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New Accurate Approximation Formula for Gamma Function

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Abstract: In this paper, a new approximation formula for the gamma function and some of its symmetric inequalities are established. We prove the superiority of our results over Yang and Tian's approximation formula for the gamma function of order v^{-9}.

Keywords: gamma function; approximation formula; completely monotonic function; digamma function; inequality; best possible constant

MSC: 33B15; 41A60; 41A21

1. Introduction

The gamma function extends the idea of a factorial to non-integer numbers. C. F. Gauss and other mathematicians investigated it further after L. Euler introduced it for the first time in the 18th century. The gamma function is defined as follows:

$$\Gamma(v) = \int_0^\infty e^{-t} t^{v-1} dt, \quad v > 0 \tag{1}$$

or by

$$\Gamma(v) = \lim_{m\to\infty} \frac{m! m^v}{v(1+v)(2+v)\dots(m+v)}, \quad v \in \mathbb{R} - \{0, -1, -2, \dots\} \tag{2}$$

and satisfies the recurrence relation $\Gamma(v+1) = v\Gamma(v)$, and hence, $m! = \Gamma(m+1)$ for $m \in \mathbb{N}$. More work has been performed by mathematicians to obtain accurate estimates of $m!$ and the gamma function. J. Stirling developed the following important formula:

$$\Gamma(v+1) = \sqrt{2\pi v}(v/e)^v \left[1 + O(v^{-1})\right] := \chi_1(v), \quad v \to \infty. \tag{3}$$

As v grows, this estimate becomes more and more accurate. For high values, Stirling's formula, which comes from Stirling's series expansion, provides a practical alternative to numerically integrating the defining integral for estimating the gamma function. It is crucial to remember that, while Stirling's approximation is frequently correct for large values of v, it might not be appropriate for small values of v or situations requiring great accuracy. Other techniques can be used to obtain more precise and comprehensive approximations.

Some sharp inequalities for the ratio of gamma functions are presented by Cao and Wang [1] by using the multiple-correction method. Alzer and Jameson [2] present a new characterization of Euler's constant γ and a concavity property of the Psi function. Yang and Tian [3] refine Windschitl's gamma function approximation formula by providing two asymptotic expansions based on a little-known power series. The authors of [4] use several classical inequalities, such as Chebychev's inequality for synchronous mappings, to propose some inequalities involving the extended gamma function and the Kummer confluent hypergeometric k-function. Qi and Guo [5] study the properties of the Bernstein

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Citation: Mahmoud, M.; Almuashi, H. New Accurate Approximation Formula for Gamma Function. *Symmetry* **2024**, *16*, 150. https://doi.org/10.3390/sym16020150

Academic Editors: Serkan Araci and Alexei Kanel-Belov

Received: 26 December 2023
Revised: 8 January 2024
Accepted: 26 January 2024
Published: 27 January 2024

function of the newly created ratio of finitely many gamma functions as well as the history, backgrounds, extensions, and applications of a series of ratios for finitely many gamma functions. For large values of the involved parameters, Reynolds and Stauffer [6] investigate the improved infinite sum for the incomplete gamma function. Tian and Yang [7] expand and generalize some previous results by presenting the necessary and sufficient conditions for a ratio involving q-gamma functions to be logarithmically completely monotonic using a new method. Zhang, Yin, and You [8] deduce some new inequalities and completely monotonic properties involving the generalized functions k-gamma and k-polygamma. Yildirim [9] uses the Bernstein–Widder theorem and some properties of the k-special function to present k-generalizations of some classical results and improvements to some bounds of recent results about the k-polygamma functions. Based on the incomplete gamma function, Castillo, Rojas, and Reyes [10] deduce a more flexible extension for the Fréchet distribution, along with applications. Mahmoud, Alsulami, and Almarashi [10] examine the monotonicity of some functions involving $\Gamma(v)$ and ascertain its bounds that they were able to derive are more precise than certain inequalities that have been published previously. The research references [1–11] present further information regarding the gamma function and the formulas, inequalities, approximations, generalizations, and applications that go along with it.

Simple and accurate gamma function approximation formulas are essential for many applications because gamma function integrals cannot be computed directly for most non-integer values. By using approximation formulas, one may evaluate $\Gamma(v)$ more quickly and with less processing effort. For instance, simple approximations for the gamma function can greatly increase the accuracy and efficiency of algorithms in numerical techniques and calculations. This is essential for areas like statistical analysis, optimization, and differential equation solutions [12,13]. Some intriguing estimates of gamma functions are as follows: Ramanujan presents the approximation formula [14] as $v \to \infty$

$$\Gamma(v+1) = \sqrt{\pi}\left(\frac{v}{e}\right)^{v} \sqrt[6]{8v^3+4v^2+v+1/30}\left[1+O(v^{-4})\right] := \chi_2(v),$$

based on some numerical calculations as conjecture.

