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Noether Symmetry of Multi-Time-Delay Non-Conservative Mechanical System and Its Conserved Quantity

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Abstract: The study of multi-time-delay dynamical systems has highlighted many challenges, especially regarding the solution and analysis of multi-time-delay equations. The symmetry and conserved quantity are two important and effective essential properties for understanding complex dynamical behavior. In this study, a multi-time-delay non-conservative mechanical system is investigated. Firstly, the multi-time-delay Hamilton principle is proposed. Then, multi-time-delay non-conservative dynamical equations are deduced. Secondly, depending on the infinitesimal group transformations, the invariance of the multi-time-delay Hamilton action is studied, and Noether symmetry, Noether quasi-symmetry, and generalized Noether quasi-symmetry are discussed. Finally, Noether-type conserved quantities for a multi-time-delay Lagrangian system and a multi-time-delay non-conservative mechanical system are obtained. Two examples in terms of a multi-time-delay non-conservative mechanical system and a multi-time-delay Lagrangian system are given.

Keywords: multi-time-delay; non-conservative system; symmetry; Noether theorem

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1. Introduction

Time-delay dynamical systems are widely present in real-world and engineering scenarios. As the demand for greater precision in complex dynamical systems increases, the influence of time-delay has received widespread attention [1–3]. Previously, the scientific phenomenon of time-delay has been utilized in applied mathematics, physics, mechanics, computer science, engineering, biology, etc. [4–10].

The time-delay differential equation [11] is the typical mathematical model used to describe a time-delay dynamical system. Challengingly, classical differential equation theory is no longer applicable, and the solution space for a time-delay equation is infinite in its dimension. Studying the dynamical characteristics of systems through variational problems is one of the most important research fields for modeling time-delay dynamical systems. In the 1960s, El'sgol'c [12] first proposed time-delay variational problems and the corresponding characterization of extrema. In 1968, Hughes deduced the sufficiency theorem for a minimum of a time-delay variational problem as well as a maximum principle for a time-delay control problem [13]. The conjugate-point conditions, sufficient conditions, and application with respect to optimal problems with delay arguments were presented [14–17]. In addition, symmetries and conserved quantities are an effective method by which to understand the behavior and basic properties of complex dynamical systems. The famous Noether theorem [18], Lie symmetry [19], and Mei symmetry [20] have previously had a profound influence on and application in optimal control and constrained mechanical systems [21–26]. Frederico and Torres [27] preliminarily introduced the classical Noether's theory to the time-delay calculus of variations. Indeed, Noether's theory has been applied to various problems involving time-delay, such as non-smooth extremals of variational problems [28], isoperimetric variational problems [29], high-order variational problems [30], non-conservative systems [31], nonholonomic systems [32], Hamiltonian systems [33], Birkhoffian systems [34], generalized Herglotz variational problems [35–37], and dynamical systems in fractional [38,39] and time-scale frameworks [6,40].

However, most of the research mentioned above was limited by taking into account a single, constant time-delay parameter. Many studies relating to controller design, stability studies, neural models, and fractional problems have addressed multi-time-delay [41–45]. Benharrat and Torres [46] studied variation problems of the optimal control problem via the penalty method by considering multi-time-delay. The variational principles on mechanical systems and the corresponding symmetry theory are still poorly studied in terms of taking into account multi-time-delay. In this paper, we present the Noether symmetry of a non-conservative mechanical system (NCMS) considering multi-time-delay and its conserved quantity, not only considering the multiple different time-delays acting on the system but also cases involving generalized coordinates and generalized velocities with different time-delays.

2. Multi-Time-Delay Non-Conservative Dynamical Equations

We study an NCMS considering multi-time-delay, the configuration of which is described by $q_s(s = 1, 2, \cdots, n)$, which are known as the generalized coordinates. The Hamilton principle of an NCMS [21] is

$$\int_{t_1}^{t_2} \left(\delta L + Q_s'' \delta q_s\right) \mathrm{d}t = 0. \tag{1}$$

Consider that the multi-time-delay exists in the system and the multi-time-delay Lagrangian is

$$L = L(t, q_s(t), q_s(t - \tau_1), \dot{q}_s(t), \dot{q}_s(t - \tau_2)) = L\left(t, q_s, q_{s\tau_1}, \dot{q}_s, \dot{q}_{s\tau_2}\right) \tag{2}$$

and the multi-time-delay generalized non-conservative force is $Q_s'' = Q_s''\left(t, q_s, q_{s\tau_1}, \dot{q}_s, \dot{q}_{s\tau_2}\right)$, subject to the following boundary conditions:

$$q_s(t) = \delta_{s_1}(t), t \in [t_1 - \tau_1, t_1 - \tau_2], \tag{3}$$

$$q_s(t) = \delta_{s_2}(t), t \in [t_1 - \tau_2, t_1], \tag{4}$$

$$q_s(t) = q_{s_2}, t = t_2, \tag{5}$$

where the time-delays are considered to be different between the generalized coordinates and the generalized velocities, τ_1 and τ_2 are assumed to be constant positive time-delays with $t_1 < \tau_1 < \tau_2 < t_2$, and the functions $\delta_{s_1}(t)$ and $\delta_{s_2}(t)$ are assumed to be piecewise smooth.

Then, principle (1) can be expressed as

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_s} \delta q_s + \frac{\partial L}{\partial q_{s\tau_1}} \delta q_{s\tau_1} + \frac{\partial L}{\partial \dot{q}_s} \delta \dot{q}_s + \frac{\partial L}{\partial \dot{q}_{s\tau_2}} \delta \dot{q}_{s\tau_2} + Q_s'' \delta q_s \right) \mathrm{d}t = 0. \tag{6}$$

Performing a linear substitution of the variables $t = \theta + \tau_1, t = \theta + \tau_2$ for the time-delay terms of Equation (6), and noting conditions (3) and (4), we have

$$\begin{aligned} &\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_{s\tau_1}}(t) \delta q_{s\tau_1} \right) \mathrm{d}t = \int_{t_1-\tau_1}^{t_1} \left(\frac{\partial L}{\partial q_{s\tau_1}}(\theta + \tau_1) \delta q_s \right) \mathrm{d}\theta + \int_{t_1}^{t_2-\tau_1} \left(\frac{\partial L}{\partial q_{s\tau_1}}(\theta + \tau_1) \delta q_s \right) \mathrm{d}\theta \\ &= \int_{t_1}^{t_2-\tau_1} \left(\frac{\partial L}{\partial q_{s\tau_1}}(\theta + \tau_1) \delta q_s \right) \mathrm{d}\theta \end{aligned} \tag{7}$$

and

$$
\begin{aligned}
&\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t)\delta\dot{q}_{s\tau_2}\right)\mathrm{d}t=\int_{t_1-\tau_2}^{t_1}\left(\frac{\partial L}{\partial \dot{q}_{s\tau_2}}(\theta+\tau_2)\delta\dot{q}_s\right)\mathrm{d}\theta+\int_{t_1}^{t_2-\tau_2}\left(\frac{\partial L}{\partial \dot{q}_{s\tau_2}}(\theta+\tau_2)\delta\dot{q}_s\right)\mathrm{d}\theta\\
&=\int_{t_1}^{t_2-\tau_2}\left(\frac{\partial L}{\partial \dot{q}_{s\tau_2}}(\theta+\tau_2)\delta\dot{q}_s\right)\mathrm{d}\theta.
\end{aligned}
\tag{8}
$$

