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Dynamics of a Stochastic SVEIR Epidemic Model with Nonlinear Incidence Rate

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Abstract: This paper delves into the analysis of a stochastic epidemic model known as the susceptible–vaccinated–exposed–infectious–recovered (SVEIR) model, where transmission dynamics are governed by a nonlinear function. In the theoretical analysis section, by suitable stochastic Lyapunov functions, we establish that when the threshold value, denoted as R_0^s, falls below 1, the epidemic is destined for extinction. Conversely, if the reproduction number R_0 of the deterministic model surpasses 1, the model manifests an ergodic endemic stationary distribution. In the numerical simulations and data interpretation section, leveraging a graphical analysis with COVID-19 data, we illustrate that random fluctuations possess the capacity to quell disease outbreaks, underscoring the role of vaccines in curtailing the spread of diseases. This study not only contributes to the understanding of epidemic dynamics but also highlights the pivotal role of stochasticity and vaccination strategies in epidemic control and management. The inherent balance and patterns observed in epidemic spread and control strategies, reflect a symmetrical interplay between stochasticity, vaccination, and disease dynamics.

Keywords: stochastic SVEIR model; stability analysis; stationary distribution

MSC: 92D30; 60H10

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1. Introduction

Over the past two decades, mathematical modeling has played a crucial role in both preventing and controlling infectious diseases, including severe acute respiratory syndrome (SARS) [1], human immunodeficiency virus infection/acquired immune deficiency syndrome (HIV/AIDS) [2], and H1N1 (swine flu) [3]. These models describe the evolution of various subpopulations over time within epidemic models. One widely used model is the SEIR (susceptible–exposed–infectious–recovered) model, which divides the population into four compartments: susceptible (S), individuals who are at risk of infection; exposed (E), individuals who have come into contact with infective individuals but show no symptoms; infectious (I), individuals displaying symptoms; and recovered (R), individuals who have recovered from the disease [4]. The SEIR model has various complex variants, including those with different control measures such as various incidence rates, constant and feedback vaccination and treatment controls, as well as models involving multiple interconnected regions or towns [5–10], and others cited within. The rate of incidence is widely recognized as playing a significant role in disease modeling, with factors such as population density and lifestyle influencing the increase and decrease in epidemics [11,12]. Many researchers have employed nonlinear incidence rates in their studies; for more in-depth information, readers are referred to [13–19] and related references. Vaccination is widely acknowledged as one of the most effective means of disease control and prevention [20], playing a pivotal role in the complete eradication of diseases like smallpox and partial control of diseases

such as measles [21]. Numerous scholarly works have explored the dynamics of epidemic models with different vaccination schedules [7,13,19,22–26].

In 2018, Gao and Huang [22] conducted a study on the model described below:

$$\begin{cases} \dfrac{dS}{dt} = A - \dfrac{\beta SI}{1+aS+bI} - (\delta_0+\mu)S + \eta V, \\ \dfrac{dE}{dt} = \dfrac{\beta SI}{1+aS+bI} - (\delta_0+\delta_1)E, \\ \dfrac{dI}{dt} = \delta_1 E - (\delta_0+\delta_2+\delta_3)I, \\ \dfrac{dR}{dt} = \delta_2 I - \delta_0 R, \\ \dfrac{dV}{dt} = \mu S - (\delta_0+\eta)V. \end{cases} \tag{1}$$

All the parameters in model (1) are positive. The variables S, E, I, R, and V represent the respective counts of susceptible, exposed, infectious, recovered, and vaccinated individuals at time t. Table 1 provides the biological interpretations of the remaining parameters.

Table 1. Biological interpretations of variables and parameters in model (1).

Parameter	Description
A	The recruitment rate of new individuals
β	The contact rate or the rate of transfer of virus from an infectious individual to the susceptible
δ_0	The natural mortality rate
η	The rate at which the vaccinated individuals lose their immunity and join the susceptible class
μ	The vaccination rate coefficient
δ_1	The rate at which exposed individuals become infectious
δ_2	The recovery rate of the infectious individuals
δ_3	The disease-related death rate of infectious individuals
a	The proportion constant related to susceptible individuals
b	The proportion constant related to infectious individuals

The results in [22] showed that the basic reproduction number of model (1) is

$$R_0 = \frac{\beta\delta_1 S_0}{(1+aS_0)(\delta_0+\delta_1)(\delta_0+\delta_2+\delta_3)}, \text{ where } S_0 = \frac{(\delta_0+\eta)A}{\delta_0^2+(\mu+\eta)\delta_0}. \tag{2}$$

It is proved that if $R_0 < 1$, the disease-free equilibrium is globally asymptotically stable and if $R_0 > 1$, model (1) has an endemic equilibrium $(S^*, E^*, I^*, R^*, V^*)$ which is globally asymptotically stable.

The inherent randomness of epidemic growth and spread, attributed to the unpredictable nature of person-to-person contacts [27] and the susceptibility of populations to various disturbances [28], suggests that stochastic models may offer a more suitable approach for modeling epidemics in many scenarios [29–42]. Stochastic models are particularly beneficial in capturing the random occurrences of infectious contacts during the latent and infectious periods [43]. Many realistic stochastic epidemic models can be derived from their deterministic counterparts. For instance, Cai [38] developed a general SIRS epidemic model with a ratio-dependent incidence rate and its corresponding stochastic differential equation version. Ball and Neal [44] explored a general stochastic SIR model within a closed finite population, deriving a threshold parameter that determines the possibility of global epidemics. Yang et al. [45] examined the ergodicity and extinction of stochastic SIR and SEIR epidemic models with saturated incidence. Zhang and Zhang [42] investigated the threshold behavior of a deterministic and a stochastic SIQS epidemic model by considering varying total population sizes.

In its investigation of the stochastic epidemic model, our analysis not only sheds light on the dynamics of disease transmission and control but also unravels the symmetrical aspects inherent in epidemic behavior. By exploring the effects of stochasticity and vaccination strategies on disease spread, we can uncover a symmetrical relationship between these factors and the patterns observed in epidemic outcomes.

This paper focuses on a stochastic SVEIR epidemic model with a nonlinear incidence rate.

$$\begin{cases} dS = \left(A - \dfrac{\beta SI}{1+aS+bI} - (\delta_0+\mu)S + \eta V\right)dt + \sigma_1 S dB_1(t), \\ dE = \left(\dfrac{\beta SI}{1+aS+bI} - (\delta_0+\delta_1)E\right)dt + \sigma_2 E dB_2(t), \\ dI = (\delta_1 E - (\delta_0+\delta_2+\delta_3)I)dt + \sigma_3 I dB_3(t), \\ dR = (\delta_2 I - \delta_0 R) + \sigma_4 R dB_4(t), \\ dV = (\mu S - (\delta_0+\eta)V) + \sigma_5 dB_5(t). \end{cases} \tag{3}$$

The parameters in model (1) remain unchanged, with $\sigma_i^2 > 0$ denoting the intensities of white noise. Additionally, $B_i(t)$ $(i = 1, 2, \ldots, 5)$ represents independent standard Brownian motions, initialized at $B_i(0) = 0$.