Windschitl finds the approximation formula [15] as $v \to \infty$

$$\Gamma(v+1) = \sqrt{2\pi v}\left(\frac{v}{e}\right)^{v}\left(v\sinh\left(\frac{1}{v}\right)\right)^{v/2}\left[1+O(v^{-5})\right] := \chi_3(v),$$

after he noticed by coincidence the relation between some power series expansions of the extended Stirling's formula and the hyperbolic sine function. Since it is accurate to more than eight decimal places for $v > 8$, he suggested using the approximation $\sqrt{2\pi v}(v/e)^{v}\left(\frac{1}{810v^6}+v\sinh\left(\frac{1}{v}\right)\right)^{v/2}$ to compute the values of the gamma function on calculators with limited program or register memory.

Smith presents the approximation formula [16] as $v \to \infty$

$$\Gamma(v+1/2) = \sqrt{2\pi}\left(\frac{v}{e}\right)^{v}\left(2v\tanh\left(\frac{1}{2v}\right)\right)^{v/2}\left[1+O(v^{-5})\right] := \chi_4(v),$$

and some new representations of the gamma function and some of its related functions. In addition, he provides new continued fractions and a new formula for the beta function. Mahmoud and Almuashi deduce the approximation formula [17] as $v \to \infty$

$$\Gamma(v+1) = \sqrt{2\pi v}\left(\frac{v}{e}\right)^{v}\left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2}\left[1+O(v^{-5})\right] := \chi_5(v).$$

Moreover, they derive some new bounds of $\Gamma(v)$ more accurately than some of its recent ones by using Padé approximants and a new asymptotic expansion of gamma that they derived.

Nemes uses a series transformation to convert the Stirling asymptotic series approximation of the gamma function into a new one with better convergence properties and deduces the approximation formula [18] as $v \to \infty$

$$\Gamma(v+1) = \sqrt{2\pi v}(v/e)^v \left(\frac{3(40v^2+3)}{120v^2-1} \right)^v \left[1 + O\left(v^{-5}\right)\right] := \chi_6(v).$$

Yang and Chu's deduce the approximation formula [19] as $v \to \infty$

$$\Gamma(v+1/2) = \sqrt{2\pi}(v/e)^v e^{-\frac{5v}{120v^2+7}} \left[1 + O\left(v^{-5}\right)\right] := \chi_7(v),$$

and some upper and lower bounds of the factorial $m!$ and the gamma function are presented as applications.

Nemes's approximation formula [18] as $v \to \infty$

$$\Gamma(v+1) = \sqrt{2\pi v}\left(\frac{v}{e}\right)^v e^{\frac{210v^2+53}{360v(7v^2+2)}} \left[1 + O(v^{-7})\right] := \chi_8(v).$$

Yang and Chu's approximation formula [19] as $v \to \infty$

$$\Gamma(v+1/2) = \sqrt{2\pi}(v/e)^v e^{\frac{-5880v^2-1517}{1440v(98v^2+31)}} \left[1 + O\left(v^{-7}\right)\right] := \chi_9(v).$$

Based on Windschitl's formula, Lu, Song, and Ma derive a generated approximation of the factorial function $m!$, prove several gamma function inequalities, and deduce the approximation formula [20] as $v \to \infty$

$$\Gamma(v+1) = \sqrt{2\pi v}\left(\frac{v}{e}\right)^v \left(v \sinh\left(\frac{1}{v} + \frac{1}{810v^7}\right)\right)^{v/2} \left[1 + O(v^{-7})\right] := \chi_{10}(v).$$

Chen presents the approximation formula [21] as $v \to \infty$

$$\Gamma(v+1) = \sqrt{2\pi v}(v/e)^v \left(\frac{168v^3+48v+7}{168v^3+48v-7} \right)^{v^2+53/210} \left[1 + O\left(v^{-7}\right)\right] := \chi_{11}(v),$$

and then creates an asymptotic expansion using this approximation formula.

Windschitl's approximation formula [15] as $v \to \infty$

$$\Gamma(v+1) = \sqrt{2\pi v}\left(\frac{v}{e}\right)^v \left(v \sinh\left(\frac{1}{v}\right) + \frac{1}{810v^6}\right)^{v/2} \left[1 + O(v^{-7})\right] := \chi_{12}(v).$$

Alzer presents the approximation formula [22] as $v \to \infty$

$$\Gamma(v+1) = \sqrt{2\pi v}\left(\frac{v}{e}\right)^v \left(v \sinh\left(\frac{1}{v}\right)\right)^{v/2} \left(\frac{1}{1620v^5} + 1 \right) \left[1 + O(v^{-7})\right] := \chi_{13}(v),$$

and a double inequality of $\Gamma(v+1)$ for $v > 0$ with the best possible constants.

