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Fixed Point and Convergence Results for Contractive-Type Self-Mappings of Metric Spaces with Graphs

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Abstract: It is known that a strict contraction on a complete metric space with a graph possesses a fixed point. In the present paper, we show that this property holds for single valued and set-valued self-mappings of metric spaces with graphs that are of the contractive type. We also show the convergence of iterates of these mappings to fixed points. In particular, our results are true for metric spaces with symmetric graphs.

Keywords: complete metric space; contractive mapping; fixed point; graph

MSC: 47H09; 47H10; 54E50

1. Introduction

For more than sixty years, the fixed point theory of nonlinear operators has been an important area of nonlinear analysis. One of its main topics is the study of the existence of fixed points of nonexpansive and contractive maps [1–12]. It should be mentioned that the analysis of nonexpansive operators acting on complete metric spaces with graphs is of great interest [13–21]. It is a well-known fact that a strict contraction on a complete metric space with a graph possesses a fixed point [22,23]. In the present paper, we show that this property holds for single valued and set-valued self-mappings of metric spaces with graphs that are of the contractive type. We also show the convergence of iterates of these mappings to fixed points. In particular, our results are true for metric spaces with symmetric graphs.

Assume that (X, d) is a complete metric space equipped with a graph G. We denote by $V(G)$ the set of its vertices, and by $E(G)$ the set of its edges. We assume that $(x, x) \in E(G)$ for any point $x \in X$.

Denote by $\mathcal{M}_{ne}$ the set of all maps $T : X \to X$ such that for every pair of points $x, y \in X$ satisfying $(x, y) \in E(G)$,

$$(T(x), T(y)) \in E(G) \text{ and } d(T(x), T(y)) \leq d(x, y).$$

In the sequel, we assume that the sum over an empty set is zero, and that the infimum of an empty set is ∞. For each point $x \in X$ and each number $r > 0$, set

$$B(x, r) := \{y \in X : d(x, y) \leq r\}.$$

For every point $x \in X$ and every set $D \subset X$, put

$$d(x, D) = \inf\{d(x, y) : y \in D\}.$$

Note that $d(x, \emptyset) = \infty$, $x \in X$. For every pair of nonempty sets $A, B \subset X$, put

$$H(A, B) = \max\{\sup\{d(x, B) : x \in A\}, \sup\{d(y, A) : y \in B\}\}.$$

Given a mapping $P : X \to X$, we define $P^0 = I$, the identity self-mapping of X, $P^1 = P$, and $P^{i+1} = P \circ P^i$ for all nonnegative integers i.

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A mapping $T \in \mathcal{M}_{ne}$ is called G-nonexpansive. If $T \in \mathcal{M}_{ne}$, $\alpha \in (0,1)$ and for every pair of points $x, y \in X$ satisfying $(x,y) \in E(G)$ the inequality

$$d(T(x), T(y)) \le \alpha d(x,y)$$

holds, then T is called a G-strict contraction.

A mapping $T \in \mathcal{M}_{ne}$ is called G-contractive (or G-Rakotch contraction [11]) if there exists a decreasing function $\phi : [0,\infty) \to [0,1]$ such that

$$\phi(t) < 1,\ t \in [0,\infty),$$

and for every pair of points $x, y \in X$ satisfying $(x,y) \in E(G)$ the inequality

$$d(T(x), T(y)) \le \phi(d(x,y))d(x,y)$$

holds.

Let $T \in \mathcal{M}_{ne}$ be a G-strict contraction. It is a well-known fact [22] that under certain mild assumptions, T possesses a unique fixed point. In [23], a very simple proof of this fact was presented by using a certain metric on X. In [24] an extension of the existence results of [22,23] to G-contractive mappings under certain assumptions was obtained. In the case where $E(G) = X \times X$, this result for Rakotch contractions is well known in the literature. It was first established in [11]. Many of its extensions and generalizations are collected in [12]. In this connection, we recall that the authors of [25] constructed an example of a so-called ϕ-contraction, to which the result of [22] cannot be extended. The results of [24] were obtained under the assumption that there exists a number $\bar{\Delta} > 0$ such that if $(x_0, x_1), (x_1, x_2) \in E(G)$ satisfy $\rho(x_0, x_1) \le \bar{\Delta},\ \rho(x_1, x_2) \le \bar{\Delta}$, then $(x_0, x_2) \in E(G)$.

In this paper, we obtain the existence of fixed points without this assumption.

Throughout the paper, we assume that $\phi : [0,\infty) \to (0,1]$ is a decreasing function, such that

$$\phi(t) < 1,\ t \in (0,\infty) \tag{1}$$

and

$$\phi(0) = 1. \tag{2}$$

2. The First Main Result

Assume that $T : X \to X$, and that the following assumption holds.