In the study of epidemic model behavior, analyzing steady states and their stability is crucial [19]. In deterministic models, this analysis involves examining the stability of the disease-free equilibrium (or endemic equilibrium) through the basic reproduction number R_0. However, in the case of the stochastic model (3), there is no endemic equilibrium. Nevertheless, Khasminskii [46] demonstrated that the presence of an ergodic stationary distribution for model (3) can reveal the persistence of the infection.

The primary objective of this paper is to examine the extinction and the existence of an ergodic stationary distribution for model (3). We establish sufficient conditions for both the extinction and the existence of an ergodic stationary distribution.

The paper is organized as follows: Section 2 introduces the necessary preliminaries for subsequent analysis. Section 3 establishes sufficient conditions for both the extinction and the existence of an ergodic stationary distribution in model (3). Section 4 presents numerical simulations based on published data from COVID-19 to support our findings. Lastly, some concluding remarks summarize the results.

2. Preliminaries

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions, where it is increasing and right-continuous, and $\{\mathscr{F}_0\}$ contains all P-null sets. Additionally, define $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_i > 0, 1 \leq i \leq d\}$.

We consider the d-dimensional stochastic differential equation (SDE) in general:

$$dx(t) = f(t, x(t))dt + g(t, x(t))dB_t, \tag{4}$$

here$f(t, x(t))$ is a function in $\mathbb{R}^d$ defined in $[t_0, \infty] \times \mathbb{R}^d$, $g(t, x(t))$ is a $d \times m$ matrix, and f and g are local Lipschitz functions in x. The term B_t represents an m-dimensional standard Brownian motion defined on the complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$. The notation $C^{2,1}\left(\mathbb{R}^d \times [t_0, \infty]; \mathbb{R}_+\right)$ refers to the family of all non-negative functions $V(x, t)$ defined on $\mathbb{R}^d \times [t_0, \infty]$ such that they are continuously twice differentiable in x and once in t. The differential operator L of Equation (4) is defined [30] as

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(t) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} \left[g^T(x, t) g(x, t)\right]_{ij} \frac{\partial^2}{\partial x_i \partial x_j}.$$

When the operator L operates on a function $V \in C^{2,1}\left(\mathbb{R}^d \times [t_0, \infty]; \mathbb{R}_+\right)$,

$$LV(x,t) = V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}trace\left[g^T(x,t)V_{xx}g(x,t)\right],$$

where $V_t(x,t) = \frac{\partial V}{\partial t}$, $V_x(x,t) = \left(\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_d}\right)$, and $V_{xx} = \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_{d\times d}$. By Itô's formula, if $x(t)$ is solution of (4), then $dV(x,t) = LV(x,t)dt + V_x(x,t)g(x,t)dB_t$.

The following lemma pertains to the existence and uniqueness of a global positive solution, which is a prerequisite for investigating the long-term behavior of system (3).

Lemma 1. *Given any initial value $(S(0), E(0), I(0), R(0), V(0)) \in \mathbb{R}^5_+$, there exists a unique solution $X(t) = (S(t), E(t), I(t), R(t), V(t))$ of system (3) on $t \geq 0$ and this solution will almost surely remain within $\mathbb{R}^5_+$.*

Proof. Since the coefficients of the system (3) satisfy the local Lipschitz condition, for any initial value $(S(0), E(0), I(0), R(0), V(0)) \in \mathbb{R}^5_+$ there is a unique local solution in $[0, \tau_e]$, where τ_e denotes the explosion time [30]. To establish global solution existence, it suffices to demonstrate that $\tau_e = \infty$ almost surely (a.s.). To achieve this, let k_0 be sufficiently large such that every component of X_0 lies within the interval $[\frac{1}{k_0}, k_0]$. For each integer $k > k_0$, define the stopping time as follows:

$$\tau_k = inf\left\{t \in [0, \tau_e) : minX(t) \leq \frac{1}{k} \text{ or } maxX(t) \geq k\right\}.$$

In this paper, we define $inf\emptyset = \infty$ (where $\emptyset$ denotes the empty set). It is evident that τ_k is increasing as $k \to \infty$. Let $\tau_\infty = \lim_{k\to\infty} \tau_k$, hence $\tau_\infty \leq \tau_e$ almost surely. If we can prove that $\tau_\infty = \infty$ almost surely, then $\tau_e = \infty$ ensuring $X(t) \in \mathbb{R}^5_+$ almost surely. for all $t \geq 0$. Therefore, demonstrating $\tau_\infty = \infty$ almost surely is crucial for completing the proof. If this assertion is contradicted there exit constants $T > 0$ and $\eta \in (0,1)$ such that $P\{\tau_\infty \leq T\} > \eta$. Consequently, there exists an integer $k_1 \geq k_0$ such that $P\{\tau_k \leq T\} > \eta$ for all $k \geq k_1$.

Define a $C^2 - function$ W: $\mathbb{R}^5_+ \to \mathbb{R}^1_+$

$$\begin{aligned} & W(S,E,I,R,V) \\ = \quad & (S - 1 - lnS) + (E - 1 - lnE) + (I - 1 - lnI) + (R - 1 - lnR) + (V - 1 - lnV). \end{aligned}$$

The non-negativity of this function follows from the inequality $u - 1 - lnu \geq 0$ *for any* $u > 0$.

Using Itô's formula, we have

$$\begin{aligned} dW(S,E,I,R,V) =& LWdt + \sigma_1(S-1)dB_1(t) + \sigma_2(E-1)dB_2(t) + \sigma_3(I-1)dB_3(t) \\ & + \sigma_4(R-1)dB_4(t) + \sigma_5(V-1)dB_5(t), \end{aligned}$$

where

$$\begin{aligned} LW =& \left(1 - \frac{1}{S}\right)\left(A - \frac{\beta SI}{1 + aS + bI} - (\delta_0 + \mu)S + \eta V\right) + \left(1 - \frac{1}{E}\right)\left(\frac{\beta SI}{1 + aS + bI} - (\delta_0 + \delta_1)E\right) \\ & + \left(1 - \frac{1}{I}\right)(\delta_1 E - (\delta_0 + \delta_2 + \delta_3)I) + \left(1 - \frac{1}{R}\right)(\delta_2 I - \delta_0 R) + \left(1 - \frac{1}{V}\right)(\mu S - (\delta_0 + \eta)V) \\ & + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2) \\ =& A + 5\delta_0 + \delta_1 + \delta_2 + \delta_3 + \eta + \mu + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2) + \frac{\beta I}{1 + aS + bI} \\ & - \frac{\beta SI}{(1 + aS + bI)E} - \delta_3 I - \frac{A}{S} - \frac{\eta V}{S} - \frac{\delta_1 E}{I} - \frac{\delta_2 I}{R} - \frac{\mu S}{V} - \delta_0(S + E + I + R + V) \\ \leq& A + 5\delta_0 + \delta_1 + \delta_2 + \delta_3 + \eta + \mu + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2) + \frac{\beta I}{1 + aS + bI}. \end{aligned}$$