Yang and Tian present the approximation formula [23] as $v \to \infty$

$$\Gamma(v+1) = \sqrt{2\pi v}\left(\frac{v}{e}\right)^v \left(v \sinh\left(\frac{1}{v}\right)\right)^{v/2} e^{\frac{7}{324v^3(35v^2+33)}} \left[1 + O(v^{-9})\right] := \chi_{14}(v)$$

and study the monotonicity of some functions involving $\Gamma(v+1)$.

In light of the aforementioned results, the purpose of this paper is to present the following most accurate Mahmoud and Almuashi-type approximation:

$$\Gamma(v+1)=\sqrt{2\pi v}\left(\frac{v}{e}\right)^{v}\left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2}e^{\frac{461}{907,200v^3\left(v^2+\frac{5197}{4610}\right)}}\left[1+O(v^{-9})\right]:=\chi_{15}(v),\quad v\to\infty \tag{4}$$

which is more accurate than Yang and Tian's approximation formula $\chi_{14}(v)$ in [23]. Also, we proved the following approximation formula

$$\Gamma(v+1)=\sqrt{2\pi v}\left(\frac{v}{e}\right)^{v}\left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2}e^{\frac{461}{907,200v^5}}\left[1+O(v^{-5})\right]:=\chi_{16}(v),\quad v\to\infty. \tag{5}$$

We have used Mathematica 10 software to perform the numerical calculations throughout this work.

2. Main Results

Recall that a real valued function g, which is defined and infinitely differentiable on $v\in(0,\infty)$, is said to be completely monontonic (CM) if for all $p\geq 0$ that $(-1)^p g^{(p)}(v)\geq 0$ on $v\in(0,\infty)$. We refer to [24–27] for further information on CM functions and their applications. The function g is CM, according to Bernstein's theorem [28], if and only if $g(v)=\int_0^\infty e^{-vt}d\varsigma(t)$, where $\varsigma(t)$ is a non-negative measure on $t\in(0,\infty)$ such that the integral converges for $v\in(0,\infty)$. If the function $g(v)$ is CM for $v\in(0,\infty)$, $\lim_{v\to\infty}g(v)\doteqdot g(\infty)$ and the function $v^{\varpi}[g(x)-g(\infty)]$ is CM for $v\in(0,\infty)$ if and only if $\varpi\in[0,\delta]$, then the real number δ is called the completely monotonic degree of $g(v)$ with respect to $v\in(0,\infty)$ (see [29,30]) and is denoted by $\deg^{v}_{\mathrm{cm}}[g(v)]=\delta$. This concept can aid in more accurate measurements of CM functions.

To obtain our first bounds of gamma functions, we first give the following theorem:

Theorem 1.

$$Q_1(v)=\frac{1}{v}+\frac{461}{181,440(v+1)^6}+\frac{1}{2v+2}+\frac{1}{20v(v+2)+19}+\frac{7}{60v(v+2)+67}$$
$$-\frac{1}{2}\ln\left(\frac{(v+1)^2((v+1)^2+\frac{7}{60})}{(v+1)^2-\frac{1}{20}}\right)+\psi(v),\quad v\geq 0 \tag{6}$$

is CM function with

$$deg^{v}_{\mathrm{cm}}[Q_1(v)]=0,$$

where $\psi(v)$ is the digamma function.

Proof. Using Gauss integral form of the digamma function [31]

$$\psi(v)=\int_0^\infty\left(\frac{1}{te^t}+\frac{e^{-vt}}{e^{-t}-1}\right)dt,\quad v>0 \tag{7}$$

we obtain

$$Q_1(v)=\int_0^\infty\frac{-e^{-t}W_1(t)}{t}e^{vt}dt,$$

where

$$\begin{aligned} W_1(t) &= -\frac{461t^6}{21{,}772{,}800} + \frac{e^{-\frac{t}{2\sqrt{5}}}t}{4\sqrt{5}} - \frac{t}{2} - \frac{e^{\frac{t}{2\sqrt{5}}}t}{4\sqrt{5}} + \frac{e^t t}{e^t-1} + \frac{1}{2}e^{\frac{t}{2\sqrt{5}}} + \frac{1}{2}e^{-\frac{t}{2\sqrt{5}}} \\ &\quad -\left(\frac{1}{2}\sqrt{\frac{7}{15}}t\sin\left(\frac{1}{2}\sqrt{\frac{7}{15}}t\right) + \cos\left(\frac{1}{2}\sqrt{\frac{7}{15}}t\right)\right) - 1. \end{aligned}$$