After the linear substitution of the variables, Equation (6) can be expressed as

$$
\begin{aligned}
&\int_{t_1}^{t_2-\tau_1}\left(\frac{\partial L}{\partial q_s}(t)+\frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1)+Q''_s(t)\right)\delta q_s\mathrm{d}t+\int_{t_2-\tau_1}^{t_2}\left(\frac{\partial L}{\partial q_s}(t)+Q''_s(t)\right)\delta q_s\mathrm{d}t\\
&+\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial \dot{q}_s}(t)\delta\dot{q}_s\right)\mathrm{d}t+\int_{t_1}^{t_2-\tau_2}\left(\frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\delta\dot{q}_s\right)\mathrm{d}t.
\end{aligned}
\tag{9}
$$

Taking into account conditions (3)–(5), we have

$$
\begin{aligned}
&\int_{t_1}^{t_2-\tau_1}\left(\frac{\partial L}{\partial q_s}(t)+\frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1)+Q''_s(t)\right)\delta q_s\mathrm{d}t\\
&=-\left[\delta q_s\int_{t}^{t_2-\tau_1}\left(\frac{\partial L}{\partial q_s}(\theta)+\frac{\partial L}{\partial q_{s\tau_1}}(\theta+\tau_1)+Q''_s(\theta)\right)\mathrm{d}\theta\right]\Bigg|_{t_1}^{t_2-\tau_1}\\
&+\int_{t_1}^{t_2-\tau_1}\delta\dot{q}_s\left[\int_{t}^{t_2-\tau_1}\left(\frac{\partial L}{\partial q_s}(\theta)+\frac{\partial L}{\partial q_{s\tau_1}}(\theta+\tau_1)+Q''_s(\theta)\right)\mathrm{d}\theta\right]\mathrm{d}t\\
&=\int_{t_1}^{t_2-\tau_1}\delta\dot{q}_s\left[\int_{t}^{t_2-\tau_1}\left(\frac{\partial L}{\partial q_s}(\theta)+\frac{\partial L}{\partial q_{s\tau_1}}(\theta+\tau_1)+Q''_s(\theta)\right)\mathrm{d}\theta\right]\mathrm{d}t
\end{aligned}
\tag{10}
$$

and

$$
\int_{t_2-\tau_1}^{t_2}\left(\frac{\partial L}{\partial q_s}(t)+Q''_s(t)\right)\delta q_s\mathrm{d}t=-\int_{t_2-\tau_1}^{t_2}\delta\dot{q}_s\left[\int_{t_2-\tau_1}^{t}\left(\frac{\partial L}{\partial q_s}(\theta)+Q''_s(\theta)\right)\mathrm{d}\theta\right]\mathrm{d}t.
\tag{11}
$$

Thus, Equation (9) can be rewritten as

$$
\begin{aligned}
&\int_{t_1}^{t_2-\tau_1}\delta\dot{q}_s\left[\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)+\int_{t}^{t_2-\tau_1}\left(\frac{\partial L}{\partial q_s}(\theta)+\frac{\partial L}{\partial q_{s\tau_1}}(\theta+\tau_1)+Q''_s(\theta)\right)\mathrm{d}\theta\right]\mathrm{d}t\\
&+\int_{t_2-\tau_1}^{t_2-\tau_2}\delta\dot{q}_s\left[\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)+\int_{t}^{t_2-\tau_1}\left(\frac{\partial L}{\partial q_s}(\theta)+Q''_s(\theta)\right)\mathrm{d}\theta\right]\mathrm{d}t\\
&+\int_{t_2-\tau_2}^{t_2}\delta\dot{q}_s\left[\frac{\partial L}{\partial \dot{q}_s}(t)+\int_{t}^{t_2-\tau_1}\left(\frac{\partial L}{\partial q_s}(\theta)+Q''_s(\theta)\right)\mathrm{d}\theta\right]\mathrm{d}t=0.
\end{aligned}
\tag{12}
$$

In fact, the integral interval $[t_1,t_2]$ should be arbitrary, and the generalized coordinates $\delta\dot{q}_s(s=1,2,\cdots,n)$ are independent of one another. Thus, we have

$$
\begin{aligned}
&\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)+\int_{t}^{t_2-\tau_1}\left(\frac{\partial L}{\partial q_s}(\theta)+\frac{\partial L}{\partial q_{s\tau_1}}(\theta+\tau_1)+Q''_s(\theta)\right)\mathrm{d}\theta=0,t\in[t_1,t_2-\tau_1],\\
&\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)+\int_{t}^{t_2-\tau_1}\left(\frac{\partial L}{\partial q_s}(\theta)+Q''_s(\theta)\right)\mathrm{d}\theta=0,t\in(t_2-\tau_1,t_2-\tau_2],\\
&\frac{\partial L}{\partial \dot{q}_s}(t)+\int_{t}^{t_2-\tau_1}\left(\frac{\partial L}{\partial q_s}(\theta)+Q''_s(\theta)\right)\mathrm{d}\theta=0,t\in(t_2-\tau_2,t_2].
\end{aligned}
\tag{13}
$$

Taking the derivative of both sides of Equation (13) for time t, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)-\frac{\partial L}{\partial q_s}(t)-\frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1)=Q_s''(t), t\in[t_1,t_2-\tau_1],\\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)-\frac{\partial L}{\partial q_s}(t)=Q_s''(t), t\in(t_2-\tau_1,t_2-\tau_2],\\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}(t)\right)-\frac{\partial L}{\partial q_s}(t)=Q_s''(t), t\in(t_2-\tau_2,t_2],(s=1,2,\cdots,n).
\end{aligned}
\tag{14}
$$

Equation (14) presents the dynamical equations for the multi-time-delay NCMS. If $Q_s''(t)=0$, then Equation (14) becomes

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)-\frac{\partial L}{\partial q_s}(t)-\frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1)=0, t\in[t_1,t_2-\tau_1],\\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)-\frac{\partial L}{\partial q_s}(t)=0, t\in(t_2-\tau_1,t_2-\tau_2],\\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}(t)\right)-\frac{\partial L}{\partial q_s}(t)=0, t\in(t_2-\tau_2,t_2],(s=1,2,\cdots,n).
\end{aligned}
\tag{15}
$$

Equation (15) presents the dynamical equations for multi-time-delay Lagrangian systems.