If $aS+1 \geq I$, then $\frac{\beta I}{1+aS+bI} = \frac{\beta}{\frac{aS+1}{I}+b} \leq \frac{\beta}{1+b}$. While if $aS+1 < I$, $0 < \frac{aS+1}{I} < 1$, and we can obtain that $\frac{\beta I}{1+aS+bI} < \frac{\beta}{b}$. Therefore, there exists a fixed positive constant F which is independent of $S(t)$, $E(t)$, $I(t)$, $R(t)$, $V(t)$, and t. Thus we obtain

$$\begin{aligned} dW \leq & Fdt + \sigma_1(S-1)dB_1(t) + \sigma_2(E-1)dB_2(t) + \sigma_3(I-1)dB_3(t) \\ & + \sigma_4(R-1)dB_4(t) + \sigma_5(V-1)dB_5(t). \end{aligned}$$

By integrating both sides from 0 to $\tau_k \wedge T$, and taking expectations, we obtain

$$EW \leq W(S(0), E(0), I(0), R(0), V(0)) + FT.$$

For any positive $k \geq k_1$, let us define $\Omega_k = \{\tau_k < T\}$, leading to $P(\Omega_k) > \frac{\eta}{2}$. Note that for every $v \in \Omega_k$, at least one of the values in the set $(S(v), E(v), I(v), R(v), V(v))$ is either $\frac{1}{k}$ or k. Therefore, we have $W \geq \frac{1}{k} - 1 - ln\frac{1}{k}$, or $W \geq k - 1 - lnk$. So we obtain

$$\begin{aligned} W(S(0), E(0), I(0), R(0), V(0)) + FT \geq & E[I_{\Omega_k} W(S(t), E(t), I(t), R(t), V(t))] \\ = & P(\Omega_k) W(S(t), E(t), I(t), R(t), V(t)) \\ > & \frac{\eta}{2}\left[\left(\frac{1}{k} - 1 - ln\frac{1}{k}\right) \wedge (k - 1 - lnk)\right], \end{aligned}$$

where I_{Ω_k} is the indicator function of Ω_k. Setting $k \to \infty$, we have

$$\infty > W(S(0), E(0), I_1(0), I_2(0), R(0)) + FT = \infty.$$

This completes the proof. □

The following lemma can be proven by employing the same arguments presented in lemma 3.1 of [47]; hence, we omit its detailed proof here.

Lemma 2. *Let $(S(t), E(t), I(t), R(t), V(t))$ be any solution of system (3) with any initial value. Assume that $\mu > \frac{\sigma_{max}^2}{2}$, then*

$$\begin{aligned} (i) & \lim_{t\to\infty} \frac{S(t)}{t} = \lim_{t\to\infty} \frac{E(t)}{t} = \lim_{t\to\infty} \frac{I(t)}{t} = \lim_{t\to\infty} \frac{R(t)}{t} = \lim_{t\to\infty} \frac{V(t)}{t} = 0 \ a.s. \ Moreover, \\ & \lim_{t\to\infty} \frac{lnS(t)}{t} = \lim_{t\to\infty} \frac{lnE(t)}{t} = \lim_{t\to\infty} \frac{lnI(t)}{t} = \lim_{t\to\infty} \frac{lnR(t)}{t} = \lim_{t\to\infty} \frac{lnV(t)}{t} = 0 \ a.s.; \end{aligned}$$

$$\begin{aligned} (ii) & \lim_{t\to\infty} \frac{1}{t}\int_0^t S(u)dB_1(u) = \lim_{t\to\infty} \frac{1}{t}\int_0^t E(u)dB_2(u) = \lim_{t\to\infty} \frac{1}{t}\int_0^t I(u)dB_3(u) \\ & = \lim_{t\to\infty} \frac{1}{t}\int_0^t R(u)dB_4(u) = \lim_{t\to\infty} \frac{1}{t}\int_0^t V(u)dB_5(u) = 0 \ a.s., \\ & where \ \sigma_{max} = \sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2. \end{aligned}$$

Consider a regular time-homogeneous Markov process $X(t)$ in $\mathbb{R}_+^d$, characterized by the following:

$$dX(t) = b(X)dt + \sum_{r=1}^{k} \sigma_r(X)dB_r(t),$$

where the diffusion matrix is given by

$$A(X) = (a_{ij}(x)), \ a_{ij}(x) = \sum_{r=1}^{k} \sigma_r^i(x)\sigma_r^j(x).$$

The lemma below is crucial for proving the existence of a stationary distribution.

Lemma 3 ([46]). *The Markov process $X(t)$ possesses a unique ergodic stationary distribution $m(\cdot)$ if there exists a bounded domain $U \in \mathbb{R}^d$ with a smooth boundary such that its closure $\tilde{U} \subset \mathbb{R}^d$ satisfies the following properties:*

(i) Within the open domain U and its neighborhood, the minimum eigenvalue of the diffusion matrix $A(t)$ remains bounded away from zero.

(ii) For any $x \in \mathbb{R}^d \setminus U$, the average time τ for a trajectory originating from x to each the set U is finite. Additionally, $\sup_{x\in K} E^x\tau < \infty$ for every compact $K \subset \mathbb{R}^d$.

Furthermore, if $f(\cdot)$ is a function integrable with respect to measure m, then

$$P\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X^x(t))dt = \int_{\mathbb{R}^d} f(X^x(t))m(dx)\right) = 1,$$

for every $x \in \mathbb{R}^d$.

Remark 1. *To establish condition (i) it is adequate to demonstrate that F exhibits uniform ellipticity in U. Here, $F(u) = b(x)u_x + \frac{1}{2}trace(A(x)u_{xx})$. This implies the existence of a positive constant M such that $\sum_{i,j=1}^{d} a_{ij}(x)\zeta_i\zeta_j \geq M|\zeta|^2$, $x \in U$, $\zeta \in \mathbb{R}_d$ [29,48]. To confirm condition (ii), it is sufficient to establish the existence of a non-negative C^2-function V and a neighborhood U such that for some $\mathcal{K} > 0$, $LV<-\mathcal{K}$, $x \in \mathbb{R}^d \setminus U$ [49].*

3. Theoretical Analysis

Stochastic Lyapunov functions play a crucial role in understanding the long-term behavior of stochastic systems and assessing the impact of randomness on system dynamics. However, the derivation and implementation of stochastic Lyapunov functions involve a rigorous mathematical process to analyze the stability of stochastic systems. In the following discussion, we shall construct suitable stochastic Lyapunov functions to obtain the extinction and stationary distribution and ergodicity results for the stochastic system.

