symmetry
Article

Lie Symmetry Analysis and Conservation Laws of Fractional Benjamin–Ono Equation

Hui Liu and Yinshan Yun *

School of Science, Inner Mongolia University of Technology, Hohhot 010051, China; 20211100055@imut.edu.cn
* Correspondence: yunyinshan@126.com or yinshan@imut.edu.cn; Tel.: +86-13948614559

Abstract: In this paper, the fractional Benjamin–Ono differential equation with a Riemann–Liouville fractional derivative is considered using the Lie symmetry analysis method. Two symmetries admitted by the equation are obtained. Then, the equation is reduced to a fractional ordinary differential equation with an Erdélyi–Kober fractional derivative by one of the symmetries. Finally, conservation laws for the equations are constructed using the new conservation theorem.

Keywords: fractional differential equation; Lie symmetry; conservation laws

check for updates

Citation: Liu, H.; Yun, Y. Lie Symmetry Analysis and Conservation Laws of Fractional Benjamin–Ono Equation. *Symmetry* **2024**, *16*, 473. https://doi.org/10.3390/sym16040473

Academic Editor: Calogero Vetro

Received: 6 March 2024
Revised: 22 March 2024
Accepted: 10 April 2024
Published: 13 April 2024

1. Introduction

The theory of fractional calculus has been developing for over three centuries, with its first official appearance in 1695 when Leibniz [1] wrote a letter to L'Hospital discussing the problem of fractional derivatives of power functions. Since then, many mathematicians have delved into the study of fractional calculus. However, the rapid development of fractional calculus has actually occurred in the last thirty years, and it has now become an important branch of mathematical theory. In addition, fractional partial differential equations play an important role in describing phenomena in the fields of physics, mechanics, biology and chemistry [2–4]. The methods of defining fractional differential derivatives are varied and complex. Currently, there are Grünwald–Letnikov fractional derivatives [4–7], Riemann–Liouville fractional derivatives [7,8], Caputo fractional derivatives [8] and so on.

The Lie symmetry theory was proposed and developed by the Norwegian mathematician Sophus Lie [9,10] in the 19th century. Lie symmetry theory has become a powerful tool for solving integer-order differential equations in the field of mathematics–physics. During the development of Lie symmetry theory, in addition to Sophus Lie, many researchers have made contributions, such as Ovsiannikov [11], lbragimov [12], Bluman [13], Cole [14], Ovler [15], Qu [16], Tian [17], Tian [18] and so on. However, its application in fractional differential equations remains insufficient. In recent years, significant progress has been made by researchers in this field. For instance, Gazizov [19,20] proposed a method to construct conservation laws for time-fractional differential programs based on symmetry groups and nonlinear self-adjoints and derived the Lie–Bäcklund transform from the abnormal linear time-fractional diffusion equation. Singla and Gupta [21–23] further extended the symmetric group analysis method to fractional differential equations. Zhang [24] provided the symmetric determining equation and nonlinear method for solving fractional nonlinear partial differential equations. Feng [25–27] investigated the symmetry and conservation laws of various classes of time-fractional nonhomogeneous nonlinear diffusion equations. Chen [28] expanded the coefficients of fractional-order equation from constant coefficients to variable coefficients while conducting Lie symmetry analysis. Zhang [29] obtained similar solutions as well as numerical solutions for the time-fractional Burgers system. Wang [30] performed Lie symmetry analysis, obtained analytical solutions and derived conservation laws for a class of sixth-order generalized time-fractional Sawada–Kotera equations. The research content of Zhang [24], Feng [25–27], Chen [28] and others is developed on the basis of Riemann–Liouville fractional derivatives. The research content of Gazizov [19], Iskenderoglu [31] and others is based on Caputo fractional derivatives.

The Benjamin–Ono equation (BO) [32] is a nonlinear partial differential equation that finds applications across various fields, initially proposed to describe nonlinear effects in water wave propagation [33]. Its application is found in many fields such as nonlinear optics, quantum field theory and nonlinear dynamics. The study of this equation holds immense significance in comprehending nonlinear phenomena and the dynamic behavior of systems. Notably, the BO equation exhibits characteristics like non-linearity, complete integrability and harmonic balance approachability, as well as being a form of partial differential equation and having soliton solutions [34,35]. Liu [36] used the Hirota bilinear method to obtain the n-order soliton solution of the Benjamin–Ono equation. In recent years, Fang [37,38] derived the Bäcklund transform and exact solution, integrability, and Darboux transform solution for the BO equation. Fang [39] obtained a novel symmetry reduction of the BO equation using the Clarkson–Kruskal direct method. Wang [32] employed the Lie symmetry method to analyze the integer-order BO equation, obtaining special solutions and conservation laws for this equation. However, there has been limited exploration by other scholars on the fractional symmetry aspects of the BO equation.