(A1) For each $(x,y) \in E(G)$,

$$(T(x), T(y)) \in E(G)$$

and

$$d(T(x), T(y)) \le \phi(d(x,y))d(x,y).$$

Theorem 1. *Assume that $\bar{x} \in X$, and that there exist a natural number q and $M > 1$ such that for each integer $n \ge 1$, there exist $x_i^{(n)} \in X$, $i = 0, \dots, q$, such that*

$$x_0^{(n)} = \bar{x},\ x_q^{(n)} = T^n(\bar{x}) \tag{3}$$

and that for each $i \in \{0, \dots, n-1\}$,

$$d(x_i^{(n)}, x_{i+1}^{(n)}) \le M \tag{4}$$

and at least one of the following relations hold:

$$(x_i^{(n)}, x_{i+1}^{(n)}) \in E(G);\ (x_{i+1}^{(n)}, x_i^{(n)}) \in E(G). \tag{5}$$

Then there exists

$$x_* = \lim_{i\to\infty} T^i(x)$$

and if T is continuous at x_*, *then* $T(x_*) = x_*$.

Proof. Let $\epsilon \in (0,1)$. Choose a natural number p such that

$$p > q^2 M\epsilon^{-1}(1 - \phi(\epsilon q^{-1}))^{-1}. \tag{6}$$

Assume that $n \geq 1$ is an integer. We show that

$$d(T^p(\bar{x}), T^{n+p}(\bar{x})) \leq \sum_{i=0}^{q-1} d(T^p(x_i^{(n)}), T^p(x_{i+1}^{(n)})) \leq \epsilon. \tag{7}$$

Assume the contrary. Then,

$$\sum_{i=0}^{q-1} d(T^p(x_i^{(n)}), T^p(x_{i+1}^{(n)})) > \epsilon. \tag{8}$$

It follows from (A1), (5) and (8) that for each integer $l \in \{0, \dots, p\}$,

$$\sum_{i=0}^{q-1} d(T^l(x_i^{(n)}), T^l(x_{i+1}^{(n)})) > \epsilon. \tag{9}$$

Let $l \in \{0, \dots, p\}$. Then, in view of (A1), for each $i \in \{0, \dots, q-1\}$,

$$d(T^{l+1}(x_i^{(n)}), T^{l+1}(x_{i+1}^{(n)})) \leq d(T^l(x_i^{(n)}), T^l(x_{i+1}^{(n)})). \tag{10}$$

By (9), there exists $j \in \{0, \dots, q-1\}$ such that

$$d(T^l(x_j^{(n)}), T^l(x_{j+1}^{(n)})) > \epsilon/q. \tag{11}$$

Assumption (A1) and Equations (5) and (11) imply that

$$d(T^{l+1}(x_j^{(n)}), T^{l+1}(x_{j+1}^{(n)})) \leq \phi(d(T^l(x_j^{(n)}), T^l(x_{j+1}^{(n)})))d(T^l(x_j^{(n)}), T^l(x_{j+1}^{(n)}))$$

$$\leq \phi(\epsilon q^{-1})d(T^l(x_j^{(n)}), T^l(x_{j+1}^{(n)})).$$

Together with (11), this implies that

$$d(T^l(x_j^{(n)}), T^l(x_{j+1}^{(n)})) - d(T^{l+1}(x_j^{(n)}), T^{l+1}(x_{j+1}^{(n)}))$$

$$\geq (1 - \phi(\epsilon/4))d(T^l(x_j^{(n)}), T^l(x_{j+1}^{(n)}))$$

$$\geq (1 - \phi(\epsilon q^{-1}))\epsilon q^{-1}. \tag{12}$$

Equations (10) and (12) imply that

$$\sum_{i=0}^{q-1} d(T^l(x_i^{(n)}), T^l(x_{i+1}^{(n)})) - \sum_{i=0}^{q-1} d(T^{l+1}(x_i^{(n)}), T^{l+1}(x_{i+1}^{(n)}))$$

$$\geq (1 - \phi(\epsilon q^{-1}))\epsilon q^{-1}. \tag{13}$$

It follows from (4) and (13) that

$$qM \geq \sum_{i=0}^{q-1} d(x_i^{(n)}, x_{i+1}^{(n)})$$

$$\geq \sum_{i=0}^{q-1} d(x_i^{(n)}), x_{i+1}^{(n)}) - \sum_{i=0}^{q-1} d(T^{p+1}(x_i^{(n)}), T^{p+1}(x_{i+1}^{(n)}))$$

$$= \sum_{l=0}^{p} (\sum_{i=0}^{q-1} d(T^l(x_i^{(n)}), T^l(x_{i+1}^{(n)})) - \sum_{i=0}^{q-1} d(T^{l+1}(x_i^{(n)}), T^{l+1}(x_{i+1}^{(n)}))$$