Consider the following functions for $t \geq 0$

$$W_2(t) = -\frac{461t^6}{21{,}772{,}800} + \frac{e^{-\frac{t}{2\sqrt{5}}}t}{4\sqrt{5}} - \frac{t}{2} - \frac{e^{\frac{t}{2\sqrt{5}}}t}{4\sqrt{5}} + \frac{1}{2}e^{\frac{t}{2\sqrt{5}}} + \frac{1}{2}e^{-\frac{t}{2\sqrt{5}}} - 1,$$

$$W_3(t) = -\frac{4799t^6}{217{,}728{,}000} - \frac{t^4}{3200} - \frac{t^2}{40} - \frac{t}{2},$$

$$W_4(t) = -\frac{e^t t}{e^t-1} + \frac{1}{2}\sqrt{\frac{7}{15}}t\sin\left(\frac{1}{2}\sqrt{\frac{7}{15}}t\right) + \cos\left(\frac{1}{2}\sqrt{\frac{7}{15}}t\right),$$

$$W_5(t) = -\frac{1}{2}\sqrt{\frac{7}{15}}t - \frac{e^t t}{e^t-1} - 1,$$

and

$$W_6(t) = -\frac{7\left(7t^2\left(7\left(t^2-300\right)t^2+324{,}000\right)-77{,}760{,}000\right)t^2}{9{,}331{,}200{,}000} - \frac{e^t t}{e^t-1} + 1.$$

For $t \geq 0$, we have

$$\begin{aligned} W_2(t) - W_3(t) &= \frac{\left(t^2+120\right)\left(t^2+240\right)t^2}{1{,}152{,}000} - \frac{t\sinh\left(\frac{t}{2\sqrt{5}}\right)}{2\sqrt{5}} + \cosh\left(\frac{t}{2\sqrt{5}}\right) - 1 \\ &= \sum_{p=4}^{\infty} -\frac{20^{-p}t^{2p}}{2p(2p-2)!} < 0. \end{aligned}$$

Then, we have

$$W_2(t) < W_3(t), \quad t \geq 0. \tag{8}$$

Using the inequality

$$\frac{1}{2}\sqrt{\frac{7}{15}}t\sin\left(\frac{1}{2}\sqrt{\frac{7}{15}}t\right) + \cos\left(\frac{1}{2}\sqrt{\frac{7}{15}}t\right) > -\frac{1}{2}\sqrt{\frac{7}{15}}t - 1, \quad \forall t,$$

we obtain

$$W_4(t) > W_5(t), \quad \forall t.$$

Now,

$$W_4(t) - W_3(t) > W_5(t) - W_3(t) = \frac{1}{217{,}728{,}000(e^t-1)}W_7(t)$$

with

$$\begin{aligned} W_7(t) &= 4799e^t t^6 - 4799t^6 + 68{,}040e^t t^4 - 68{,}040t^4 + 5{,}443{,}200e^t t^2 - 5{,}443{,}200t^2 \\ &\quad -7{,}257{,}600\sqrt{105}e^t t - 108{,}864{,}000e^t t + 7{,}257{,}600\sqrt{105}t - 108{,}864{,}000\, t \\ &\quad -217{,}728{,}000e^t + 217{,}728{,}000 \\ &= W_8(t) + \sum_{n=7}^{\infty} \frac{a_n e^8 (t-8)^n}{n!} > 0, \quad t \geq 8 \end{aligned}$$

where

$$\begin{aligned} a_n &= 4799n^6 + 158{,}367n^5 + 2{,}779{,}475n^4 + 30{,}251{,}105n^3 + 213{,}687{,}206n^2 \\ &+ \left(814{,}213{,}960 - 7{,}257{,}600\sqrt{105}\right)n + 1024\left(777{,}779 - 56{,}700\sqrt{105}\right) > 0,\ n = 7, 8, 9, \ldots \end{aligned}$$

and the polynomial

$$\begin{aligned} W_8(t) &= 9.90552 \times 10^{10}t^6 - 4.3738 \times 10^{12}t^5 + 8.10277 \times 10^{13}t^4 - 8.05282 \times 10^{14}t^3 \\ &+ 4.52427 \times 10^{15}t^2 - 1.36148 \times 10^{16}t + 1.71352 \times 10^{16} \end{aligned}$$

is positive and has no real zeros for $t \geq 8$. Then,

$$W_4 > W_3, \quad t \geq 8. \tag{9}$$

Using the two inequalities [32]

$$\cos(t) > \left(-\frac{t^6}{720} + \frac{t^4}{24} - \frac{t^2}{2} + 1\right), \quad t > 0$$

and

$$\sin(t) < \left(-\frac{t^7}{5040} + \frac{t^5}{120} - \frac{t^3}{6} + t\right), \quad t > 0$$

we obtain

$$W_4(t) > W_6(t), \quad t > 0.$$

Then,

$$\begin{aligned} &W_4(t) - W_3(t) > W_4(t) - W_6(t) \\ &= \frac{-2401t^8 + 2{,}160{,}000t^6 - 90{,}720{,}000t^4 + 5{,}443{,}200{,}000t^2 + 65{,}318{,}400{,}000}{65{,}318{,}400{,}000} \\ &-\frac{1}{2}t\coth\left(\frac{t}{2}\right). \end{aligned}$$