Remark 1. *If $\tau_1=\tau_2\neq 0$, Equation (14) reduces to the dynamical equations for the NCMS with the same time-delay [31] between the generalized coordinate and the generalized velocity.*

Remark 2. *If $\tau_1=\tau_2=0$, then Equation (15) reduces to classical Lagrange equations.*

3. Variations in Multi-Time-Delay Hamilton Action

The multi-time-delay Hamilton action is given by

$$S(\gamma)=\int_{t_1}^{t_2}L\left(t,q_s,q_{s\tau_1},\dot{q}_s,\dot{q}_{s\tau_2}\right)\mathrm{d}t. \tag{16}$$

We assume that Equation (16) undergoes the following infinitesimal transformation

$$t^*=t+\Delta t, q_s^*(t^*)=q_s(t)+\Delta q_s \tag{17}$$

and the corresponding expansion with infinitesimal parameters $\varepsilon_\sigma(\sigma=1,2,\cdots,r)$ and infinitesimal generators ξ_0^σ, ξ_s^σ

$$t^*=t+\varepsilon_\sigma\xi_0^\sigma\left(t,q_s,\dot{q}_s\right), q_s^*(t^*)=q_s(t)+\varepsilon_\sigma\xi_s^\sigma\left(t,q_s,\dot{q}_s\right) \tag{18}$$

and becomes

$$S(\gamma^*)=\int_{t_1^*}^{t_2^*}L^*\left(t^*,q_s^*(t^*),q_s^*(t^*-\tau_1),\dot{q}_s^*(t^*),\dot{q}_s^*(t^*-\tau_2)\right)\mathrm{d}t^*. \tag{19}$$

Considering the main linear part of the difference $S(\gamma^*)-S(\gamma)$ with respect to ε, we have

$$
\begin{aligned}
\Delta S=\textstyle\int_{t_1}^{t_2}&\left(\frac{\partial L}{\partial t}(t)\Delta t+\frac{\partial L}{\partial q_s}(t)\Delta q_s+\frac{\partial L}{\partial q_{s\tau_1}}(t)\Delta q_{s\tau_1}\right.\\
&\left.+\frac{\partial L}{\partial \dot{q}_s}(t)\Delta\dot{q}_s+\frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t)\Delta\dot{q}_{s\tau_2}+L(t)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)\right)\mathrm{d}t.
\end{aligned}
\tag{20}
$$

Taking note of the following relations,

$$\Delta\dot{q}_s=\frac{\mathrm{d}}{\mathrm{d}t}\Delta q_s-\dot{q}_s\frac{\mathrm{d}}{\mathrm{d}t}\Delta t, \delta q_s=\Delta q_s-\dot{q}_s\Delta t, \tag{21}$$

Equation (20) becomes

$$\begin{aligned}\Delta S = \int_{t_1}^{t_2} \bigg(& \frac{\mathrm{d}}{\mathrm{d}t}(L(t)\Delta t) + \frac{\partial L}{\partial q_s}(t)\delta q_s + \frac{\partial L}{\partial q_{s\tau_1}}(t)\delta q_{s\tau_1} \\ & + \frac{\partial L}{\partial \dot{q}_s}(t)\frac{\mathrm{d}}{\mathrm{d}t}(\delta q_s) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t)\frac{\mathrm{d}}{\mathrm{d}t}(\delta q_{s\tau_2}) \bigg)\mathrm{d}t.\end{aligned} \tag{22}$$

Using the linear substitution of variables for the time-delay terms in Equations (20) and (22), Equations (20) and (22) can be rewritten as

$$\begin{aligned}\Delta S = & \int_{t_1}^{t_2-\tau_1} \left(\frac{\partial L}{\partial t}(t)\Delta t + \left(\frac{\partial L}{\partial q_s}(t) + \frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1) \right)\Delta q_s \right. \\ & \left. + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2) \right)\Delta \dot{q}_s + L(t)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) \right)\mathrm{d}t \\ & + \int_{t_2-\tau_1}^{t_2-\tau_2} \left(\frac{\partial L}{\partial t}(t)\Delta t + \frac{\partial L}{\partial q_s}(t)\Delta q_s + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2) \right)\Delta \dot{q}_s + L(t)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) \right)\mathrm{d}t \\ & + \int_{t_2-\tau_2}^{t_2} \left(\frac{\partial L}{\partial t}(t)\Delta t + \frac{\partial L}{\partial q_s}(t)\Delta q_s + \frac{\partial L}{\partial \dot{q}_s}(t)\Delta \dot{q}_s + L(t)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) \right)\mathrm{d}t \\ & + \int_{t_1-\tau_1}^{t_1-\tau_2} \left(\frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1)\dot{q}_s\Delta t \right)\mathrm{d}t + \int_{t_1-\tau_2}^{t_1} \left(\frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1)\dot{q}_s\Delta t + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\ddot{q}_s\Delta t \right)\mathrm{d}t\end{aligned} \tag{23}$$

and

$$\begin{aligned}\Delta S = & \int_{t_1}^{t_2-\tau_1} \varepsilon_\sigma \left\{ \frac{\mathrm{d}}{\mathrm{d}t}\left(L(t)\xi_0^\sigma + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2) \right)\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right) \right) \right. \\ & \left. + \left[\frac{\partial L}{\partial q_s}(t) + \frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2) \right) \right]\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right) \right\}\mathrm{d}t \\ & + \int_{t_2-\tau_1}^{t_2-\tau_2} \varepsilon_\sigma \left\{ \frac{\mathrm{d}}{\mathrm{d}t}\left(L(t)\xi_0^\sigma + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2) \right)\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right) \right) \right. \\ & \left. + \left[\frac{\partial L}{\partial q_s}(t) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2) \right) \right]\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right) \right\}\mathrm{d}t \\ & + \int_{t_2-\tau_2}^{t_2} \varepsilon_\sigma \left\{ \frac{\mathrm{d}}{\mathrm{d}t}\left(L(t)\xi_0^\sigma + \frac{\partial L}{\partial \dot{q}_s}(t)\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right) \right) + \left[\frac{\partial L}{\partial q_s}(t) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}(t) \right) \right]\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right) \right\}\mathrm{d}t.\end{aligned} \tag{24}$$

Thus, two basic formulas, Equations (23) and (24), for the variation in the multi-time-delay Hamilton Equation (16) are obtained.

4. Multi-Time-Delay Noether Symmetry

Based on classical Noether-type symmetries [21], three kinds of Noether-type symmetries with multi-time-delay are introduced.

First, we introduce the multi-time-delay Noether symmetry.

Definition 1. *Undergoing the transformations in (17), if the multi-time-delay Hamilton action (16) is invariant, namely,*

$$\Delta S = 0 \tag{25}$$

the following formulas hold:

$$
\begin{aligned}
&\frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1)\dot{q}_s\Delta t = 0, t \in [t_1-\tau_1, t_1-\tau_2),\\
&\frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1)\dot{q}_s\Delta t + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\ddot{q}_s\Delta t = 0, t \in [t_1-\tau_2, t_1),\\
&\frac{\partial L}{\partial t}(t)\Delta t + \left(\frac{\partial L}{\partial q_s}(t) + \frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1)\right)\Delta q_s + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)\Delta\dot{q}_s\\
&+L(t)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) = 0, t \in [t_1, t_2-\tau_1],\\
&\frac{\partial L}{\partial t}(t)\Delta t + \frac{\partial L}{\partial q_s}(t)\Delta q_s + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)\Delta\dot{q}_s + L(t)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) = 0, t \in (t_2-\tau_1, t_2-\tau_2],\\
&\frac{\partial L}{\partial t}(t)\Delta t + \frac{\partial L}{\partial q_s}(t)\Delta q_s + \frac{\partial L}{\partial \dot{q}_s}(t)\Delta\dot{q}_s + L(t)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) = 0, t \in (t_2-\tau_2, t_2].
\end{aligned}
\tag{26}
$$

This is called the multi-time-delay Noether symmetry.

Second, we introduce the multi-time-delay Noether quasi-symmetry.