3.1. Extinction of the Stochastic Epidemic Model (3)

In this subsection, we will explore the sufficient criteria for the extinction of the infected individuals. Set

$$\mathcal{R}_0^s \doteq \frac{2\beta\delta_1(\delta_0+\delta_1)}{a[(\delta_0+\delta_1)^2(\delta_0+\delta_2+\delta_3+\frac{1}{2}\sigma_3^2)\wedge\frac{1}{2}\delta_1^2\sigma_2^2]}.$$

Theorem 1. *Let $(S(t), E(t), I(t), R(t), V(t))$ be the solution of model (3) with any initial value $(S(0), E(0), I(0), R(0), V(0)) \in \mathbb{R}^5_+$. If $\mathcal{R}_0^s < 1$ and $\delta_0 > \frac{\sigma_{max}^2}{2}$, then the solution of model (3) satisfies*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t S(u)du = \frac{A(\delta_0+\eta)}{\delta_0^2+\eta\delta_0+\mu\delta_0}, \qquad \lim_{t\to\infty}\frac{1}{t}\int_0^t V(u)du = \frac{A\mu}{\delta_0^2+\eta\delta_0+\mu\delta_0},$$
$$\lim_{t\to\infty}\frac{1}{t}\int_0^t E(u)du = \lim_{t\to\infty}\frac{1}{t}\int_0^t I(u)du = \lim_{t\to\infty}\frac{1}{t}\int_0^t R(u)du = 0.$$

Proof. Let $Q(t) = \delta_1 E(t) + (\delta_0+\delta_1)I(t)$. By applying Itô′s formula, we obtain

$$\begin{aligned} d\ln Q(t) =& \left[\frac{\frac{\delta_1\beta SI}{1+aS+bI} - (\delta_0+\delta_1)(\delta_0+\delta_2+\delta_3)I}{\delta_1 E+(\delta_0+\delta_1)I} - \frac{\delta_1^2\sigma_2^2E^2+(\delta_0+\delta_1)^2\sigma_3^2I^2}{2[\delta_1 E+(\delta_0+\delta_1)^2 I]^2}\right]dt \\ &+ \frac{\delta_1\sigma_2 E}{\delta_1 E+(\delta_0+\delta_1)I}dB_2(t) + \frac{(\delta_0+\delta_1)\sigma_3 I}{\delta_1 E+(\delta_0+\delta_1)I}dB_3(t) \\ \leq& \frac{\delta_1\beta}{a(\delta_0+\delta_1)}dt - \frac{1}{[\delta_1 E+(\delta_0+\delta_1)I]^2}\left[((\delta_0+\delta_1)^2(\delta_0+\delta_2+\delta_3)+\frac{1}{2}(\delta_0+\delta_1)^2\sigma_3^2)I^2\right. \\ &\left.+\frac{1}{2}\delta_1^2\sigma_2^2E^2\right]dt + \frac{\delta_1\sigma_2 E}{\delta_1 E+(\delta_0+\delta_1)I}dB_2(t) + \frac{(\delta_0+\delta_1)\sigma_3 I}{\delta_1 E+(\delta_0+\delta_1)I}dB_3(t) \\ \leq& \frac{\delta_1\beta}{a(\delta_0+\delta_1)}dt - \frac{1}{2(\delta_0+\delta_1)^2}\left[(\delta_0+\delta_1)^2(\delta_0+\delta_2+\delta_3+\frac{1}{2}\sigma_3^2)\wedge\frac{1}{2}\delta_1^2\sigma_2^2\right]dt \\ &+ \frac{\delta_1\sigma_2 E}{\delta_1 E+(\delta_0+\delta_1)I}dB_2(t) + \frac{(\delta_0+\delta_1)\sigma_3 I}{\delta_1 E+(\delta_0+\delta_1)I}dB_3(t). \end{aligned} \tag{5}$$

Integrating (5) from 0 to t, combining with Lemma 2, and noting $\mathcal{R}_0^s < 1$, we have

$$\lim_{t\to\infty} sup\frac{lnQ(t)}{t} \leq \frac{\delta_1\beta}{a(\delta_0+\delta_1)} - \frac{1}{2(\delta_0+\delta_1)^2}\left[(\delta_0+\delta_1)^2(\delta_0+\delta_2+\delta_3+\frac{1}{2}\sigma_3^2)\wedge\frac{1}{2}\delta_1^2\sigma_2^2\right] < 0 \quad a.s.,$$

which implies that

$$\lim_{t\to\infty} E(t) = 0, \quad \lim_{t\to\infty} I(t) = 0 \quad a.s..$$

In other words, the susceptible individuals $E(t)$ and the infectious individuals $I(t)$ will both exponentially approach zero with probability one. From model (3), it is evident that $\lim_{t\to\infty} R(t) = 0$ a.s. This implies that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t E(u)du = \lim_{t\to\infty}\frac{1}{t}\int_0^t I(u)du = \lim_{t\to\infty}\frac{1}{t}\int_0^t R(u)du = 0 \ \ a.s.$$

On the other hand, according to model (3), we have

$$dS(t) = \left(A - \frac{\beta SI}{1+aS+bI} - (\delta_0+\mu)S + \eta V\right)dt + \sigma_1 S(t)dB_1(t) \tag{6}$$

Integrating (6) from 0 to t on both sides, we can derive

$$S(t) - S(0) = At - \int_0^t \frac{\beta S(u)I(u)}{1+aS(u)+bI(u)}du - \int_0^t(\delta_0+\mu)S(u)du + \int_0^t \eta V(u)du + \int_0^t \sigma_1 S(u)dB_1(u).$$

And then, we can obtain

$$(\delta_0+\mu)\int_0^t S(u)du \leq At - \int_0^t(\delta_0+\mu)S(u)du + \int_0^t \eta V(u)du + \int_0^t \sigma_1 S(u)dB_1(u). \tag{7}$$

In the same way, we can obtain

$$(\delta_0+\eta)\int_0^t V(u)du \leq \int_0^t \mu S(u)du + \int_0^t \sigma_5 V(u)dB_5(u). \tag{8}$$

According to (7) and (8), we can obtain

$$(\delta_0+\mu-\frac{\eta\mu}{\delta_0+\eta})\int_0^t S(u)du \leq At + \frac{\eta\sigma_5}{\delta_0+\eta}\int_0^t V(u)dB_5(u) + \int_0^t \sigma_1 S(u)dB_1(u). \tag{9}$$

By dividing both sides of (9) by t, taking the limit superior, and combining this result with Lemma 2, one can deduce that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t S(u)du = \frac{A(\delta_0+\eta)}{\delta_0^2+\eta\delta_0+\mu\delta_0}\ a.s.$$

And similarly, we can obtain

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t V(u)du = \frac{A\mu}{\delta_0^2+\eta\delta_0+\mu\delta_0}\ a.s.$$