Using

$$\left(\frac{t^3}{42} + t\right)(e^t + 1) - \left(\frac{t^4}{840} + \frac{3t^2}{14} + 2\right)(e^t - 1) = -\sum_{p=9}^{\infty} \frac{(p-8)(p-7)(p-6)(p-5)}{840p!}t^p,$$

we obtain

$$\coth\left(\frac{t}{2}\right) < \frac{\frac{t^4}{840} + \frac{3t^2}{14} + 2}{\frac{t^3}{42} + t}, \quad t > 0$$

and hence,

$$W_4(t) - W_3(t) > W_4(t) - W_6(t) > \frac{t^8(2{,}059{,}158 - 2401t^2)}{65{,}318{,}400{,}000(t^2 + 42)} > 0, \quad 0 < t < \frac{\sqrt{2{,}059{,}158}}{49}.$$

Then,

$$W_4(t) > W_3(t), \quad 0 < t \leq 8. \tag{10}$$

From inequalities (8)–(10), we obtain

$$W_1(t) < 0, \quad t \geq 0$$

and therefore, $Q_1(v)$ is CM function.

Now, if we suppose that $v^{\alpha}Q_1(v)$ is a CM function for $v > 0$, then it is a decreasing function, that is

$$\alpha \le -\frac{vQ_1'(v)}{Q_1(v)}.$$

Using the following asymptotic expansion and its derivative

$$\psi(v) \sim -\frac{1}{v} - \gamma + \frac{\pi^2 v}{6} - \zeta(3)v^2 + \frac{\pi^4 v^3}{90} - \zeta(5)v^4 + \dots, \quad v \to 0$$

where $\zeta(v)$ is the Riemann zeta function for $v > 1$, we obtain

$$\alpha \le \frac{\left(\frac{81{,}155{,}027{,}309}{49{,}004{,}796{,}960} - \frac{\pi^2}{6}\right)v + v^2\left(2\zeta(3) - \frac{22{,}195{,}771{,}076{,}117}{8{,}911{,}872{,}361{,}440}\right) + O(v^3)}{\left(\frac{152{,}361{,}413}{230{,}973{,}120} - \gamma - \frac{1}{2}\log\left(\frac{67}{57}\right)\right) + \left(\frac{\pi^2}{6} - \frac{81{,}155{,}027{,}309}{49{,}004{,}796{,}960}\right)v + O(v^2)} \to 0 \quad \text{as} \quad v \to 0.$$

Hence, $\alpha = 0$ or

$$\deg_{\mathrm{cm}}^{v}[Q_1(v)] = 0.$$

□

Theorem 2. *The function*

$$Q_2(v) = \frac{\Gamma(v+1)}{\sqrt{2\pi v}\left(\frac{v}{e}\right)^v \left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2} e^{\frac{461}{907200v^5}}}, \quad v \ge 1$$

is increasing. Furthermore, we have the following symmetric inequality:

$$a_1\sqrt{2\pi v}\left(\frac{v}{e}\right)^v \left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2} e^{\frac{461}{907{,}200v^5}} < \Gamma(v+1) < a_2\sqrt{2\pi v}\left(\frac{v}{e}\right)^v \left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2} e^{\frac{461}{907{,}200v^5}}, \quad v \ge 1 \tag{11}$$

with best possible constants $a_1 = Q_2(1) \approx 0.999733$ and $a_2 = 1$.

Proof. Using the relation

$$Q_1(v) = \frac{d}{dv}\ln Q_2(v+1), \quad v > 0$$

and with $Q_1(v)$ a CM function for $v > 0$, then $Q_2(v)$ is an increasing function for $v \ge 1$. Also, using the asymptotic expansion [17]

$$\frac{\Gamma(v+1)}{\sqrt{2\pi v}(v/e)^v \left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2}} \sim 1 + \sum_{r=1}^{\infty} \lambda_r v^{-r}, \qquad v \to \infty \tag{12}$$