Definition 2. *Supposing that L_1 is another multi-time-delay Lagrangian, the transformations in (17), which are accurate to an infinitesimal of the first order, satisfy the relation*

$$
\begin{aligned}
&\int_{t_1}^{t_2} L\left(t, q_s, q_{s\tau_1}, \dot{q}_s, \dot{q}_{s\tau_2}\right)\mathrm{d}t\\
&= \int_{t_1^*}^{t_2^*} L_1\left(t^*, q_s^*(t^*), q_s^*(t^*-\tau_1), \dot{q}_s^*(t^*), \dot{q}_s^*(t^*-\tau_2)\right)\mathrm{d}t^*
\end{aligned}
\tag{27}
$$

namely,

$$
\Delta S = -\int_{t_1}^{t_2}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G)\right)\mathrm{d}t \tag{28}
$$

where $G = G\left(t, q_s, q_{s\tau_1}, \dot{q}_s, \dot{q}_{s\tau_2}\right)$, and the following formulas hold:

$$
\begin{aligned}
&\frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1)\dot{q}_s\Delta t = 0, t \in [t_1-\tau_1, t_1-\tau_2),\\
&\frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1)\dot{q}_s\Delta t + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\ddot{q}_s\Delta t = 0, t \in [t_1-\tau_2, t_1),\\
&\frac{\partial L}{\partial t}(t)\Delta t + \left(\frac{\partial L}{\partial q_s}(t) + \frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1)\right)\Delta q_s + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)\Delta\dot{q}_s\\
&+L(t)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) = -\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G), t \in [t_1, t_2-\tau_1],\\
&\frac{\partial L}{\partial t}(t)\Delta t + \frac{\partial L}{\partial q_s}(t)\Delta q_s + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)\Delta\dot{q}_s\\
&+L(t)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) = -\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G), t \in (t_2-\tau_1, t_2-\tau_2],\\
&\frac{\partial L}{\partial t}(t)\Delta t + \frac{\partial L}{\partial q_s}(t)\Delta q_s + \frac{\partial L}{\partial \dot{q}_s}(t)\Delta\dot{q}_s + L(t)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) = -\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G), t \in (t_2-\tau_2, t_2].
\end{aligned}
\tag{29}
$$

This is called the multi-time-delay Noether quasi-symmetry.

The Lagrangian L_1 thus determined has the same dynamical equations as the Lagrangian L.

Third, we introduce the generalized multi-time-delay Noether quasi-symmetry.

Definition 3. *If a multi-time-delay mechanical system is under the action of non-potential force Q_s'' and the transformations in (17), which are accurate to an infinitesimal of first order, satisfy the relation*

$$\begin{aligned}&\int_{t_1}^{t_2} L\left(t, q_s, q_{s\tau_1}, \dot{q}_s, \dot{q}_{s\tau_2}\right)\mathrm{d}t \\ &= \int_{t_1^*}^{t_2^*} L_1\left(t^*, q_s^*(t^*), q_s^*(t^*-\tau_1), \dot{q}_s^*(t^*), \dot{q}_s^*(t^*-\tau_2)\right)\mathrm{d}t^* + \int_{t_1}^{t_2} Q_s''(t)\delta q_s \mathrm{d}t,\end{aligned} \tag{30}$$

namely,

$$\Delta S = -\int_{t_1}^{t_2}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G) + Q_s''\delta q_s\right)\mathrm{d}t \tag{31}$$

the following formulas hold:

$$\begin{aligned}
&\frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1)\dot{q}_s\Delta t = 0, t\in[t_1-\tau_1, t_1-\tau_2),\\
&\frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1)\dot{q}_s\Delta t + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\ddot{q}_s\Delta t = 0, t\in[t_1-\tau_2, t_1),\\
&\frac{\partial L}{\partial t}(t)\Delta t + \left(\frac{\partial L}{\partial q_s}(t) + \frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1)\right)\Delta q_s + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)\Delta\dot{q}_s\\
&+L(t)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) + Q_s''(t)\left(\Delta q_s - \dot{q}_s\Delta t\right) = -\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G), t\in[t_1, t_2-\tau_1],\\
&\frac{\partial L}{\partial t}(t)\Delta t + \frac{\partial L}{\partial q_s}(t)\Delta q_s + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)\Delta\dot{q}_s + L(t)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)\\
&+Q_s''(t)\left(\Delta q_s - \dot{q}_s\Delta t\right) = -\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G), t\in(t_2-\tau_1, t_2-\tau_2],\\
&\frac{\partial L}{\partial t}(t)\Delta t + \frac{\partial L}{\partial q_s}(t)\Delta q_s + \frac{\partial L}{\partial \dot{q}_s}(t)\Delta\dot{q}_s + L(t)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) + Q_s''(t)\left(\Delta q_s - \dot{q}_s\Delta t\right) = -\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G), t\in(t_2-\tau_2, t_2].
\end{aligned} \tag{32}$$

This is called the generalized multi-time-delay Noether quasi-symmetry.

Remark 3. *Since $\Delta t = \varepsilon_\sigma \xi_0^\sigma$ and $\Delta q_s = \varepsilon_\sigma \xi_s^\sigma$, Equations (29) and (32) can be expressed as*

$$\begin{aligned}
&\frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1)\dot{q}_s\xi_0^\sigma = 0, t\in[t_1-\tau_1, t_1-\tau_2),\\
&\frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1)\dot{q}_s\xi_0^\sigma + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\ddot{q}_s\xi_0^\sigma = 0, t\in[t_1-\tau_2, t_1),\\
&\frac{\partial L}{\partial t}(t)\xi_0^\sigma + \left(\frac{\partial L}{\partial q_s}(t) + \frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1)\right)\xi_s^\sigma + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)\left(\dot{\xi}_s^\sigma - \dot{q}_s(t)\dot{\xi}_0^\sigma\right)\\
&+L(t)\dot{\xi}_0^\sigma = -\dot{G}^\sigma, t\in[t_1, t_2-\tau_1],\\
&\frac{\partial L}{\partial t}(t)\xi_0^\sigma + \frac{\partial L}{\partial q_s}(t)\xi_s^\sigma + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)\left(\dot{\xi}_s^\sigma - \dot{q}_s(t)\dot{\xi}_0^\sigma\right)\\
&+L(t)\dot{\xi}_0^\sigma = -\dot{G}^\sigma, t\in(t_2-\tau_1, t_2-\tau_2],\\
&\frac{\partial L}{\partial t}(t)\xi_0^\sigma + \frac{\partial L}{\partial q_s}(t)\xi_s^\sigma + \frac{\partial L}{\partial \dot{q}_s}(t)\left(\dot{\xi}_s^\sigma - \dot{q}_s(t)\dot{\xi}_0^\sigma\right) + L(t)\dot{\xi}_0^\sigma = -\dot{G}^\sigma, t\in(t_2-\tau_2, t_2],
\end{aligned} \tag{33}$$