This completes the proof. □

3.2. Stationary Distribution and Ergodicity of the Stochastic Model (3)

Theorem 2. *Assume $R_0 > 1$, where R_0 is defined in (2). If the following conditions are satisfied:*
(i)

$$\begin{aligned}
m_1 &= 2\delta_0 - \sigma_1^2 > 0,\\
m_2 &= 2\delta_0 - \sigma_2^2 > 0,\\
m_3 &= \frac{4\delta_1(\delta_0+\delta_3)+2(2\delta_0+\delta_3(\delta_0+\delta_2+\delta_3))-(2\delta_0+2\delta_1+\delta_3)\sigma_3^2}{2\delta_1} > 0,\\
m_4 &= \frac{4\delta_0\delta_2+2\delta_0(2\delta_0+\delta_3)-(2\delta_0+2\delta_2+\delta_3)\sigma_4^2}{2\delta_2} > 0,\\
m_5 &= \frac{2\delta_0\mu+2\delta_0(\delta_0+\eta)-(\mu+\delta_0)\sigma_5^2}{\mu} > 0;
\end{aligned}$$

(ii)

$$0 < F < \min\left(m_1\theta_1^2S^{*2}, m_2\theta_2^2E^{*2}, m_3\theta_3^2I^{*2}, m_4\theta_4^2R^{*2}, m_5\theta_5^2V^{*2}\right),$$

where

$$\begin{aligned}
F =& \frac{2\delta_0\sigma_1^2S^{*2}}{2\delta_0-\sigma_1^2} + \frac{2\delta_0\sigma_2^2E^{*2}}{2\delta_0-\sigma_2^2} + \frac{[4\delta_1(\delta_0+\delta_3)+2(2\delta_0+\delta_3)(\delta_0+\delta_2+\delta_3)](2\delta_0+2\delta_1+\delta_3)\sigma_3^2I^{*2}}{\delta_1[4\delta_1(\delta_0+\delta_3)+2(2\delta_0+\delta_3)(\delta_0+\delta_2+\delta_3)-(2\delta_0+2\delta_1+\delta_3)\sigma_3^2]}\\
&+ \frac{[4\delta_0\delta_2+2\delta_0(2\delta_0+\delta_3)](2\delta_0+2\delta_2+\delta_3)\sigma_4^2R^{*2}}{\delta_2(4\delta_0\delta_2+2\delta_0(2\delta_0+\delta_3)-(2\delta_0+2\delta_2+\delta_3)\sigma_4^2)} + \frac{2\delta_0(\mu+\eta+\delta_0)(\delta_0+\mu)\sigma_5^2V^{*2}}{\mu[2\delta_0(\mu+\eta+\delta_0)-(\delta_0+\mu)\sigma_5^2]},
\end{aligned}$$

$$\begin{aligned}
\theta_1 &= \frac{2\delta_0}{2\delta_0-\sigma_1^2},\ \theta_2 = \frac{2\delta_0}{2\delta_0-\sigma_2^2},\\
\theta_3 &= \frac{4\delta_1(\delta_0+\delta_3)+2(2\delta_0+\delta_3)(\delta_0+\delta_2+\delta_3)}{4\delta_1(\delta_0+\delta_3)+2(2\delta_0+\delta_3)(\delta_0+\delta_2+\delta_3)-(2\delta_0+2\delta_1+\delta_3)\sigma_3^2},\\
\theta_4 &= \frac{4\delta_0\delta_2+2\delta_0(2\delta_0+\delta_3)}{4\delta_0\delta_2+2\delta_0(2\delta_0+\delta_3)-(2\delta_0+2\delta_2+\delta_3)\sigma_4^2},\ \theta_5 = \frac{2\delta_0(\mu+\eta+\delta_0)}{2\delta_0(\mu+\eta+\delta_0)-(\delta_0+\mu)\sigma_5^2}
\end{aligned}$$

and S^*, E^*, I^*, R^*, V^* are defined as the endemic equilibrium of the deterministic model (1), then there exists a unique stationary distribution $\pi(\cdot)$ for the stochastic model (3) with any initial value $X_0 \in \mathbb{R}_+^5$ and it has the ergodic property.

Proof. Since $R_0 > 1$, there is an endemic equilibrium $P* = (S^*, E^*, I^*, R^*, V^*)$ of the deterministic model (1). Then, we have

$$\begin{aligned} &A = \frac{\beta S^* I^*}{1 + aS^* + bI^*} + (\delta_0 + \mu)S^* - \eta V^*, \quad \frac{\beta S^* I^*}{1 + aS^* + bI^*} = (\delta_0 + \delta_1)E^*, \\ &\delta_1 E^* = (\delta_0 + \delta_2 + \delta_3)I^*, \quad \delta_2 I^* = \delta_0 R^*, \qquad \mu S^* = (\delta_0 + \eta)V^*. \end{aligned} \tag{10}$$

Define

$$\mathcal{V}(X) = \mathcal{V}_1(X) + \frac{2\delta_0 + \delta_3}{2\delta_1}\mathcal{V}_2(X) + \frac{2\delta_0 + \delta_3}{2\delta_2}\mathcal{V}_3(X) + \frac{\delta_0}{\mu}\mathcal{V}_4(X), \tag{11}$$

where

$$\begin{aligned} &\mathcal{V}_1(X) = (S - S^* + E - E^* + I - I^* + R - R^* + V - V^*)^2, \\ &\mathcal{V}_2(X) = (I - I^*)^2, \quad \mathcal{V}_3(X) = (R - R^*)^2, \quad \mathcal{V}_4(X) = (V - V^*)^2. \end{aligned}$$

Then, $\mathcal{V}$ is positive definite and $\lim_{|X|\to\infty} \mathcal{V}(X) = \infty$. By virtue of (10), we can obtain

$$\begin{aligned} L\mathcal{V}_1 =& 2[(S - S^*) + (E - E^*) + (I - I^*) + (R - R^*) + (V - V^*)][A - \delta_0(S + E + I + R + V) \\ &- \delta_0 I] + \sigma_1^2 S^2 + \sigma_2^2 E^2 + \sigma_3^2 I^2 + \sigma_4^2 R^2 + \sigma_5^2 V^2 \\ =& 2[(S - S^*) + (E - E^*) + (I - I^*) + (R - R^*) + (V - V^*)][-\delta_0(S - S^*) - \delta_0(E - E^*) \\ &- (\delta_0 + \delta_3)(I - I^*) - \delta_0(R - R^*) - \delta_0(V - V^*)] + \sigma_1^2 S^2 + \sigma_2^2 E^2 + \sigma_3^2 I^2 + \sigma_4^2 R^2 + \sigma_5^2 V^2 \\ \leq& 2[-\delta_0(S - S^*)^2 - \delta_0(E - E^*)^2 - (\delta_0 + \delta_3)(I - I^*)^2 - \delta_0(R - R^*)^2 - \delta_0(V - V^*)^2 \\ &- (2\delta_0 + \delta_3)(E - E^*)(I - I^*) - (2\delta_0 + \delta_3)(I - I^*)(R - R^*) - 2\delta_0(S - S^*)(V - V^*)] \\ &+ \sigma_1^2 S^2 + \sigma_2^2 E^2 + \sigma_3^2 I^2 + \sigma_4^2 R^2 + \sigma_5^2 V^2, \end{aligned}$$

