β parameter of the $RW(\alpha, \beta)$ distribution based on progressive type-II censored samples under the squared error loss function are obtained, respectively, as follows:

$$\hat{\alpha}_{BL} = \hat{\alpha}_{MLE} + \left(\frac{a_1 - 1}{\hat{\alpha}_{MLE}} - b_1\right)\sigma_{11} + \left(\frac{a_2 - 1}{\hat{\beta}_{MLE}} - b_2\right)\sigma_{12} + \frac{1}{2}[\sigma_{11} d + \sigma_{21} e] \tag{17}$$

$$\hat{\beta}_{BL} = \hat{\beta}_{MLE} + \left(\frac{a_1 - 1}{\hat{\alpha}_{MLE}} - b_1\right)\sigma_{21} + \left(\frac{a_2 - 1}{\hat{\beta}_{MLE}} - b_2\right)\sigma_{22} + \frac{1}{2}[\sigma_{12} d + \sigma_{22} e] \tag{18}$$

3.2. Tierney–Kadane Approximation

The Tierney–Kadane approximation, proposed by Tierney and Kadane [50], is a method alternative to Lindley's approximation. This method uses second derivatives of a function composed of the log-likelihood function and the log-prior function, and has an error of order $O\left(n^{-2}\right)$. Therefore, the Tierney–Kadane approximation is more advantageous than Lindley's approximation. The Tierney–Kadane approximation has been used by many authors, such as Gencer and Gencer [51], Kim and Han [52], Elshahhat and Rastogi [53], Singh et al. [54], to compute the approximate Bayes estimators of different lifetime distributions based on the censored samples. This approximation can be defined as follows:

$$\eta(\alpha,\beta) = \frac{1}{n}[L(\alpha,\beta) + \rho(\alpha,\beta)]$$

$$\eta^*(\alpha,\beta) = \frac{1}{n}\ln[u(\alpha,\beta)] + \eta(\alpha,\beta)$$

where $L(\alpha,\beta)$ denotes the log-likelihood function, and $\rho(\alpha,\beta)$ denotes the log of the joint prior density. Thus, by means of the Tierney–Kadane approximation, Equation (15) can be written as follows:

$$\begin{aligned} \hat{u}_{BTK}(\alpha,\beta) &= E[u(\alpha,\beta)] = \frac{\int e^{n\eta^*(\alpha,\beta)} d(\alpha,\beta)}{\int e^{n\eta(\alpha,\beta)} d(\alpha,\beta)} \\ &\approx \left[\frac{\det\Sigma^*}{\det\Sigma}\right]^{1/2} \exp\left\{n\left[\eta^*\left(\hat{\alpha}^*,\hat{\beta}^*\right) - \eta\left(\hat{\alpha},\hat{\beta}\right)\right]\right\} \end{aligned}$$

where $\left(\hat{\alpha}^*,\hat{\beta}^*\right)$ and $\left(\hat{\alpha},\hat{\beta}\right)$ maximize $\eta^*(\alpha,\beta)$ and $\eta(\alpha,\beta)$, respectively. Σ^* and Σ are minus the inverse Hessians of $\eta^*(\alpha,\beta)$ and $\eta(\alpha,\beta)$ at $\left(\hat{\alpha}^*,\hat{\beta}^*\right)$ and $\left(\hat{\alpha},\hat{\beta}\right)$, respectively.

In this case, $\eta(\alpha,\beta)$, $\eta^*(\alpha,\beta)$, and Σ are given as follows:

$$\eta(\alpha,\beta) = \tfrac{1}{n}[mln(\alpha) + 2mln(\beta) + (2\alpha - 1)\sum_{i=1}^{m} ln(x_{i:m:n}) - \beta^2 \sum_{i=1}^{m} (1+R_i)x_{i:m:n}^{2\alpha} +$$

$$(a_1 - 1)ln(\alpha) + (a_2 - 1)ln(\beta) - b_1\alpha - b_2\beta + a_1 ln(a_1) +$$

$$a_2 ln(b_2) - ln[\Gamma(a_1)] - ln[\Gamma(a_2)]]$$

$$\eta^*(\alpha,\beta) = \tfrac{1}{n}[ln[u(\alpha,\beta)] + mln(\alpha) + 2mln(\beta) + (2\alpha - 1)\sum_{i=1}^{m} ln(x_{i:m:n}) -$$