The fractional-order BO differential equation considered in this paper has the following form:

The fractional-order BO differential equation considered in this paper has the following form:

$$D_t^\alpha u = -\beta\left(u^2\right)_{xx} - \gamma u_{xxx}, 1 < \alpha < 2, \tag{1}$$

where β, γ are nonzero constants.

The content of this paper is arranged as follows: In Section 2, the basic knowledge of fractional derivatives and Lie symmetry are reviewed. In Section 3, the Lie symmetry analysis of BO equation is given. In Section 4, the symmetry reduction of the BO equation is discussed. In Section 5, conservation laws of the equation are given.

2. Fractional Derivatives and Lie Symmetry

Definition 1. *For a function $u(x,t)$, the Riemann–Liouville fractional derivative is defined [24] as*

$$D_t^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t u(x,s)(t-s)^{n-\alpha-1} ds, & n-1 < \alpha < n, \\ \frac{\partial^n}{\partial t^n} u(x,t), & \alpha = n, \end{cases} \tag{2}$$

where $n \in N$. When $u = u(t)$,

$$D_t^\alpha u(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t u(s)(t-s)^{n-\alpha-1} ds, & n-1 < \alpha < n, \\ \frac{d^n}{dt^n} u(t), & \alpha = n, \end{cases} \tag{3}$$

where the gamma function is defined as $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ at $z > 0$.

The Riemann–Liouville fractional derivative of the power function and an arbitrary constant [40,41] take the following form:

$$\begin{aligned} &D_t^\alpha C = \frac{C}{\Gamma(1-\alpha)} t^{-\alpha} \\ &D_t^\alpha t^\gamma = \begin{cases} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha}, & \gamma > \alpha - 1, \\ 0, & \gamma = \alpha - 1, \end{cases} \end{aligned} \tag{4}$$

where C is an arbitrary constant.

Definition 2. *Assuming $u(x,t)$ and $v(x,t)$ are two continuous functions in the domain of definition, then the Riemann–Liouville fractional derivative of the product of the two functions is as follows:*

$$D_t^{\alpha}(u(x,t)v(x,t)) = \sum_{n=0}^{\infty}\begin{pmatrix}\alpha\\ n\end{pmatrix} D_t^{\alpha-n}v(x,t)D_t^{n}u(x,t), \tag{5}$$

where $\begin{pmatrix}\alpha\\ n\end{pmatrix} = \frac{(-1)^{\text{n-1}}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}$; *Formula (5) is also known as the generalized Leibniz [38] Rule.*

Definition 3. *The Erdélyi–Kober fractional differential operators [24] are defined as follows:*

$$\begin{aligned}&\left(P_{\delta}^{\tau,\alpha}g\right)(r) := \prod_{j=0}^{m-1}\left(\tau + j - \tfrac{1}{\delta}r\tfrac{d}{dr}\right)\left(K_{\delta}^{\tau+\alpha,m-\alpha}g\right)(r), \quad r>0,\delta>0,\alpha>0\\ &m = \begin{cases}[\alpha]+1, & \alpha\notin N,\\ \alpha, & \alpha\in N,\end{cases}\end{aligned} \tag{6}$$

where

$$\left(K_{\delta}^{\tau,\alpha}g\right)(r) := \begin{cases}\frac{1}{\Gamma(\alpha)}\int_1^{\infty}(v-1)^{\alpha-1}v^{-(\tau+\alpha)}g\left(rv^{\frac{1}{\delta}}\right)dv, & \alpha>0,\\ g(r), & \alpha=0,\end{cases} \tag{7}$$

is called the Erdélyi–Kober fractional integration operator.

In order to introduce the main ideas of the Lie symmetry method, consider a time-fractional partial differential equation with a Riemann–Liouville fractional partial derivative

$$D_t^{\alpha}u = f(x,t,u,u_1,u_2,\cdots u_l), \quad 1<\alpha<2 \tag{8}$$

where $u = u(x,t)$ and $u_i = \partial^i u/\partial x^i, i = 1,\ldots,l$.