$$\geq p(1-\phi(\epsilon q^{-1}))\epsilon q^{-1}$$

and

$$p \leq Mq^2(1-\phi(\epsilon q^{-1}))^{-1}\epsilon^{-1}.$$

This contradicts (6), and proves that (7) holds. Since ϵ is any number from $(0,1)$, $\{T^n(\bar{x})\}_{n=0}^{\infty}$ is a Cauchy sequence. Therefore, there exists

$$x_* = \lim_{n\to\infty} T^n(\bar{x}).$$

Combined with Equation (7), this implies that

$$d(T^p(\bar{x}), x_*) \leq \epsilon$$

and

$$d(T^{n+p}(\bar{x}), x_*) \leq 2\epsilon$$

for each integer $n \geq 0$. Therefore,

$$x_* = \lim_{i\to\infty} T^i(\bar{x}).$$

Evidently, if T is continuous at x_*, then $T(x_*) = x_*$. Theorem 1 is proved. □

3. The Second Main Result

Assume that $T : X \to X$ satisfies (A1).

We prove the following convergence result.

Theorem 2. *Assume that $x_* \in X$ satisfies*

$$T(x_*) = x_*, \tag{14}$$

q is a natural number, $M > 0$ and that

$$X_{M,q} = \{x \in X : \ \textit{there exist } x_i \in X,\ i = 0, \dots, q$$

$$\textit{such that } x_0 = x,\ x_q = x_* \textit{ and for each } i \in \{0, \dots, n-1\},$$

$$d(x_i, x_{i+1}) \leq M \textit{ and at least one of the following relations holds:}$$

$$(x_i, x_{i+1}) \in E(G);\ (x_{i+1}, x_i) \in E(G)\}. \tag{15}$$

Then, $d(T^n(x), x_*) \to 0$ as $n \to \infty$ uniformly on $X_{q,M}$.

Proof. Let $\epsilon \in (0,1)$. Choose an integer p such that

$$p > q^2(M+1)\epsilon^{-1}(1-\phi(\epsilon q^{-1}))^{-1}. \tag{16}$$

Assume that

$$x \in X(M,q). \tag{17}$$

In view of (15) and (17), there exist $x_i \in X,\ i = 0,\dots,q$ such that

$$x_0 = x,\ x_q = x_* \tag{18}$$

for each $i \in \{0,\dots,n-1\}$,

$$d(x_i, x_{i+1}) \le M \tag{19}$$

and at least one of the following relations holds:

$$(x_i, x_{i+1}) \in E(G);\ (x_{i+1}, x_i) \in E(G). \tag{20}$$

By (A1) and (19), (20), for each integer $i \in \{0,\dots,q-1\}$ and each integer $n \ge 0$,

$$d(T^{n+1}(x_i), T^{n+1}(x_{i+1})) \le \phi(d(T^n(x_i), T^n(x_{i+1}))d(T^n(x_i), T^n(x_{i+1})) \tag{21}$$

and at least one of the following relations holds:

$$(T^n(x_i), T^n(x_{i+1})) \in E(G);\ (T^n(x_{i+1}), T^n(x_i)) \in E(G). \tag{22}$$

Assume that an integer $n \ge 0$ and

$$\sum_{i=0}^{q-1} d(T^n(x_i), T^n(x_{i+1})) > \epsilon. \tag{23}$$

In view of (23), there exists an integer $j \in \{0,\dots,q-1\}$ such that

$$d(T^n(x_j), T^n(x_{j+1})) > \epsilon/q. \tag{24}$$

By (A1), (22) and (24),

$$d(T^{n+1}(x_j), T^{n+1}(x_{j+1})) \le \phi(d(T^n(x_j), T^n(x_{j+1})))d(T^n(x_j), T^n(x_{j+1}))$$

$$\le \phi(q^{-1}\epsilon)d(T^n(x_j), T^n(x_{j+1}))$$

and

$$d(T^n(x_j), T^n(x_{j+1})) - d(T^{n+1}(x_j), T^{n+1}(x_{j+1}))$$

$$\ge (1-\phi(q^{-1}\epsilon))d(T^n(x_j), T^n(x_{j+1})) \ge (1-\phi(q^{-1}\epsilon))\epsilon/q. \tag{25}$$

Equations (21) and (25) imply that

$$\sum_{i=0}^{q-1} d(T^n(x_i), T^n(x_{i+1})) - \sum_{i=0}^{q-1} d(T^{n+1}(x_i), T^{n+1}(x_{i+1}))$$

$$\ge d(T^n(x_j), T^n(x_{j+1})) - d(T^{n+1}(x_j), T^{n+1}(x_{j+1})) \ge (1-\phi(\epsilon/q))\epsilon/q.$$