$$\beta^2 \sum_{i=1}^{m} (1+R_i)x_{i:m:n}^{2\alpha} + (a_1 - 1)ln(\alpha) + (a_2 - 1)ln(\beta) - b_1\alpha - b_2\beta +$$

$$a_1 ln(b_1) + a_2 ln(b_2) - ln[\Gamma(a_1)] - ln[\Gamma(a_2)]]$$

and

$$\Sigma = \begin{bmatrix} -\frac{\partial^2\eta(\alpha,\beta)}{\partial\alpha^2} & -\frac{\partial^2\eta(\alpha,\beta)}{\partial\alpha\partial\beta} \\ -\frac{\partial^2\eta(\alpha,\beta)}{\partial\alpha\partial\beta} & -\frac{\partial^2\eta(\alpha,\beta)}{\partial\beta^2} \end{bmatrix}^{-1}$$

respectively.

Through the Tierney–Kadane approximation, the approximate Bayes estimators of the α and β parameters of the $RW(\alpha,\beta)$ distribution based on the PTR-II censored samples under the squared error loss function are obtained as follows:

$$\eta^*_\alpha(\alpha,\beta) = \frac{ln(\alpha)}{n} + \eta(\alpha,\beta)$$

and

$$\Sigma^*_\alpha = \begin{bmatrix} -\frac{\partial^2\eta^*_\alpha(\alpha,\beta)}{\partial\alpha^2} & -\frac{\partial^2\eta^*_\alpha(\alpha,\beta)}{\partial\alpha\partial\beta} \\ -\frac{\partial^2\eta^*_\alpha(\alpha,\beta)}{\partial\alpha\partial\beta} & -\frac{\partial^2\eta^*_\alpha(\alpha,\beta)}{\partial\beta^2} \end{bmatrix}^{-1}$$

$$\hat{\alpha}_{BTK} = \left[\frac{\det\Sigma^*_\alpha}{\det\Sigma}\right]^{1/2} \exp\left\{n\left[\eta^*_\alpha\left(\hat{\alpha}^*,\hat{\beta}^*\right) - \eta\left(\hat{\alpha},\hat{\beta}\right)\right]\right\} \tag{19}$$

$$\eta^*_\beta(\alpha,\beta) = \frac{ln(\beta)}{n} + \eta(\alpha,\beta)$$

and

$$\Sigma^*_\beta = \begin{bmatrix} -\frac{\partial^2 \eta^*_\beta(\alpha,\beta)}{\partial \alpha^2} & -\frac{\partial^2 \eta^*_\beta(\alpha,\beta)}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 \eta^*_\beta(\alpha,\beta)}{\partial \alpha \partial \beta} & -\frac{\partial^2 \eta^*_\beta(\alpha,\beta)}{\partial \beta^2} \end{bmatrix}^{-1}$$

$$\hat{\beta}_{BTK} = \left[\frac{\det\Sigma^*_\beta}{\det\Sigma}\right]^{1/2} \exp\left\{n\left[\eta^*_\beta(\hat{\alpha}^*, \hat{\beta}^*) - \eta(\hat{\alpha}, \hat{\beta})\right]\right\} \tag{20}$$

3.3. Markov Chain Monte Carlo (MCMC)

The Markov Chain Monte Carlo (MCMC) method is one of the best and most useful method for Bayesian estimation and has been used by many authors such as Aljuaid [21], Singh et al. [55], Lv et al. [56], and Zhou et al. [57]. The MCMC method provides sampling from the posterior distribution. The Metropolis–Hastings (MH) algorithm introduced by Metropolis et al. [58] is one of the most popular algorithms for the MCMC method. The random-walk Metropolis–Hastings algorithm is a type of the Metropolis–Hastings algorithm. For more details about the random-walk Metropolis–Hastings algorithm steps, see Junnumtuam et al. [59]. In this study, the random-walk Metropolis algorithm to obtain Bayes estimates of α and β parameters was applied using the rwmetrop function in R programme LearnBayes library (Albert and Albert [60]).

4. Simulation Study

In this section, Monte Carlo simulation studies for different sample sizes (n and m) and different censoring schemes are done. In the aspect of the estimated risks, the performances of the approximate Bayes estimates computed with Lindley and Tierney–Kadane approximation and MCMC methods under the squared error loss function for the α and β parameters of $RW(\alpha, \beta)$ based on PTR-II censored sample are compared with those of the ML. Informative priors for $a_1 = 1$, $b_1 = 1$, $a_2 = 3$, $b_2 = 3$ are used while computing the approximate Bayes estimates. The ER for the estimate of the α parameter can be computed with the $\hat{\alpha}_{ER} = E(\hat{\alpha}_i - \alpha)^2,\ i = 1, 2, \ldots, 10{,}000$, where $\hat{\alpha}$ is the ML or the approximate Bayes estimation, and α is generated from the Gamma distribution with parameters (a_1, b_1). In addition, the ER for the estimate of the β parameter is computed in the same way. All the computations are based on 10.000 replications in MATLAB.