Suppose that the one-parameter Lie group is

$$\begin{aligned}t^* &= t + \varepsilon\tau(x,t,u) + o(\varepsilon^2),\\ x^* &= x + \varepsilon\xi(x,t,u) + o(\varepsilon^2),\\ u^* &= u + \varepsilon\eta(x,t,u) + o(\varepsilon^2),\end{aligned} \tag{9}$$

where ε is the group parameter, and

$$\xi(x,t,u) = \left.\frac{\partial x}{\partial\varepsilon}\right|_{\varepsilon=0}, \tau(x,t,u) = \left.\frac{\partial t}{\partial\varepsilon}\right|_{\varepsilon=0}, \eta(x,t,u) = \left.\frac{\partial u}{\partial\varepsilon}\right|_{\varepsilon=0}, \tag{10}$$

The corresponding infinitesimal operator is

$$X = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{11}$$

If structure of the fractional derivative operator remains invariant under the transformation group, then $\tau(x,t,u)|_{t=0} = 0$ (see details in [24]).

Lemma 1. *Equation (8) admits the one-parameter Lie group (9) if and only if [42]*

$$\left.\mathrm{Pr}^{(\alpha,l)}X(\Delta)\right|_{\Delta=0} = 0, \tag{12}$$

where $\Delta = D_t^{\alpha}u - f(x,t,u,u_1,u_2,\cdots u_l) = 0$, $\mathrm{Pr}^{(\alpha,l)}X$ is (α,l)-order prolongation of X and the extension expression is as follows:

$$\mathrm{Pr}^{(\alpha,l)}X = X + \eta^{\alpha}\frac{\partial}{\partial(\partial_t^{\alpha}u)} + \sum_{i=1}^{l}\eta^{i}\frac{\partial}{\partial u_i}, \tag{13}$$

$$\eta^i = D_x^i(\eta - \xi u_x - \tau u_t) + \xi D_x^i(u_x) + \tau D_x^i(u_t), i = 1, 2, \ldots, l, \tag{14}$$

$$\begin{aligned}\eta^\alpha = \; & D_t^\alpha \eta + [\eta_u - \alpha D_t(\tau)] D_t^\alpha u - u D_t^\alpha(\eta_u) + \mu \\ & + \sum_{k=1}^{\infty} \left[\binom{\alpha}{k} \partial_t^k(\eta_u) - \binom{\alpha}{k+1} D_t^{k+1}(\tau) \right] \partial_t^{\alpha-k} u - \sum_{k=1}^{\infty} \binom{\alpha}{k} D_t^k(\xi) \partial_t^{k-\alpha}(u_x),\end{aligned} \tag{15}$$

where

$$\mu = \sum_{n=2}^{\infty} \sum_{m=2}^{n} \sum_{k=2}^{m} \sum_{r=0}^{k} \binom{\alpha}{n} \binom{n}{m} \binom{k}{r} \frac{(-1)^r}{k!} u^r \partial_t^\alpha u^{k-r} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \frac{\partial^{n-m+k}\eta}{\partial^{n-m} t \partial^k u}, \tag{16}$$

where D_x as well as D_t below are the total derivative operators of x and t, as follows:

$$\begin{aligned} D_x &= \partial_x + u_x \partial_u + u_{tx} \partial_{u_t} + u_{xx} \partial_{u_x} + \cdots, \\ D_t &= \partial_t + u_t \partial_u + u_{tx} \partial_{u_x} + u_{tt} \partial_{u_t} + \cdots. \end{aligned} \tag{17}$$

Definition 4. *A vector field $C = (C^x, C^t)$ is called a conserved vector for (8) if the following equality holds for all solutions of (8):*

$$\left[D_t C^t + D_x C^x \right] \big|_{(8)} = 0, \tag{18}$$

Equation (18) is called a conservation law for Equation (8).

Definition 5. *The Euler–Lagrange operator [43] is defined as follows:*

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (D_t^\alpha)^* \frac{\partial}{\partial D_t^\alpha u} - D_x \frac{\partial}{\partial u_x} + D_{xx} \frac{\partial}{\partial u_{xx}} - D_{xxx} \frac{\partial}{\partial u_{xxx}} + D_{xxxx} \frac{\partial}{\partial u_{xxxx}}, \tag{19}$$

where $(D_t^\alpha)^*$ denotes the conjugate operator of the D_t^α, as follows:

$$(D_t^\alpha)^* = (-1)^n I_r^{n-\alpha}(D_t^n) = {}^C D_r^\alpha, \tag{20}$$

where ${}^C D_r^\alpha$ is the Caputo fractional differential operator.