Thus we have shown that the following property holds:

(i) if $n \geq 0$ is an integer and (23) holds, then

$$\sum_{i=0}^{q-1} d(T^n(x_i), T^n(x_{i+1})) - \sum_{i=0}^{q-1} d(T^{n+1}(x_i), T^{n+1}(x_{i+1})) \geq (1 - \phi(\epsilon/q))\epsilon/q.$$

We show that there exists an integer $n \in \{0, \dots, p\}$ such that

$$\sum_{i=0}^{q-1} d(T^n(x_i), T^n(x_{i+1})) \leq \epsilon.$$

Assume the contrary. Then, for each $n \in \{0, \dots, p\}$,

$$\sum_{i=0}^{q-1} d(T^n(x_i), T^n(x_{i+1})) > \epsilon$$

and in view of property (i),

$$\sum_{i=0}^{q-1} d(T^n(x_i), T^n(x_{i+1})) - \sum_{i=0}^{q-1} d(T^{n+1}(x_i), T^{n+1}(x_{i+1})) \geq (1 - \phi(\epsilon/q))\epsilon/q. \tag{26}$$

It follows from (19) and (26) that

$$qM \geq \sum_{i=0}^{q-1} d(x_i, x_{i+1})$$

$$\geq \sum_{i=0}^{q-1} d(x_i, x_{i+1}) - \sum_{i=0}^{q-1} d(T^p(x_i), T^p(x_{i+1}))$$

$$= \sum_{n=0}^{p} \sum_{i=0}^{q-1} (d(T^n(x_i), T^n(x_{i+1})) - d(T^{n+1}(x_i), T^{n+1}(x_{i+1})))$$

$$\geq p(1 - \phi(\epsilon q^{-1}))\epsilon q^{-1}$$

and

$$p \leq Mq^2(1 - \phi(\epsilon q^{-1}))^{-1}\epsilon^{-1}.$$

This contradicts (16). The contradiction we have reached proves that there exists an integer $n \in \{0, \dots, p\}$ such that

$$d(T^n(x), T^n(x_*)) \leq \sum_{i=0}^{q-1} d(T^n(x_i), T^n(x_{i+1})) \leq \epsilon.$$

By (18), (21) and the relation above, for each integer $s > n$,

$$d(T^s(x), x_*) \leq d(T^s(x), T^s(x_*)) \leq \sum_{i=0}^{q-1} d(T^s(x_i), T^s(x_{i+1}))$$

$$\leq \sum_{i=0}^{q-1} d(T^n(x_i), T^n(x_{i+1})) \leq \epsilon.$$

Theorem 2 is proved. □

4. Set-Valued Rakotch Contractions

We prove the following convergence result.

Theorem 3. *Let $T : X \to 2^X \setminus \{\emptyset\}$, $x_* \in X$ satisfy*

$$x_* \in T(x_*) \tag{27}$$

$q \geq 2$ be natural number, $M > 0$ and let

$$X_{M,q} = \{x \in X : \text{ there exist } y_i \in X,\ i = 1, \ldots, q \text{ such that}$$

$$y_1 = x_*,\ y_q = x,\ d(y_i, y_{i+1}) \leq M \text{ and } (y_i, y_{i+1}) \in E(G),\ i \in \{1, \ldots, q-1\}\}. \tag{28}$$

Assume that the following assumption holds
(A2) For each $(x, y) \in E(G)$ and each $z \in T(x)$, the set

$$\{\xi \in T(y) : (z, \xi) \in E(G)\}$$

is nonempty and

$$d(z, \{\xi \in T(y) : (z, \xi) \in E(G)\}) \leq \phi(d(x, y))d(x, y).$$

Then, for each $x \in X(M, q)$, there exists a sequence $\{x_i\}_{i=1}^{\infty}$ such that

$$x_1 = x,\ x_{i+1} \in T(x_i),\ i = 1, 2, \ldots,\ \lim_{i \to \infty} x_i = x_*$$

and for each $\epsilon > 0$, there exists a natural number $p(\epsilon)$ depending only on ϵ, such that $d(x_i, x_*) \leq \epsilon$ for each integer $i \geq p(\epsilon)$.