and

$$
\begin{aligned}
&\frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1)\dot{q}_s\xi_0^\sigma = 0, t\in[t_1-\tau_1, t_1-\tau_2),\\
&\frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1)\dot{q}_s\xi_0^\sigma + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\ddot{q}_s\xi_0^\sigma = 0, t\in[t_1-\tau_2, t_1),\\
&\frac{\partial L}{\partial t}(t)\xi_0^\sigma + \left(\frac{\partial L}{\partial q_s}(t) + \frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1)\right)\xi_s^\sigma + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)\left(\dot{\xi}_s^\sigma - \dot{q}_s(t)\dot{\xi}_0^\sigma\right)\\
&+L(t)\dot{\xi}_0^\sigma + Q_s''(t)\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right) = -\dot{G}^\sigma, t\in[t_1, t_2-\tau_1],\\
&\frac{\partial L}{\partial t}(t)\xi_0^\sigma + \frac{\partial L}{\partial q_s}(t)\xi_s^\sigma + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)\left(\dot{\xi}_s^\sigma - \dot{q}_s(t)\dot{\xi}_0^\sigma\right) + L(t)\dot{\xi}_0^\sigma\\
&+Q_s''(t)\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right) = -\dot{G}^\sigma, t\in(t_2-\tau_1, t_2-\tau_2],\\
&\frac{\partial L}{\partial t}(t)\xi_0^\sigma + \frac{\partial L}{\partial q_s}(t)\xi_s^\sigma + \frac{\partial L}{\partial \dot{q}_s}(t)\left(\dot{\xi}_s^\sigma - \dot{q}_s(t)\dot{\xi}_0^\sigma\right) + L(t)\dot{\xi}_0^\sigma\\
&+Q_s''(t)\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right) = -\dot{G}^\sigma, t\in(t_2-\tau_2, t_2], (\sigma = 1, 2, \cdots, r),
\end{aligned}
\tag{34}
$$

respectively.

Equations (26), (29), and (32) can each be used as the criterion equation for the three symmetries above, respectively. Indeed, Equations (33) and (34) are often referred to as the Noether identity when $r = 1$.

Remark 4. *In the intervals $[t_1-\tau_1, t_1-\tau_2)$ and $[t_1-\tau_2, t_1)$, when $\tau_1 = \tau_2$, the first two formulas of Equations (33) and (34) are not obtained in Ref. [31] because of the calculation problems with respect to the non-isochronous variation.*

5. Multi-Time-Delay Noether Theorem

The intrinsic connection between symmetry and conserved quantity can be revealed by the multi-time-delay Noether theorems below.

Theorem 1. *The multi-time-delay Lagrangian system (15) exists with the multi-time-delay conserved quantities if the transformations in (18) correspond to the multi-time-delay Noether symmetries, which are*

$$
\begin{aligned}
&I^\sigma = L(t)\xi_0^\sigma + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right) = c^\sigma, t\in[t_1, t_2-\tau_1],\\
&I^\sigma = L(t)\xi_0^\sigma + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right) = c^\sigma, t\in(t_2-\tau_1, t_2-\tau_2],\\
&I^\sigma = L(t)\xi_0^\sigma + \frac{\partial L}{\partial \dot{q}_s}(t)\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right) = c^\sigma, t\in(t_2-\tau_2, t_2].
\end{aligned}
\tag{35}
$$

Proof. Due to the Noether symmetric transformations, Equation (25) holds. Using Equation (24), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(L(t)\xi_0^\sigma + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right)\right)\\
&+\left[\frac{\partial L}{\partial q_s}(t) + \frac{\partial L}{\partial q_{s\tau_1}}(t+\tau_1) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)\right]\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right) = 0, t \in [t_1, t_2-\tau_1],\\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(L(t)\xi_0^\sigma + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right)\right)\\
&+\left[\frac{\partial L}{\partial q_s}(t) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)\right]\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right) = 0, t \in (t_2-\tau_1, t_2-\tau_2],\\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(L(t)\xi_0^\sigma + \frac{\partial L}{\partial \dot{q}_s}(t)\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right)\right) + \left[\frac{\partial L}{\partial q_s}(t) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}(t)\right)\right]\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right) = 0,\\
&t \in (t_2-\tau_2, t_2],
\end{aligned} \tag{36}
$$

noting that the integral interval $[t_1, t_2]$ is arbitrary, and the parameters ε_σ are independent of each other.

For the multi-time-delay Lagrangian system, we have Equation (15), and Equation (36) becomes

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(L(t)\xi_0^\sigma + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right)\right) = 0, t \in [t_1, t_2-\tau_1],\\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(L(t)\xi_0^\sigma + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right)\right) = 0, t \in (t_2-\tau_1, t_2-\tau_2],\\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(L(t)\xi_0^\sigma + \frac{\partial L}{\partial \dot{q}_s}(t)\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right)\right) = 0, t \in (t_2-\tau_2, t_2].
\end{aligned} \tag{37}
$$

Integrating Equation (37), the theorem is proven. □

Theorem 2. *The multi-time-delay Lagrangian system (15) exists with the multi-time-delay conserved quantities if the transformations in (18) correspond to the multi-time-delay Noether quasi-symmetries, which are*

$$
\begin{aligned}
&I^\sigma = L(t)\xi_0^\sigma + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right) + G^\sigma = c^\sigma, t \in [t_1, t_2-\tau_1],\\
&I^\sigma = L(t)\xi_0^\sigma + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau_2}}(t+\tau_2)\right)\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right) + G^\sigma = c^\sigma, t \in (t_2-\tau_1, t_2-\tau_2],\\
&I^\sigma = L(t)\xi_0^\sigma + \frac{\partial L}{\partial \dot{q}_s}(t)\left(\xi_s^\sigma - \dot{q}_s(t)\xi_0^\sigma\right) + G^\sigma = c^\sigma, t \in (t_2-\tau_2, t_2].
\end{aligned} \tag{38}
$$

Proof. Due to the Noether quasi-symmetric transformations, Equation (28) holds. By substituting Equation (24) into Equation (28) and using Equation (15), we obtain Theorem 2 in a similar manner to Theorem 1. □

Theorem 3. *The multi-time-delay NCMS (14) exists with the multi-time-delay conserved quantities in (38) if the transformations in (18) correspond to the generalized multi-time-delay Noether quasi-symmetry.*

Proof. Due to the generalized Noether quasi-symmetric transformations, Equation (31) holds. By substituting Equation (24) into Equation (31) and using Equation (14), we obtain Theorem 3 in a similar manner to Theorem 1. □

Remark 5. *Theorems 1–3 reveal that, although there are differences among criterion Equations (26), (29), and (32) of different symmetries, the conserved quantities have the same form of expression when $t \in [t_1, t_2 - \tau_1]$ and $t \in (t_2 - \tau_1, t_2 - \tau_2]$.*

Remark 6. *The variable that affects the form of expression of the conserved quantities (38) for a multi-time-delay NCMS is the delay τ_2 of generalized velocities, but this is not related to the delay τ_1 of generalized coordinates only if the generators ξ_0 and ξ_s are the same in three intervals.*

6. Examples

Example 1. *A multi-time-delay NCMS is considered. The multi-time-delay Lagrangian is*

$$L = \frac{1}{2} m \dot{q}^2(t) - kq(t - \tau_1) \tag{39}$$

and the multi-time-delay generalized non-potential force is

$$Q'' = -c\dot{q}(t - \tau_2), \tag{40}$$

where τ_1 and τ_2 are constant time-delays, and m, k, and c are constant physical quantities. The system satisfies boundary conditions (3)–(5). We will now study the conserved quantity by applying the generalized Noether quasi-symmetry.