$$\begin{aligned} L\mathcal{V}_2 &= 2(I - I^*)[\delta_1(E - E^*) - (\delta_0 + \delta_2 + \delta_3)(I - I^*)] + \sigma_3^2 I^2 \\ &= -2(\delta_0 + \delta_2 + \delta_3)(I - I^*)^2 + 2\delta_1(E - E^*)(I - I^*) + \sigma_3^2 I^2, \end{aligned}$$

$$\begin{aligned} L\mathcal{V}_3 &= 2(R - R^*)[\delta_2(I - I^*) - \delta_0(R - R^*)] + \sigma_4^2 R^2 \\ &= -2\delta_0(R - R^*)^2 + 2\delta_2(I - I^*)(R - R^*) + \sigma_4^2 R^2, \end{aligned}$$

$$\begin{aligned} L\mathcal{V}_4 &= 2(V - V^*)[\mu(S - S^*) - (\delta_0 + \eta)(V - V^*)] + \sigma_5^2 V^2 \\ &= -2(\delta_0 + \eta)(V - V^*)^2 + 2\mu(S - S^*)(V - V^*) + \sigma_5^2 V^2. \end{aligned}$$

This implies that

$$
\begin{aligned}
L\mathcal{V} =& L\mathcal{V}_1 + \frac{2\delta_0+\delta_3}{2\delta_1}L\mathcal{V}_2 + \frac{2\delta_0+\delta_3}{2\delta_2}L\mathcal{V}_3 + \frac{\delta_0}{\mu}L\mathcal{V}_4 \\
\leq& -2\delta_0(S-S^*)^2 - 2\delta_0(E-E^*)^2 - \left[2(\delta_0+\delta_3) + \frac{(2\delta_0+\delta_3)(\delta_0+\delta_2+\delta_3)}{\delta_1}\right](I-I^*)^2 \\
& - 2\left[2\delta_0 + \frac{\delta_0(2\delta_0+\delta_3)}{\delta_2}\right](R-R^*)^2 - \left[2\delta_0 + \frac{2\delta_0(\delta_0+\eta)}{\mu}\right](V-V^*)^2 + \sigma_1^2S^2 + \sigma_2^2E^2 \\
& + \frac{(2\delta_0+2\delta_1+\delta_3)\sigma_3^2}{2\delta_1}I^2 + \frac{(2\delta_0+2\delta_2+\delta_3)\sigma_4^2}{2\delta_2}R^2 + \frac{(\delta_0+\mu)\sigma_5^2}{\mu}V^2 \\
=& -(2\delta_0-\sigma_1^2)\left(S - \frac{2\delta_0}{2\delta_0-\sigma_1^2}S^*\right)^2 - (2\delta_0-\sigma_2^2)\left(E - \frac{2\delta_0}{2\delta_0-\sigma_2^2}E^*\right)^2 \\
& - \left(\frac{2\delta_1(\delta_0+\delta_3)}{\delta_1} + \frac{(2\delta_0+\delta_3)(\delta_0+\delta_2+\delta_3)}{\delta_1} - \frac{2\delta_0+\delta_2+\delta_3}{2\delta_1}\sigma_3^2\right)\times \\
& \left(I - \frac{2\delta_1[2\delta_1(\delta_0+\delta_3)+(2\delta_0+\delta_3)(\delta_0+\delta_2+\delta_3)]}{4\delta_1(\delta_0+\delta_3)+2(2\delta_0+\delta_3)(\delta_0+\delta_2+\delta_3)-(2\delta_0+2\delta_1+\delta_3)\sigma_3^2}I^*\right)^2 \\
& - \left(\frac{2\delta_0\delta_2}{\delta_2} + \frac{2\delta_0(2\delta_0+\delta_3)-(2\delta_0+2\delta_2+\delta_3)}{2\delta_2}\right)\times \\
& \left(R - \frac{4\delta_0\delta_2+2\delta_0(2\delta_0+\delta_3)}{4\delta_0\delta_2+2\delta_0(2\delta_0+\delta_3)-(2\delta_0+2\delta_2+\delta_3)\sigma_4^2}R^*\right)^2 \\
& - \left(\frac{2\delta_0\mu+2\delta_0(\delta_0+\eta)-(\mu+\delta_0)\sigma_5^2}{\mu}\right)\left(V - \frac{2\delta_0(\mu+\eta+\delta_0)}{2\delta_0(\mu+\eta+\delta_0)-(\delta_0+\mu)\sigma_5^2}V^*\right)^2 + F \\
=& -m_1(S-\theta_1S^*)^2 - m_2(E-\theta_2E^*)^2 - m_3(I-\theta_3I^*)^2 - m_4(R-\theta_4R^*)^2 \\
& - m_5(V-\theta_5V^*)^2 + F
\end{aligned}
$$

If F satisfies the following criteria

$$0 < F < \min\left(m_1\theta_1^2S^{*2}, m_2\theta_2^2E^{*2}, m_3\theta_3^2I^{*2}, m_4\theta_4^2R^{*2}, m_5\theta_5^2V^{*2}\right),$$

then the ellipsoid

$$m_1(S-\theta_1S^*)^2 + m_2(E-\theta_2E^*)^2 + m_3(I-\theta_3I^*)^2 + m_4(R-\theta_4R^*)^2 + m_5(V-\theta_5V^*)^2 = F$$

is completely contained within $\mathbb{R}^5_+$. One can choose U to be any neighborhood of the ellipsoid such that $\bar{U} \subset \mathbb{R}^5_+$, where $\bar{U}$ is the closure of U. Thus, we get $L\mathcal{V}(X) < -1$ for $X \in \mathbb{R}^5_+ \setminus U$, thus indicating the condition in Lemma 3 hold. Therefore, the stochastic model (3) possesses a stationary distribution and exhibits ergodicity. □

4. Numerical Simulations and Data Interpretation

Since the emergence of COVID-19 in late December 2019, the pandemic remains a substantial concern. However, the situation appears to have improved with the availability of vaccines. The purpose of this section is to provide a numerical example illustrating our main results, and offer recommendations for controlling pandemic based on data from COVID-19.

4.1. Model Validation

Based on official data and the research of various scholars, we have conducted a simple data analysis to obtain important parameters, presented in Table 2.

Table 2. Relevant variables and parameter values.