Proof. Let

$$x \in X_{M,q}.$$

There exist $y_i \in X,\ i = 1, \ldots, q$ such that (28) holds. Assume that $s \geq 1$ is an integer and we defined a sequence $\{y_i\}_{i=1}^{sq} \subset X$ such that

$$y_{(s-1)q+1} = x_*, \tag{29}$$

$$(y_i, y_{i+1}) \in E(G),\ i = (s-1)q + 1, \ldots, sq - 1. \tag{30}$$

(Clearly, for $s = 1$ our assumption holds.) By induction, using (27), (29) and (30), we define $y_i \in X$, $i = sq + 1, \ldots, s(q+1)$ such that

$$y_{sq+1} = x_* \tag{31}$$

and that for each $i \in \{(s-1)q + 1, \ldots, sq - 1\}$,

$$y_{i+q} \in T(y_i), \tag{32}$$

$$(y_{i+q}, y_{i+q+1}) \in E(G) \tag{33}$$

and

$$d(y_{i+q}, y_{i+q+1})$$

$$\leq d(y_{i+q}, \{\xi \in T(y_{i+1}) : (y_{i+q}, \xi) \in E(G)\})(1 + \phi(d(y_i, y_{i+1}))^{-1})/2. \tag{34}$$

(Note that if $y_i = y_{i+1}$, then $y_{i+q} = y_{i+q+1}$.) Thus, by induction we defined the sequence $\{y_i\}_{i=0}^{\infty} \subset X$ such that for each integer $s \geq 1$, relations (29), (30) hold and that for each $i \in \{(s-1)q+1, \dots, sq-1\}$, Equations (32)–(34) are true. For each natural number s set

$$x_s = y_{sq}. \tag{35}$$

Assumption (A2) and Equations (30), (32) and (35) imply that

$$x_{s+1} \in T(x_s) \tag{36}$$

and that for each integer $s \geq 1$ and each $i \in \{(s-1)q+1, \dots, sq-1\}$,

$$d(y_{i+q}, y_{i+q+1}) \leq \phi(d(y_i, y_{i+1}))d(y_i, y_{i+1})(1 + \phi(d(y_i, y_{i+1}))^{-1})/2$$

$$\leq 2^{-1}(1 + \phi(d(y_i, y_{i+1})))d(y_i, y_{i+1}). \tag{37}$$

Let s be a natural number. By (37),

$$\sum_{i=sq+1}^{(s+1)q-1} d(y_i, y_{i+1}) = \sum_{i=(s-1)q+1}^{sq-1} d(y_{i+q}, y_{i+q+1})$$

$$\leq \sum_{i=(s-1)q+1}^{sq-1} d(y_i, y_{i+1})(1 + \phi(d(y_i, y_{i+1})))/2 \tag{38}$$

and

$$\sum_{i=sq+1}^{(s+1)q-1} d(y_i, y_{i+1}) \leq \sum_{i=(s-1)q+1}^{sq-1} d(y_i, y_{i+1}). \tag{39}$$

Let $\epsilon \in (0, 1)$ and

$$p > 2q^2 M\epsilon^{-1}(1 - \phi(\epsilon q^{-1}))^{-1} \tag{40}$$

be an integer. Assume that s is a natural number and

$$\sum_{i=(s-1)q+1}^{sq-1} d(y_i, y_{i+1}) > \epsilon. \tag{41}$$

By (41), there exists $j \in \{(s-1)q+1, \dots, sq-1\}$ such that

$$d(y_i, y_{i+1}) > \epsilon/q. \tag{42}$$

By (37) and (42),

$$d(y_{j+q}, y_{j+q+1}) \leq 2^{-1}(1 + \phi(d(y_j, y_{j+1})))d(y_j, y_{j+1})$$

$$\leq 2^{-1}(1 + \phi(\epsilon/q))d(y_j, y_{j+1})$$

and

$$d(y_j, y_{j+1}) - d(y_{j+q}, y_{j+q+1})$$

$$\geq 2^{-1}d(y_j, y_{j+1})(1 - \phi(\epsilon/q)) \geq 2^{-1}\epsilon q^{-1}(1 - \phi(\epsilon/q)).$$

Together with (37), this implies that

$$\sum_{i=(s-1)q+1}^{sq-1} d(y_i, y_{i+1}) - \sum_{i=sq+1}^{(s+1)q-1} d(y_i, y_{i+1}))$$

$$\geq 2^{-1}\epsilon q^{-1}(1-\phi(\epsilon/q)). \tag{43}$$

Thus, we have shown that the following property holds:
(i) if s is a natural number, then (41) implies (43).
We show that there exists $s \in \{0,\ldots,p(\epsilon)\}$ such that

$$\sum_{i=(s-1)q+1}^{sq-1} d(y_i,y_{i+1}) \leq \epsilon. \tag{44}$$

Assume the contrary.
Then, for each $s \in \{0,\ldots,p(\epsilon)\}$,

$$\sum_{i=(s-1)q+1}^{sq-1} d(y_i,y_{i+1}) \geq \epsilon. \tag{45}$$

Property (i) and (45) imply that for each $s \in \{0,\ldots,p(\epsilon)\}$,

$$\sum_{i=(s-1)q+1}^{sq-1} d(y_i,y_{i+1}) - \sum_{i=sq+1}^{(s+1)q-1} d(y_i,y_{i+1})$$