3. Lie Symmetry of BO Equation (1)

3.1. Lie Point Symmetries

By applying the extension $\mathrm{Pr}^{(\alpha,l)} X$ to the fractional-order BO Equation (1), one obtains

$$\mathrm{Pr}^{(\alpha,4)} X[D_t^\alpha u + \beta \left(u^2\right)_{xx} + \gamma u_{xxxx}] = \eta^\alpha + \gamma \eta^{xxxx} + 2\beta \eta^{xx} u + 4\beta \eta^x u_x + 2\beta \eta u_{xx} = 0, \tag{21}$$

where

$$\begin{aligned} \eta^x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi), \\ \eta^{xx} &= D_x(\eta^x) - u_{xt} D_x(\tau) - u_{xx} D_x(\xi), \\ \eta^{xxx} &= D_x(\eta^{xx}) - u_{xxt} D_x(\tau) - u_{xxx} D_x(\xi), \\ \eta^{xxxx} &= D_x(\eta^{xxx}) - u_{xxxt} D_x(\tau) - u_{xxxx} D_x(\xi). \end{aligned} \tag{22}$$

By substituting (15) and (22) into (21), the determining equations of the Lie symmetry of the BO Equation (1) can be obtained as follows:

$$\begin{aligned}
&\xi_u = \xi_t = \tau_u = \tau_x = \eta_{uu} = 0, \\
&-3\xi_{xx} + 2\eta_{tu} = 0, \\
&\alpha\tau_t - 4\xi_x = 0, \\
&(1-\alpha)\alpha\tau_{tt} + 2\eta_{tu} = 0, \\
&-\gamma\xi_{xxxx} + 4\beta\eta_x + 4\gamma\eta_{xxxu} = 0 \\
&\alpha\beta\tau_t - 2\beta\xi_x + \beta\eta_u = 0, \\
&\eta + u(2\xi_x - 2\gamma\xi_{xxx} + 3\gamma\eta_{xxu}) = 0 \\
&(\alpha-2)\tau_{ttt} - 3\eta_{ttu} = 0, \\
&D_t^\alpha\eta - uD_t^\alpha\eta_u + 2\beta u\eta_{xx} + \gamma\eta_{xxxx} = 0.
\end{aligned} \tag{23}$$

By solving the determining Equation (23), one obtains

$$\begin{aligned}
&\tau = \tfrac{4c_1}{\alpha}t + c_3, \\
&\xi = c_2 + c_1 x, \\
&\eta = -2c_1 u,
\end{aligned} \tag{24}$$

where c_1 and c_2 are arbitrary constants, and $c_3 = 0$. Thus, the operators are

$$\begin{aligned}
&X_1 = \tfrac{\partial}{\partial x}, \\
&X_2 = \tfrac{4t}{\alpha}\tfrac{\partial}{\partial t} + x\tfrac{\partial}{\partial x} - 2u\tfrac{\partial}{\partial u}.
\end{aligned} \tag{25}$$

The generators X_1 and X_2 form a Lie algebra, because $[X_1, X_2] = X_1$, $[X_2, X_1] = -X_1$.

3.2. One-Parameter Lie Transformation Group of BO Equation

In order to obtain the one-parameter Lie transform group, we only need to solve the following initial value problem:

$$\begin{aligned}
&\tfrac{d(x^*(\varepsilon))}{d\varepsilon} = \xi_x(x^*(\varepsilon), t^*(\varepsilon), u^*(\varepsilon)), x^*(0) = x, \\
&\tfrac{d(t^*(\varepsilon))}{d\varepsilon} = \tau_t(x^*(\varepsilon), t^*(\varepsilon), u^*(\varepsilon)), t^*(0) = t, \\
&\tfrac{d(u^*(\varepsilon))}{d\varepsilon} = \eta_u(x^*(\varepsilon), t^*(\varepsilon), u^*(\varepsilon)), u^*(0) = u,
\end{aligned} \tag{26}$$

where $|\varepsilon| << 1$ is an infinitesimal group parameter.

The one-parameter Lie group can be written in the following form:

$$g : (x, t, u) \to (x^*, t^*, u^*), \tag{27}$$

Using the operator (21) to solve the initial problem (26), the Lie symmetric transformation of the fractional BO equation is obtained as follows:

$$\begin{aligned}
&g_1(\varepsilon) : (x, t, u) \to (x + \varepsilon, t, u), \\
&g_2(\varepsilon) : (x, t, u) \to \left(e^\varepsilon x, e^{\frac{4}{\alpha}\varepsilon}t, e^{-2\varepsilon}u\right).
\end{aligned} \tag{28}$$

Therefore, through the Lie symmetric transformation group (28), the other two solutions of the Benjamin–Ono equation (1) can be obtained as follows:

$$\begin{aligned}
&u_1 = f(x - \varepsilon, t), \\
&u_2 = e^{2\varepsilon} f(e^{-\varepsilon}x, e^{-\frac{4}{\alpha}\varepsilon}t),
\end{aligned} \tag{29}$$

if $u = f(x, t)$ is a solution of (1).