where

$$\begin{cases} \lambda_r = \frac{1}{r}\sum_{j=1}^{r} j\lambda_{r-j}\rho_j \\ \rho_r = \frac{B_{r+1}}{r(r+1)} - \frac{1}{2}\omega_{r+1} \\ \omega_r = \xi_r - \frac{1}{r}\sum_{j=1}^{r-1} j\omega_j\xi_{r-j} \end{cases}, \quad r = 1,2,3,\dots$$

with

$$\xi_0 = 1, \quad \xi_{2s} = \frac{10}{3(20)^s}, \quad \xi_{2s-1} = 0, \quad s = 1,2,3,\dots$$

we obtain

$$Q_2(v) \sim e^{-\frac{461}{907{,}200v^5}}\left(1 + \frac{461}{907{,}200v^5} - \frac{5197}{9{,}072{,}000v^7} + \frac{1{,}436{,}249}{1{,}710{,}720{,}000v^9} + \dots\right), \quad v \to \infty \tag{13}$$

and

$$a_2 = \lim_{v \to \infty} Q_2(v) = 1.$$

□

Remark 1. *From the expansion (13), we conclude that*

$$\Gamma(v+1) = \sqrt{2\pi v}\left(\frac{v}{e}\right)^v \left(\frac{v^2 + \frac{7}{60}}{v^2 - \frac{1}{20}}\right)^{v/2} e^{\frac{461}{907{,}200 v^5}}\left[1 + O(v^{-5})\right] := \chi_{16}(v), \quad v \to \infty.$$

Now, we present our second bounds of gamma function.

Theorem 3. *The function*

$$G_1(v) = \frac{\Gamma(v+1)}{\sqrt{2\pi v}\left(\frac{v}{e}\right)^v \left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2} e^{\frac{461}{907{,}200 v^3\left(v^2+\frac{5197}{4610}\right)}}}, \quad v \geq \frac{13}{10}$$

is decreasing. Furthermore, we have the following symmetric inequality:

$$a_3 < \frac{\Gamma(v+1)}{\sqrt{2\pi v}\left(\frac{v}{e}\right)^v \left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2} e^{\frac{461}{907{,}200 v^3\left(v^2+\frac{5197}{4610}\right)}}} < a_4, \quad v \geq \frac{13}{10} \tag{14}$$

with best possible constants $a_3 = 1$ *and* $a_4 = G_1\left(\frac{13}{10}\right) \approx 1.00000004$.

Proof. Consider the function

$$K_1(v) = \frac{d}{dv}\ln G_1\left(v + \frac{13}{10}\right) - \frac{d}{dv}\ln G_1\left(v + \frac{23}{10}\right), \quad v > 0$$

then,

$$\begin{aligned}
K_1(v) \quad &= -\frac{5}{10v+13} - \frac{2{,}449{,}304{,}525}{61{,}255{,}978{,}812(10v+13)^2} + \frac{26{,}565{,}125}{1{,}964{,}466(10v+13)^4} - \frac{5}{10v+23} \\
&+ \frac{2{,}449{,}304{,}525}{61{,}255{,}978{,}812(10v+23)^2} - \frac{26{,}565{,}125}{1{,}964{,}466(10v+23)^4} + \frac{5}{4(5v(5v+13)+41)} \\
&+ \frac{35}{60v(5v+13)+542} + \frac{1{,}129{,}129{,}386{,}025}{61{,}255{,}978{,}812(9220v(5v+13)+129{,}879)} \\
&- \frac{5{,}645{,}646{,}930{,}125}{2{,}946{,}699(9220v(5v+13)+129{,}879)^2} - \frac{5}{20v(5v+23)+524} - \frac{35}{2(30v(5v+23)+811)} \\
&- \frac{1{,}129{,}129{,}386{,}025}{61{,}255{,}978{,}812(9220v(5v+23)+295{,}839)} + \frac{5{,}645{,}646{,}930{,}125}{2{,}946{,}699(9220v(5v+23)+295{,}839)^2} \\
&- \log\left(v + \frac{13}{10}\right) + \log\left(v + \frac{23}{10}\right) - \frac{1}{2}\log\left(\frac{25}{6(5v(5v+13)+41)} + 1\right) \\
&+ \frac{1}{2}\log\left(\frac{25}{30v(5v+23)+786} + 1\right)
\end{aligned}$$