$$\geq 2^{-1}\epsilon q^{-1}(1-\phi(\epsilon/q)). \tag{46}$$

In view of (28) and (46),

$$Mq \geq \sum_{i=1}^{q-1} d(y_i,y_{i+1}) \geq \sum_{i=1}^{q-1} d(y_i,y_{i+1}) - \sum_{i=p(\epsilon)q+1}^{(p(\epsilon)+1)q-1} d(y_i,y_{i+1})$$

$$= \sum_{s=0}^{p(\epsilon)} \Big(\sum_{i=(s-1)q+1}^{sq-1} d(y_i,y_{i+1}) - \sum_{i=sq+1}^{(s+1)q-1} d(y_i,y_{i+1}) \Big)$$

$$\geq 2^{-1}p(\epsilon)\epsilon q^{-1}(1-\phi(\epsilon/q))$$

and

$$p(\epsilon) \leq 2Mq^2\epsilon^{-1}(1-\phi(\epsilon/q))^{-1}.$$

This contradicts (40). The contradiction we have reached proves that there exists $s \in \{0,\ldots,p(\epsilon)\}$ such that (44) holds. By (29), (35), (39) and (44), for each integer $m \geq s$,

$$d(x_*,x_m) = d(y_{(m-1)q+1},y_{mq}) \leq \sum_{i=(m-1)q+1}^{mq-1} d(y_i,y_{i+1}) \leq \epsilon.$$

Theorem 3 is proved. □

5. Set-Valued Strict Contractions

Assume that $E(G)$ is a closed set in $X \times X$, $c \in (0,1)$, $T : X \to 2^X \setminus \{\emptyset\}$, $T(x)$ is closed for each $x \in X$, and that the following assumption holds.
(A3) For each $(x,y) \in E(G)$ and each $\xi \in T(x)$,

$$d(\xi, \{u \in T(y) : \ (\xi,u) \in E(G)\}) \leq cd(x,y).$$

We prove the following result.

Theorem 4. *Assume that* $x_0 \in X$,

$$\{u \in T(x_0) : (x_0, u) \in E(G)\} \neq \emptyset \tag{47}$$

and that $\{\epsilon_i\}_{i=0}^{\infty} \subset (0,1)$ *satisfies*

$$\sum_{i=1}^{\infty} \epsilon_i < \infty. \tag{48}$$

Then, there exists a sequence $\{x_i\}_{i=1}^{\infty}$ *such that for each integer* $i \geq 0$,

$$(x_i, x_{i+1}) \in E(G),\ x_{i+1} \in T(x_i) \tag{49}$$

and

$$d(x_i, x_{i+1}) \leq d(x_i, \{u \in T(x_i) : (x_i, u) \in E(G)\}) + \epsilon_i. \tag{50}$$

Moreover, if a sequence $\{x_i\}_{i=1}^{\infty}$ *satisfies (49) and (50) for each integer* $i \geq 0$*, then it converges to a point* $x_* \in X$ *and if, in addition, the graph of* T *is closed, then* $x_* \in T(x_*)$.

Proof. By (47), there exists

$$x_1 \in T(x_0), \tag{51}$$

such that

$$(x_0, x_1) \in E(G), \tag{52}$$

$$d(x_0, x_1) \leq d(x_0, \{u \in T(x_0) : (x_0, x_1) \in E(G)\}) + \epsilon_0. \tag{53}$$

Assume that $n \geq 1$ is an integer and that a finite sequence $\{x_i\}_{i=1}^{n}$ was defined such that (49) and (50) are valid for each integer $i \in \{0, \ldots, n-1\}$ (Note that for $n = 1$ our assumption holds). By (49),

$$(x_{n-1}, x_n) \in E(G),\ x_n \in T(x_{n-1}). \tag{54}$$

In view of (54), there exists

$$x_{n+1} \in T(x_n)$$

such that

$$(x_n, x_{n+1}) \in E(G)$$

and

$$d(x_n, x_{n+1}) \leq d(x_n, \{u \in T(x_n) : (x_n, u) \in E(G)\}) + \epsilon_n.$$

Thus, by induction, we constructed the sequence $\{x_i\}_{i=1}^{\infty}$ which satisfies (49) and (50) for each integer $i \geq 0$.

Assume that a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ satisfies (49) and (50) for each integer $i \geq 0$. We show that it is a Cauchy sequence.