4. Similar Reduction of the BO Equation

Theorem 3. *Under the operator X_2, the BO equation is reduced to the following fractional ordinary differential equation:*

$$\left(P_{\frac{4}{\alpha}}^{1-\frac{3\alpha}{2},\alpha}g\right)(r)=-\beta\left(g^2(r)\right)''-\gamma g'''{}'(r). \tag{30}$$

Proof. The characteristic equation corresponding to X_2 in operator (21) is

$$\frac{dx}{x}=\frac{\alpha dt}{4t}=\frac{du}{-2u}. \tag{31}$$

The characteristic Equation (31) is solved to obtain the similarity variables r and $g(r)$

$$r=xt^{-(\frac{\alpha}{4})},u(x,t)=t^{-(\frac{\alpha}{2})}g(r). \tag{32}$$

When $n-1<\alpha<n, n=1,2,3,\ldots$, through the similarity variables (32), the fractional Riemann–Liouville derivative becomes

$$D_t^\alpha u=\frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}s^{-\frac{\alpha}{2}}f\left(xs^{-\frac{\alpha}{4}}\right)ds\right]. \tag{33}$$

Letting $v=\frac{t}{s}$, Equation (33) can be written as

$$D_t^\alpha u=\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha-\frac{\alpha}{2}}\frac{1}{\Gamma(n-\alpha)}\int_1^\infty(v-1)^{n-\alpha-1}v^{-(n-\alpha-\frac{\alpha}{2}+1)}f\left(rv^{\frac{\alpha}{4}}\right)dv\right], \tag{34}$$

According to (7), the above equation is rewritten as follows:

$$D_t^\alpha u=\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{3\alpha}{2}}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha}g\right)(r)\right]. \tag{35}$$

Applying the formula $t\frac{\partial}{\partial t}\phi(r)=tx\left(-\frac{\alpha}{4}\right)t^{-\frac{\alpha}{4}-1}\phi'(r)=-\frac{\alpha}{4}r\frac{d}{dr}\phi(r)$, the fractional derivative is rewritten as follows

:

$$\begin{aligned}D_t^\alpha u&=\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{3\alpha}{2}}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha}g\right)(r)\right]\\&=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\frac{3\alpha}{2}}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha}g\right)(r)\right)\right]\\&=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{3\alpha}{2}-1}\left(n-\frac{3\alpha}{2}\right)\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha}g\right)(r)+t^{n-\frac{3\alpha}{2}-1}\left(-\frac{\alpha}{4}r\frac{d}{dr}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha}g\right)(r)\right)\right]\\&=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{3\alpha}{2}-1}\left(n-\frac{3\alpha}{2}-\frac{\alpha}{4}r\frac{d}{dr}\right)\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha}g\right)(r)\right].\end{aligned} \tag{36}$$

Repeat the above process $n-1$ times to obtain

$$\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{3\alpha}{2}}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha}g\right)(r)\right]=t^{-\frac{3\alpha}{2}}\prod_{j=0}^{n-1}\left(1-\frac{3\alpha}{2}+j-\frac{\alpha}{4}r\frac{d}{dr}\right)\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha}g\right)(r). \tag{37}$$

According to the definition of the Erdélyi–Kober fractional differential operator (6), Equation (37) becomes

$$\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{3\alpha}{2}}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha}g\right)(r)\right]=t^{-\frac{3\alpha}{2}}\left(P_{\frac{4}{\alpha}}^{1-\frac{3\alpha}{2},n-\alpha}g\right)(r). \tag{38}$$

Equation (33) can be written as

$$D_t^{\alpha} u = t^{-\frac{3\alpha}{2}} \left(P_{\frac{4}{\alpha}}^{1-\frac{3\alpha}{2}, n-\alpha} g \right)(r). \tag{39}$$

where $u_{xx} = t^{-\left(\frac{\alpha}{2}\right)} g''(r), u_{xxxx} = t^{-\left(\frac{\alpha}{2}\right)} g'''{}'(r)$. We can reduce the fractional BO equation to the fractional ordinary differential in Equation (30). □

5. Conservation Laws of the BO Equation

According to the concept of nonlinear self-conjugation [44], the formal Lagrangian of the BO Equation (1) can be expressed as

$$L = v(x,t) \left[D_t^{\alpha} u + \beta \left(u^2 \right)_{xx} + \gamma u_{xxxx} \right], \tag{40}$$

where $v(x,t)$ is a new dependent variable with x, t.