In this simulation study, in order to produce the PTR-II censored samples from the $RW(\alpha, \beta)$ distribution, we have benefited from the algorithm presented in Balakrishnan and Sandhu [61]. The algorithm for the $RW(\alpha, \beta)$ distribution is given as follows:

1. Let $W_1, W_2, \ldots, W_m$ be m-sized samples generated from the Uniform$(0, 1)$ distribution.
2. $V_i = W_i^{\left(i + \sum_{j=m-i+1}^{m} R_j\right)^{-1}}$ is defined by replacing $i = 1, 2, \ldots, m$.
3. $U^{R}_{i:m:n} = 1 - V_m V_{m-1} \cdots V_{m-i+1}$ is obtained by replacing $i = 1, 2, \ldots, m$.

Thus $U^{\mathbf{R}}_{1:m:n} < U^{\mathbf{R}}_{2:m:n} < \cdots < U^{\mathbf{R}}_{m:m:n}$ are progressively Type-II censored samples with the censoring scheme $\mathbf{R} = (R_1, R_2, \ldots, R_m)$ taken from the Uniform$(0, 1)$ distribution. Finally, $X^{R}_{i:m:n} = \left[-\frac{\ln\left(1 - \ln U^{R}_{i:m:n}\right)}{\beta^2}\right]^{\left(\frac{1}{2\alpha}\right)},\quad i = 1, 2, \ldots, m$ is the progressively Type-II censored i^{th} order statistic with the censoring scheme $\mathbf{R} = (R_1, R_2, \ldots, R_m)$ taken from the $RW(\alpha, \beta)$ distribution. The estimated risks of the approximate Bayes estimates computed with Lindley and Tierney–Kadane approximations and MCMC methods under the squared-error loss function and ML estimates for the α and β parameters of $RW(\alpha, \beta)$ based on progressive type-II censored sample are tabulated in Table 1. α and β parameters are generated from Gamma distribution parameters (a_1, b_1) and (a_2, b_2) in each replicate, respectively.

Table 1. ER values of the ML and the Approximate Bayes Estimates of the α and β parameters for $a_1 = 1$, $b_1 = 1$, $a_2 = 3$, $b_2 = 3$.