From the dynamical Equation (14) of the multi-time-delay NCMS, we obtain

$$\begin{array}{l} m\ddot{q} + k + c\dot{q}_{\tau_2} = 0, t \in [t_1, t_2 - \tau_1], \\ m\ddot{q} + c\dot{q}_{\tau_2} = 0, t \in (t_2 - \tau_1, t_2 - \tau_2], \\ m\ddot{q} + c\dot{q}_{\tau_2} = 0, t \in (t_2 - \tau_2, t_2], \end{array} \tag{41}$$

and Equation (34) gives

$$\begin{array}{l} -k\xi_1 + m\left(\dot{\xi}_1 - \dot{q}\dot{\xi}_0\right)\dot{q} + \left(\frac{1}{2}m\dot{q}^2 - kq_{\tau_1}\right)\dot{\xi}_0 - c(\xi_1 - \dot{q}\xi_0)\dot{q}_{\tau_2} = -\dot{G}, t \in [t_1, t_2 - \tau_1], \\ m\left(\dot{\xi}_1 - \dot{q}\dot{\xi}_0\right)\dot{q} + \left(\frac{1}{2}m\dot{q}^2 - kq_{\tau_1}\right)\dot{\xi}_0 - c(\xi_1 - \dot{q}\xi_0)\dot{q}_{\tau_2} = -\dot{G}, t \in (t_2 - \tau_1, t_2 - \tau_2], \\ m\left(\dot{\xi}_1 - \dot{q}\dot{\xi}_0\right)\dot{q} + \left(\frac{1}{2}m\dot{q}^2 - kq_{\tau_1}\right)\dot{\xi}_0 - c(\xi_1 - \dot{q}\xi_0)\dot{q}_{\tau_2} = -\dot{G}, t \in (t_2 - \tau_2, t_2], \end{array} \tag{42}$$

where $\dot{q}(t - \tau_1) \triangleq \dot{q}_{\tau_1}$, and $\dot{q}(t - \tau_2) \triangleq \dot{q}_{\tau_2}$.

Equation (42) presents the following solutions:

$$\begin{array}{l} \xi_0 = 0, \xi_1 = 1, G = kt + cq_{\tau_2}, t \in [t_1, t_2 - \tau_1], \\ \xi_0 = 0, \xi_1 = 1, G = cq_{\tau_2}, t \in (t_2 - \tau_1, t_2 - \tau_2], \\ \xi_0 = 0, \xi_1 = 1, G = cq_{\tau_2}, t \in (t_2 - \tau_2, t_2]. \end{array} \tag{43}$$

Evidently, $\xi_0 = 0$ satisfies Equation (34) in the intervals $[t_1 - \tau_1, t_1 - \tau_2)$ and $[t_1 - \tau_2, t_1)$. The results generated from (43) correspond to generalized Noether quasi-symmetry. Applying Theorem 3, we obtain

$$\begin{array}{l} I = m\dot{q} + cq_{\tau_2} + kt = \text{const.}, t \in [t_1, t_2 - \tau_1], \\ I = m\dot{q} + cq_{\tau_2} = \text{const.}, t \in (t_2 - \tau_1, t_2 - \tau_2], \\ I = m\dot{q} + cq_{\tau_2} = \text{const.}, t \in (t_2 - \tau_2, t_2]. \end{array} \tag{44}$$

Therefore, Equation (44) presents Noether conserved quantities for the multi-time-delay NCMS (41).

Example 2. *The Lagrangian L of a two-degrees-of-freedom multi-time-delay oscillator system is*

$$L = \frac{m}{2}\sum_{s=1}^{2}\left[\dot{q}_s(t) + \dot{q}_s(t-\tau_2)\right]^2 - \frac{k}{2}\sum_{s=1}^{2}[q_s(t) + q_s(t-\tau_1)]^2 \tag{45}$$

where τ_1 and τ_2 are constant time-delays, and m and k are constant physical quantities.

The dynamical equations of the two-degrees-of-freedom multi-time-delay system are

$$m\left[\ddot{q}_s(t-\tau_2) + 2\ddot{q}_s(t) + \ddot{q}_s(t+\tau_2)\right] + k[q_s(t-\tau_1) + 2q_s(t) + q_s(t+\tau_1)] = 0, (s = 1, 2) \tag{46}$$

for $t \in [t_1, t_2 - \tau_1]$, and

$$m\left[\ddot{q}_s(t-\tau_2) + 2\ddot{q}_s(t) + \ddot{q}_s(t+\tau_2)\right] + k[q_s(t-\tau_1) + q_s(t)] = 0, (s = 1, 2) \tag{47}$$

for $t \in (t_2 - \tau_1, t_2 - \tau_2]$, and

$$m\left[\ddot{q}_s(t-\tau_2) + \ddot{q}_s(t)\right] + k[q_s(t-\tau_1) + q_s(t)] = 0, (s = 1, 2) \tag{48}$$

for $t \in (t_2 - \tau_2, t_2]$.

When $t \in [t_1, t_2 - \tau_1]$, Equation (33) gives

$$\begin{aligned} &m\sum_{s=1}^{2}\left[\dot{q}_s(t-\tau_2) + 2\dot{q}_s(t) + \dot{q}_s(t+\tau_2)\right]\left(\dot{\xi}_s - \dot{q}_s(t)\dot{\xi}_0\right) \\ &-k\sum_{s=1}^{2}[q_s(t-\tau_1) + 2q_s(t) + q_s(t+\tau_1)]\xi_s + L\dot{\xi}_0 = -\dot{G}. \end{aligned} \tag{49}$$

Equation (49) has the following solution:

$$\begin{aligned} &\xi_0 = 0, \xi_1 = \dot{q}_1(t-\tau_2) + 2\dot{q}_1(t) + \dot{q}_1(t+\tau_2), \xi_2 = \dot{q}_2(t-\tau_2) + 2\dot{q}_2(t) + \dot{q}_2(t+\tau_2), \\ &G = -\frac{m}{2}\sum_{s=1}^{2}\left[\dot{q}_s(t-\tau_2) + 2\dot{q}_s(t) + \dot{q}_s(t+\tau_2)\right]^2 \\ &+k\sum_{s=1}^{2}\int_{t_1}^{t_2-\tau_1}\left(\dot{q}_s(t-\tau_2) + 2\dot{q}_s(t) + \dot{q}_s(t+\tau_2)\right)(q_s(t-\tau_1) + 2q_s(t) + q_s(t+\tau_1))\mathrm{d}t. \end{aligned} \tag{50}$$

From Theorem 2, we obtain

$$\begin{aligned} &I = \frac{m}{2}\sum_{s=1}^{2}\left[\dot{q}_s(t-\tau_2) + 2\dot{q}_s(t) + \dot{q}_s(t+\tau_2)\right]^2 \\ &+k\sum_{s=1}^{2}\int_{t_1}^{t_2-\tau_1}\left(\dot{q}_s(t-\tau_2) + 2\dot{q}_s(t) + \dot{q}_s(t+\tau_2)\right)(q_s(t-\tau_1) + 2q_s(t) + q_s(t+\tau_1))\mathrm{d}t = \text{const.}. \end{aligned} \tag{51}$$

When $t \in (t_2 - \tau_1, t_2 - \tau_2]$, we have

$$m\sum_{s=1}^{2}\left[\dot{q}_s(t-\tau_2) + 2\dot{q}_s(t) + \dot{q}_s(t+\tau_2)\right]\left(\dot{\xi}_s - \dot{q}_s(t)\dot{\xi}_0\right) - k\sum_{s=1}^{2}[q_s(t-\tau_1) + q_s(t)]\xi_s + L\dot{\xi}_0 = -\dot{G}. \tag{52}$$