The conjugate Euler–Lagrange equation of the fractional-order BO equation is

$$\frac{\delta L}{\delta u} = 0, \tag{41}$$

The conjugate equation for the fractional BO equation can be written as

$$(D_t^{\alpha})^* v + 2\beta v u_{xx} - 2\beta u v_x + 2\beta u v_{xx} + \gamma v_{xxxx} = 0, \tag{42}$$

For the operator (25) of fractional B-O equation (1), it satisfies the conservation law equation $D_t C^t + D_x C^x = 0$, where C^x, C^t are represented by the following equation:

$$C^t = \tau L + \sum_{k=0}^{n-1} (-1)^k D_t^{\alpha-1-k}(W_i) D_t^k \frac{\partial L}{\partial (D_t^{\alpha} u)} - (-1)^n J\left(W_i, D_t^n \frac{\partial L}{\partial (D_t^{\alpha} u)} \right), \tag{43}$$

$$\begin{aligned} C^x = {} & W_i \left[\tfrac{\partial L}{\partial u_x} - D_x \tfrac{\partial L}{\partial u_{xx}} + D_{xx} \tfrac{\partial L}{\partial u_{xxx}} - D_{xxxx} \tfrac{\partial L}{\partial u_{xxxx}} \right] \\ & + D_x(W_i) \left[\tfrac{\partial L}{\partial u_{xx}} - D_x \tfrac{\partial L}{\partial u_{xxx}} + D_{xx} \tfrac{\partial L}{\partial u_{xxxx}} \right] \\ & + D_{xx}(W_i) \left[\tfrac{\partial L}{\partial u_{xxx}} - D_x \tfrac{\partial L}{\partial u_{xxxx}} \right] + D_{xxx}(W_i) \tfrac{\partial L}{\partial u_{xxxx}}, \end{aligned} \tag{44}$$

where

$$J(f,g) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \int_t^T \frac{f(\tau,x) g(\mu,x)}{(\mu-\tau)^{\alpha+1-n}} d\mu d\tau, \tag{45}$$

$$W = \eta - \tau u_t - \xi u_x. \tag{46}$$

One can find the components of the conserved vector of the BO equation in the following two cases

Case 1: For $X = \frac{\partial}{\partial x}$, one obtains $W = -u_x$. Therefore, the conserved vector of (1) can be obtained as

$$\begin{aligned} C^t &= -v D_t^{\alpha-1}(u_x) - J(u_x, v_t), \\ C^x &= -u_x(4\beta u v - 2\beta u v_x - \gamma v_{xxx}) - u_{xx}(2\beta u v + \gamma v_{xx}) + 2\gamma u_{xx} v_x - \gamma x u_{xxxx} v. \end{aligned} \tag{47}$$

Case 2: For $X = \frac{4t}{\alpha} \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2u \frac{\partial}{\partial u}$, one obtains $W = -2u - \frac{4t}{\alpha} u_t - x u_x$. Therefore, the conserved vector of (1) can be obtained as

$$\begin{aligned} C^t = {} & v D_t^{\alpha-1}\left(-2u - \tfrac{4t}{\alpha} u_t - x u_x \right) + J\left(-2u - \tfrac{4t}{\alpha} u_t - x u_x, v_t \right), \\ C^x = {} & -(2u + \tfrac{4t}{\alpha} u_t + x u_x)(4\beta u v - 2\beta u v_x - \gamma v_{xxx}) + \gamma v_x \left(4u_{xx} + \tfrac{4t}{\alpha} u_{txx} + x u_{xxx} \right) \\ & - \left(3u_x + \tfrac{4t}{\alpha} u_{tx} + x u_{xx} \right)(2\beta u v + \gamma v_{xx}) - \gamma v \left(5u_{xxx} + \tfrac{4t}{\alpha} u_{txxx} + x u_{xxxx} \right). \end{aligned} \tag{48}$$

6. Conclusions

In this paper, Lie symmetry analysis of the BO equation with Riemann–Liouville derivatives is carried out, and the Lie symmetry structure of the BO equation is obtained. Using the infinitesimal operator $X_2 = \frac{4t}{\alpha}\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}$, the BO equation can be reduced to a fractional ordinary differential equation $\left(P^{1-\frac{3\alpha}{2},\alpha}_{\frac{4}{\alpha}} g\right)(r) = -\beta\left(g^2(r)\right)'' - \gamma g''''(r)$. So, one can see that the Lie symmetry analysis method is an efficient method for fractional differential equations. Similarly, Lie symmetry analysis is also applied to the BO equation with Caputo fractional derivatives or other type fractional derivatives.

Compared with the results of the literature [34], the fractional BO equation considered in this paper allows two fewer symmetric infinitesimal operators than the integer BO equation considered in [34]. It is worth considering whether the same fractional differential equation allows less or no more symmetry than its integer-order equation.