Parameter	Value	Reference
A	100,000	Estimate
a	0.5	[50]
b	0.5	[50]
δ_0	7.14×10^{-3}	[51]
μ	0.4	Estimate
η	0.3	[52]
δ_1	0.2	[53]
δ_2	0.02	[53]
δ_3	0.2	Estimate
σ_1	0.06	Estimate
σ_2	0.115	Estimate
σ_3	0.03	Estimate
σ_4	0.008	Estimate
σ_5	0.008	Estimate
$S(0)$	11,081,000	[53]
$E(0)$	600	[53]
$I(0)$	410	[53]
$R(0)$	30	[53]
$V(0)$	0	[53]

We set $\beta = 1.398 \times 10^{-3}$, and obtain $\mathcal{R}_0^s = 0.8759 < 1$ and $\delta_0 = 7.14 \times 10^{-3} > \frac{\sigma_{max}^2}{2} = 6.6125 \times 10^{-3}$ by using these parameter values. The dynamics of model (3) is presented in Figure 1. It shows that the disease will become extinct, which supports the results stated in Theorem 1.

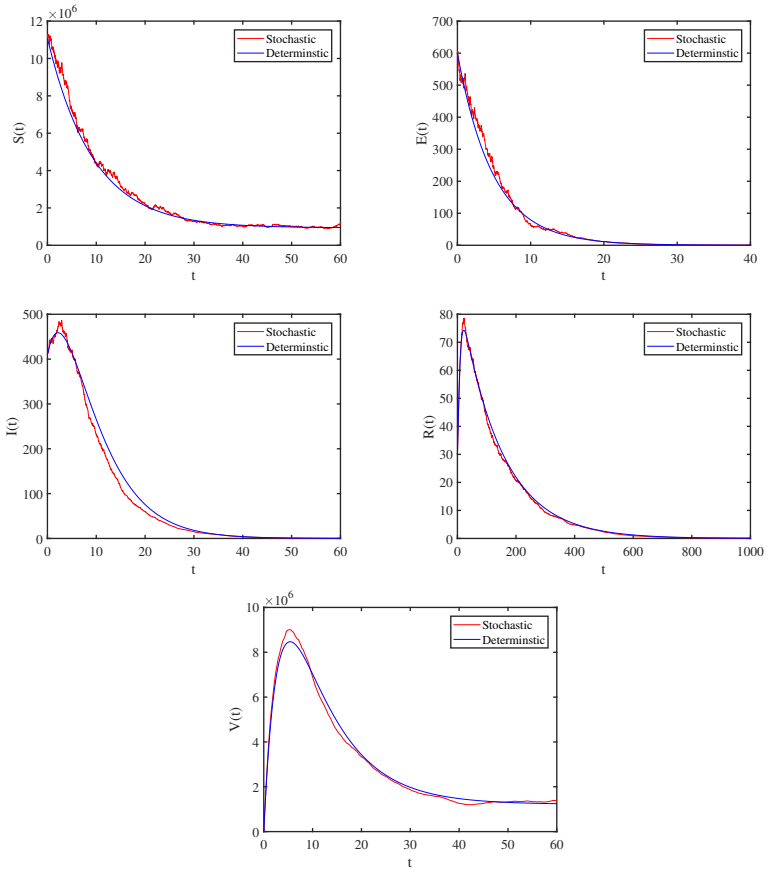

Figure 1. The spread of COVID-19 when $\mathcal{R}_0^s < 1$.

We set $\beta = 0.1898$ and $\sigma_2 = 0.07$, and the other parameter values used are those given in Table 2. Then, we obtain $R_0 = 2.1929 > 1$, and $m_1 = 0.0107$, $m_2 = 0.0094$, $m_3 = 0.5098$, $m_4 = 0.131$, $m_5 = 0.0252$, $0 < F = 1.9007 \times 10^9 < \min\left(m_1\theta_1^2 S^{*2}, m_2\theta_2^2 E^{*2}, m_3\theta_3^2 I^{*2}, m_4\theta_4^2 R^{*2}, m_5\theta_5^2 V^{*2}\right) = 1.9410 \times 10^9$. The dynamics of model (3) is presented in Figure 2. This indicates that the disease will prevail by our analytical results stated in Theorem 2.

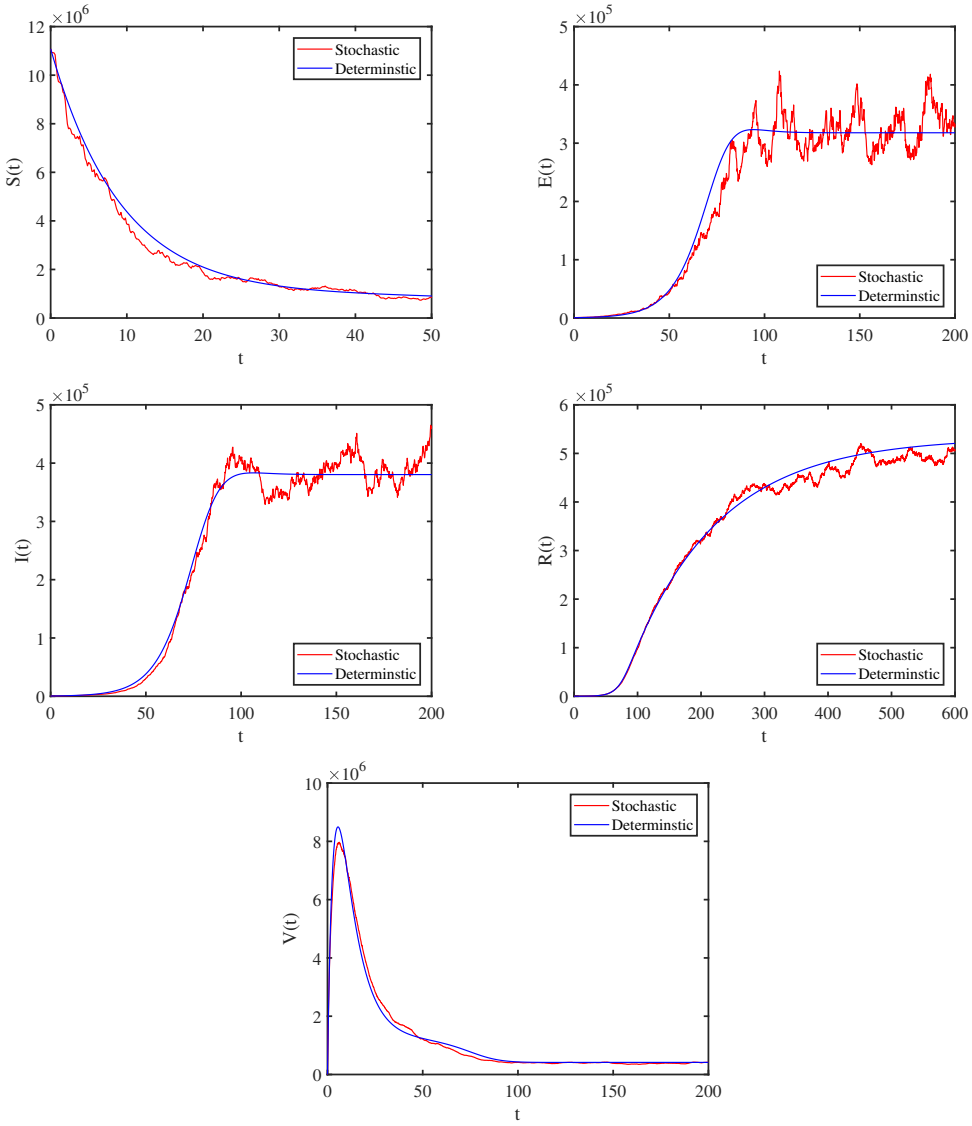

Figure 2. The spread of COVID-19 when $\mathcal{R}_0 > 1$.