Equation (52) has the following solution:

$$\begin{aligned} &\xi_0 = 0, \xi_1 = \dot{q}_1(t-\tau_2) + 2\dot{q}_1(t) + \dot{q}_1(t+\tau_2), \xi_2 = \dot{q}_2(t-\tau_2) + 2\dot{q}_2(t) + \dot{q}_2(t+\tau_2), \\ &G = -\frac{m}{2}\sum_{s=1}^{2}\left[\dot{q}_s(t-\tau_2) + 2\dot{q}_s(t) + \dot{q}_s(t+\tau_2)\right]^2 \\ &+k\sum_{s=1}^{2}\int_{t_2-\tau_1}^{t_2-\tau_2}\left(\dot{q}_s(t-\tau_2) + 2\dot{q}_s(t) + \dot{q}_s(t+\tau_2)\right)(q_s(t-\tau_1) + 2q_s(t) + q_s(t+\tau_1))\mathrm{d}t. \end{aligned} \tag{53}$$

From Theorem 2, we have

$$I = \frac{m}{2}\sum_{s=1}^{2}\left[\dot{q}_s(t-\tau_2) + 2\dot{q}_s(t) + \dot{q}_s(t+\tau_2)\right]^2 \\ +k\sum_{s=1}^{2}\int_{t_2-\tau_1}^{t_2-\tau_2}\left(\dot{q}_s(t-\tau_2) + 2\dot{q}_s(t) + \dot{q}_s(t+\tau_2)\right)(q_s(t-\tau_1) + 2q_s(t) + q_s(t+\tau_1))\mathrm{d}t = \text{const.} \tag{54}$$

When $t \in (t_2 - \tau_2, t_2]$, we have

$$m\sum_{s=1}^{2}\left[\dot{q}_s(t-\tau_2) + \dot{q}_s(t)\right]\left(\dot{\xi}_s - \dot{q}_s(t)\dot{\xi}_0\right) - k\sum_{s=1}^{2}[q_s(t-\tau_1) + q_s(t)]\xi_s + L\dot{\xi}_0 = -\dot{G}. \tag{55}$$

Equation (55) has the following solution:

$$\xi_0 = 0, \xi_1 = \dot{q}_1(t-\tau_2) + \dot{q}_1(t), \xi_2 = \dot{q}_2(t-\tau_2) + \dot{q}_2(t), \\ G = -\frac{m}{2}\sum_{s=1}^{2}\left[\dot{q}_s(t-\tau_2) + \dot{q}_s(t)\right]^2 + k\sum_{s=1}^{2}\int_{t_2-\tau_2}^{t_2}\left(\dot{q}_s(t-\tau_2) + \dot{q}_s(t)\right)(q_s(t-\tau_1) + q_s(t))\mathrm{d}t. \tag{56}$$

From Theorem 2, we have

$$I = \frac{m}{2}\sum_{s=1}^{2}\left[\dot{q}_s(t-\tau_2) + \dot{q}_s(t)\right]^2 + k\sum_{s=1}^{2}\int_{t_2-\tau_2}^{t_2}\left(\dot{q}_s(t-\tau_2) + \dot{q}_s(t)\right)(q_s(t-\tau_1) + q_s(t))\mathrm{d}t = \text{const.} \tag{57}$$

Evidently, $\xi_0 = 0$ satisfies Equation (34) in the intervals $[t_1 - \tau_1, t_1 - \tau_2)$ and $[t_1 - \tau_2, t_1)$. Equations (51), (54), and (57) are Noether conserved quantities of the system (46)–(48).

Compared to the classical case, there is an obvious conserved quantity in the oscillator system without time-delay parameters, namely,

$$I = \frac{1}{2}m\left[\dot{q}_1^2(t) + \dot{q}_2^2(t)\right] + \frac{1}{2}k\left[q_1^2(t) + q_2^2(t)\right]. \tag{58}$$

7. Conclusions

In this study, the Noether symmetry of a multi-time-delay NCMS and its conserved quantity were investigated. The variational principle (1) and multi-time-delay Equation (14) of the system were obtained. Dynamical Equation (15) of the multi-time-delay Lagrangian systems is a special case of Equation (14). Three kinds of Noether symmetric transformations and corresponding criterion Equations (26), (29), and (32) were established. Thus, the Noether theorems of the multi-time-delay Lagrangian system and the multi-time-delay NCMS were established. The results show that the delay τ_2 of generalized velocities affects the form of the conserved quantities in (38) for a multi-time-delay NCMS, but this is not related to the delay τ_1 of generalized coordinates only if the generators ξ_0 and ξ_s are the same in three intervals.

Compared with some previous studies on time-delay mechanical systems [31–40], this paper not only takes into account the more realistic description of different time-delays between the generalized coordinates and the generalized velocities, but also achieves the more general Noether-type conserved quantities.

The time-delays discussed in this paper are only constants. As a more general case, time-varying time-delay can be further discussed and the two time-delays described in this paper can be expanded to n time-delays, which will make the research on practical problems related to objective mechanics more accurate. Further studies could include Hamiltonian systems, Birkhoffian systems, nonholonomic systems, and the corresponding complex dynamical models with a consideration of multi-time-delay. Since the solution space for a time-delay equation is infinite in its dimensions, it is worth looking forward to the effective numerical methods and structure-preserving numerical methods for studying multi-time-delay equations.