n	m	Censoring Scheme	ML		LINDLEY		TIERNEY-KADANE		MCMC	
		R	$\hat{\alpha}_{ER}$	$\hat{\beta}_{ER}$	$\hat{\alpha}_{ER}$	$\hat{\beta}_{ER}$	$\hat{\alpha}_{ER}$	$\hat{\beta}_{ER}$	$\hat{\alpha}_{ER}$	$\hat{\beta}_{ER}$
10	5	A	0.0256	0.0290	0.0243	0.0258	0.0220	0.0253	0.0399	0.0221
		B	0.0338	0.0256	0.0311	0.0234	0.0292	0.0229	0.0221	0.0263
		C	0.0267	0.0262	0.0244	0.0229	0.0218	0.0227	0.0258	0.0234
	10	D	0.0154	0.0207	0.0142	0.0183	0.0133	0.0182	0.0154	0.0181
15	5	A	0.0196	0.0260	0.0179	0.0228	0.0169	0.0227	0.0426	0.0238
		B	0.0375	0.0241	0.0354	0.0224	0.0337	0.0222	0.0199	0.0246
		C	0.0188	0.0204	0.0179	0.0190	0.0165	0.0188	0.0220	0.0216
	10	A	0.0142	0.0177	0.0126	0.0163	0.0122	0.0162	0.0195	0.0147
		B	0.0208	0.0149	0.0189	0.0136	0.0184	0.0135	0.0136	0.0158
		C	0.0132	0.0145	0.0114	0.0134	0.0112	0.0133	0.0132	0.0132
	15	D	0.0111	0.0139	0.0101	0.0129	0.0100	0.0129	0.0103	0.0126
20	10	A	0.0211	0.0135	0.0196	0.0125	0.0190	0.0125	0.0209	0.0120
		B	0.0124	0.0160	0.0113	0.0148	0.0111	0.0147	0.0115	0.0154
		C	0.0139	0.0133	0.0122	0.0123	0.0121	0.0123	0.0122	0.0130
	20	D	0.0082	0.0097	0.0074	0.0090	0.0073	0.0090	0.0079	0.0096
30	10	A	0.0218	0.0134	0.0206	0.0130	0.0202	0.0129	0.0221	0.0134
		B	0.0110	0.0149	0.0106	0.0141	0.0103	0.0141	0.0115	0.0138
		C	0.0229	0.0144	0.0220	0.0137	0.0214	0.0137	0.0103	0.0111
	15	A	0.0140	0.0085	0.0129	0.0081	0.0128	0.0081	0.0142	0.0088
		B	0.0085	0.0112	0.0078	0.0106	0.0078	0.0106	0.0087	0.0107
		C	0.0141	0.0088	0.0134	0.0085	0.0133	0.0084	0.0084	0.0086
30	20	A	0.0106	0.0077	0.0099	0.0073	0.0099	0.0073	0.0099	0.0073
		B	0.0070	0.0091	0.0066	0.0086	0.0066	0.0086	0.0068	0.0085
		C	0.0071	0.0077	0.0065	0.0073	0.0065	0.0072	0.0071	0.0071
	25	A	0.0072	0.0069	0.0068	0.0065	0.0068	0.0065	0.0071	0.0066
30		B	0.0059	0.0077	0.0055	0.0073	0.0055	0.0073	0.0061	0.0075
		C	0.0065	0.0075	0.0063	0.0072	0.0061	0.0072	0.0062	0.0068
	30	D	0.0054	0.0068	0.0051	0.0064	0.0051	0.0064	0.0052	0.0065
50	20	A	0.0110	0.0068	0.0104	0.0066	0.0104	0.0066	0.0106	0.0066
		B	0.0062	0.0079	0.0059	0.0076	0.0059	0.0076	0.0062	0.0079
		C	0.0061	0.0060	0.0057	0.0059	0.0057	0.0059	0.0056	0.0061
	30	A	0.0072	0.0048	0.0068	0.0046	0.0068	0.0046	0.0061	0.0075
		B	0.0046	0.0063	0.0045	0.0061	0.0045	0.0061	0.0062	0.0068
		C	0.0045	0.0048	0.0043	0.0047	0.0043	0.0047	0.0052	0.0065
	40	A	0.0048	0.0041	0.0046	0.0039	0.0046	0.0039	0.0106	0.0066
		B	0.0038	0.0048	0.0036	0.0047	0.0036	0.0047	0.0062	0.0079
		C	0.0036	0.0042	0.0034	0.0041	0.0034	0.0041	0.0056	0.0061
	50	D	0.0030	0.0039	0.0028	0.0038	0.0028	0.0038	0.0061	0.0075
70	30	A	0.0076	0.0045	0.0074	0.0044	0.0073	0.0043	0.0076	0.0043
		B	0.0043	0.0056	0.0041	0.0055	0.0040	0.0054	0.0043	0.0055
		C	0.0039	0.0042	0.0037	0.0041	0.0037	0.0040	0.0038	0.0039
	40	A	0.0057	0.0032	0.0056	0.0031	0.0055	0.0030	0.0053	0.0035
		B	0.0034	0.0031	0.0032	0.0030	0.0031	0.0029	0.0035	0.0047
		C	0.0043	0.0028	0.0042	0.0027	0.0041	0.0027	0.0032	0.0035
	50	A	0.0038	0.0031	0.0036	0.0030	0.0035	0.0029	0.0037	0.0032
		B	0.0030	0.0038	0.0029	0.0037	0.0028	0.0036	0.0029	0.0037
		C	0.0028	0.0030	0.0027	0.0029	0.0026	0.0028	0.0027	0.0031
	70	D	0.0022	0.0029	0.0021	0.0028	0.0020	0.0026	0.0023	0.0027
100	25	A	0.0095	0.0070	0.0092	0.0068	0.0091	0.0067	0.0100	0.0076
		B	0.0049	0.0065	0.0048	0.0063	0.0046	0.0062	0.0045	0.0063
		C	0.0038	0.0046	0.0037	0.0045	0.0036	0.0044	0.0038	0.0047