4.2. Sensitivity Analysis of R_0 and $\mathcal{R}_0^s$

To effectively control infectious diseases, it is crucial to investigate the impact of various factors on disease transmission. Therefore, in this study, we examine the relationship between certain parameters and $\mathcal{R}_0^s$ and R_0, as well as potential measures to mitigate the spread of disease.

Firstly, we demonstrate the influence of the pertinent parameters on the threshold $\mathcal{R}_0^s$, as depicted in Figure 3a,b.

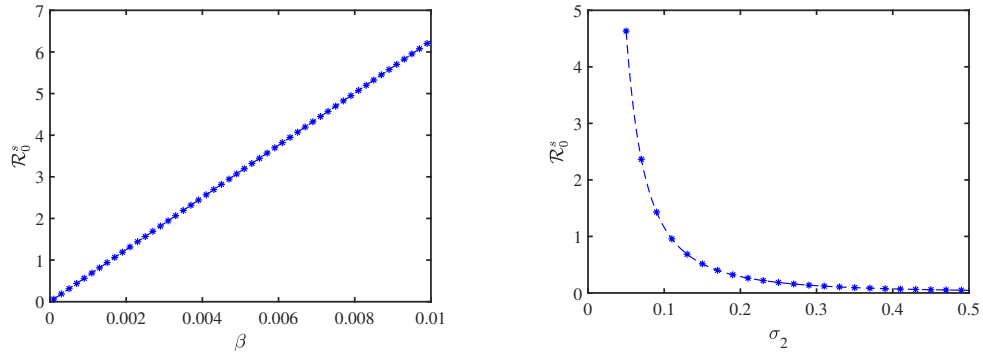

Figure 3. The relationship between $\mathcal{R}_0^s$ and related parameters. (**a**) Relationship between $\mathcal{R}_0^s$ and β. (**b**) Relationship between $\mathcal{R}_0^s$ and σ_2.

Upon setting $\beta \in [10^{-4}, 10^{-2}]$ while keeping the remaining parameter values unchanged from those presented in Table 2, we observe that $\mathcal{R}_0^s$ exhibits a decreasing trend as β decreases (refer to Figure 3a). Similarly, under the constant parameter values specified in Table 2, when σ_2 varies within the range [0, 0.5] and $\beta = 1.398 \times 10^{-3}$, we note that $\mathcal{R}_0^s$ displays a decreasing pattern as σ_2 increases (see Figure 3b). This observation indicates that the introduction of random fluctuations in our stochastic model can effectively suppress disease outbreaks.

We further investigate the relationship between several parameters and the value of R_0, as depicted in Figure 4. According to the definition of R_0 in Equation (2), it is evident that R_0 decreases as β decreases or μ increases. To clearly illustrate the impact of β and μ on R_0, we fix $A = 10$, $\delta_0 = 0.01$, $\delta_1 = 0.1$, $\delta_2 = 0.01$, $\delta_3 = 0.15$, $a = 0.01$, and vary β within the range [0.01, 0.1] and μ within the range [0, 0.8]. Through the analysis of Figure 4, it is evident that by reducing interpersonal contact and promoting vaccination efforts, the spread of the disease can be effectively controlled, aligning with the current strategies implemented in response to the ongoing pandemic.

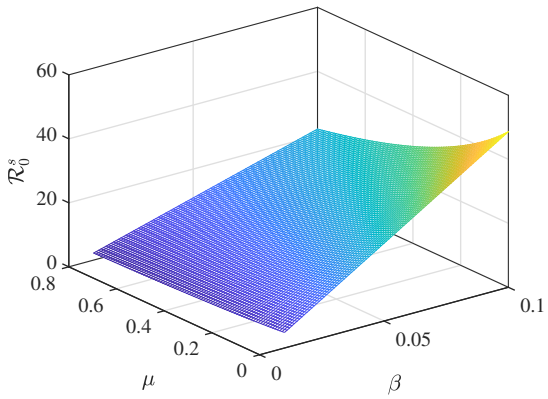

Figure 4. Relationship between $\mathcal{R}_0$ and β, μ.

5. Concluding Remarks

In this paper, we considered a stochastic SVEIR epidemic model with a nonlinear incidence rate. We utilized two key values to determine the system dynamics: one is defined as $\mathcal{R}_0^s$, and the other is the reproduction number R_0 of the corresponding deterministic model. We demonstrated that when $\mathcal{R}_0^s < 1$, the disease will become extinct. On the other hand, if $R_0 > 1$ and the other parameter values satisfy the conditions in Theorem 2, the disease will persist.

We extracted feasible coefficients from published studies on COVID-19 transmission to exemplify our findings. Through our sensitivity analysis, we revealed that the stochastic model, with the introduction of random fluctuations, can effectively mitigate disease outbreaks. Specifically, the contact transmission rate β and the vaccination rate coefficient μ exert substantial influence on the value of $\mathcal{R}_0^s$. These results indicate that reducing interpersonal contact and increasing vaccine usage are effective strategies for controlling epidemic spread.

The utilization of stochastic Lyapunov functions and numerical simulations with COVID-19 data accentuates the symmetrical interplay between random fluctuations, vaccination efficacy, and disease containment. This symmetrical perspective enhances our understanding of epidemic dynamics and underscores the importance of balanced strategies in mitigating disease outbreaks. As such, this study aligns with the principles of symmetry, emphasizing the harmonious interactions and equilibrium present in epidemic modeling and control efforts.

Finally, it should be noted that there are several areas that warrant further investigation in the field of stochastic epidemic modeling. For instance, (1) in the model we assumed Brownian noise, but in reality, some cases involve Lévy noise; (2) in the numerical simulation part, we assumed certain parameter values, but the actual parameter values are still uncertain; (3) the conditions outlined in Theorem 2 are intricate; (4) the nonlinear incidence rate may vary when modeling different diseases; (5) since most vaccines have a time limit, it is essential to incorporate this limitation into the model. In our future work, we will focus on addressing these questions.

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