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References

1. Hu, H.Y.; Wang, Z.H. Review on nonlinear dynamic systems involving time delays. *Adv. Mech.* **1999**, *29*, 501–512.
2. Richard, J. Time-delay systems: An overview of some recent advances and open problems. *Automatica* **2003**, *39*, 1667–1694. [CrossRef]
3. Xu, J.; Pei, L.J. Advances in dynamics for delayed systems. *Adv. Mech.* **2006**, *36*, 17–30.
4. Hu, H.Y.; Wang, Z.H. Singular perturbation methods for nonlinear dynamic systems with time delays. *Chaos Solitons Fractals* **2009**, *40*, 13–27. [CrossRef]
5. Naumkin, I.; Weder, R. Time delay for the Dirac equation. *Lett. Math. Phys.* **2016**, *106*, 1345–1378. [CrossRef]
6. Zhai, X.H.; Zhang, Y. Noether theorem for non-conservative systems with time delay on time scales. *Commun. Nonlinear Sci. Numer. Simul.* **2017**, *52*, 32–43. [CrossRef]
7. Sweilam, N.; AL-Mekhlafi, S.; Shatta, S.; Baleanu, D. Numerical study for a novel variable-order multiple time delay awareness programs mathematical model. *Appl. Numer. Math.* **2020**, *158*, 212–235. [CrossRef]
8. Rihan, F.A.; Velmurugan, G. Dynamics of fractional-order delay differential model for tumor-immune system. *Chaos Solitons Fractals* **2020**, *132*, 109592. [CrossRef]
9. Allen, B.C.; Stubbs, K.J.; Dixon, W.E. Robust cadence tracking for switched FES-cycling using a time-varying estimate of the electromechanical delay. *Automatica* **2022**, *144*, 110466. [CrossRef]
10. Budzinski, R.C.; Nguyen, T.T.; Benigno, G.B.; Đoàn, J.; Mináč, J.; Sejnowski, T.J.; Muller, L.E. Analytical prediction of specific spatiotemporal patterns in nonlinear oscillator networks with distance-dependent time delays. *Phys. Rev. Res.* **2023**, *5*, 013159. [CrossRef]
11. Driver, R.D. *Ordinary and Delay Differential Equations*; Springer: New York, NY, USA, 1977.
12. El'sgol'c, L.E. *Qualitative Methods in Mathematical Analysis*; American Mathematical Society: Providence, RI, USA, 1964.
13. Hughes, D.K. Variational and optimal control problems with delayed argument. *J. Optim. Theory Appl.* **1968**, *2*, 1–14. [CrossRef]
14. Palm, W.J.; Schmitendorf, W.E. Conjugate-point conditions for variational problems with delay argument. *J. Optim. Theory Appl.* **1974**, *14*, 599–612. [CrossRef]
15. Rosenblueth, J.F. Systems with time delay in the calculus of variations: A variational approach. *IMA J. Math. Control Inform.* **1988**, *5*, 125–145. [CrossRef]
16. Chan, W.L.; Yung, S.P. Sufficient conditions for variational problems with delayed argument. *J. Optim. Theory Appl.* **1993**, *76*, 131–144. [CrossRef]
17. Lee, C.H.; Yung, S.P. Sufficient conditions for optimal control problems with time delay. *J. Optim. Theory Appl.* **1996**, *88*, 157–176. [CrossRef]
18. Noether, A.E. Invariante variationsprobleme. *Nachr. Akad. Wiss. Gott. Math. Phys.* **1918**, *2*, 235–237.
19. Lutzky, M. Dynamical symmetries and conserved quantities. *J. Phys. A Math. Gen.* **1979**, *12*, 973–981. [CrossRef]
20. Mei, F.X. *Symmetries and Conserved Quantities of Constrained Mechanical Systems*; Beijing Institute of Technology Press: Beijing, China, 2004.
21. Mei, F.X. *Applications of Lie Groups and Lie Algebras to Constrained Mechanical Systems*; Science Press: Beijing, China, 1999.
22. Torres, D.F.M. Conservation laws in optimal control. In *Dynamics, Bifurcations, and Control. Lecture Notes in Control and Information Sciences*, 2002nd ed.; Colonius, F., Grüne, L., Eds.; Springer: Berlin/Heidelberg, Germany, 2002.
23. Dong, X.C.; Zhang, Y. Herglotz type conservation laws for nonconservative nonholonomic systems. *AIP Adv.* **2023**, *13*, 105014. [CrossRef]
24. Liu, J.; Song, C.J. Symmetries and perturbations of time-scale nonshifted singular systems. *AIP Adv.* **2023**, *13*, 115317. [CrossRef]
25. Zhai, X.H.; Zhang, Y. Mei symmetry and new conserved quantities of time-scale Birkhoff's equations. *Complexity* **2020**, *2020*, 1691760. [CrossRef]
26. Shen, S.L.; Song, C.J. Disturbation to Lie symmetry for constrained Hamiltonian system within Agrawal's operators. *AIP Adv.* **2024**, *14*, 015118. [CrossRef]
27. Frederico, G.S.F.; Torres, D.F.M. Noether's symmetry theorem for variational and optimal control problems with time delay. *Numer. Algebra Control Optim.* **2012**, *2*, 619–630. [CrossRef]

28. Frederico, G.S.F.; Odzijewicz, T.; Torres, D.F.M. Noether's theorem for non-smooth extremals of variational problems with time delay. *Appl. Anal.* **2014**, *93*, 153–170. [CrossRef]
29. Frederico, G.S.F.; Lazo, M.J.; Barreto, M.N.F.; Paiva, J. Constants of motion for isoperimetric variational problems with time delay. *Appl. Anal.* **2022**, *101*, 2707–2725. [CrossRef]
30. Frederico, G.S.F.; Lazo, M.J.; Barreto, M.N.F.; Sousa, J.V.C. Higher-order Noether's theorem for isoperimetric variational problems. *J. Optim. Theory Appl.* **2023**, *199*, 541–568. [CrossRef]
31. Zhang, Y.; Jin, S.X. Noether symmetries of dynamics for non-conservative systems with time delay. *Acta Phys. Sin.* **2013**, *62*, 234502. [CrossRef]
32. Jin, S.X.; Zhang, Y. Noether theorem for nonholonomic systems with time delay. *Math. Probl. Eng.* **2015**, *2015*, 539276. [CrossRef]
33. Jin, S.X.; Zhang, Y. Noether symmetry and conserved quantity for a Hamilton system with time delay. *Chin. Phys. B* **2014**, *23*, 054501. [CrossRef]
34. Zhai, X.H.; Zhang, Y. Noether symmetries and conserved quantities for Birkhoffian systems with time delay. *Nonlinear Dyn.* **2014**, *77*, 73–86. [CrossRef]
35. Santos, S.P.S.; Martins, N.; Torres, D.F.M. Variational problems of Herglotz type with time delay: DuBois-Reymond condition and Noether's first theorem. *Discrete Cont. Dyn. Syst.* **2015**, *35*, 4593–4610. [CrossRef]
36. Santos, S.P.S.; Martins, N.; Torres, D.F.M. Higher-order variational problems of Herglotz type with time delay. *Pure Appl. Funct. Anal.* **2016**, *1*, 291–307.
37. Zhang, Y. Herglotz's variational problem for non-conservative system with delayed arguments under Lagrangian framework and its Noether's theorem. *Symmetry* **2020**, *12*, 845. [CrossRef]
38. Jin, S.X.; Zhang, Y. Noether symmetries for non-conservative Lagrange systems with time delay based on fractional model. *Nonlinear Dyn.* **2015**, *79*, 1169–1183. [CrossRef]
39. Zhai, X.H.; Zhang, Y. Noether symmetries and conserved quantities for fractional Birkhoffian systems with time delay. *Commun. Nonlinear Sci. Numer. Simul.* **2016**, *36*, 81–97. [CrossRef]
40. Zhai, X.H.; Zhang, Y. Conservation laws for a delayed Hamiltonian system in a time scales version. *Symmetry* **2018**, *10*, 668. [CrossRef]
41. Chen, L.X.; Cai, G.P. Design method of multiple time-delay controller for active structural vibration control. *Appl. Math. Mech.* **2009**, *30*, 1405–1414. [CrossRef]
42. Chen, C.W. A criterion of robustness intelligent nonlinear control for multiple time-delay systems based on fuzzy Lyapunov methods. *Nonlinear Dyn.* **2014**, *76*, 23–31. [CrossRef]
43. Pakzad, M.A.; Nekoui, M.A. Stability map of multiple time delayed fractional order systems. *Int. J. Control Autom. Syst.* **2014**, *12*, 37–43. [CrossRef]
44. Ge, J. Multi-delay-induced bifurcation singularity in two-neuron neural models with multiple time delays. *Nonlinear Dyn.* **2022**, *108*, 4357–4371. [CrossRef]
45. Chidouh, A.; Atmania, R.; Torres, D.F.M. Study of a fractional creep problem with multiple delays in terms of Boltzmann's superposition principle. *Fractal Fract.* **2022**, *6*, 434. [CrossRef]
46. Benharrat, M.; Torres, D.F.M. Optimal control with time delays via the penalty method. *Math. Probl. Eng.* **2014**, *2014*, 250419. [CrossRef]