Table 1. *Cont.*

n	m	Censoring Scheme	ML		LINDLEY		TIERNEY-KADANE		MCMC	
		R	$\hat{\alpha}_{ER}$	$\hat{\beta}_{ER}$	$\hat{\alpha}_{ER}$	$\hat{\beta}_{ER}$	$\hat{\alpha}_{ER}$	$\hat{\beta}_{ER}$	$\hat{\alpha}_{ER}$	$\hat{\beta}_{ER}$
100	40	A	0.0057	0.0032	0.0056	0.0032	0.0056	0.0032	0.0055	0.0034
		B	0.0034	0.0043	0.0032	0.0042	0.0031	0.0041	0.0034	0.0043
		C	0.0028	0.0031	0.0027	0.0030	0.0027	0.0030	0.0028	0.0031
	50	A	0.0043	0.0025	0.0042	0.0025	0.0042	0.0025	0.0044	0.0027
		B	0.0027	0.0037	0.0026	0.0036	0.0026	0.0036	0.0029	0.0036
		C	0.0025	0.0027	0.0024	0.0026	0.0024	0.0026	0.0025	0.0027
	70	A	0.0028	0.0024	0.0026	0.0022	0.0025	0.0021	0.0029	0.0022
		B	0.0022	0.0027	0.0021	0.0026	0.0021	0.0025	0.0022	0.0026
		C	0.0020	0.0024	0.0019	0.0021	0.0019	0.0020	0.0019	0.0021
	90	A	0.0019	0.0020	0.0018	0.0019	0.0018	0.0019	0.0021	0.0020
		B	0.0018	0.0022	0.0017	0.0021	0.0016	0.0020	0.0016	0.0022
		C	0.0017	0.0021	0.0016	0.0020	0.0016	0.0020	0.0055	0.0034
	100	D	0.0015	0.0019	0.0014	0.0018	0.0013	0.0018	0.0034	0.0043

Where A: The censoring at the end of the experiment $R = (0, 0, \ldots, n - m)$, B: The censoring at the beginning of the experiment $R = (n - m, 0, \ldots, 0)$, C: Other censoring schemes $R = (0, 0, \ldots, (n - m), \ldots, 0)$, D: Complete sample $R = (0, 0, \ldots, 0)$.

In Tables 2 and 3, coverage probabilities, lengths, lower and upper bounds for the asymptotic confidence intervals (ACI) and bootstrap confidence intervals for the α and β parameters are given.

Table 2. Confidence average width and coverage probability for the asymptotic confidence interval and the bootstrap confidence interval of the parameter α ($\alpha = 0.5$).

(n,m)	R	ML Estimates	Lower Limit	Upper Limit	ACI Width	Probability of Coverage	Boot ML Estimates	Boot Lower Limit	Boot Upper Limit	Boot ACI Width	Boot Probability of Coverage
20, 10	A	0.6108	0.2598	0.9618	0.7020	0.9618	0.7301	0.0403	1.0654	1.0251	0.9700
20, 10	B	0.5618	0.3117	0.8118	0.5001	0.9540	0.4987	0.0403	0.8309	0.7906	0.9560
20, 10	C	0.5740	0.3184	0.8297	0.5112	0.9460	0.8986	0.0446	0.8945	0.8498	0.9490
20, 20	D	0.5318	0.3478	0.7159	0.3681	0.9300	0.6347	0.4034	0.8236	0.4203	0.9100
50, 30	A	0.5414	0.3656	0.7172	0.3515	0.9400	0.5393	0.4112	0.7922	0.3811	0.9000
50, 30	B	0.5316	0.3822	0.6810	0.2988	0.9200	0.5465	0.4209	0.7432	0.3223	0.9000
50, 30	C	0.5217	0.3885	0.6549	0.2664	0.9600	0.5138	0.4223	0.7106	0.2882	0.9100
50, 50	D	0.5129	0.4012	0.6246	0.2234	0.9400	0.6025	0.4269	0.6568	0.2299	0.9300
100, 50	A	0.5170	0.3844	0.6501	0.2657	0.9420	0.7353	0.4174	0.8799	0.4625	0.9100
100, 50	B	0.5126	0.4068	0.6183	0.2114	0.9560	0.6427	0.4300	0.7583	0.3283	0.9220
100, 50	C	0.5149	0.4174	0.6124	0.1950	0.9480	0.7634	0.4666	0.8495	0.3829	0.9590
100, 70	A	0.5107	0.4046	0.6167	0.2121	0.9420	0.7674	0.4575	0.8715	0.4140	0.9270
100, 70	B	0.5110	0.4195	0.6025	0.1830	0.9510	0.7105	0.4637	0.7769	0.3132	0.9380
100, 70	C	0.5110	0.4228	0.5992	0.1765	0.9600	0.7092	0.4658	0.7935	0.3277	0.9520
100, 100	D	0.5059	0.4285	0.5834	0.1549	0.9520	0.6822	0.4650	0.7321	0.2671	0.9380