and

$$K_1'(v) = \frac{K_2(v)}{K_3(v)} > 0, \quad v > 0$$

where

$$K_2(v) = 9.76563 \times 10^6(1.40931 \times 10^{47}v^{26} + 6.59557 \times 10^{48}v^{25} + 1.48138 \times 10^{50}v^{24} + 2.12565 \times 10^{51}v^{23} + 2.1883 \times 10^{52}v^{22} + 1.72065 \times 10^{53}v^{21} + 1.0741 \times 10^{54}v^{20} + 5.46199 \times 10^{54}v^{19} + 2.30366 \times 10^{55}v^{18} + 8.16187 \times 10^{55}v^{17} + 2.45138 \times 10^{56}v^{16} + 6.28063 \times 10^{56}v^{15} + 1.37817 \times 10^{57}v^{14} + 2.59529 \times 10^{57}v^{13} + 4.19512 \times 10^{57}v^{12} + 5.81144 \times 10^{57}v^{11} + 6.8751 \times 10^{57}v^{10} + 6.90658 \times 10^{57}v^{9} + 5.84297 \times 10^{57}v^{8} + 4.1146 \times 10^{57}v^{7} + 2.37302 \times 10^{57}v^{6} + 1.09554 \times 10^{57}v^{5} + 3.91652 \times 10^{56}v^{4} + 1.03052 \times 10^{56}v^{3} + 1.83331 \times 10^{55}v^{2} + 1.87037 \times 10^{54}v + 7.09549 \times 10^{52})$$

and

$$K_3(v) = 378(10v+13)^5(10v+23)^5\left(25v^2+65v+41\right)^2\left(25v^2+115v+131\right)^2 \left(150v^2+390v+271\right)^2\left(150v^2+690v+811\right)^2\left(46{,}100v^2+119{,}860v+129{,}879\right)^3 \left(46{,}100v^2+212{,}060v+295{,}839\right)^3.$$

Then, $K_1(v)$ is increasing function with $\lim_{v\to\infty} K_1(v) = 0$ since

$$K_1(v) \sim -\frac{33{,}094{,}699{,}909{,}052{,}147}{9{,}022{,}995{,}596{,}160{,}000v^{13}} + \frac{213{,}922{,}547}{619{,}584{,}000v^{12}} - \frac{213{,}922{,}547}{12{,}267{,}763{,}200v^{11}} + \dots, \quad v \to \infty.$$

Then, $K_1(v) < 0$ or

$$\frac{d}{dv}\ln G_1\left(v+\frac{13}{10}\right) < \frac{d}{dv}\ln G_1\left(v+\frac{23}{10}\right), \quad v > 0.$$

Using expansion (12), we obtain

$$\frac{d}{dv}\ln G_1(v) \sim -\frac{212{,}521\left(23{,}050v^2+15{,}591\right)}{90{,}720v^4\left(4610v^2+5197\right)^2}\left(-\frac{461}{181{,}440v^6} + \frac{5197}{1{,}296{,}000v^8} - \frac{1{,}436{,}249}{190{,}080{,}000v^{10}} + \frac{26{,}863{,}154{,}077}{1{,}273{,}708{,}800{,}000v^{12}} + \dots\right), \quad v \to \infty$$

and then $\lim_{v\to\infty} \frac{d}{dv}\ln G_1\left(v+\frac{13}{10}\right) = 0$. But, if a function $g : (v_0,\infty) \to \mathbb{R}$ satisfies $\lim_{v\to\infty} g(v) = 0$ and $g(v) < g(v+k)$, $k \in \mathbb{N}$, then $g(v) < 0$ for $v > v_0$ (see [33]). Then, $\frac{d}{dv}\ln G_1\left(v+\frac{13}{10}\right) < 0$ or $\ln G_1\left(v+\frac{13}{10}\right)$ is decreasing for $v > 0$. Hence, $G_1(v)$ is decreasing for $v > \frac{13}{10}$. Using the expansion

$$G_1(v) \sim e^{\frac{461}{907{,}200\left(\frac{5197}{4610v^2}+1\right)v^5}}\left(1 + \frac{461}{907{,}200v^5} - \frac{5197}{9{,}072{,}000v^7} + \frac{1{,}436{,}249}{1{,}710{,}720{,}000v^9} + \dots\right), \quad v \to \infty \tag{15}$$

we obtain $\lim_{v\to\infty} G_1(v) = 1$. Then, the function $G_1(v)$ is bounded by the two best possible constants $a_3 = 1$ and $a_4 = G_1(\frac{13}{10})$. □

Remark 2. *From the expansion (15), we conclude that*

$$\Gamma(v+1) = \sqrt{2\pi v}\left(\frac{v}{e}\right)^v\left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2} e^{\frac{461}{907{,}200v^3\left(v^2+\frac{5197}{4610}\right)}}\left[1+O(v^{-9})\right] = \chi_{15}(v), \quad v \to \infty.$$