Let $i \geq 1$ be an integer. Assumption (A3), (49) and (50) imply that

$$d(x_{i+1}, x_{i+2}) \leq d(x_{i+1}, \{u \in T(x_{i+1}) : (x_{i+1}, u) \in E(G)\}) + \epsilon_{i+1}$$

$$\leq cd(x_i, x_{i+1}) + \epsilon_{i+1}. \tag{55}$$

In view of (55),

$$d(x_1, x_2) \leq cd(x_0, x_1) + \epsilon_1,$$

$$d(x_2, x_3) \leq cd(x_1, x_2) + \epsilon_2 \leq c^2 d(x_0, x_1) + c\epsilon_1 + \epsilon_2.$$

Now, we show by induction that for every integer $n \geq 1$,

$$d(x_n, x_{n+1}) \leq c^n d(x_0, x_1) + \sum_{i=0}^{n-1} c^i \epsilon_{n-i}. \tag{56}$$

(Note that by the relations above (56) holds for $n = 1, 2$).

Assume that $k \geq 1$ is an integer and that (56) holds for $n = k$. When combined with (55), this implies that

$$d(x_{k+1}, x_{k+2}) \leq cd(x_k, x_{k+1}) + \epsilon_{k+1}$$

$$\leq c^{k+1} d(x_0, x_1) + \sum_{i=0}^{k-1} c^{i+1} \epsilon_{k-i} + \epsilon_{k+1}$$

$$= c^{k+1} d(x_0, x_1) + \sum_{i=0}^{k} c^i \epsilon_{k+1-i}.$$

Thus, (56) holds with $n = k + 1$ too. Thus, we have showed by induction that (56) holds for all integers $n \geq 1$. By (48) and (56),

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) \leq \sum_{n=1}^{\infty} (c^n d(x_0, x_1) + \sum_{i=1}^{n} c^{n-i} \epsilon_i)$$

$$\leq d(x_0, x_1) \sum_{n=1}^{\infty} c^n + \sum_{i=1}^{\infty} (\sum_{j=0}^{\infty} c^j) \epsilon_i$$

$$\leq (\sum_{n=0}^{\infty} c^n)[d(x_0, x_1) + \sum_{n=1}^{\infty} \epsilon_n] < \infty.$$

Thus $\{x_n\}_{n=0}^{\infty}$ is indeed a Cauchy sequence and there exists

$$x_* = \lim_{n \to \infty} x_n.$$

Clearly, if the graph of T is closed, then $x_* \in T(x_*)$. □

Theorem 5. *Assume that the graph of T is closed and that $\epsilon > 0$. Then, there exists $\delta > 0$ such that if $x \in X$ and*

$$d(x, \{u \in T(x) : (x, u) \in E(G)\}) < \delta, \tag{57}$$

then there is $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$ and $d(x, \bar{x}) \leq \epsilon$.

Proof. Choose

$$\delta \in (0, 4^{-1}(1-c)\epsilon) \tag{58}$$

Let $x \in X$ and (57) hold. Set

$$x_0 = x. \tag{59}$$

By (57) and (59), there is

$$x_1 \in T(x_0) \tag{60}$$

such that

$$(x_0, x_1) \in E(G),\ d(x_0, x_1) < \delta, \tag{61}$$

$$d(x_0, x_1) \leq d(x_0, \{u \in T(x_0) : (x_0, u) \in E(G)\})(1+c)(2c)^{-1}. \tag{62}$$

By induction, we construct a sequence $\{x_n\}_{n=2}^{\infty} \subset X$ such that for each integer $n \geq 0$,

$$x_{n+1} \in T(x_n),\ (x_n, x_{n+1}) \in E(G), \tag{63}$$

$$d(x_{n+1}, x_n) \leq d(x_n, \{u \in T(x_n) :\ (x_n, u) \in E(G)\})(1+c)(2c)^{-1}. \tag{64}$$

(Note that by (61), (62) the relations above hold for $n = 0$).

Assume that k is a natural number number and (63) and (64) hold for all integers $n = 0, \dots, k-1$. Then,

$$(x_{k-1}, x_k) \in E(G),\ x_k \in T(x_{k-1}).$$

By (A3),

$$\{u \in T(x_k) :\ (x_k, u) \in E(G)\} \neq \emptyset$$

and there is

$$x_{k+1} \in \{u \in T(x_k) :\ (x_k, u) \in E(G)\}$$

such that

$$d(x_k, x_{k+1}) \leq d(x_k, \{u \in T(x_k) :\ (x_k, u) \in E(G)\})(1+c)(2c)^{-1}.$$

Thus, by induction we constructed the sequence $\{x_n\}_{n=0}^{\infty} \subset X$, such that, for each integer $n \geq 0$, relations (63) and (64) hold. Assumption (A3), (63) and (64) imply that for each integer $n \geq 1$,

$$d(x_n, x_{n+1}) \leq ((1+c)/2) d(x_n, x_{n-1})$$

and

$$d(x_n, x_{n+1}) \leq [(1+c)/2]^n d(x_0, x_1) \leq [(1+c)/2]^n \delta. \tag{65}$$