As shown in Table 1, for all censoring schemes, the performances of the Tierney–Kadane approximate Bayes estimates outdo those of both the ML estimates, the Lindley and MCMC approximate Bayes estimates. Additionally, the approximate Bayes estimates of Lindley, Tierney–Kadane and MCMC methods are approximate as expected. For all the estimation methods, it is observed that for the same n and all censoring schemes as $\frac{m}{n} \to 1$, the ER values of the ML and the approximate Bayes estimates tend to decrease. Additionally, in complete sample case ($n = m$), the ER values of the ML and the approximate Bayes estimates are the smallest, as expected. In addition, as seen from Tables 2 and 3, when the n and m values increase, the coverage probabilities reach the desired level as expected. In different n and m values, the coverage probabilities of the ACIs and the Bootstrap confidence intervals are approximately $1 - \alpha = 0.95$.

Table 3. Confidence average width and coverage probability for the asymptotic confidence interval and the bootstrap confidence interval of the parameter β ($\beta = 0.8$).

(n,m)	R	ML Estimates	Lower Limit	Upper Limit	ACI Width	Probability of Coverage	Boot ML Estimates	Boot Lower Limit	Boot Upper Limit	Boot ACI Width	Boot Probability of Coverage
20, 10	A	0.8456	0.5594	1.1317	0.5723	0.9500	0.6020	0.0628	1.2287	1.1659	0.9090
20, 10	B	0.8161	0.5293	1.1028	0.5735	0.9430	1.2557	0.0588	1.0514	0.9926	0.8790
20, 10	C	0.8320	0.5624	1.1017	0.5394	0.9230	0.6366	0.0620	1.1208	1.0588	0.8780
20, 20	D	0.8215	0.6064	1.0365	0.4301	0.9500	0.6937	0.5947	1.0367	0.4690	0.9500
50, 30	A	0.8082	0.5238	0.3592	0.3293	0.9600	0.5375	0.4021	0.7620	0.3599	0.9100
50, 30	B	0.7983	0.6248	0.9718	0.3470	0.9300	0.8135	0.6215	0.9815	0.3600	0.9300
50, 30	C	0.8046	0.6531	0.9562	0.3031	0.9500	0.7046	0.6641	0.9831	0.3190	0.9400
50, 50	D	0.7952	0.6612	0.9293	0.2681	0.9800	0.7776	0.6571	0.9312	0.2741	0.9700
100, 50	A	0.8087	0.6947	0.9227	0.2280	0.9300	1.0515	0.7127	1.1498	0.4371	0.9220
100, 50	B	0.8005	0.6701	0.9309	0.2608	0.9480	1.0057	0.6774	1.0668	0.3894	0.9440
100, 50	C	0.8072	0.6926	0.9219	0.2293	0.9410	1.0665	0.7409	1.1613	0.4204	0.9220
100, 70	A	0.8022	0.7043	0.9002	0.1959	0.9590	0.9923	0.7421	1.0638	0.3216	0.9540
100, 70	B	0.7979	0.6861	0.9096	0.2285	0.9490	1.0143	0.7277	1.0827	0.3550	0.9210
100, 70	C	0.7987	0.6974	0.9000	0.2026	0.9530	0.9865	0.7361	1.0494	0.3133	0.9480
100, 100	D	0.8000	0.7052	0.8948	0.1896	0.9480	0.9645	0.7395	1.0215	0.2820	0.9410

5. Real Data Analysis

In this section, real lifetime datasets of bladder cancer, head and neck cancer, and leukemia are used. Observing the survival times of cancer patients during their treatment course may sometimes not be feasible. For instance, patients may relocate or discontinue treatment. Due to such reasons, survival analysis is conducted using censored sample designs. Progressive censoring, being a generalization of complete and Type-II censoring, is preferred in this study. The parameter estimates for the four estimation methods are obtained and then the performances of ML and Bayes estimation methods are compared using three different real datasets. We applied the goodness-of-fit of censored data for the RW distribution using approximate KS test statistics proposed by Pakyari and Balakrishnan [62]. The test statistics KS and the corresponding p-values are calculated using the R program using parametric bootstrap for censored datasets.