3. Comparison among Some Approximation Formulas of the Gamma Function

The approximation formula $\chi_{14}(v)$ is of course better than the other mentioned formulas from $\chi_1(v)$ to $\chi_{13}(v)$ based on the rate of convergence. Also, the two approximation formulas $\chi_{14}(v)$ and $\chi_{15}(v)$ are converging to $\Gamma(v+1)$ with a rate like v^{-9} as $v \to \infty$. To compare between the formulas $\chi_{14}(v)$ and $\chi_{15}(v)$, consider the function

$$K_4(v) = \ln\left(\frac{3v(20v^2-1)\sinh\left(\frac{1}{v}\right)}{60v^2+7}\right) + \frac{1{,}597{,}365v^2 + 3{,}172{,}927}{45{,}360v^4(35v^2+33)(4610v^2+5197)}, \quad v \geq 1$$

and then

$$\begin{aligned} K_4'\left(\frac{1}{v}\right) = & \left(\frac{14}{81(33v^2+35)} - \frac{212{,}521}{11{,}340(5197v^2+4610)}\right)v^5 - \frac{1}{v} + \coth(v) \\ & + \left(\frac{245}{81\left(\frac{35}{v^2}+33\right)^2} - \frac{97{,}972{,}181}{2268\left(\frac{4610}{v^2}+5197\right)^2} - \frac{400}{-7v^4+80v^2+1200}\right)v, \quad 0 < v \leq 1. \end{aligned}$$

Using the expansion

$$\begin{aligned} & \frac{21(e^{2v}+1)v(v^4+60v^2+495) - 2e^v(v^6+210v^4+4725v^2+10{,}395)\sinh(v)}{10{,}395} \\ & = \sum_{p=13}^{\infty} \frac{2^{p-6}(p-12)(p-11)(p-10)(p-9)(p-8)(p-7)}{10{,}395p!}v^p > 0, \quad v > 0 \end{aligned}$$

we obtain

$$\coth(v) < \frac{\frac{v^6}{10{,}395} + \frac{2v^4}{99} + \frac{5v^2}{11} + 1}{\frac{v^5}{495} + \frac{4v^3}{33} + v}, \quad v > 0.$$

Then,

$$K_4'\left(\frac{1}{v}\right) < \frac{v^9 K_5(v)}{K_6(v)} < 0, \quad 0 < v \leq 1$$

where

$$\begin{aligned} K_5(v) & = 3{,}809{,}121{,}073{,}989v^{10} + 308{,}281{,}165{,}523{,}325v^8 + 19{,}393{,}944{,}515{,}758{,}250v^6 \\ & +272{,}405{,}703{,}557{,}635{,}250v^4 + 603{,}315{,}377{,}277{,}995{,}625v^2 + 344{,}328{,}389{,}211{,}515{,}625 \end{aligned}$$

and

$$K_6(v) = 11{,}340\left(v^2-20\right)\left(7v^2+60\right)\left(33v^2+35\right)^2\left(5197v^2+4610\right)^2\left(v^4+60v^2+495\right).$$

Hence, $K_4(v)$ is an increasing function for $v \geq 1$. Also,

$$K_4(v) \sim -\frac{758{,}772{,}059}{3{,}864{,}345{,}408{,}000v^{10}} + \frac{2{,}050{,}016{,}121{,}570{,}689}{4{,}863{,}394{,}626{,}330{,}240{,}000v^{12}} + \ldots, \quad v \to \infty$$

and hence, $\lim_{v\to\infty} K_4(v) = 0$. Then, $K_4(v) < 0$ for $v \geq 1$ or

$$e^{\frac{7}{324v^3(35v^2+33)}}\left(v\sinh\left(\frac{1}{v}\right)\right)^{v/2} < \left(\frac{v^2+\frac{7}{60}}{v^2-\frac{1}{20}}\right)^{v/2} e^{\frac{461}{907{,}200v^3\left(v^2+\frac{5197}{4610}\right)}}, \quad v \geq 1. \tag{16}$$

Then, using the inequalities (14) and (16), we obtain

$$\chi_{14}(v) < \chi_{15}(v) < \Gamma(v), \quad v > 13/10.$$

Therefore, the approximation formula $\chi_{15}(v)$ is better than $\chi_{14}(v)$ for $v \geq \frac{13}{10}$.

4. Conclusions

Applications of the gamma function may be found in a wide range of real-world fields, from finance and economics (e.g., option pricing models) to medical research (e.g., modeling disease spread). Approximation formulae can improve these studies' precision and speed. The main conclusions of this paper are stated in Theorems (2) and (3). Concretely speaking, based on Mahmoud and Almuashi's formula, the authors studied the monotonicity and complete monotonicity of some functions related to $\Gamma(v)$ to present the Formulas (4) and (5) and some symmetric inequalities for $\Gamma(v)$. Our new approximation formula $\chi_{15}(v)$ and Yang and Tian's approximation $\chi_{14}(v)$ are of the same order v^{-9} but the superiority of our results is proven.

Author Contributions: Writing of original draft, M.M. and H.A. All authors contributed equally to the writing of this paper. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

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