Author Contributions: Conceptualization, Y.Y. and H.L.; methodology, Y.Y. and H.L.; formal analysis, Y.Y. and H.L.; writing—original draft, Y.Y. and H.L.; writing—review and editing, Y.Y. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by National Natural Science Foundation of China, No. 12161064.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Wu, Q.; Huang, J.H. *Fractional Calculus*; Tsinghua University Press: Beijing, China, 2016; pp. 1–10.
2. Kiryakova, V.S. *Generalized Fractional Calculus and Applications*; Pitman Research Notes in Mathematics Longman & J. Wiley: New York, NY, USA, 1994.
3. Sun, H.; Zhang, Y.; Baleanu, D.; Chen, W.; Chen, Y. A new collection of real world applications of fractional calculus in science and engineering. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *64*, 213–231. [CrossRef]
4. Podlubny, I. *Fractional Differential Equations*; Academic Press: London, UK, 1999.
5. Origenian, M.D. Machado. Fractional calculus applications in signals and systems. *Signal Process.* **2006**, *83*, 2503–2504.
6. Li, X.R. *Fractional Calculus Fractal Geometry and Stochastic Process*; The University of Western Ontario USA: Ontario, CA, USA, 2003.
7. Origenian, M.D. Introduction to fractional linear systems Part1: Continuous-time case. *J. IEEE Proc. Vis. Image Signal Process.* **2000**, *147*, 63–65.
8. Hashem, M.S. Invariant subspace admitted by fractional differential equations with conformable derivatives. *Chaos Solitons Fractals* **2018**, *10*, 161–169. [CrossRef]
9. Lie, S. On integration of a class of linear partial differential equations by means ofdefnite integrals. *Arch. Math. VI* **1881**, *3*, 328–368.
10. Lie, S. *Sophus Lie's 1880 Transformation Group Paper*; Ackerman, M.; Hermann, R., Translators; Mathematical Science Press: Brookline, MA, USA, 1975.
11. Ovsjannikov, L.V. *Group Properties of Differential Equations*; Siberian Section of the Academy of Science of USSR: Siberian, Russia, 1962.
12. Ibragimov, N.K. Group analysis of ordinary differential equations and the invariance principle in mathematical physics (for the 150th anniversary of Sophus Lie). *Russ. Math. Surv.* **1992**, *47*, 89–156. [CrossRef]
13. Bluman, G.W.; Cole, J. The general similarity solution of the heat equation. *J. Math Mech.* **1969**, *18*, 1025–1042.
14. Olver, P.J. Evolution equations possessing infinitely many symmetries. *J. Math. Phys.* **1977**, *18*, 1212–1215. [CrossRef]
15. Bluman, G.W.; Reid, G.J.; Kumei, S. New classes of symmetries for partial differential equations. *J. Math. Phys.* **1988**, *29*, 806–811. [CrossRef]
16. Qu, C.Z. Group classification and generalized conditional symmetry reduction of the non-linear diffusion-convection equation with a nonlinear source. *J. Stud. Appl. Math.* **1997**, *99*, 107–136. [CrossRef]
17. Tian, C. Two sets of symmetries of MKdV Equation and their Geometric Significance. *J. Appl. Math. Univ. Ser. A (Middle Ed.)* **1989**, *2*, 179–186.
18. Tian, S.F. Lie symmetry analysis, conservation laws and solitary wave solutions to a fourth-order nonlinear generalized Boussinesq water wave equation. *J. Appl. Math. Lett.* **2020**, *100*, 106056. [CrossRef]
19. Gazizov, R.K.; Ibragimov, N.H.; Lukashchuk, S.Y. Nonlinear self-adjointness, conservation laws and exact solutions of time-fractional Kompaneets equations. *J. Commun. Nonlinear Sci. Numer. Simul.* **2015**, *23*, 153–163. [CrossRef]
20. Gazizov, R.K.; Lukashchuk, S.Y. Higher-order symmetries of a time-fractional anomalous diffusion equation. *J. Math.* **2021**, *9*, 216. [CrossRef]