The Real data-1 set represents the remission times (in months) of a random sample of 128 bladder cancer patients [26]. The real data-1 set is given in Table 4.

Table 4. Real data-1 set, $n = 128$.

0.08	0.2	0.4	0.5	0.51	0.81	0.9	1.05	1.19	1.26	1.35	1.4	1.46	1.76	2.02	2.02
2.07	2.09	2.23	2.26	2.46	2.54	2.62	2.64	2.69	2.69	2.75	2.83	2.87	3.02	3.25	3.31
3.36	3.36	3.48	3.52	3.57	3.64	3.7	3.82	3.88	4.18	4.23	4.26	4.33	4.34	4.4	4.5
4.51	4.87	4.98	5.06	5.09	5.17	5.32	5.32	5.34	5.41	5.41	5.49	5.62	5.71	5.85	6.25
6.31	6.54	6.76	6.93	6.94	6.97	7.09	7.26	7.28	7.32	7.39	7.59	7.62	7.63	7.66	7.87
7.93	8.26	8.37	8.53	8.65	8.66	9.02	9.22	9.47	9.74	10.06	10.34	10.66	10.75	11.25	11.64
11.79	11.98	12.02	12.03	12.07	12.63	13.11	13.29	13.8	14.24	14.76	14.77	14.83	15.96	16.62	17.12
17.14	17.36	18.1	19.13	20.28	21.73	22.69	23.63	25.74	25.82	32.15	34.26	36.66	43.01	46.12	79.05

Censored Data-1 based on real data-1 set were obtained according to the censoring schemes-(19*0.108). Censored data-1 is given in Table 5.

The approximate KS and the corresponding p-value (in parentheses) for censored data-1 set are 0.4276 (1.000). Accordingly, it is seen that the censored data-1 set fit the RW distribution with $\hat{\alpha} = 0.9007$ and $\hat{\beta} = 0.7275$ (ML estimates). Then, the following ML and approximate Bayes estimates for α and β parameters under PTR-II censoring are acquired. In Table 6, ML, Lindley, Tierney–Kadane and MCMC estimates are given. Additionally, in Table 7, bootstrap confidence intervals for α and β parameters are given as (0.6771–1.3656) and (0.5046–0.9459), respectively.

Table 5. Censored data-1, $m = 20$.

0.08	0.2	0.4	0.5	0.51	0.81	0.9	1.05	1.19	1.26	1.35	1.4	1.46
1.76	2.02	2.02	2.07	2.09	2.23	2.26						

Table 6. The ML and approximate Bayes estimates for α and β parameters in real data-1 set.

(n,m)	Censoring Scheme	MLE		LINDLEY		TIERNEY–KADANE		MCMC	
	R	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
(128, 20)	(19*0.108)	0.9007	0.7275	0.8878	0.7363	0.8880	0.7355	0.9751	0.6929

Table 7. The bootstrap confidence intervals for α and β parameters in real data-1 set.

(n,m)	Censoring Scheme	α			β		
	R	**Boot ML Estimate**	**Boot Lower Limit**	**Boot Upper Limit**	**Boot ML Estimate**	**Boot Lower Limit**	**Boot Upper Limit**
(128, 20)	(19*0.108)	0.9598	0.6771	1.3656	0.7242	0.5046	0.9459

The Real data-2 set represents the remission times (in days) of 51 leukemia patients [63]. The real data-2 set is given in Table 8.

Table 8. Real data-2 set, $n = 51$.

24	46	57	57	64	65	82	89	90	90	111	117	128	143	148	152
166	171	186	191	197	209	223	230	239	247	254	264	269	273	284	294
304	304	332	341	393	395	487	510	516	518	518	534	608	642	697	955
1160															

Censored data-2 based on real data-2 are obtained according to the censoring schemes-(19*0.31). Censored data-2 is given in Table 9.

Table 9. Censored data-2, $m = 20$.

24	46	57	57	64	65	82	89	90	90	111	117	128
143	148	152	166	171	186	191						

The approximate KS and the corresponding p-value (in parentheses) for censored data-2 are 0.4939 (1.000). Accordingly, it is seen that the censored data-2 set fit the RW distribution with $\hat{\alpha} = 1.2166$ and $\hat{\beta} = 0.0029$ (ML estimates). Then, the following ML and approximate Bayes estimates for α and β parameters under PTR-II censoring are acquired. In Table 10, ML, Lindley, Tierney–Kadane and MCMC estimates are given. Additionally, in Table 11, bootstrap confidence intervals for α and β parameters are given as (0.9278–1.8922) and (0.0001–0.0124), respectively.