Therefore,

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty,$$

$\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence and there exists $\bar{x} \in X$ such that

$$\bar{x} = \lim_{n \to \infty} x_n.$$

Since the graph of T is closed, we have

$$\bar{x} \in T(\bar{x}).$$

By (58) and (65),

$$d(x_0, \bar{x}) = \lim_{n \to \infty} d(x_0, x_n) \leq \sum_{n=0}^{\infty} d(x_n, x_{n+1})$$

$$\leq \sum_{i=0}^{\infty} [(1+c)/2]^i \delta = 2\delta/(1-c) < \epsilon.$$

This completes the proof of Theorem 5. □

Theorem 6. *Let $\epsilon \in (0,1)$, $M > 0$,*

$$\delta \in (0, 2^{-1}(1-c)\epsilon) \tag{66}$$

and a natural number n_0 satisfy

$$c^{n_0}(2M+1) < \epsilon/2. \tag{67}$$

Then, for each sequence $\{x_i\}_{i=0}^{\infty} \subset X$*, such that*

$$\{u \in T(x_0) : (x_0, u) \in E(G)\} \neq \emptyset,$$

$$d(x_0, \{u \in T(x_0) : (x_0, u) \in E(G)\}) \leq M \tag{68}$$

and that for each integer $n \geq 0$*,*

$$x_{n+1} \in T(x_n), \ (x_n, x_{n+1}) \in E(G), \tag{69}$$

$$d(x_n, x_{n+1}) \leq d(x_n, \{u \in T(x_n) : (x_n, u) \in E(G)\}) + \delta \tag{70}$$

the inequality

$$d(x_{n+1}, x_n) \leq \epsilon \text{ for all integers } n \geq n_0.$$

Proof. Let $\{x_n\}_{n=0}^{\infty} \subset X$ satisfy (68), (69) and (70) for each integer $i \geq 0$. By (68) and (69),

$$d(x_0, x_1) \leq \delta + d(x_0, \{u \in T(x_0) : (x_0, u) \in E(G)\}) \leq M + 1. \tag{71}$$

Assumption (A3), (69) and (70) imply that for each integer $n \geq 1$,

$$d(x_{n+1}, x_{n+2})$$

$$\leq d(x_{n+1}, \{u \in T(x_{n+1}) : (x_{n+1}, u) \in E(G)\}) + \delta \leq cd(x_n, x_{n+1}) + \delta. \tag{72}$$

By induction, we show that for each integer $n \geq 1$,

$$d(x_{n+1}, x_n) \leq \delta \sum_{i=0}^{n-1} c^i + c^n d(x_0, x_1). \tag{73}$$

Clearly, for $n = 1$, (73) holds. Assume that $k \geq 1$ is an integer and that (73) holds with $n = k$. Together with (72), this implies that

$$d(x_{k+2}, x_{k+1}) \leq cd(x_k, x_{k+1}) + \delta \leq \delta \sum_{i=0}^{k} c^i + c^{k+1} d(x_0, x_1)$$

and (73) holds for $n = k + 1$. Thus, (73) holds for each integer $n \geq 0$. By (66), (67) and (71), for each integer $n \geq n_0$,

$$d(x_n, x_{n+1}) \leq \delta(1 - c)^{-1} + c^n(2M + 1) \leq \epsilon.$$

Theorem 6 is proved. □

Theorems 5 and 6 imply the following additional result.

Theorem 7. *Assume that the graph of T is closed and let positive numbers* ϵ *and* M *be given. Then, there exist* $\delta > 0$ *and an integer* $n_0 \geq 1$ *such that if a sequence* $\{x_i\}_{i=0}^{\infty} \subset X$ *satisfies*

$$d(x_0, \{u \in T(x_0) : (x_0, u) \in E(G)\}) \leq M$$

and that for each integer $n \geq 0$*,*

$$(x_n, x_{n+1}) \in E(G), \ x_{n+1} \in T(x_n),$$

$$d(x_n, x_{n+1}) \leq d(x_n, \{u \in T(x_n) : (x_n, u) \in E(G)\}) + \delta$$

for each integer $n \geq n_0$ there is a point $y \in X$ such that $y \in T(y)$ and $d(y, x_n) \leq \epsilon$.

6. Conclusions

In out work, we show the existence of a fixed point for single valued and set-valued self-mappings of metric spaces with graphs which are of the contractive type. We also show the convergence of iterates of these mappings to fixed points. In particular, our results are true for metric spaces with symmetric graphs. They are extensions of the results of [24], which were obtained under some additional restrictive assumption on a graph which is not used here. Our results can be helpful if one needs to find an approximate solution of a set-valued inclusion. The further development of our research is in generalizing our results under the presence of computational errors when the next iterate x_{t+1} does not belong to $T(x_t)$, but to its small neighborhood.

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