21. Singla, K.; Gupta, R.K. On invariant analysis of some time fractional nonlinear systems of partial differential equations. I. *J. Math. Phys.* **2016**, *57*, 101504. [CrossRef]
22. Singla, K.; Gupta, R.K. On invariant analysis of space-time fractional nonlinear systems of partial differential equations. II. *J. Math. Phys.* **2017**, *58*, 051503. [CrossRef]
23. Singla, K.; Gupta, R.K. Generalized Lie symmetry approach for fractional order systems of differential equations. III. *J. Math. Phys.* **2017**, *58*, 061501. [CrossRef]
24. Zhang, Z.Y. Symmetry determination and nonlinearization of a nonlinear time-fractional partial differential equation. *J. Proc. R. Soc. A* **2020**, *476*, 20190564. [CrossRef] [PubMed]
25. Fen, W.; Zhao, S. Time-fractional in homogeneous nonlinear diffusion equation:symmetries, conservation laws, invariant subspaces, and exact solutions. *J. Mod. Phys. Lett. B* **2018**, *32*, 1850401.
26. Feng, W. On symmetry groups and conservation laws for space-time fractional in homogeneous nonlinear diffusion equation. *J. Rep. Math. Phys.* **2019**, *84*, 375–392. [CrossRef]
27. Feng, W. Exact solutions and conservation laws of time-fractional Levi equation. *J. Symmetry* **2020**, *12*, 1074. [CrossRef]
28. Chen, C.; Jiang, Y.L.; Wang, X.T. Lie Symmetry Analysis of the Time Fractional Generalized KdV Equations with Variable Coefficients. *Symmetry* **2019**, *11*, 1281. [CrossRef]
29. Zhang, X.Z.; Zhang, Y.F. Some Similarity Solutions and Numerical Solutions to the Time-Fractional Burgers System. *Symmetry* **2019**, *11*, 112. [CrossRef]
30. Wang, Y.; Li, L. Lie Symmetry Analysis, Analytical Solution, and Conservation Laws of a Sixth-Order Generalized Time-Fractional Sawada-Kotera Equation. *Symmetry* **2019**, *11*, 1436. [CrossRef]
31. Iskenderoglu, G.; Kaya, D. Symmetry analysis of initial and boundary value problems for fractional differential equations in Caputo sense. *Chaos Solitons Fractals* **2020**, *134*, 109684. [CrossRef]
32. Wang, Z.; Sun, L.; Hua, R.; Zhang, L.; Wang, H. Lie Symmetry Analysis, Particular Solutions and Conservation Laws of Benjiamin Ono Equation. *Symmetry* **2022**, *14*, 1315. [CrossRef]
33. Mikhailov, A.V.; Novikov, V.S. Classification of Integrable Benjamin-Ono-type equations. *Mosc. Math. J.* **2003**, *3*, 1293–1305. [CrossRef]
34. Benci, V.; Fortunato, D. Hylomorphic solitons for the Benjamin-Ono and the fractional KdV equations. *Nonlinear Anal. -Theory Methods Appl.* **2016**, *144*, 41–57. [CrossRef]
35. Schippa, R. Local and global well-posedness for dispersion generalized Benjamin-Ono equations on the circle. *Nonlinear Anal.-Theory Methods Appl.* **2020**, *196*, 111777. [CrossRef]
36. Liu, Y.K.; Li, B. Dynamics of rogue waves on multisoliton background in the Benjamin Ono equation. *Pramana-J. Phys.* **2017**, *88*, 4. [CrossRef]
37. Fang, C.M.; Xue, L.H.; Tian, S.F. Integrability and Darboux transformation solutions of Deformed Boussinesq Equation and Benjamin-Ono Equation. *J. Hubei Univ. Natl. (Nat. Sci. Ed.)* **2016**, *34*, 127–130.
38. Fang, C.M. New Backlund transformation and exact Solution of Benjamin-Ono Equation. *J. Chifeng Univ. (Nat. Sci. Ed.)* **2013**, *15*, 22–23.
39. Fang, C.M. Symmetry reduction of Benjamin-Ono Equation. *J. Hubei Univ. Natl. (Nat. Sci. Ed.)* **2013**, *31*, 190–193.
40. Samko, S.; Kilbas, A.A.; Marichev, O. Fractional integral and derivatives: Theory and applications. *J. Gordon Breach Science. Geneva.* **1933**.
41. Wang, X.D. Riemann-Liouville Fractional Calculus and Its Property Proof. Ph.D. Thesis, Taiyuan University of Technology, Taiyuan, China, 2008.
42. Huang, Q.; Zhdanov, R. Symmetries and exact solutions of the time fractional harry-dym equation with riemann–liouville derivative. *Phys. A Stat. Mech. Its Appl.* **2014**, *409*, 110–118. [CrossRef]
43. Zhang, Z.Y.; Zheng, J. Symmetry structure of multi-dimensional timefractional partial differential equations. *J. Nonlinearity.* **2021**, *34*, 5186–5212. [CrossRef]
44. Liu, J.G.; Wang, J.Q. Invariant analysis of the linear time-space fractional (2+1)-dimensional Burgers equation. *Comp. J. Appl. Math.* **2023**, *42*, 199. [CrossRef]