Table 10. The ML and approximate Bayes estimates for α and β parameters in real data-2 set.

(n,m)	Censoring Scheme	MLE		LINDLEY		TIERNEY–KADANE		MCMC	
	R	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
(51, 20)	(19*0.31)	1.2166	0.0029	0.9139	0.0089	0.9599	0.0150	0.9062	0.0158

Table 11. The bootstrap confidence intervals for α and β parameters in real data-2 set.

(n,m)	Censoring Scheme	α			β		
	R	**Boot ML Estimate**	**Boot Lower Limit**	**Boot Upper Limit**	**Boot ML Estimate**	**Boot Lower Limit**	**Boot Upper Limit**
(51, 20)	(19*0.31)	1.3196	0.9278	1.8922	0.0031	0.0001	0.0124

The Real data-3 set represents survival times of 45 patients suffering from head and neck cancer treated with combined radiotherapy and chemotherapy [64]. The real data-3 set is given in Table 12.

Table 12. Real Data-3 set, $n = 45$.

12.20	23.56	23.74	25.87	31.98	37	41.35	47.38	55.46	58.36	63.47	68.46	78.26	74.47	81	43
84	92	94	110	112	119	127	130	133	140	146	155	159	173	179	194
195	209	249	281	319	339	432	469	519	633	725	817	1776			

Censored data-3 based on real data-3 are obtained according to the censoring schemes-(19*0.25). Censored data-3 is given in Table 13.

Table 13. Censored Data-3, $m = 20$.

12.20	23.56	23.74	25.87	31.98	37	41.35	47.38	55.46	58.36	63.47
68.46	78.26	74.47	81	83	84	92	94	110		

The approximate KS and the corresponding p-value (in parentheses) for censored data-3 are 0.3852 (1.000). Accordingly, it is seen that the censored dataset fit the RW distribution with $\hat{\alpha} = 1.1476$ and $\hat{\beta} = 0.0083$ (ML estimates). Then, the following ML and approximate Bayes estimates for α and β parameters under PTR-II censoring are acquired. In Table 14, ML, Lindley, Tierney–Kadane and MCMC estimates are given. Additionally, in Table 15, bootstrap confidence intervals for α and β parameters are given as (0.8764–1.7795) and (0.0006–0.0287), respectively.

Table 14. The ML and approximate Bayes estimates for α and β parameters in real data-3 set.

(n,m)	Censoring Scheme	MLE		LINDLEY		TIERNEY–KADANE		MCMC	
	R	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
(45, 20)	(19*0.25)	1.1476	0.0083	0.9062	0.0205	0.9420	0.0269	0.8879	0.0289

Table 15. The bootstrap confidence intervals for α and β parameters in real data-3 set.

(n,m)	Censoring Scheme	α			β		
	R	**Boot ML Estimate**	**Boot Lower Limit**	**Boot Upper Limit**	**Boot ML Estimate**	**Boot Lower Limit**	**Boot Upper Limit**
(45, 20)	(19*0.25)	1.2359	0.8764	1.7795	0.0086	0.0006	0.0287

6. Conclusions

In this article, the MLE and approximate Bayes estimators for unknown parameters of the RW distribution based on PTR-II censored samples were evaluated. The maximum likelihood estimators of the parameters were obtained by using the Newton–Raphson method. Because the Bayes estimators of the parameters cannot be obtained in explicit forms, we obtained the approximate Bayes estimators using Lindley, Tierney–Kadane, and MCMC methods under squared-error loss functions. We have compared the performance of the approximate Bayes estimates with the ML estimates by means of Monte Carlo simulations, and it has been observed that the performances of approximate Bayes estimates are better than those of ML estimates. Further, the ER values of the estimates of α and β parameters obtained by using Tierney and Kadane's approximation method were lower than those obtained by using both Lindley's approximation, and MCMC method and also MLE. It is also seen that the width of the asymptotic confidence intervals and the bootstrap confidence intervals decreases and the coverage possibilities approach to 0.95 when (n, m) values increase. In future research, estimators of the parameters of the new discrete distributions in the literature proposed for modeling discrete data in medical studies can be obtained. There are very few studies on its estimation for parameters of discrete distribution in medicine.

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